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INDUCING FUNCTIONS BETWEEN INVERSE LIMITS WITH UPPER SEMICONTINUOUS BONDING FUNCTIONS

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Inducing functions between inverse limits with upper semicontinuous bonding functions

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Abstract

In this paper we introduce the category $\mathcal{CU}$ in which the compact metric spaces are objects and upper semicontinuous functions from X to 2^Y are morphisms from X to Y. We also introduce the category $\mathcal{ICU}$ of inverse sequences in $\mathcal{CU}$. Then we investigate the induced functions between inverse limits of compact metric spaces with upper semicontinuous bonding functions. We provide criteria for their existence and prove that under suitable assumptions they have surjective graphs. We also show that taking such inverse limits is very close to being a functor (but is not a functor) from $\mathcal{ICU}$ to $\mathcal{CU}$, if morphisms are mapped to induced functions. At the end of the paper we give a useful application of the mentioned results.

Keywords: Inverse limits, Upper semi-continuous functions, Induced functions, Induced morphisms
2000 Mathematics Subject Classification: 54F15,54C60

1 Introduction

A function $f : X \to 2^Y$, where X and Y are compact metric spaces, is *upper semicontinuous* function from X to 2^Y (abbreviated *u.s.c.* function) if for each open set $V \subseteq Y$ the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X. We will interpret such a function $f : X \to 2^Y$ as a morphism $f : X \multimap Y$ and thus obtain a category, which we will denote by $\mathcal{CU}$. Then $\mathcal{ICU}$ is the standard category of inverse sequences in $\mathcal{CU}$.

Consider two inverse sequences $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ of compact metric spaces and morphisms of $\mathcal{CU}$. Let for each positive integer i, φ_i be a u.s.c. function from X_i to 2^{Y_i}. In this paper we study the question which u.s.c. functions from $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ to $2^{\varprojlim\{Y_n,g_n\}_{n=1}^{\infty}}$ can be interpreted as induced by φ_1, φ_2, φ_3, …; we also study the problem of existence of such induced functions as well as their properties. The special case when φ_1, φ_2, φ_3, … are single-valued continuous functions was studied in [8, 15]. In their paper [15], Ingram and Mahavier showed that if X and Y are inverse limits of inverse sequences with u.s.c. bonding functions and each $\varphi_i : X_i \to Y_i$ is a homeomorphism, then the function induced by the functions φ_i is a homeomorphism. In [8] it was shown under suitable assumptions, if each of the functions φ_i are surjective, one-to-one, or a homeomorphism respectively, then also the induced mapping is surjective, one-to-one, or a homeomorphism, respectively.

In the present paper we assume that each φ_i is a u.s.c. function from X_i to 2^{Y_i} and show that even in such more general situation the notion of an induced function

$$\Phi : \varprojlim\{X_n, f_n\}_{n=1}^{\infty} \to 2^{\varprojlim\{Y_n,g_n\}_{n=1}^{\infty}}$$

can be defined in such a way that it is a u.s.c. function (if certain mild conditions are satisfied). Further we show that if each of the φ_i's has a surjective graph, then under certain additional conditions also Φ has a surjective graph.

It is a well-known fact that if $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ are inverse sequences of compact metric spaces and continuous single-valued bonding functions, and if for each positive integer i, φ_i is a continuous single-valued function from X_i to Y_i, then the transformation

$$\{X_n, f_n\}_{n=1}^{\infty} \longmapsto \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$$

and

$$(\varphi_1, \varphi_2, \varphi_3, \ldots) \longmapsto \varprojlim \varphi_i$$

is a functor from the category of inverse sequences in the category of compact metric spaces with continuous single-valued functions (i.e. from the category in which inverse sequences of compact metric spaces with continuous single-valued bonding functions are objects, and sequences $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ of single-valued mappings having certain commutativity property are morphisms), to the category of compact metric spaces and continuous functions.

In the present paper we prove that in the case when the inverse sequences $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ are inverse sequences in $\mathcal{CU}$ (i.e. X_n, Y_n are compact metric spaces, and $f_n : X_{n+1} \multimap X_n$, $g_n : Y_{n+1} \multimap Y_n$ are morphisms in $\mathcal{CU}$, meaning that $f_n : X_{n+1} \to 2^{X_n}$, $g_n : Y_{n+1} \to 2^{Y_n}$ are u.s.c. functions) and each φ_i is a u.s.c. function from X_i to 2^{Y_i}, the transformation

$$\{X_n, f_n\}_{n=1}^{\infty} \longmapsto \varprojlim \{X_n, f_n\}_{n=1}^{\infty}$$

and

$$(\varphi_1, \varphi_2, \varphi_3, \ldots) \longmapsto \varprojlim \varphi_i$$

is not a functor from the category $\mathcal{ICU}$ of inverse sequences in $\mathcal{CU}$ to the category $\mathcal{CU}$, but is very close to being one.

In the last section we give a useful application of the mentioned results.

2 Definitions and notation

Our definitions and notation mostly follow Nadler [19], Ingram [14], and Ingram and Mahavier [15].

A *map* is a continuous function.

A *continuum* is a nonempty, compact and connected metric space.

If (X, d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X. Let for each $\varepsilon > 0$ and each $A \in 2^X$

$$N_d(\varepsilon, A) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

The set 2^X will be always equipped with the *Hausdorff metric* H_d, which is defined by

$$H_d(H, K) = \inf\{\varepsilon > 0 \mid H \subseteq N_d(\varepsilon, K), K \subseteq N_d(\varepsilon, H)\},$$

for $H, K \in 2^X$. Then $(2^X, H_d)$ is a metric space, called the *hyperspace* of the space (X, d). For more details see [13, 19].

For a function $f : X \to 2^Y$ and a subset $A \subseteq X$, we use $f[A] = \bigcup_{x \in A} f(x)$ to denote the image of A under f.

The *graph* $\Gamma(f)$ of a function $f : X \to 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$ (as defined in [14, p. 2]).

A function $f : X \to 2^Y$ *has a surjective graph* if for each $y \in Y$ there is an $x \in X$, such that $y \in f(x)$, i.e. if $f[X] = Y$ (as defined in [14, p. 2]).

A function $f : X \to 2^Y$, where X and Y are compact metric spaces, is *upper semicontinuous* function from X to 2^Y (abbreviated *u.s.c.*) if for each open set $V \subseteq Y$ the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X.

The following is a well-known characterization of u.s.c. functions (see [15, p. 120, Theorem 2.1]).

Theorem 2.1. *Let X and Y be compact metric spaces and $f : X \to 2^Y$ a function. Then f is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

A sequence $\{X_k, f_k\}_{k=1}^{\infty}$ is an *inverse sequence* of compact metric spaces and u.s.c. bonding functions if each X_k is a compact metric space and each f_k is a u.s.c. function $f_k : X_{k+1} \to 2^{X_k}$.

The *inverse limit* of an inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ of compact metric spaces and u.s.c. bonding functions is defined to be the subspace of the product space $\prod_{k=1}^{\infty} X_k$ of all $x = (x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X_k$, such that $x_k \in f_k(x_{k+1})$ for each k. The inverse limit of $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted by $\varprojlim \{X_k, f_k\}_{k=1}^{\infty}$.

In the present paper we will interpret inverse sequences $\{X_k, f_k\}_{k=1}^{\infty}$ of compact metric spaces and u.s.c. bonding functions as inverse sequences in $\mathcal{CU}$ and study $\varprojlim$ as a possible functor from $\mathcal{ICU}$ to $\mathcal{CU}$.

The notion of the inverse limit of an inverse sequence with u.s.c. bonding functions was introduced by Mahavier in [18] and Ingram and Mahavier in [15]. Since the introduction of such inverse limits, there has been much interest in the subject and many papers appeared [1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 17, 11, 18, 20, 21, 22, 23, 24], as well as the book [14].

On the product space $\prod\limits_{n=1}^{\infty} X_n$, where (X_n, d_n) is a compact metric space for each n, and the set of all diameters of (X_n, d_n) is majorized by 1, we use the metric

$$D(x, y) = \sup_{n \in \{1,2,3,\ldots\}} \left\{ \frac{d_n(x_n, y_n)}{2^n} \right\},$$

where $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$. It is well known that the metric D induces the product topology on $\prod\limits_{n=1}^{\infty} X_n$ [10, p. 190].

The set $\mathbb{N}$ denotes the set of all positive integers $\{1, 2, 3, \ldots\}$.

3 The categories $\mathcal{CU}$ and $\mathcal{ICU}$

In this section we give detailed descriptions of the following two categories:

1. $\mathcal{CU}$: the category of compact metric spaces and u.s.c. functions;

2. $\mathcal{ICU}$: the category of inverse sequences in $\mathcal{CU}$.

For any category $\mathcal{K}$, the class of objects of $\mathcal{K}$ is denoted by $Ob(\mathcal{K})$, and for any $X, Y \in Ob(\mathcal{K})$, the set of morphisms of $\mathcal{K}$ from X to Y is denoted by $Mor(\mathcal{K})(X, Y)$.

3.1 $\mathcal{CU}$

The category $\mathcal{CU}$ of compact metric spaces and u.s.c. functions consists of the following objects and morphisms:

1. $Ob(\mathcal{CU})$ is the class of compact metric spaces;

2. The set of morphisms $Mor(\mathcal{CU})(X, Y)$ from X to Y is the set of u.s.c. functions $X \to 2^Y$. The u.s.c. function $f : X \to 2^Y$ as a morphism of $\mathcal{CU}$ is denoted by $f : X \multimap Y$. This can be reformulated as

$$Mor(\mathcal{CU})(X, Y) = \{f : X \multimap Y \mid f : X \to 2^Y \text{ is u.}s.c.\}.$$

We also define the partial binary operation $\circ$ (the composition) on the class of morphisms of $\mathcal{CU}$ as follows. For each $f \in Mor(\mathcal{CU})(X, Y)$ and each $g \in Mor(\mathcal{CU})(Y, Z)$ we define $g \circ f \in Mor(\mathcal{CU})(X, Z)$ by

$$(g \circ f)(x) = g[f(x)] = \bigcup_{y \in f(x)} g(y)$$

for each $x \in X$. It is easy to see that for each u.s.c. function $f : X \to 2^Y$ and each u.s.c. function $g : Y \to 2^Z$, the composition $g \circ f$ defined above is again a u.s.c. function $X \to 2^Z$ [14, p. 4]. That means that if $f : X \multimap Y$ and $g : Y \multimap Z$, then $g \circ f : X \multimap Z$. Therefore $\circ$ is well-defined. It is also easy to see that $\circ$ is associative.

Theorem 3.1. *$\mathcal{CU}$ is a category.*

Proof. All that is left to show is that for each $X \in Ob(\mathcal{CU})$ there is a morphism $1_X : X \multimap X$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any morphisms $f : Y \multimap X$ and $g : X \multimap Z$. We easily see that the u.s.c. function $1_X : X \to 2^X$, defined by $1_X(x) = \{x\}$ for each $x \in X$ satisfies the above conditions. □

3.2 $\mathcal{ICU}$

The sequence $\{X_n, f_n\}_{n=1}^{\infty}$ is an inverse sequence in $\mathcal{CU}$ if each X_n is an object of $\mathcal{CU}$ and each f_n is a morphism from $Mor(\mathcal{CU})(X_{n+1}, X_n)$ (i.e. $f_n : X_{n+1} \multimap X_n$, meaning that f_n is a u.s.c. function $f_n : X_{n+1} \to 2^{X_n}$).

The category $\mathcal{ICU}$ of inverse sequences in $\mathcal{CU}$ consists of the following objects and morphisms:

1. $Ob(\mathcal{ICU})$ is the class of inverse sequences $\{X_n, f_n\}_{n=1}^{\infty}$ in $\mathcal{CU}$;

2. For any two objects $\underline{X} = \{X_n, f_n\}_{n=1}^{\infty}$ and $\underline{Y} = \{Y_n, g_n\}_{n=1}^{\infty}$ of $\mathcal{ICU}$, the set $Mor(\mathcal{ICU})(\underline{X}, \underline{Y})$ consists of all sequences $\varphi = (\varphi_1, \varphi_2, \varphi_3, \ldots)$, where each φ_i is a morphism in $\mathcal{CU}$ from X_i to Y_i (i.e. it is a u.s.c. function from X_i to 2^{Y_i}), such that $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ for each positive integer i.

We also define the partial binary operation $\Box$ (the composition) as follows. Let $\underline{X} = \{X_n, f_n\}_{n=1}^{\infty}$, $\underline{Y} = \{Y_n, g_n\}_{n=1}^{\infty}$, $\underline{Z} = \{Z_n, h_n\}_{n=1}^{\infty}$ be any objects of $\mathcal{ICU}$, and let $\varphi = (\varphi_1, \varphi_2, \varphi_3, \ldots) \in Mor(\mathcal{ICU})(\underline{X}, \underline{Y})$, and $\psi = (\psi_1, \psi_2, \psi_3, \ldots) \in Mor(\mathcal{ICU})(\underline{Y}, \underline{Z})$ be any morphisms of $\mathcal{ICU}$. Then we define $\psi \Box \varphi$ by

$$\psi \Box \varphi = (\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2, \psi_3 \circ \varphi_3, \ldots).$$

Theorem 3.2. *$\mathcal{ICU}$ is a category.*

Proof. Using the above notation we see that for each i, $\psi_i \circ \varphi_i : X_i \to 2^{Z_i}$ is a u.s.c. function, i.e. $\psi_i \circ \varphi_i : X_i \multimap Z_i$ is a morphism of $\mathcal{CU}$.

Next we see that for each positive integer i,

$$h_i \circ (\psi_{i+1} \circ \varphi_{i+1}) = (\psi_i \circ \varphi_i) \circ f_i,$$

since

$$h_i \circ (\psi_{i+1} \circ \varphi_{i+1}) = (h_i \circ \psi_{i+1}) \circ \varphi_{i+1} = (\psi_i \circ g_i) \circ \varphi_{i+1} =$$

$$\psi_i \circ (g_i \circ \varphi_{i+1}) = \psi_i \circ (\varphi_i \circ f_i) = (\psi_i \circ \varphi_i) \circ f_i.$$

That proves that $\psi \Box \varphi \in Mor(\mathcal{ICU})(\underline{X}, \underline{Z})$, as required. Since $\circ$ is associative, it easily follows that $\Box$ is also associative.

Also, from the properties of functions 1_{X_i} it easily follows that for any inverse sequence $\underline{X} = \{X_n, f_n\}_{n=1}^{\infty}$ the morphism $1_{\underline{X}} = (1_{X_1}, 1_{X_2}, 1_{X_3}, \ldots) : \underline{X} \to \underline{X}$ satisfies the conditions

$$1_{\underline{X}} \Box \varphi = \varphi \quad \text{and} \quad \psi \Box 1_{\underline{X}} = \psi,$$

for any morphisms $\varphi : \{Y_n, g_n\}_{n=1}^{\infty} \to \underline{X}$ and $\psi : \underline{X} \to \{Z_n, h_n\}_{n=1}^{\infty}$. $\Box$

4 Induced functions and induced morphisms

We begin this section with the definition of functions induced by sequences of u.s.c. functions.

Definition 4.1. *Let $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ be any inverse sequences of compact metric spaces and u.s.c. bonding functions (i.e. inverse sequences in $\mathcal{CU}$), and let for each n, $\varphi_n : X_n \to 2^{Y_n}$ be a u.s.c. function. Let for each $(x_1, x_2, x_3, \ldots) \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$,*

$$\Phi(x_1, x_2, x_3, \ldots) = (\varphi_1(x_1) \times \varphi_2(x_2) \times \varphi_3(x_3) \times \cdots) \cap \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}. \quad (1)$$

If Φ is a u.s.c. function from $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ to $2^{\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}}$, then we say that it is induced *by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$.*

Theorem 4.2 provides a simple criterion for recognizing induced functions.

Theorem 4.2. *We use the notation from Definition 4.1. Then Φ is induced by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ if and only if $\Phi(x_1, x_2, x_3, \ldots) \neq \emptyset$ for each $(x_1, x_2, x_3, \ldots) \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.*

Proof. Let us denote

$$H = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$$

and

$$K = \varprojlim\{Y_i, g_i\}_{i=1}^{\infty}.$$

If $\Phi : H \to 2^K$ is induced by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$, then for each $(x_1, x_2, x_3, \ldots) \in H$, $\Phi(x_1, x_2, x_3, \ldots) \in 2^K$. Therefore $\Phi(x_1, x_2, x_3, \ldots) \neq \emptyset$.

Now assume that $\Phi(x_1, x_2, x_3, \ldots) \neq \emptyset$ for each $(x_1, x_2, x_3, \ldots) \in H$. Since $\Phi(x_1, x_2, x_3, \ldots)$ is compact it is closed in K. Therefore the function $\Phi : H \to 2^K$ is well-defined.

Next we prove that Φ is a u.s.c. function. It is sufficient to prove that the graph $\Gamma(\Phi)$ of Φ is a closed subset of $H \times K$. It follows from the definition of Φ that

$$\Gamma(\Phi) = \{((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)) \in H \times K \mid \forall i \in \mathbb{N}, y_i \in \varphi_i(x_i)\}.$$

Let $\alpha : H \times K \to (X_1 \times Y_1) \times (X_2 \times Y_2) \times \cdots$ be defined by

$$\alpha(x, y) = ((x_1, y_1), (x_2, y_2), \ldots),$$

for all $x = (x_1, x_2, x_3, \ldots) \in H$ and $y = (y_1, y_2, y_3, \ldots) \in K$.

Note that $A : H \times K \to \operatorname{Im}\alpha$ defined by $A(x, y) = \alpha(x, y)$ is a homeomorphism.

We prove that $A(\Gamma(\Phi))$ is closed in $\operatorname{Im} A = A(H \times K)$.

$$\begin{aligned}
A(\Gamma(\Phi)) &= \{((x_1, y_1), (x_2, y_2), \ldots) \mid x \in H, y \in K, \forall i \in \mathbb{N}, y_i \in \varphi_i(x_i)\} \\
&= \{((x_1, y_1), (x_2, y_2), \ldots) \mid x \in H, y \in K, \forall i \in \mathbb{N}, (x_i, y_i) \in \Gamma(\varphi_i)\} \\
&= \{((x_1, y_1), (x_2, y_2), \ldots) \in \Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \cdots \mid x \in H, y \in K\} \\
&= (\Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \cdots) \cap \{((x_1, y_1), (x_2, y_2), \ldots) \mid x \in H, y \in K\} \\
&= (\Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \cdots) \cap A(H \times K).
\end{aligned}$$

The product

$$\Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \Gamma(\varphi_3) \times \cdots$$

is a closed subset of $(X_1 \times Y_1) \times (X_2 \times Y_2) \times \cdots$, therefore $A(\Gamma(\Phi))$ is closed in $A(H \times K)$. It follows that $\Gamma(\Phi)$ is closed in $H \times K$. $\square$

The next theorem presents a commutativity-like condition under which Φ is induced.

Theorem 4.3. *Let $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ be any inverse sequences of compact metric spaces and u.s.c. bonding functions (i.e. inverse sequences in $\mathcal{CU}$), and let for each n, $\varphi_n : X_n \to 2^{Y_n}$ be a u.s.c. function. If*

$$\varphi_n[f_n(x)] \subseteq g_n[\varphi_{n+1}(x)]$$

for each positive integer n and each $x \in X_{n+1}$, then Φ defined by (1) is induced by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$.

Proof. By Theorem 4.2 it suffices to prove that $\Phi(x)$ is nonempty for arbitrary $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.

For arbitrary $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ we construct a point $y = (y_1, y_2, y_3, \ldots) \in \Phi(x)$ by an inductive construction of coordinates y_i. More precisely, by induction on $i \in \mathbb{N}$, we construct a sequence $y_i \in Y_i$ satisfying $y_i \in \varphi_i(x_i)$ and $y_i \in g_i(y_{i+1})$ for each i.

We choose any $y_1 \in \varphi_1(x_1)$; it can be done since $\varphi_1(x_1)$ is nonempty.

Assume next that we have already constructed $y_i \in \varphi_i(x_i)$. Now we construct $y_{i+1} \in \varphi_{i+1}(x_{i+1})$ such that $y_i \in g_i(y_{i+1})$.

It follows from $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ that $x_i \in f_i(x_{i+1})$. Therefore

$$y_i \in \varphi_i(x_i) \subseteq \varphi_i[f_i(x_{i+1})] \subseteq g_i[\varphi_{i+1}(x_{i+1})] = \bigcup_{t \in \varphi_{i+1}(x_{i+1})} g_i(t).$$

Hence, there exists a point $t_0 \in \varphi_{i+1}(x_{i+1})$ such that $y_i \in g_i(t_0)$. We take any such t_0 for y_{i+1}. $\square$

This immediately leads to the following corollary.

Corollary 4.4. *Let $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ be any objects of $\mathcal{ICU}$, and let the sequence $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ be any morphism of $\mathcal{ICU}$ from $\{X_n, f_n\}_{n=1}^{\infty}$ to $\{Y_n, g_n\}_{n=1}^{\infty}$. Then $\Phi : \varprojlim\{X_n, f_n\}_{n=1}^{\infty} \to 2^{\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}}$, defined by (1), is induced by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$, meaning that $\Phi : \varprojlim\{X_n, f_n\}_{n=1}^{\infty} \multimap \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ is a morphism in $\mathcal{CU}$.* $\square$

Definition 4.5. *The function* Φ *from Corollary 4.4 is called the morphism of* $\mathcal{CU}$ *induced by the morphism* $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ *of* $\mathcal{ICU}$ *and is denoted by* $\Phi = \varprojlim \varphi_i$.

Note that if $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ is not a morphism of $\mathcal{ICU}$ but the induced function Φ is a morphism of $\mathcal{CU}$ (that happens when $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ satisfies the conditions of Theorem 4.2) $\varprojlim \varphi_i$ does not exist.

The induced morphism $\varprojlim \varphi_i : \varprojlim\{X_n, f_n\}_{n=1}^{\infty} \multimap \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ cannot be defined simply by the formula

$$\varprojlim \varphi_i(x_1, x_2, x_3, \ldots) = \varphi_1(x_1) \times \varphi_2(x_2) \times \varphi_3(x_3) \times \cdots \tag{2}$$

since the right hand side product of (2) is not necessarily a subset of $\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$, as shown by Example 4.6.

Example 4.6. *Let* $X_i = Y_i = [0,1]$*, let* $f_i = g_i = 1_{[0,1]}$*, where* $1_{[0,1]} : [0,1] \to 2^{[0,1]}$ *is the u.s.c. function, defined by* $1_{[0,1]}(x) = \{x\}$ *for each* $x \in [0,1]$*, and let* $\varphi_i : [0,1] \to 2^{[0,1]}$ *be defined by its graph:* $\Gamma(\varphi_i) = [0,1] \times [0,1]$*, for each positive integers* i*. Then* $\varphi_1(x_1) \times \varphi_2(x_2) \times \varphi_3(x_3) \times \cdots$ *is not a subset of* $\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$.

Proof. Obviously, $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ holds true for any positive integer i, and therefore $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ is a morphism of $\mathcal{ICU}$.

Also, $\varprojlim\{X_n, f_n\}_{n=1}^{\infty} = \varprojlim\{Y_n, g_n\}_{n=1}^{\infty} = \{(t, t, t, \ldots) \mid t \in [0,1]\}$, and therefore

$$\varphi(x_1, x_2, x_3, \ldots) = \varphi_1(x_1) \times \varphi_2(x_2) \times \varphi_3(x_3) \times \cdots = [0,1] \times [0,1] \times [0,1] \times \cdots$$

is not a subset of $\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ (and therefore it is not an element of $2^{\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}}$). $\square$

This example shows also that (2) cannot replace (1) in the definition of induced functions.

In the following theorem we prove that if each of the φ_i's has a surjective graph, then also $\varprojlim \varphi_i$ has a surjective graph. Note that it is not required that any of the functions f_n and g_n has a surjective graph.

Theorem 4.7. *Let* $\{X_n, f_n\}_{n=1}^{\infty}$ *and* $\{Y_n, g_n\}_{n=1}^{\infty}$ *be any objects of* $\mathcal{ICU}$ *and let the sequence* $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ *be a morphism of* $\mathcal{ICU}$ *from* $\{X_n, f_n\}_{n=1}^{\infty}$ *to* $\{Y_n, g_n\}_{n=1}^{\infty}$ *such that* $\varphi_i : X_i \to 2^{Y_i}$ *has a surjective graph for each positive integer* i*. Then* $\varprojlim \varphi_i$ *has a surjective graph.*

Proof. Let $y = (y_1, y_2, y_3, \ldots) \in \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ be arbitrary. We construct a point $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ such that $y \in \varprojlim \varphi_i(x)$.

Let n be any positive integer. Since $\varphi_n : X_n \to 2^{Y_n}$ has a surjective graph, there is a point $x_n^n \in X_n$ such that $y_n \in \varphi_n(x_n^n)$. We choose and fix such an x_n^n. Then by downwards induction we prove that for any $k \in \{1, 2, 3, \ldots, n-1\}$ there is $x_k^n \in X_k$ such that $y_k \in \varphi_k(x_k^n)$ and $x_k^n \in f_k(x_{k+1}^n)$.

Let k be any integer from $\{1, 2, 3, \ldots, n-1\}$. Assume that x_{k+1}^n has already been chosen in such a way that $y_{k+1} \in \varphi_{k+1}(x_{k+1}^n)$. Note that this assumption is fulfilled for $k = n-1$.

Since $y_k \in g_k(y_{k+1})$ and $y_{k+1} \in \varphi_{k+1}(x_{k+1}^n)$, it follows that

$$y_k \in g_k[\varphi_{k+1}(x_{k+1}^n)] = \varphi_k[f_k(x_{k+1}^n)].$$

Therefore there is a point $x_k^n \in X_k$ such that $x_k^n \in f_k(x_{k+1}^n)$ and $y_k \in \varphi_k(x_k^n)$ and we fix one such x_k^n.

This construction yields

$$x^n = (x_1^n, x_2^n, x_3^n, \ldots, x_{n-1}^n, x_n^n, z_{n+1}^n, z_{n+2}^n, z_{n+3}^n, \ldots) \in \prod_{i=1}^{\infty} X_i,$$

where $z_i^n \in X_i$ is arbitrarily chosen for each $i > n$. Then $\{x^n\}_{n=1}^{\infty}$ is a sequence in the compact metric space $(\prod_{i=1}^{\infty} X_i, D)$. Let $x = (x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X_i$ be any accumulation point of the sequence $\{x^n\}_{n=1}^{\infty}$.

Next we prove that $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ and that $y \in \varprojlim \varphi_i(x)$. Let $\{i_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers such that

$$\lim_{n\to\infty} x^{i_n} = x.$$

First we prove that $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. Let m be any positive integer. Then $(x_{m+1}^{i_k}, x_m^{i_k}) \in \Gamma(f_m)$ for each positive integer $i_k > m$. Since $\lim\limits_{k\to\infty}(x_{m+1}^{i_k}, x_m^{i_k}) = (x_{m+1}, x_m)$ and since the graph $\Gamma(f_m)$ is closed in $X_{m+1} \times X_m$, it follows that $(x_{m+1}, x_m) \in \Gamma(f_m)$. Therefore $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.

Finally we prove that $y \in \varprojlim \varphi_i(x)$. Let m be any positive integer. Then $y_m \in \varphi_m(x_m^{i_k})$ for each positive integer $i_k > m$. Therefore $(x_m^{i_k}, y_m) \in \Gamma(\varphi_m)$ for each $i_k > m$. Since $\lim\limits_{k\to\infty}(x_m^{i_k}, y_m) = (x_m, y_m)$ and since the graph $\Gamma(\varphi_m)$ is closed in $X_m \times Y_m$, it follows that $(x_m, y_m) \in \Gamma(\varphi_m)$, and therefore $y_m \in \varphi_m(x_m)$.

It follows that $(y_1, y_2, y_3, \ldots) \in \varprojlim \varphi_i(x_1, x_2, x_3, \ldots)$ and hence $\varprojlim \varphi_i$ has a surjective graph. $\square$

Next example shows that the function Φ induced by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ need not have a surjective graph if $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ is not a morphism of $\mathcal{ICU}$, even if each φ_i, f_i, and g_i has a surjective graph and if $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ holds true for any positive integer i but $i = 1$.

Example 4.8. *Let for each positive integer i and $j > 1$, $X_i = Y_i = [0, 1]$, $f_i = g_j = \varphi_i = 1_{[0,1]}$, and let $g_1 : [0, 1] \to 2^{[0,1]}$ be defined by its graph: $\Gamma(g_1) = [0, 1] \times [0, 1]$. Then the function Φ induced by $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ does not have a surjective graph.*

Proof. Obviously, $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ holds true for any positive integer $i > 1$, and $\varphi_1 \circ f_1(t) \subseteq g_1 \circ \varphi_2(t)$ for any $t \in [0, 1]$. Therefore $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ induces Φ defined by (1) according to Theorem 4.3.

Obviously $(0, 1, 1, 1, \ldots) \in \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ and $\varprojlim\{X_n, f_n\}_{n=1}^{\infty} = \{(t, t, t, \ldots) \mid t \in [0, 1]\}$. But $\Phi(t, t, t, \ldots) = \{(t, t, t, \ldots)\}$, and therefore $(0, 1, 1, 1, \ldots) \notin \Phi(t, t, t, \ldots)$ for any $t \in [0, 1]$. $\square$

In the rest of the section we study the transformation $F : \mathcal{ICU} \to \mathcal{CU}$, defined by

$$\{X_n, f_n\}_{n=1}^{\infty} \overset{F}{\longmapsto} \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$$
$$(\varphi_1, \varphi_2, \varphi_3, \ldots) \overset{F}{\longmapsto} \varprojlim \varphi_n \,.$$

In Theorem 4.9 we show that the transformation F is very close to being a functor from $\mathcal{ICU}$ to $\mathcal{CU}$. Example 4.10 follows after the theorem to show that F is not a functor from $\mathcal{ICU}$ to $\mathcal{CU}$.

Theorem 4.9. *Let $\{X_n, f_n\}_{n=1}^{\infty}$, $\{Y_n, g_n\}_{n=1}^{\infty}$ and $\{Z_n, h_n\}_{n=1}^{\infty}$ be any objects of $\mathcal{ICU}$, and*

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \ldots) : \{X_n, f_n\}_{n=1}^{\infty} \to \{Y_n, g_n\}_{n=1}^{\infty}$$

and

$$\psi = (\psi_1, \psi_2, \psi_3, \ldots) : \{Y_n, g_n\}_{n=1}^{\infty} \to \{Z_n, h_n\}_{n=1}^{\infty}$$

its morphisms. Then

1. *$F(1_{X_1}, 1_{X_2}, 1_{X_3}, \ldots) = 1_{\varprojlim\{X_n, f_n\}_{n=1}^{\infty}}$;*
2. *$(F(\psi) \circ F(\varphi))(x) \subseteq F(\psi \Box \varphi)(x)$ for all $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.*

Proof. To prove (1), choose arbitrary $x = (x_1, x_2, x_3, \ldots) \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. Then

$$F(1_{X_1}, 1_{X_2}, 1_{X_3}, \ldots)(x) = (1_{X_1}(x_1) \times 1_{X_2}(x_2) \times 1_{X_3}(x_3) \times \ldots) \cap \varprojlim\{X_n, f_n\}_{n=1}^{\infty} =$$
$$\{x\} = 1_{\varprojlim\{X_n, f_n\}_{n=1}^{\infty}}(x).$$

To prove (2), let $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ and let

$$z \in (F(\psi) \circ F(\varphi))(x) = F(\psi)[F(\varphi)(x)] = \bigcup_{y \in F(\varphi)(x)} F(\psi)(y)$$

be arbitrary. Then

$$z \in \bigcup_{y \in (\varphi_1(x_1) \times \varphi_2(x_2) \times \cdots) \cap \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}} (\psi_1(y_1) \times \psi_2(y_2) \times \cdots) \cap \varprojlim\{Z_n, h_n\}_{n=1}^{\infty}$$

and therefore there is a point $y \in \varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ such that $y_n \in \varphi_n(x_n)$ and $z_n \in \psi_n(y_n)$ for each positive n. It follows that $z_n \in \bigcup_{t \in \varphi_n(x_n)} \psi_n(t) = \psi_n[\varphi_n(x_n)] = (\psi_n \circ \varphi_n)(x_n)$ for each positive integer n and hence $z \in F(\psi \Box \varphi)(x)$. □

F is a functor if and only if $(F(\psi) \circ F(\varphi))(x) = F(\psi \Box \varphi)(x)$ holds true for all $x \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ and all objects $\{X_n, f_n\}_{n=1}^{\infty}$, $\{Y_n, g_n\}_{n=1}^{\infty}$ and $\{Z_n, h_n\}_{n=1}^{\infty}$ and all morphisms $\varphi = (\varphi_1, \varphi_2, \varphi_3, \ldots) : \{X_n, f_n\}_{n=1}^{\infty} \to \{Y_n, g_n\}_{n=1}^{\infty}$ and $\psi = (\psi_1, \psi_2, \psi_3, \ldots) : \{Y_n, g_n\}_{n=1}^{\infty} \to \{Z_n, h_n\}_{n=1}^{\infty}$ of $\mathcal{ICU}$. Example 4.10 shows that this is not the case, hence F is not a functor.

Example 4.10. *We use the notation from Theorem 4.9. Let for each positive integer n, $X_n = Y_n = Z_n = [0,1]$ and let $f, g : [0,1] \to 2^{[0,1]}$ be u.s.c. functions defined by $f(t) = \{t\}$ and $g(t) = [0,1]$ for each $t \in [0,1]$. Also let $f_1 = h_1 = \psi_1 = \varphi_n = g$ for each $n \geq 2$ and let $\varphi_1 = f_{n+1} = g_n = h_{n+1} = \psi_{n+1} = f$ for each $n \geq 1$. Let $x = (1,0,0,0,\ldots) \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. Then $(F(\psi) \circ F(\varphi))(x) \neq F(\psi \Box \varphi)(x)$.*

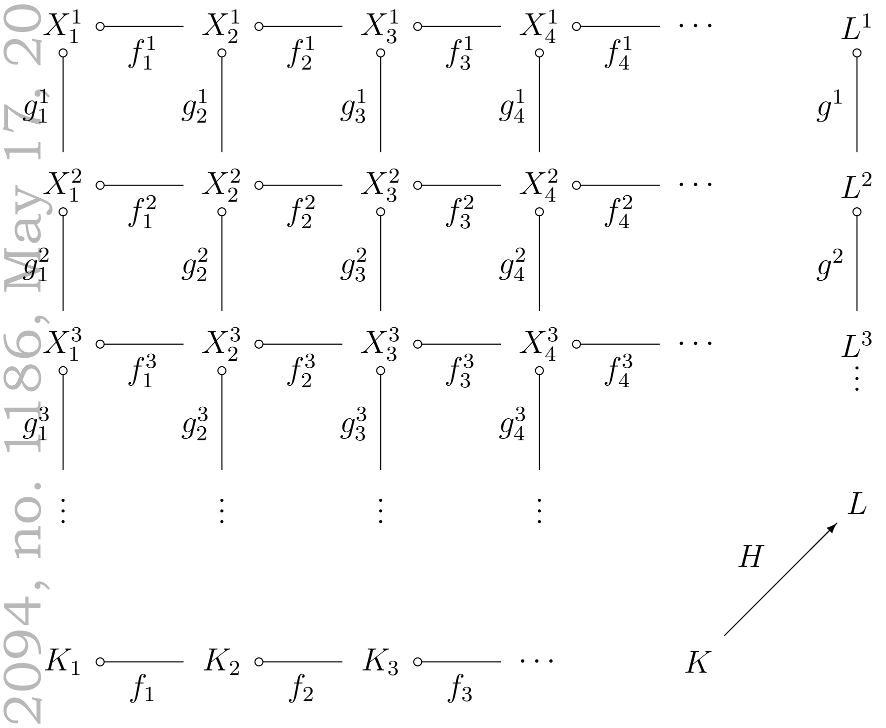
Proof. Let $z = (1,0,0,0,\ldots) \in \varprojlim\{Z_n, h_n\}_{n=1}^{\infty}$. Obviously $z \in F(\psi \Box \varphi)(x)$. Then, since $\varphi_1(t) = \{t\}$ for each $t \in [0,1]$, $y = (1,1,1,\ldots)$ is the only element in $\varprojlim\{Y_n, g_n\}_{n=1}^{\infty}$ such that $y \in F(\varphi)(x)$. But, since $F(\psi)(y) = [0,1] \times \{1\} \times \{1\} \times \cdots$ and $z_2 = 0$ it follows that $z \notin F(\psi)(y)$. Therefore $z \notin (F(\psi) \circ F(\varphi))(x)$ and hence $(F(\psi) \circ F(\varphi))(x) \neq F(\psi \Box \varphi)(x)$. □

5 An application

In the final section we study the following diagram.

Theorem 5.1. *Let X_i^j be compact metric spaces, and let $f_i^j : X_{i+1}^j \to 2^{X_i^j}$, $g_i^j : X_{i+1}^j \to 2^{X_i^j}$ be u.s.c. functions, for all positive integers i and j. Let also for each j*

$$L^j = \varprojlim\{X_i^j, f_i^j\}_{i=1}^{\infty}$$



and for each i

$$K_i = \varprojlim\{X_i^j, g_i^j\}_{j=1}^{\infty}.$$

If for each integer n, f_n is the function induced by $(f_n^1, f_n^2, f_n^3, \ldots)$ and g^n is the function induced by $(g_1^n, g_2^n, g_3^n, \ldots)$, then

$$L = \varprojlim\{L^j, g^j\}_{j=1}^{\infty}$$

and

$$K = \varprojlim\{K_i, f_i\}_{i=1}^{\infty}.$$

are homeomorphic.

Proof. Define the function $H : K \to L$ as follows:

$$H\left(\left(x_1^1, x_1^2, x_1^3, \ldots\right), \left(x_2^1, x_2^2, x_2^3, \ldots\right), \left(x_3^1, x_3^2, x_3^3, \ldots\right), \ldots\right) = \left(\left(x_1^1, x_2^1, x_3^1, \ldots\right), \left(x_1^2, x_2^2, x_3^2, \ldots\right), \left(x_1^3, x_2^3, x_3^3, \ldots\right), \ldots\right), \quad (3)$$

where $(x_i^1, x_i^2, x_i^3, \ldots) \in K_i$ and $(x_i^1, x_i^2, x_i^3, \ldots) \in f_i\left(x_{i+1}^1, x_{i+1}^2, x_{i+1}^3, \ldots\right)$ for each positive integer i.

We will prove that H is a homeomorphism.

First, we prove that H is well-defined. We need to show that the right side of (3) is a point of L. The proof is in the following steps.

1. $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in L^j$, for arbitrary $j \in \mathbb{N}$;
2. $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in g^j\left(x_1^{j+1}, x_2^{j+1}, x_3^{j+1}, \ldots\right)$, for arbitrary $j \in \mathbb{N}$.

Let us prove (1).

Since $(x_i^1, x_i^2, x_i^3, \ldots) \in f_i\left(x_{i+1}^1, x_{i+1}^2, x_{i+1}^3, \ldots\right) = \left(\prod_{j=1}^{\infty} f_i^j(x_{i+1}^j)\right) \cap K_i$, it follows that $x_i^j \in f_i^j\left(x_{i+1}^j\right)$ for each i and j. Hence, $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in L^j$.

It remains to prove (2).

Since $(x_i^1, x_i^2, x_i^3, \ldots) \in K_i$, it follows that for each i and j $x_i^j \in g_i^j(x_i^{j+1})$. Therefore $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in \left(\prod_{i=1}^{\infty} g_i^j(x_i^j)\right) \cap L^j = g^j\left(x_1^{j+1}, x_2^{j+1}, x_3^{j+1}, \ldots\right)$ for all j.

Hence, $\left((x_1^1, x_2^1, x_3^1, \ldots), (x_1^2, x_2^2, x_3^2, \ldots), (x_1^3, x_2^3, x_3^3, \ldots), \ldots\right) \in L$. So we have proved that $H : K \to L$ is well defined.

In the same manner we prove that $H' : L \to K$ defined by

$$\begin{gathered} H'\left(\left(x_1^1, x_2^1, x_3^1, \ldots\right), \left(x_1^2, x_2^2, x_3^2, \ldots\right), \left(x_1^3, x_2^3, x_3^3, \ldots\right), \ldots\right) = \\ \left(\left(x_1^1, x_1^2, x_1^3, \ldots\right), \left(x_2^1, x_2^2, x_2^3, \ldots\right), \left(x_3^1, x_3^2, x_3^3, \ldots\right), \ldots\right), \end{gathered}$$

is well defined. Since obviously H and H' are both continuous and inverses to each other, it follows that they are homeomorphisms. □

Corollary 5.2. *We use the notation of Theorem 5.1. If for all positive integers i and j*

$$g_i^j \circ f_i^{j+1} = f_i^j \circ g_{i+1}^j,$$

then the spaces L and K are homeomorphic.

Proof. The claim follows by Theorem 5.1 since by Corollary 4.4 there are induced functions f_n and g^n for each n. □

We conclude the paper with the following example.

Example 5.3. *Let X be any compact metric space and let $f : X \to X$ be a surjective single-valued mapping. Let $L' = \varprojlim\{X, f^{-1}\}_{n=1}^{\infty}$, where f^{-1} is the u.s.c. function $f^{-1} : X \to 2^X$ defined by its graph*

$$\Gamma(f^{-1}) = \{(x, y) \in X \times X \mid (y, x) \in \Gamma(f)\}.$$

Let $\sigma : L' \to L'$ be the shift map, defined by

$$\sigma(t_1, t_2, t_3, \ldots) = (t_2, t_3, t_4, \ldots)$$

for each $(t_1, t_2, t_3, \ldots) \in L'$.

Then the inverse limit $\varprojlim\{L', \sigma\}_{n=1}^{\infty}$ is homeomorphic to $\varprojlim\{X, f\}_{n=1}^{\infty}$.

Proof. We show first that the mapping $(t_1, t_2, t_3, \dots) \mapsto \{\sigma(t_1, t_2, t_3, \dots)\}$ can be interpreted as an induced function and then we use Theorem 5.1 to prove that the inverse limit $\varprojlim\{L', \sigma\}_{n=1}^{\infty}$ is homeomorphic to $\varprojlim\{X, f\}_{n=1}^{\infty}$.

We use the notation that is used in Theorem 5.1. Let for all positive integers i, j, $X_i^j = X$, $g_i^j(t) = \{f(t)\}$, and $f_i^j(t) = f^{-1}(t)$ for each $t \in X$.

Then $g^n(t_1, t_2, t_3, \dots) = (\{f(t_1)\} \times \{f(t_2)\} \times \{f(t_3)\} \times \dots) \cap L' = \{(t_2, t_3, t_4, \dots)\} = \{\sigma(t_1, t_2, t_3, \dots)\}$ for any $(t_1, t_2, t_3, \dots) \in L'$. It follows that $L = \varprojlim\{L^n, g^n\}_{n=1}^{\infty} = \varprojlim\{L', \sigma\}_{n=1}^{\infty}$.

Let $K' = K_n = \varprojlim\{X, f\}_{n=1}^{\infty}$ for each positive integer n. Next we show that $K = \varprojlim\{K_n, f_n\}_{n=1}^{\infty} = \varprojlim\{K', \sigma'^{-1}\}_{n=1}^{\infty}$, where σ' is the shift map from K' to K'. Note that σ' is a homeomorphism, since f is single-valued, and that $\sigma'^{-1}(t_1, t_2, t_3, \dots) = (f(t_1), t_1, t_2, t_3, \dots)$ for each $(t_1, t_2, t_3, \dots) \in K'$.

Then $f_n(t_1, t_2, t_3, \dots) = (\{f^{-1}(t_1)\} \times \{f^{-1}(t_2)\} \times \{f^{-1}(t_3)\} \times \dots) \cap K' = \{(f(t_1), t_1, t_2, t_3, \dots)\} = \{\sigma'^{-1}(t_1, t_2, t_3, \dots)\}$ for any $(t_1, t_2, t_3, \dots) \in K'$. It follows that $K = \varprojlim\{K_n, f_n\}_{n=1}^{\infty} = \varprojlim\{K', \sigma'^{-1}\}_{n=1}^{\infty}$.

Since σ'^{-1} is a homeomorphism it follows that $K = \varprojlim\{K', \sigma'^{-1}\}_{n=1}^{\infty}$ is homeomorphic to $K' = \varprojlim\{X, f\}_{n=1}^{\infty}$. By Theorem 5.1 K is homeomorphic to L, and that proves that $\varprojlim\{X, f\}_{n=1}^{\infty}$ is homeomorphic to $\varprojlim\{L', \sigma\}_{n=1}^{\infty}$. $\square$

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