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A simplified collective model of pion ⋆

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Abstract. In order to test the accuracy of the aproximate methods commonly used for the Nambu – Jona-Lasinio model we study a simpler model which can be solved exactly. We find that the Random Phase Approximation gives reasonably good results if used in combination with the Hartree ground state (vacuum). On the other hand, the Tamm-Dancoff and Hermitian Operator Methods give even better results, but for the price of requiring a better approximation of the ground state.

1 Introduction

In the Nambu – Jona-Lasinio model (NJL), the vacuum properties and the pion excitation are usually calculated using the Hartree-Fock (HF) and Random Phase Approximations (RPA). We propose a simplied version of NJL which is appropriate to test the accuracy of these aproximate methods. The model preserves the main features of NJL and is simple enough to be solved exactly. For simplicity we limit ourselves to one flavour of quarks.

Since we shall deal with a finite number of quarks, it is convenient to start with the one-flavour NJL Hamiltonian written in the first-quantized form [1] and with a momentum cutoff Λ

$$
\begin{aligned}
H = & \sum_{k=1}^{N} \big(\gamma_5(k) h(k) p(k) + m_0 \beta(k)\big) \\
& - \frac{g}{2} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l\neq k}}^{N} \Big(\beta(k)\beta(l) + \big(i\beta(k)\gamma_5(k)\big)\cdot\big(i\beta(l)\gamma_5(l)\big)\Big)\cdot \\
& \cdot \sum_{\mathbf{p}'_k}^{\Lambda} \sum_{\mathbf{p}'_l}^{\Lambda} \sum_{\mathbf{p}_k}^{\Lambda} \sum_{\mathbf{p}_l}^{\Lambda} \delta_{\mathbf{p}'_k+\mathbf{p}'_l,\,\mathbf{p}_k+\mathbf{p}_l} |\mathbf{p}'_k,\ \mathbf{p}'_l\rangle\langle \mathbf{p}_k,\ \mathbf{p}_l| \quad .
\end{aligned}
$$

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2 The simple model

We made four approximations:

1. We confined quarks in a finite volume $\mathcal{V}$ with periodic boundary conditions so that their momenta become discrete. Because the absolute values of momenta are limited, there is only a finite number of momenta avaliable. Therefore we have only a finite number $2N$ of single-particle states occupied by a finite number N of quarks.
2. In the kinetic term of the Hamiltonian we take an average absolute value of momenta ($P = \frac{3}{4}\Lambda$) instead of their true values.
3. The interaction changes only the quark's chirality and preserves its helicity, color and momentum which then label the quark. Therefore the quarks can be treated as distinguishable and Hartree is equivalent to Hartree-Fock.
4. We include a quark selfinteraction so that the double sumations can be replaced by two single sumations. This contributes a trivial constant $-gN$.

The simplified Hamiltonian is:

$$H = \sum_{k=1}^{N}\Big(\gamma_5(k)h(k)P + m_0\beta(k)\Big) - \frac{g}{2}\Big(\sum_{k=1}^{N}\beta(k)\sum_{l=1}^{N}\beta(l) + \sum_{k=1}^{N} i\beta(k)\gamma_5(k)\sum_{l=1}^{N} i\beta(l)\gamma_5(l)\Big) \; .$$

We can introduce the operators:

$$j_x = \frac{1}{2}\beta\,, \quad j_y = \frac{1}{2}i\beta\gamma_5\,, \quad j_z = \frac{1}{2}\gamma_5\,,$$

which obey (quasi)spin commutation relations and allow us to make full use of the angular momentum algebra.

Also separate sums over quarks with right and left helicity

$$L_\alpha = \sum_{k=1}^{N}\frac{1+h(k)}{2}j_\alpha(k)\,, \quad S_\alpha = \sum_{k=1}^{N}\frac{1-h(k)}{2}j_\alpha(k)$$

as well as the total sum

$$J_\alpha = L_\alpha + S_\alpha = \sum_{k=1}^{N} j_\alpha(k)$$

obey the (quasi)spin commutation relations ($\alpha = x,\, y,\, z$).

The model Hamiltonian can then be written as

$$H = 2P(L_z - S_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2)\,.$$

It commutes with $\mathbf{L}^2$ and $\mathbf{S}^2$ but not with L_z and S_z. Nevertheless, it is convenient to work in the basis $|L, S, L_z, S_z\rangle$.The Hamiltonian matrix elements can be easily calculated using the angular momentum algebra. By diagonalisation we then obtain the energy spectrum of the system.

The model has three model parameters: P, m_0 and g. Instead of g we prefer to take $G = g\mathcal{V}/2$ where $\mathcal{V} = \pi^2 N/\Lambda^3$ is the normalization volume since g decreases with increasing number of quarks while G stabilizes at large N.

We want to study the simple model in a physically interesting regime. Therefore we choose the three model parameters so that we fit three observable

1. We calculate the *mass of dressed quark* ($M = 335$ MeV) from the difference between the ground state energies (E_g) of the systems of N and $N-1$ quarks. For the momentum of quark we take the average value (P) and we obtain

$$M = \sqrt{(E_g(N) - E_g(N-1))^2 - P^2}\,.$$

2. The *mass of pion* m_π should be 138 MeV. The pion corresponds to the first excited state of the system,

$$E_\pi = E_1 - E_g \quad \Rightarrow \quad m_\pi = \sqrt{E_\pi^2 - p_\pi^2}\,,$$

where E_1 is the energy of the first excited state and E_π is the pion energy. We determined the effective pion momentum p_π by the requirement, that the pion behaves as a Goldstone boson in the chiral limit and that p_π does not change much when the small quark mass term is introduced:

$$p_\pi = E_1(m_0 = 0) - E_g(m_0 = 0)\,.$$

3. Instead of the *pionic decay constant* ($f_\pi = 93$ MeV) we prefer to fit the *chiral condensate* Q which is related to f_π through the Gell-Mann – Oakes – Renner relation

$$-Q = f_\pi^2 m_\pi^2 / m_0\,.$$

In this way we avoid the ambiguity how to introduce f_π in a one-flavour model, as well as the ambiguities with the effective momentum of the pion in a finite volume. In our model, the chiral condensate is

$$Q = \frac{1}{\mathcal{V}}\langle g|\sum_{i=1}^{N} \beta_i|g\rangle = \frac{2}{\mathcal{V}}\langle g|J_x|g\rangle\,.$$

We compare the fitted values of model parameters for several values of N (Table 1). It is amusing that they are rather close to NJL parameters corresponding to two flavours and infinite number of quarks in the system [2].

3 Test of approximate methods – the vacuum

We compare the ground state (vacuum) energy E_g and the chiral condensate Q of the Hartree approximation with the exact solution.

The vacuum energy (Table 2) for N=48 and for the physically interesting value $G = 40.1\,\mathrm{MeV\,fm}^3$ deviates only by 1.2%. The deviation decreases with

Table 1. Model parameters (above) fitted to reproduce the observables (below).

N	12	24	36	48	NJL	exper.
G (MeV fm^3)	69.9	55.9	46.5	40.1	42.2	
m_0 (MeV)	26.0	15.9	11.8	9.6	5.5	
P (MeV)	484	557	613	659	473	
M (MeV)	335	335	335	335	335	335
m_π (MeV)	138	138	138	138	138	138
f_π (MeV)	93.0	93.0	93.0	93.0	93.0	93.0

Table 2. The energies E_g of the ground state for 48 quarks for $P = 659\,\text{MeV}$ and $m_0 = 9.6\,\text{MeV}$ and three values of G.

G (MeV fm^3)	20.0	40.1	60.0
Exact	–32058.96	–32970.80	–37028.30
Hartree	–31991.62	–32586.51	–36565.25

increasing N which hints that Hartre is a good large-N limit (we could not test it yet for large enough N). One should take some care, however, since in spite of the good agreement the Hartree ground state is still above the first (few) exact excited states in some of the studied cases.

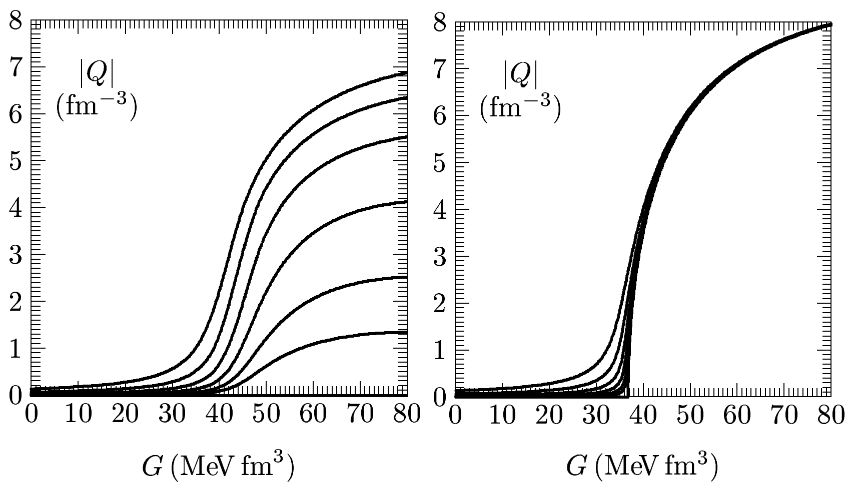

Fig. 1. Dependence of absolute value of the chiral condensate on the strength of interaction for 48 particles and $P = 659\,\text{MeV}$. From above follow the lines for $m_0 = 9.6, 4.8, 2.4, 1.2, 0.6, 0.3$ and $0\,\text{MeV}$. Exact (left) and Hartree (right) results are compared.

For a finite system we do not expect a sharp transition from the chirally symmetric to the chirally broken phase as a function of the interaction strength G. As a matter of fact, for $m_0 = 0$ the system remains chirally symmetric, the order parameter Q remains zero. For a small but finite explicit symmetry breaking term m_0 the system responds first with a small Q proportional to m_0. For G larger than some critical value, however, Q starts to rise sharply (Fig.1). This is the analogue for spontaneous symmetry breaking in the case of a finite system. One expects a sharp phase transition if one makes the limit $N \to \infty$ faster than the limit $m_0 \to 0$.

On the other hand, one gets in the Hartree approximation a sharp phase transition already in the chiral limit $m_0 = 0$ and a slightly larger chiral condensate for $m_0 > 0$. This shows that the Hartree approximation strongly exaggerates the chiral symmetry breaking and in this way immitates the situation at $N \to \infty$ even at smaller N.

4 Test of approximate methods – π and σ mesons.

The first excited state (negative parity) corresponds to pion and the second excited state (positive parity) corresponds to sigma meson. As approximate methods we study several *particle-hole methods* in which the ground state is excited by a one-body ("particle-hole") excitation operator.

In our case the low-lying states are symmetric under permutation of quark labels. Therefore the one-body excitation operators can be expressed as combination of quasispin operators L_x , S_x , iL_y , iS_y , L_z and S_z which we denote jointly by B_i, $i = 1, \ldots 6$. Then we expand the excited states in the basis $|i\rangle$

$$|\text{exc}\rangle = \sum_i c_i|i\rangle\,, \quad |i\rangle = B_i^\dagger|g\rangle.$$

The calculation is formulated in terms of a secular equation for the excitation energy ω and expansion coefficients c_i

$$\underline{\underline{\mathcal{H}}}\ \underline{c} = \omega\, \underline{\underline{\mathcal{N}}}\ \underline{c}\,.$$

Different approximate methods differ in the proposition for the hamiltonian and overlap matrices

1. In the *Tamm-Dancoff method* (TD) the basis $|i\rangle$ is taken literally and one obtains

$$\begin{aligned} \mathcal{H}_{j\,i} &= \langle j|(H - E_g)|i\rangle = \langle g|B_j\,(H - E_g)\,B_i^\dagger|g\rangle \quad \text{and} \\ \mathcal{N}_{j\,i} &= \langle j|i\rangle = \langle g|B_j\,B_i^\dagger|g\rangle\,. \end{aligned}$$

2. The *Hermitian Operator Method* (HOM) [3] is an approximation to TD which restricts the excitation operator to be either hermitian or antihermitian and relies on $|g\rangle$ being an exact ground state. This simplifies the evaluation of the matrix elements, but it makes a restriction to a smaller model space by decoupling the spaces generated by real hermitian (L_x , S_x , L_z and S_z) and real antihermitian (iL_y and iS_y) operators.

$$\begin{aligned} \mathcal{H}_{j\,i} &= \frac{1}{2}\langle g|\,[\,B_j\,,\,[\,H,\,B_i^\dagger\,]\,]\,|g\rangle \quad \text{and} \\ \mathcal{N}_{j\,i} &= \left\{ \begin{array}{ccc} \langle j|i\rangle & = & \langle g|B_j\,B_i^\dagger|g\rangle \\ 0 & & \end{array} \right. , \end{aligned}$$

where the upper line in $\mathcal{N}_{j\,i}$ applies if B_i and $B_j^\dagger$ are both hermitian or both antihermitian and the lower line (0) otherwise.

3. The *Simple Operator Method* (SOM) is even a more restrictive approximation to TD, it chooses only one of the listed one-body operators, iJ_y. Its succes in the description of the pion is based on the observation that such state is very close to the pionic excitation: $\langle\pi|\, iJ_y|g\rangle/\sqrt{\langle g|J_y^2|g\rangle} = 0.990$ (for $N = 48$). It is even useful to calculate the two-pion excitation $|2\pi\rangle = -J_y^2\,|g\rangle - \langle g|{-J_y^2}|g\rangle\,|g\rangle$.
4. In the *Random Phase Apoproximation* one makes a risky but often sucessful assumption that there exists an excitation operator $A^\dagger = \sum_i c_i B_i^\dagger$ whose adjoint kills the ground state

$$A^\dagger|g\rangle = |\text{exc}\rangle\,, \quad A|g\rangle = 0\,.$$

The inspiration comes from the creation and annihilation operators of the harmonic oscillator and it is a promissing approximation when one observes harmonic vibrational spectra. One then gets

$$\mathcal{H}_{j\,i} = \langle g|\,[\,B_j\,,\,[\,H,\,B_i^\dagger\,]\,]\,|g\rangle \quad \text{and}$$
$$\mathcal{N}_{j\,i} = \langle g|\,[\,B_j\,,\,B_i^\dagger\,]\,|g\rangle\,.$$

5 Conclusion

We found that although the Hartree ground state energy differs from the exact ground state energy only by a small percentage for realistic model parameters, the energy difference between the Hartree and the exact ground state is comparable to the energy differences between the lowest exact excited states and the exact ground state. In spite of this, the Random Phase Approximation (RPA) gives a rather good approximation of the pion energy if used with the Hartree ground state (Table 3); as a matter of fact, it gives better results when used with the Hartree ground state than when used with the exact ground state. The condition $A|g\rangle = 0$ for a one-body operator A is namely better fulfilled in the case of the Hartree ground state than in the case of the exact ground state. On the other hand, the Hermitian Operator Method (HOM), the Simple Operator Method (SOM) and the Tamm-Dancoff (TD) method fail for the Hartree ground state, but give very good results when used with the exact ground state.

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Table 3. Exact excitation energies compared to several approximate methods for 48 quarks, $P = 659$ MeV and $m_0 = 9.6$ MeV and three values of G.

G (MeV fm^3)		20.0		40.1		60.0	
	state	ω(MeV)	parity	ω(MeV)	parity	ω(MeV)	parity
		exact solutions					
		1870.76	1.00	879.21	1.00	947.76	–1.00
		1870.76	–1.00	788.36	–1.00	647.98	1.00
		1848.51	1.00	788.33	1.00	579.88	–1.00
	$\|\sigma\rangle$	916.91	1.00	365.20	1.00	401.18	1.00
	$\|\pi\rangle$	916.46	–1.00	319.65	–1.00	214.59	–1.00
	$\|g\rangle$	0.00	1.00	0.00	1.00	0.00	1.00
		approximations of low-lying states computed from the **exact** ground state					
RPA	$\|\sigma\rangle$	917.06	1.00	538.36	1.00	1630.42	1.00
	$\|\pi\rangle$	916.59	–1.00	423.80	–1.00	591.55	–1.00
TD	$\|\sigma\rangle$	917.00	1.00	423.54	1.00	1365.81	1.00
	$\|\pi\rangle$	916.53	–1.00	337.51	–1.00	246.99	–1.00
	$\|g\rangle$	0.48	1.00	4.74	1.00	9.01	1.00
HOM	$\|\sigma\rangle$	916.96	1.00	413.93	1.00	1223.81	1.00
	$\|\pi\rangle$	916.49	–1.00	333.98	–1.00	243.37	–1.00
	$\|g\rangle$	0.48	1.00	4.76	1.00	9.07	1.00
SOM	$\|2\pi\rangle$	1859.35	1.00	843.56	1.00	609.14	1.00
	$\|\pi\rangle$	916.49	–1.00	333.98	–1.00	243.37	–1.00
		approximations of low-lying states computed from the **Hartree** ground state					
RPA	$\|\sigma\rangle$	908.25	1.00	656.92	1.00	1691.41	1.00
	$\|\pi\rangle$	907.75	–1.00	260.35	–1.00	229.05	–1.00
TD	$\|\sigma\rangle$	976.01	1.00	886.01	1.00	1744.26	1.00
	$\|\pi\rangle$	975.67	–1.00	760.63	–1.00	1083.67	–1.00
	$\|g\rangle$	0.00	1.00	0.00	1.00	0.00	1.00
HOM	$\|\sigma\rangle$	975.70	–0.01	763.78	0.11	1881.07	–0.34
	$\|\pi\rangle$	773.19	0.55	584.56	0.40	1097.31	0.00
	$\|g\rangle$	0.00	1.00	0.00	1.00	0.00	1.00
SOM	$\|2\pi\rangle$	1965.87	1.00	1540.17	1.00	2156.63	1.00
	$\|\pi\rangle$	975.67	–1.00	760.63	–1.00	1083.67	–1.00

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