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CYCLIC BUNDLE HAMILTONICITY

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ABSTRACT. Cyclic bundle Hamiltonicity $cbH(G)$ of a graph G is the minimal n for which there is an automorphism α of G such that the graph bundle $C_n \Box^\alpha G$ is Hamiltonian. We define $\nabla(\tilde{G}_\alpha)_{\min}$, an invariant that is related to the maximal vertex degree of spanning trees suitably involving the symmetries of G and prove $cbH(G) \leq \nabla(\tilde{G}_\alpha)_{min} \leq cbH(G)+1$ for any non-trivial connected graph G.

1. Introduction

In 1982, Batagelj and Pisanski [1] proved that the Cartesian product of a tree T and a cycle C_n has a Hamiltonian cycle if and only if $n \geq \Delta(T)$, where $\Delta(T)$ denotes the maximum valence (or vertex degree) of T. They introduced the cyclic Hamiltonicity $cH(G)$ of G as the smallest integer n for which the Cartesian product $C_n \Box G$ is Hamiltonian. More than twenty years later, Dimakopoulos, Palios and Paulakidas [2] proved that $cH(G) \leq \mathcal{D}(G) \leq cH(G)+1$, as conjectured already in [1]. (Here $\mathcal{D}(G)$ denotes the minimum of $\Delta(T)$ over all spanning trees T of G.) These results can be extended in a certain way to graph bundles. Recently, Pisanski and Žerovnik [3] proved that the graph bundle $C_n \Box^\alpha T$ has a Hamiltonian cycle if and only if $n \geq h(T,\alpha)$, where $h(T,\alpha)$ is the maximum value of $\left\lceil \frac{d(v,\alpha)}{o(v,\alpha)} \right\rceil$ over all vertices $v \in V(T)$ and $o(v,\alpha)$ denotes the number of elements in the orbit of v under the automorphism α while $d(v,\alpha)$ is the degree of the vertex corresponding to the orbit of v in the tree T/α.

In this note, we show that the results for general graphs can naturally be generalized from Cartesian graph products to Cartesian graph bundles. As an analog of the cyclic Hamiltonicity $cH(G)$ we define cyclic bundle Hamiltonicity $cbH(G)$ of the graph G as the minimal n such that there exists $\alpha \in \mathrm{Aut}(G)$ and $C_n \Box^\alpha G$ has a Hamiltonian cycle. We prove that

$$cbH(G) \leq \nabla(\tilde{G}_\alpha)_{\min} \leq cbH(G)+1 \qquad (1)$$

where
$\nabla(\tilde{G}_\alpha)_{\min} = \min \left\{ \nabla(\tilde{G}_\alpha) \mid \alpha \in \mathrm{Aut}(G) \right\}$,
$\nabla(\tilde{G}_\alpha) = \min \left\{ h(\tilde{T}_\alpha) \mid \tilde{T}_\alpha \text{ is a spanning tree of } \tilde{G}_\alpha \right\}$ and
$h(\tilde{T}_\alpha) = \max \left\{ \left\lceil \frac{\tilde{d}(w)}{\tilde{o}(w)} \right\rceil \mid w \in V(\tilde{T}_\alpha) \right\}$ where $\tilde{d}(w)$ denotes the degree of the vertex w in $\tilde{T}_\alpha$ and $\tilde{o}(w)$ is the number of elements in the orbit of the vertex w in $\tilde{T}_\alpha$.

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As the result given in this paper directly generalizes the result of [3] that is valid for trees, it is natural to ask whether a brief argument is sufficient for the conclusion. On one hand, a Hamiltonian cycle of $C_n\Box^\alpha T$ is clearly also a Hamiltonian cycle of $C_n\Box^\alpha G$ for any spanning tree T of G, and similarly, a Hamiltonian cycle of $C_n\Box^\alpha T_\alpha$ is also a Hamiltonian cycle of $C_n\Box^\alpha G_\alpha$. However, on the other hand, taking a spanning tree T of G, the sets of automorphisms of G and of T are in general different, hence direct application of theorem for trees is not straightforward. After a section with basic terminology and notation, a detailed proof is given in Section 3, and the result is illustrated with several examples.

2. Terminology and notation

Here we will study hamiltonicity of a graph bundle $C_n\Box^\alpha G$. An arbitrary connected graph G is said to be Hamiltonian if it contains a spanning cycle called a Hamiltonian cycle.

The *Cartesian product* of graphs G and H is the graph $G\Box H$ with vertex set $V(G\Box H) = V(G)\times V(H)$. The edges of $G\Box H$ are given by

(1) for any $g_1g_2 \in E(G)$ and $h \in V(H)$, (g_1,h) and (g_2,h) are adjacent in $G\Box H$ and
(2) for any $h_1h_2 \in E(H)$ and $g \in V(G)$, (g,h_1) and (g,h_2) are adjacent in $G\Box H$.

Let B and G be graphs and $\mathrm{Aut}(G)$ be the set of automorphisms of G. For any pair of adjacent vertices $u,v \in V(B)$ we will assign an automorphism of G to the ordered pairs of vertices. Formally, let $\sigma : V(B)\times V(B) \to \mathrm{Aut}(G)$. For brevity, we will write $\sigma(u,v) = \sigma_{u,v}$ and assume that $\sigma_{v,u} = \sigma_{u,v}^{-1}$ and $\sigma_{u,u} = id$ for any $u,v \in V(B)$.

Now we construct the graph X as follows. The vertex set of X is $V(X) = V(B)\times V(G)$. The edges of X are given by:

(1) for any $g_1g_2 \in E(G)$ and $b \in V(B)$, (b,g_1) and (b,g_2) are adjacent in X and
(2) for any $b_1b_2 \in E(B)$ and $g \in V(G)$, (b_1,g) and $(b_2,\sigma_{b_1,b_2}(g))$ are adjacent in X.

We call $X = B\Box^\sigma G$ *Cartesian graph bundle* with base B and fibre G. Other standard graph products [4] can be generalized to graph bundles. Graph bundles were first studied in [6], and received considerable interest in the literature, see, for example [5, 7, 8, 9, 10, 11] and the references there.

Clearly, if all $\sigma_{u,v}$ are identity automorphisms, graph bundle is the Cartesian product $X = B\Box^\sigma G = B\Box G$. Furthermore, it is well-known [6] that if the base graph is a tree, then the graph bundle is isomorphic to the Cartesian product, i.e. $X = T\Box^\sigma G \simeq T\Box G$ for any graph G, any tree T and any assignment of automorphisms σ.

A graph bundle over a cycle can always be constructed in a way that all but at most one automorphism are identities. Fixing $V(C_n) = \{0,1,2,\dots,n-1\}$ we denote $\sigma_{n-1,0} = \alpha$, $\sigma_{i-1,i} = id$ for $i = 1,2,\dots,n-1$, and $C_n\Box^\alpha G = C_n\Box^\sigma G$.

There are two interesting cases where the construction above may not result in a simple graph. For $n = 2$, $C_2\Box^\alpha G$ has double edges whenever α fixes a vertex. For $n = 1$, $C_1\Box^\alpha G$ has a loop whenever α fixes a vertex and has double edges whenever vertices v and $\alpha(v)$ are adjacent in G. However, when interested in Hamiltonian properties, we can consider $C_2\square^\alpha G$ and $C_1\square^\alpha G$ as simple graphs by ignoring loops and multiple edges because a Hamiltonian cycle traverses each multiple edge at most once and never uses a loop.

Let G be an arbitrary connected graph. Let $\alpha \in \operatorname{Aut} G$ be an arbitrary automorphism of G. It partitions the vertex set $V(G)$ into disjoint orbits. For a given vertex $v \in V(G)$ let $O(v,\alpha)$ denote such an orbit whose size is denoted by $o(v,\alpha)$. To simplify the notation, let $\tilde{G}_\alpha$ denote the quotient graph obtained from the graph G by vertex identification of each vertex orbit $O(v,\alpha)$ and with two orbits $O(u,\alpha)$ and $O(v,\alpha)$ being adjacent in $\tilde{G}_\alpha$ if and only if there are two representatives $u \in O(u,\alpha)$ and $v \in O(v,\alpha)$ that are adjacent in G. By $\tilde{T}_\alpha$ we will denote a spanning tree of the quotient graph $\tilde{G}_\alpha$.

3. Cyclic bundle Hamiltonicity

Cyclic Hamiltonicity $cH(G)$ of the graph G is the minimal n for which $C_n\square G$ is Hamiltonian, where C_n is the cycle on n vertices [1]. As an analog we define here the *cyclic bundle Hamiltonicity* $cbH(G)$ of the graph G as the minimal n for which there is an automorphism $\alpha \in \operatorname{Aut}(G)$ such that $C_n\square^\alpha G$ is Hamiltonian.

If G is Hamiltonian, then $cbH(G) = 1$. In this case, $C_1\square^\alpha G$ is Hamiltonian for every automorphism α of G. But the converse is not true. We may have $cbH(G) = 1$, yet G is not necessarily Hamiltonian. In Figure 1, G is not Hamiltonian while $C_1\square^\alpha G$ where $\alpha = (1,2)(3,4)$ is. An example of a Hamiltonian cycle in $C_1\square^\alpha G$ is drawn with thick lines. Figure 2 shows an example of Hamiltonian cycle in the graph bundle $C_2\square^\alpha G$, where $\alpha = (2,3)(4,6)(5,7)$.

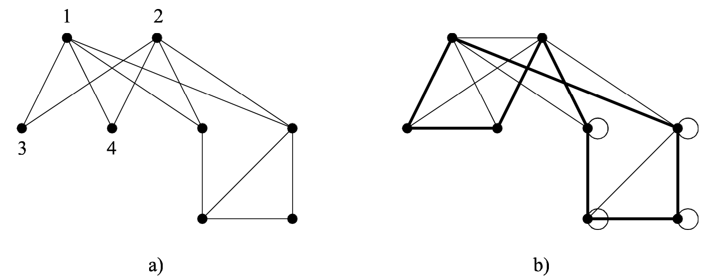

FIGURE 1. a)The graph G, b)The graph bundle $C_1\square^\alpha G$ where $\alpha = (1,2)(3,4)$

Let G be an arbitrary connected graph with the maximum degree $\Delta(G) \geq 2$ and α an arbitrary automorphism of G. Let $\tilde{T}_\alpha$ be a spanning tree of the quotient graph $\tilde{G}_\alpha$. Let $\tilde{d}(w)$ be the degree of the vertex $w = O(v,\alpha)$ in $\tilde{T}_\alpha$ and let $\tilde{o}(w) = o(v,\alpha)$ denote the number of elements in the orbit of the vertex w in $\tilde{T}_\alpha$. Define

$$h(\tilde{T}_\alpha) = \max\left\{ \left\lceil \frac{\tilde{d}(w)}{\tilde{o}(w)} \right\rceil \mid w \in V(\tilde{T}_\alpha) \right\}$$

and

$$\nabla(\tilde{G}_\alpha) = \min\left\{ h(\tilde{T}_\alpha) \mid \tilde{T}_\alpha \text{ is a spanning tree of } \tilde{G}_\alpha \right\}.$$

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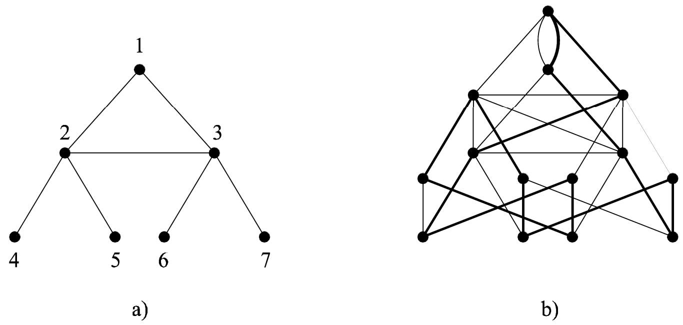

FIGURE 2. a)The graph G, b)The graph bundle $C_2\square^{\alpha}G$ where $\alpha=(2,3)(4,6)(5,7)$

In the sequel we will assume that $\Delta(G)\geq 2$. Among connected graphs this excludes only P_2, the path with one edge, and the graph with one vertex P_1. These two cases are easy and will be treated separately.

First we prove a technical lemma.

Lemma 3.1. *Let $\alpha\in Aut(G)$ and $\tilde{T}_\alpha$ be a spanning tree of the quotient graph $\tilde{G}_\alpha$. Denote the elements of the orbit w with $w_1,w_2,\ldots,w_{\tilde{o}(w)}$. If the orbits w and x are adjacent in $\tilde{T}_\alpha$, let*

$$E(w,x)=\{w_ix_j \mid w_i\in w, x_j\in x\}\subseteq E(G)$$

be the set of edges between the elements of the two orbits. There is a subset

$$F\subseteq \bigcup_{wx\in E(\tilde{T}_\alpha)} E(w,x)\subseteq E(G)$$

such that:

- *if w,x are adjacent in $\tilde{T}_\alpha$ then $F\cap E(w,x)\neq\emptyset$,*
- *each w_i meets at most $\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil$ edges of F.*

Furthermore, there is a proper edge coloring of the subgraph induced on F with at most $h(\tilde{T}_\alpha)$ colors.

Proof. First we will build a set of subforests of $\tilde{T}_\alpha$ as follows. Start with $T_0=\tilde{T}_\alpha$. Take a maximal subforest F_1 of $\tilde{T}_\alpha$ such that the degree of a vertex $w\in V(F_1)$ in F_1 is $\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil$. (This can be done for example using a breadth first search of the graph choosing the first edges met.) Note that when $\tilde{d}(w)\leq\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil$ for every vertex $w\in T_0$, then $F_1=T_0$.

If $F_1\neq T_0$ then continue taking $T_1=T_0-F_1$. Let F_2 be a maximal subforest of T_1 (constructed as above) such that the degree of a vertex $w\in V(F_2)$ in F_2 is at most $\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil$. If $\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil\leq\tilde{d}_{T_1}(w)$, then $\tilde{d}_{F_2}(w)=\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil$. If $\left\lceil\frac{\tilde{d}(w)}{\tilde{o}(w)}\right\rceil>\tilde{d}_{T_1}(w)$, then $\tilde{d}_{F_2}(w)=\tilde{d}_{T_1}(w)$.

Construct in this way further subforests F_i until $T_i=\emptyset$.

Observe that, by construction $\cup F_i=\tilde{T}_\alpha$.

Recall that at each step (except at the least perhaps) the degree of the vertex $w \in T_i$ is decreased by $\left\lceil \frac{\tilde{d}(w)}{\tilde{o}(w)} \right\rceil$. When $\tilde{d}_{T_i}(w) \leq \left\lceil \frac{\tilde{d}(w)}{\tilde{o}(w)} \right\rceil$ for every vertex $w \in T_i$, then $F_{i+1} = T_i$.

Hence, because of

$$\tilde{d}(w) \leq \tilde{o}(w) \left\lceil \frac{\tilde{d}(w)}{\tilde{o}(w)} \right\rceil$$

the vertex w appears in at most $\tilde{o}(w)$ subforests F_i.

Finally we use the subforests to define a subset F of edges of G. Let

$$V_i = \{v \in V(G) \mid v = w_i \text{ for some orbit } w \in V(F_i)\}.$$

For any edge wx of F_i put the edge $w_i x_i$ of G in the set F. By construction the set of edges F is a subforest of G, and maximal degree of a vertex in F is $h(\tilde{T}_\alpha)$. As F is bipartite, edges of F can be properly colored by $h(\tilde{T}_\alpha)$ colors as claimed. □

Theorem 3.2. *Let $X = C_n \square^\alpha G$ be a graph bundle. If $n \geq \nabla(\tilde{G}_\alpha)$, then X is Hamiltonian.*

Proof. Assume $n \geq \nabla(\tilde{G}_\alpha)$. Let $\tilde{T}_\alpha$ be a spanning tree of $\tilde{G}_\alpha$ such that $h(\tilde{T}_\alpha) = \nabla(\tilde{G}_\alpha)$. We construct a Hamilton cycle in X as in Theorem 4.2 of [3]. More precisely, recall that the vertices of the spanning tree $\tilde{T}_\alpha$ are the orbits $O(v, \alpha)$. If $O(v, \alpha)$ is a set of k vertices, then the subgraph of $C_n \square^\alpha G$ induced on the vertex set $V(C_n) \times O(v, \alpha)$ is isomorphic to a cycle C_{nk} or a Möbius ladder, i.e. a cycle C_{nk} with diagonals (as proved in [3]). Start with cycles corresponding to the orbits. Recall that given $\tilde{T}_\alpha$, by Lemma 3.1 there exists a partial proper coloring of edges of G with $h(\tilde{T}_\alpha)$ colors. Now use the edges of F with the edge coloring to construct a Hamiltonian cycle as follows. If two orbits are adjacent, then by construction there is an edge $e = w_i x_j$ of F that meets both orbits. If $c \neq h(\tilde{T}_\alpha) - 1$, then the edge e and its color c define a 4-cycle in X which has one edge in each of the cycles and the other two edges correspond to the edge e in c-th copy and in $(c+1)$-th copy of the fibre. If $c = h(\tilde{T}_\alpha) - 1$, then the other two edges correspond to the edge e in c-th copy of the fibre and to the edge $\alpha(w_i)\alpha(x_j)$ in $(c+1)$-th copy of the fibre. Replacing the two edges of the 4-cycle by the other two parallel edges we get a cycle covering both orbits. Repeating this operation joins all the orbits and consequently covers all the cycles because $\tilde{T}_\alpha$ is a spanning tree. □

We illustrate the construction used in the proof by two examples.

Example 3.3. *Figure 3a) shows the graph G and the quotient graph $\tilde{G}_\alpha$ (Figure 3b) if we take the automorphism $\alpha = (2,3)(4,6)(5,7)$. Therefore $\nabla(\tilde{G}_\alpha) = h(\tilde{G}_\alpha) = \max\left\{1, \left\lceil \frac{3}{2} \right\rceil, \left\lceil \frac{1}{2} \right\rceil\right\} = 2$ and there exists a Hamiltonian cycle in bundle with $n \geq 2$.*

Note that the vertices of $\tilde{G}_\alpha$ represent orbits, i.e. cycles in $C_2 \square^\alpha G$, one of them being isomorphic to C_2, two to C_4 and one to C_4 with diagonals. We join this cycles into a Hamiltonian cycle using the three disjoint 4-cycles as in the proof of theorem above. For example: in Figure 3c) we have two subforests F_1 and F_2 of $\tilde{T}_\alpha$. If the subset F of edges of G is defined with the edges $(2,4), (2,5) \in E(G)$ corresponding to F_1 and $(1,3) \in E(G)$ coresponding to F_2 then we can color the edge $(2,4)$ and $(1,3)$ with color 0 while the edge $(2,5)$ with color 1 (Figure 3d)). In this situation we get the Hamiltonian cycle in Figure 2b).

Example 3.4. *Figure 4 shows: a)the graph G, b)the quotient graph $\tilde{G}_\alpha$ with the automorphism $\alpha = (5,6)(7,8)$ and c)the choosen spanning tree $\tilde{T}_\alpha$. In this case is $h(\tilde{T}_\alpha) = \max\left\{\left\lceil \frac{1}{2} \right\rceil, 2\right\} = 2$. A spanning path in Figure 4c) can be used to construct a Hamiltonian cycle in $C_2\square^\alpha G$ in Figure 4d).*

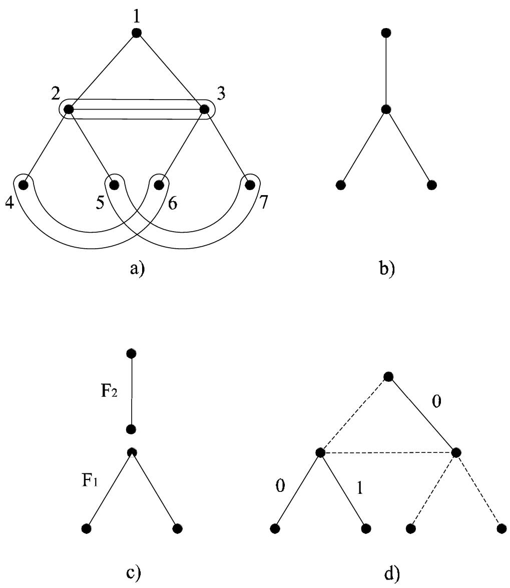

FIGURE 3. a)The graph G with orbits of $\alpha = (2,3)(4,6)(5,7)$, b)The quotient graph $\tilde{G}_\alpha$, c)The two subforests F_1 and F_2 of $\tilde{T}_\alpha$, d)The coloring of F in G is indicated with colors 0 and 1

The example in Figure 1 shows that the converse of Theorem 3.2 is not true. In Figure 5a) it is shown the same graph G once more, on 5b) the quotient graph $\tilde{G}_\alpha$ with the automorphism $\alpha = (1,2)(3,4)$ and on 5c) the choosen spanning tree $\tilde{T}_\alpha$ with $h(\tilde{T}_\alpha) = 2$. It holds $\nabla(\tilde{G}_\alpha) = 2$ but a Hamiltonian cycle exists already in the bundle $C_1\square^\alpha G$ (see Figure 1b)).

Construction of a spanning tree of quotient graph $\tilde{G}_\alpha$ from a Hamiltonian cycle of graph bundle $X = C_n\square^\alpha G$. Let $X = C_n\square^\alpha G$ be a Hamiltonian graph and let $\mathcal{H}$ be a Hamiltonian cycle of X. We form the set A of edges of $\tilde{G}_\alpha$ as follows. Pick an arbitrary vertex $(u,v) \in X (u \in C_n, v \in G)$ and the orbit $O(v,\alpha) \in \tilde{G}_\alpha$ mark as "encountered". X is Hamiltonian so $(u,v) \in \mathcal{H}$. Then walk along $\mathcal{H}$ and do the following: for every two consecutive vertices $(u_1,v_1),(u_2,v_2) \in X$ look at the orbits $O(v_1,\alpha)$ and $O(v_2,\alpha)$. If $O(v_1,\alpha) \neq O(v_2,\alpha)$ and $O(v_2,\alpha)$ has not been marked yet, mark $O(v_2,\alpha)$ as "encountered" and insert the pair $[O(v_1,\alpha), O(v_2,\alpha)]$ in A. We continue with this procedure until we come along all cycle $\mathcal{H}$.

Lemma 3.5. *Let $X = C_n\square^\alpha G$ be a Hamiltonian graph and $\mathcal{H}$ a Hamiltonian cycle of X. The set A described above induces a spanning tree $\tilde{T}_\alpha$ of the quotient graph $\tilde{G}_\alpha$.*

Proof. A pair $[O(v_1,\alpha), O(v_2,\alpha)]$ $(O(v_2,\alpha) \neq O(v,\alpha))$ is inserted in the set A when the first time an element of orbit $O(v_2,\alpha)$ is visited and the orbit $O(v_2,\alpha)$ is marked.

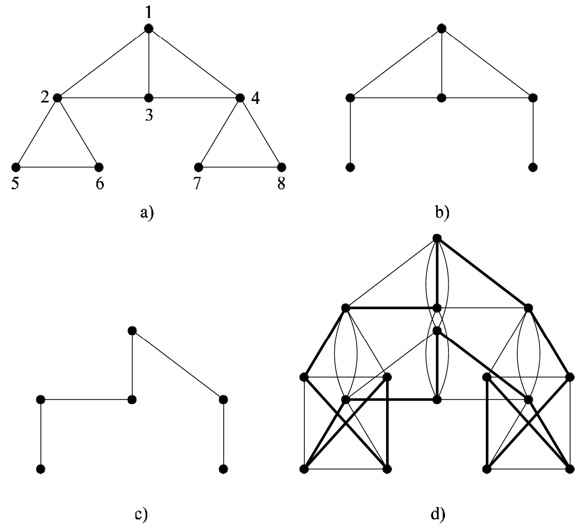

FIGURE 4. a)The graph G, b)The quotient graph $\tilde{G}_\alpha$ with $\alpha = (5,6)(7,8)$, c)The choosen spanning tree $\tilde{T}_\alpha$, d)The Hamiltonian cycle in $C_2 \square^\alpha G$

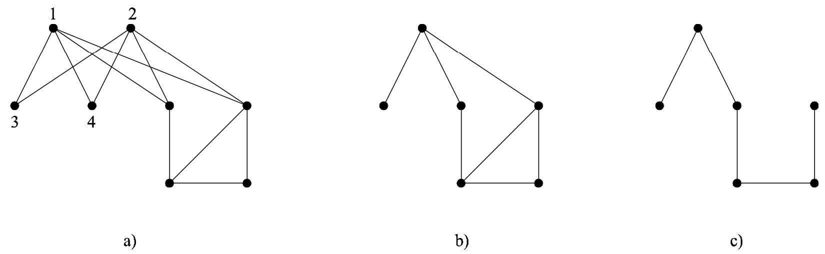

FIGURE 5. a)The graph G, b)The quotient graph $\tilde{G}_\alpha$ with $\alpha = (1,2)(3,4)$, c)The choosen spanning tree $\tilde{T}_\alpha$

Since $\mathcal{H}$ is a Hamiltonian cycle, all vertices of G are visited, so all vertices of quotient graph $\tilde{G}_\alpha$ are marked as "encountered" and the corresponding pairs inserted in A. This fact implies that the pairs in A are distinct and their number is $|V(\tilde{G}_\alpha)| - 1$. Moreover, the pairs in A do not induce a cycle. Therefore, the edges corresponding to the pairs in A form a spanning tree $\tilde{T}_\alpha$ of quotient graph $\tilde{G}_\alpha$. □

Now we can prove the second theorem

Theorem 3.6. *If* $\nabla(\tilde{G}_\alpha) > n+1$, *then* $C_n \square^\alpha G$ *is not Hamiltonian.*

Proof. Suppose for contradiction that $\nabla(\tilde{G}_\alpha) > n+1$, the bundle $X = C_n \square^\alpha G$ is Hamiltonian and let $\mathcal{H}$ be a Hamiltonian cycle of X. Construct the set A as above to obtain a spanning tree $\tilde{T}_\alpha$ as shown in Lemma 3.5. By construction, each orbit is "encountered" at most once, and each time a vertex in the orbit is visited along the walk, another orbit may be "encountered", so the maximal valence of a vertex $w \in \tilde{T}_\alpha$ is $mn+1$, where $m = \tilde{o}(w)$. Hence $\left\lceil \frac{\tilde{d}(w)}{\tilde{o}(w)} \right\rceil = \left\lceil \frac{mn+1}{m} \right\rceil = \left\lceil n + \frac{1}{m} \right\rceil = n+1$

and the valence $\left\lceil \frac{\tilde{d}}{\delta} \right\rceil$ of the spanning tree $\tilde{T}_\alpha$ of $\tilde{G}_\alpha$ is at most $n+1$. Therefore $\nabla(\tilde{G}_\alpha) \leq n+1$, contradiction. □

Let G be an arbitrary connected graph with maximum degree $\Delta(G) \geq 2$ and let $\mathrm{Aut}(G)$ be the group of automorphisms of G. We define the number $\nabla(\tilde{G}_\alpha)_{\min}$ as

$$\nabla(\tilde{G}_\alpha)_{\min} = \min\left\{\nabla(\tilde{G}_\alpha) \mid \alpha \in \mathrm{Aut}(G)\right\}.$$

Using this notation, the statements of Theorems 3.2 and 3.6 can be written as

Theorem 3.7. *For any non-trivial connected graph G with $\Delta(G) \geq 2$,*

$$cbH(G) \leq \nabla(\tilde{G}_\alpha)_{min} \leq cbH(G) + 1.$$

Example 3.8. *Now we turn back to Figure 3. All automorphisms of G are: $\alpha_1 = (2,3)(4,6)(5,7)$, $\alpha_2 = (4,5)$, $\alpha_3 = (6,7)$ (which were already mentioned) and $\alpha_4 = (4,5)(6,7), \alpha_5 = (2,3)(4,7)(5,6)$ and the identity (which were not mentioned yet). For identity we have $\nabla(\tilde{G}_{id}) = 3$, which means that the Cartesian product $C_n \square G$ has a Hamiltonian cycle if $n \geq 3$. Since $\nabla(\tilde{G}_{\alpha_1}) = \nabla(\tilde{G}_{\alpha_4}) = \nabla(\tilde{G}_{\alpha_5}) = 2$ and $\nabla(\tilde{G}_{\alpha_2}) = \nabla(\tilde{G}_{\alpha_3}) = \nabla(\tilde{G}_{id}) = 3$ hence $\nabla(\tilde{G}_\alpha)_{min} = 2$. Hence $cbH(G) = 2$.*

Cyclic bundle Hamiltonicity $cbH(G)$ for the graph G in Figure 4 is 1. Namely, minimal value of $\nabla(\tilde{G}_\alpha)$, where $\alpha \in \mathrm{Aut}(G)$ is when $\alpha = (1,3)(2,4)(5,7)(6,8)$. Then is $\nabla(\tilde{G}_\alpha) = 1$ hence $\nabla(\tilde{G}_\alpha)_{\min} = 1$.

In Figure 5 the graph G has the next automorphisms: $\alpha_1 = (1,2)(3,4), \alpha_2 = (1,2), \alpha_3 = (3,4)$ and identity. In this case $\nabla(\tilde{G}_{\alpha_1}) = \nabla(\tilde{G}_{\alpha_2}) = \nabla(\tilde{G}_{\alpha_3}) = \nabla(\tilde{G}_{id}) = 2$, so $\nabla(\tilde{G}_\alpha)_{\min} = 2$. But as we already mentioned there exists a Hamiltonian cycle in bundle $C_1 \square^{\alpha_1} G$ therefore $cbH(G) = 1$.

Remark 3.9. *Clearly, $C_n \square^\alpha P_1 = C_n \square P_1 = C_n$ is Hamiltonian for any n. There are two automorphisms of P_2: id and exchange of the two vertices, α. $C_n \square P_2$ is Hamiltonian for any $n \geq 2$ and $C_n \square^\alpha P_2$ is Hamiltonian for any n. Hence $cbH(P_2) = 1$ and $cbH(P_1) = 1$.*

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