University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 47 (2009), 1087

INDEPENDENT DOMINATING SETS AND IDOMATIC PARTITIONS IN DIRECT PRODUCTS OF FOUR COMPLETE GRAPHS

Sandi Klavžar Gašper Mekiš

ISSN 1318-4865

Ljubljana, April 14, 2009

Independent dominating sets and idomatic partitions in direct products of four complete graphs

Sandi Klavžar* Gašper Mekiš†

Abstract

Independent dominating sets in the direct product of four complete graphs are considered. Possible types of such sets are classified. The sets in which every pair of vertices agree in exactly one coordinate, called T_1-sets, are explicitly described. It is proved that the direct product of four complete graphs admits a partition into T_1-sets if and only if each factor has at least three vertices and the orders of at least two factors are divisible by 3.

Key words: independent set; dominating set; idomatic partition; direct product of graph; complete graph

1 Introduction

A fall coloring of a graph G is a partition of $V(G)$ into color classes that are (as usual) independent sets, and every vertex u has at least one neighbor in each of the other color classes. Not every graph contains a fall coloring, consider for instance the 5-cycle, so the basic question here is which graphs admit such colorings. It is not difficult to see that fall colorings of G coincide with partitions of the vertex of G into independent dominating sets. Such a partition is in this context known as an *idomatic partition*.

Fall colorings were introduced in [2], but a closely related concept was studied back in 1976 by Cockayne and Hedetniemi [1]: they were interested in the largest number of independent dominating sets contained in a given graph. Fall colorings were further studied in [4] where it is proved that a strongly chordal graph G has a fall coloring if and only if the clique number of G equals the minimum degree in G plus one.

*Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia; Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška 160, 2000 Maribor, Slovenia; Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana. E-mail: `sandi.klavzar@fmf.uni-lj.si`.

†Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana. E-mail: `gasper.mekis@gmail.com`.

In [2] fall colorings of the Cartesian product of graphs were studied. It was further proved that the only idomatic partitions of the direct product of two complete graphs consist from parts in which one coordinate is fixed and the other arbitrary. The authors also posed the problem of characterizing idomatic partitions of direct products of at least three complete graphs. The problem for three factors was solved in [6]. Roughly speaking, there are only two types of independent dominating sets that can be combined into idomatic partitions. The first type has analogous structure as for two factors (that is, fixing one coordinate and having arbitrary the other two coordinates), the other type consists of sets of size four. While the first type always gives idomatic partitions, there are certain restrictions on the size of factors that guarantee such partitions in the second case. Moreover, the two types can also be combined to obtain additional idomatic partitions. There are no other such partitions.

In this paper we focus on independent dominating sets and idomatic partitions of direct products of four complete graphs. Now the variety of types (see the next section for the definition of a type) is already bigger, there are seven possible types. In Section 2 we first fix the notation and terminology and then show that among the seven possible types, three are not possible and for three there exist independent dominating sets. We don't know, however, about the existence in the remaining case. In Section 3 we characterize independent dominating T_1-sets, that is, sets in which every pair of vertices agree in exactly one coordinate. Each such set necessary contains nine vertices whose coordinates can be explicitly described. Then, in Section 4, we prove that the direct product of four complete graphs has an idomatic partition into T_1-sets if and only if the order of at least two factors is divisible by 3.

2 Preliminaries

The *direct product* $G \times H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$ and edges $(g_1, h_1)(g_2, h_2)$, where $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. This graph product is commutative and associative, hence it extends naturally to more than two factors. The direct product of graphs $G_1, \ldots, G_n$ will be denoted $\times_{i=1}^{n} G_i$. It is well-known (cf. [3]) that the direct product of vertex-transitive graphs is vertex-transitive. In the rest we will frequently and implicitly use this fact.

For a complete graph K_n, $n \geq 1$, we will always assume $V(K_n) = [n] = \{0, 1, \ldots, n-1\}$. Let $G = \times_{i=1}^{k} K_{n_i}$ and let $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ be vertices of G. Then let

$$e(u, v) = |\{i \mid u_i = v_i\}|$$

be the number of coordinates in which u and v coincide. With this notation we can state that u and v are adjacent in $G = \times_{i=1}^{k} K_{n_i}$ if and only if $e(u, v) = 0$.

Therefore $I \subseteq V(G)$ is independent if and only if $e(u,v) > 0$ for any $u, v \in I$. Note also that $e(u,v) \leq k-1$ holds for any $u \neq v$.

Now comes the key definition. Let $X \subseteq V(G)$, where $G = \times_{i=1}^{k} K_{n_i}$, and let

$$\{e(u,v) \mid u, v \in X, u \neq v\} = \{j_1, \ldots, j_r\}\,.$$

Then we say that X is a *$T_{j_1,\ldots,j_r}$-set.*

Let $I \subset G = \times_{i=1}^{3} K_{n_i}$ be an independent and dominating set of G. Then I can only be a T_1-set, a T_2-set, or a $T_{1,2}$-set. Valencia-Pabon [6] showed that there is no such T_2-set and described the other two types as well as determined when they form idomatic partitions. We also wish to add that the sporadic example from [2] is a T_1-set.

From now on we will consider direct products of four complete graphs and will index the factors from 0 to 3. We first exclude the following four types.

Proposition 2.1 *Let I be an independent dominating set of $G = \times_{i=0}^{3} K_{n_i}$, $n_i \geq 2$. Then I is not T_2, T_3, $T_{2,3}$ nor $T_{1,3}$.*

Proof. Assume on the contrary that I is a T_2-set or a $T_{2,3}$-set. Since I is dominating, $|I| > 2$. Assume $(0,0,0,0) \in I$. Using vertex-transitivity and the commutativity of the direct product we may also assume that $(0,0,1,1) \in I$. Since $e((0,0,1,1),(1,0,0,0)) = 1$, we get $(1,0,0,0) \notin I$. Hence there exists a vertex $(a,b,c,d) \in I$ such that (a,b,c,d) dominates $(1,0,0,0)$. This in particular implies that $b,c,d \neq 0$. But then $e((a,b,c,d),(0,0,0,0)) \leq 1$, a contradiction.

Suppose next that I is a T_3-set. We may assume that $\{(0,0,0,0),(0,0,0,1)\} \subseteq I$. Clearly, $(0,0,1,1) \notin I$, hence there exists a vertex $(a,b,c,d) \in I$ that dominates $(0,0,1,1)$. But then $a,b \neq 0$ and thus $e((a,b,c,d),(0,0,0,0)) \leq 2$.

In the last case let I be a $T_{1,3}$ set. Again, assume that $\{(0,0,0,0),(0,0,0,1)\} \subseteq I$. From $e((0,0,0,1),(0,0,1,0)) = 2$ we deduce that $(0,0,1,0) \notin I$. Hence there is a vertex $(a,b,c,d) \in I$ that dominates $(0,0,1,0)$. The elements a, b, and d are different from 0, therefore $c = 0$ due to the independence of I. Moreover, $e((0,0,0,1),(a,b,0,1)) = 2$, hence $d \neq 1$. Since we already know that $d \neq 0$, the fourth factor must contain more than 2 vertices, for otherwise no element from I would dominate $(0,0,1,0)$. We next consider the vertex $(0,0,0,d)$, $d \notin \{0,1\}$. Suppose that $(0,0,0,d) \notin I$. I is dominating, so there exists a vertex $(e,f,g,h) \in I$, such that $e,f,g \neq 0$ and $h \neq d$. But then the vertex (e,f,g,h) is adjacent with at least one of the vertices $(0,0,0,0),(0,0,0,1) \in I$, a contradiction. Thus we must have $(0,0,0,d) \in I$. The proof is concluded by observing that $e((0,0,0,d),(a,b,0,d)) = 2$. $\square$

So we are left with the possible types $T_1, T_{1,2}, T_{1,2,3}$ and all three of them are achievable.

We have already mentioned that for two and three factors one can construct independent dominating sets by fixing one coordinate. This is true for any number of factors, in particular the vertex subset of $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$, $n_i \geq 2$, defined with

$$I = [n_0] \times [n_1] \times [n_2] \times \{i\} ,$$

where $i \in [n_3]$, is independent and dominating. Moreover, I is a $T_{1,2,3}$-set. Of course, we could fix any of the four coordinates in the above construction. But there are additional sporadic independent dominating sets that are $T_{1,2,3}$. Consider $K_2 \times K_2 \times K_2 \times K_2$, then

$$\begin{aligned} I_1 &= \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0), \\ &\quad (0,0,0,1),(1,1,0,0),(1,0,1,0),(1,0,0,1)\} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \{(1,1,1,1),(0,1,1,1),(1,0,1,1),(1,1,0,1), \\ &\quad (1,1,1,0),(0,0,1,1),(0,1,0,1),(0,1,1,0)\} \end{aligned}$$

are both independent dominating $T_{1,2,3}$-sets. In addition, they form an idomatic partition.

We conclude the section with an idomatic partition into $T_{1,2}$-sets. Let $G = K_2 \times K_2 \times K_2 \times K_4$ and consider the set

$$\begin{aligned} I_1 &= \{(0,0,0,0),(1,1,1,0),(0,1,1,1),(1,0,0,1), \\ &\quad (1,0,1,2),(0,1,0,2),(1,1,0,3),(0,0,1,3)\}. \end{aligned}$$

Clearly, I_1 is an independent $T_{1,2}$-set. But it is also dominating. To see it, note first that G consists of four connected components isomorphic to $K_2 \times K_4$ which is in turn isomorphic to the 3-cube Q_3. Then consecutive pairs of vertices dominate the four copies of Q_3 respectively. Finally, the sets I_1, I_2, I_3, I_4, where

$$\begin{aligned} I_2 &= \{(u_1 + 1 \bmod 2, u_2, u_3, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\} , \\ I_3 &= \{(u_1, u_2 + 1 \bmod 2, u_3, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\} , \\ I_4 &= \{(u_1, u_2, u_3 + 1 \bmod 2, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\} , \end{aligned}$$

form an idomatic partition of G.

3 Independent dominating T_1-sets

In this section we prove the following:

Theorem 3.1 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$, $n_i \geq 2$, and let I be an independent T_1-set of G. Then I is a dominating set if and only if $n_i \geq 3$ and I is of the form*

$$
\begin{aligned}
I = \{ & (\alpha_0, \alpha_1, \alpha_2, \alpha_3), (\alpha_0, \beta_1, \beta_2, \beta_3), (\alpha_0, \gamma_1, \gamma_2, \gamma_3), \\
& (\beta_0, \alpha_1, \beta_2, \gamma_3), (\beta_0, \gamma_1, \alpha_2, \beta_3), (\beta_0, \beta_1, \gamma_2, \alpha_3), \\
& (\gamma_0, \alpha_1, \gamma_2, \beta_3), (\gamma_0, \beta_1, \alpha_2, \gamma_3), (\gamma_0, \gamma_1, \beta_2, \alpha_3)\}\,,
\end{aligned}
$$

where $\alpha_i, \beta_i, \gamma_i \in K_{n_i}$ are pairwise different, $0 \leq i \leq 3$.

Proof. We will prove Theorem 3.1 in two steps. In the first, major step, we assume that $n_i \geq 3$, $0 \leq i \leq 3$.

So let I be an independent T_1-set of G and without loss of generality assume that $(0, 0, 0, 0) \in I$. Clearly, $|I| > 1$, hence we may further assume that $(0, \beta_1, \beta_2, \beta_3) \in I$, where $\beta_1, \beta_2, \beta_3 \neq 0$.

If $w \neq 0$ then $(0, 0, 0, w) \notin I$, hence there exists a vertex $(\beta_0, b, c, d) \in I$, where $\beta_0, b, c \neq 0$ and $d \neq w$. Since the latter vertex is not adjacent to $(0, 0, 0, 0)$ we get $d = 0$. In addition, it is also not adjacent to $(0, \beta_1, \beta_2, \beta_3)$, hence either $b = \beta_1$ and $c \neq \beta_2$, or $b \neq \beta_1$ and $c = \beta_2$. In the first case set $c = \gamma_2$, in the second $b = \gamma_1$. This gives the following two possibilities:

$$A_1 = \{(0, 0, 0, 0), (0, \beta_1, \beta_2, \beta_3), (\beta_0, \beta_1, \gamma_2, 0)\} \subseteq I$$

and

$$A_2 = \{(0, 0, 0, 0), (0, \beta_1, \beta_2, \beta_3), (\beta_0, \gamma_1, \beta_2, 0)\} \subseteq I.$$

Similarly, the vertex $(0, y, 0, 0)$ does not belong to I hence there exists $(a, b, c, d) \in I$ (for some a, b, c, d different as above) that dominates $(0, y, 0, 0)$. As $(0, 0, 0, 0) \in I$, we infer that $b = 0$. Now comparing $(a, 0, c, d)$ with $(0, \beta_1, \beta_2, \beta_3)$ we obtain either (i) $(a, 0, c, d) = (a, 0, \beta_2, d), d \neq \beta_3$, or (ii) $(a, 0, c, d) = (a, 0, c, \beta_3), c \neq \beta_2$.

Consider case (i). Comparing $(a, 0, \beta_2, d)$ $(d \neq \beta_3)$ with the third vertex of A_1 we get $a = \beta_0$. Since $0 \neq d \neq \beta_3$ we can set $d = \gamma_3$. Similarly, comparing $(a, 0, \beta_2, d)$ with the third vertex of A_2 we obtain $0 \neq a \neq \beta_0$ and $0 \neq d \neq \beta_3$, hence we can set $a = \gamma_0$ and $d = \gamma_3$.

Consider case (ii). In this case $(a, 0, c, \beta_3)$ $(c \neq \beta_2)$ is a candidate for a vertex from I. In that case, we must have either $a \neq \beta_0$ and $c = \gamma_2$, or $a = \beta_0$ and $\beta_2 \neq c \neq \gamma_2$. In the latter case set $c = \delta_2$. Note that when $n_2 = 3$ this is not possible there are only three coordinates available for the K_{n_2} factor. Comparing $(a, 0, c, \beta_3)$ with the third vertex of A_2 we find that $a = \beta_0$. Since $0 \neq c \neq \beta_2$ we can set $c = \gamma_2$. Hence we have altogether obtained 5 possible subsets of I:

$$\begin{aligned}
B_1 &= A_1 \cup \{(\beta_0, 0, \beta_2, \gamma_3)\},\\
B_2 &= A_2 \cup \{(\gamma_0, 0, \beta_2, \gamma_3)\},\\
B_3 &= A_1 \cup \{(\gamma_0, 0, \gamma_2, \beta_3)\},\\
B_4 &= A_1 \cup \{(\beta_0, 0, \delta_2, \beta_3)\},\\
B_5 &= A_2 \cup \{(\beta_0, 0, \gamma_2, \beta_3)\}.
\end{aligned}$$

We next take into account that $(0, 0, z, 0) \notin I$ for any $z \neq 0$. Hence there is a vertex $(a, b, c, d) \in I$, $a, b, d \neq 0$, $c \neq z$. Comparing it with $(0, 0, 0, 0)$ we get $c = 0$. Since it is not adjacent to $(0, \beta_1, \beta_2, \beta_3)$, either (i) $b = \beta_1$ and $d \neq \beta_3$, or (ii) $b \neq \beta_1$ and $d = \beta_3$.

We first consider case (i), that is, we will compare the vertex $(a, \beta_1, 0, d)$ with $B_1, \ldots, B_5$. Since $e((a, \beta_1, 0, d), (\beta_0, \beta_1, \gamma_2, 0)) = 1$ we infer $a \neq \beta_0$. Hence, as in the set B_1 no γ_0 is present we can set $a = \gamma_0$. Considering further the vertex $(\beta_0, 0, \beta_2, \gamma_3) \in B_1$ we find that $d = \gamma_3$, for otherwise $e((\gamma_0, \beta_1, 0, d), (\beta_0, 0, \beta_2, \gamma_3)) = 0$. Set

$$C_1 = B_1 \cup \{(\gamma_0, \beta_1, 0, \gamma_3)\}.$$

Next, compare $(a, \beta_1, 0, d)$ with B_2. Then $a = \beta_0$ because it is not adjacent to $(\beta_0, \gamma_1, \beta_2, 0)$. In addition, $d = \gamma_3$ as $(\beta_0, \beta_1, 0, d)$ is not adjacent to $(\gamma_0, 0, \beta_2, \gamma_3)$. So we have the following possibility:

$$C_2 = B_2 \cup \{(\beta_0, \beta_1, 0, \gamma_3)\}.$$

Next, compare $(a, \beta_1, 0, d)$ with $(\gamma_0, 0, \gamma_2, \beta_3) \in B_3$. Since $d \neq \beta_3$ we must have $a = \gamma_0$, while d can be denoted with γ_3, yielding

$$C_3 = B_3 \cup \{(\gamma_0, \beta_1, 0, \gamma_3)\}.$$

Now compare $(a, \beta_1, 0, d)$ with $(\beta_0, 0, \delta_2, \beta_3)$. Since $d \neq \beta_3$ we have $a = \beta_0$. But then $e((\beta_0, \beta_1, 0, d), (\beta_0, \beta_1, \gamma_2, 0)) = 2$, a contradiction. Finally, compare $(a, \beta_1, 0, d)$ with $(\beta_0, \gamma_1, \beta_2, 0) \in B_5$. Since $e((a, \beta_1, 0, d), (\beta_0, \gamma_1, \beta_2, 0)) = 1$, we have $a = \beta_0$. As $0 \neq d \neq \beta_3$ and γ_3 is not present in B_5 we can set $d = \gamma_3$ thus yielding

$$C_4 = B_5 \cup \{(\beta_0, \beta_1, 0, \gamma_3)\}.$$

We next consider case (ii) by comparing $(a, b, 0, \beta_3)(b \neq \beta_1)$ with the sets $B_1, \ldots, B_5$. The arguments are similar as above. In particular, also this vertex is not compatible with B_4. The other four cases give the following possibilities:

$$\begin{aligned}
C_5 &= B_1 \cup \{(\beta_0, \gamma_1, 0, \beta_3)\},\\
C_6 &= B_2 \cup \{(\gamma_0, \gamma_1, 0, \beta_3)\},\\
C_7 &= B_3 \cup \{(\beta_0, \gamma_1, 0, \beta_3)\},\\
C_8 &= B_5 \cup \{(\gamma_0, \gamma_1, 0, \beta_3)\},
\end{aligned}$$

where γ_1 is introduced into C_5 and C_7, and γ_0 into C_8.

To complete the proof of necessity part of the proof we claim that in any of the above 8 cases the set I is as stated in the lemma. More precisely, if one of the sets C_1, C_3, C_5, C_7 is contained in I, then the set

$$\begin{gathered}\{(0,0,0,0),(0,\beta_1,\beta_2,\beta_3),(0,\gamma_1,\gamma_2,\gamma_3),\\(\beta_0,0,\beta_2,\gamma_3),(\beta_0,\beta_1,\gamma_2,0),(\beta_0,\gamma_1,0,\beta_3),\\(\gamma_0,0,\gamma_2,\beta_3),(\gamma_0,\beta_1,0,\gamma_3),(\gamma_0,\gamma_1,\beta_2,0)\}\end{gathered}$$

is contained in I. And if one of C_2, C_4, C_6, C_8 is contained in I, then eventually the set

$$\begin{gathered}\{(0,0,0,0),(0,\beta_1,\beta_2,\beta_3),(0,\gamma_1,\gamma_2,\gamma_3),\\(\beta_0,0,\gamma_2,\beta_3),(\beta_0,\beta_1,0,\gamma_3),(\beta_0,\gamma_1,\beta_2,0),\\(\gamma_0,0,\beta_2,\gamma_3),(\gamma_0,\beta_1,\gamma_2,0),(\gamma_0,\gamma_1,0,\beta_3)\}\end{gathered}$$

is contained in I. Note that both of these sets are as claimed in the statement of the lemma, which can be seen by observing that exchanging β_0 with γ_0 in the second set yields the first one. Observe also that such a set is a maximal independent set with respect to the property that $e(u,v)=1$ for any of its different vertices u and v.

We are going to prove the above claim for C_1. The still missing vertices are

$$(0,\gamma_1,\gamma_2,\gamma_3),(\beta_0,\gamma_1,0,\beta_3),(\gamma_0,0,\gamma_2,\beta_3) \text{ and } (\gamma_0,\gamma_1,\beta_2,0),$$

where $\gamma_1 \in V(K_{n_1}) - \{0,\beta_1\}$ has not yet been introduced.

Assume $(\gamma_0,0,\gamma_2,\beta_3) \notin I$. Then there exists $(a,b,c,d) \in I$ such that $a \neq \gamma_0, b \neq 0, c \neq \gamma_2$ and $d \neq \beta_3$. Since $e((a,b,c,d),(0,0,0,0)) = 1$, exactly one of a, c and d equals 0. If $a=0$, then we compare $(0,b,c,d)$ with $(\beta_0,\beta_1,\gamma_2,0) \in C_1$ to realize that they can only be equal in the second coordinate, that is, $b=\beta_1$. It follows that $(0,\beta_1,c,d) = (0,\beta_1,\beta_2,\beta_3) \in C_1$, which is not possible because $d \neq \beta_3$. Suppose $c=0$. Since $e((a,b,0,d),(0,\beta_1,\beta_2,\beta_3)) = 1$, we have $b=\beta_1$. Then $(a,\beta_1,0,d) = (\gamma_0,\beta_1,0,\gamma_3) \in C_1$, another contradiction. Suppose finally $d=0$. Since $e((a,b,c,0),(\gamma_0,\beta_1,0,\gamma_3)) = 1$, we get $b=\beta_1$ and thus $(a,b,c,0) = (\beta_0,\beta_1,\gamma_2,0) \in C_1$, the final contradiction. We conclude that $(\gamma_0,0,\gamma_2,\beta_3) \in I$.

To prove that the remaining three vertices necessarily lie in I, some more efforts are needed. Clearly, $(0,0,\gamma_2,\gamma_3) \notin I$. Note also that this vertex is not dominated with any of the vertices from $C_1 \cup \{(\gamma_0,0,\gamma_2,\beta_3)\}$. Hence there exists $(a,b,c,d) \in I$ with $a,b \neq 0, c \neq \gamma_2$, and $d \neq \gamma_3$. Since $e((a,b,c,d),(0,0,0,0)) = 1$, either $c=0$ or $d=0$. Assume first $c=0$. Comparing $(a,b,0,d)$ with $(\beta_0,0,\beta_2,\gamma_3)$ yields $a=\beta_0$. Since $(\beta_0,b,0,d)$ and $(\beta_0,\beta_1,\gamma_2,0)$ already coincide in one position and differ in two positions, we find that $b \neq \beta_1$. Hence we can set $b=\gamma_1$.

Comparing $(\beta_0, \gamma_1, 0, d)$ with $(0, \beta_1, \beta_2, \beta_3)$ gives $d = \beta_3$. We conclude that

$$C_{1.1} = C_1 \cup \{(\gamma_0, 0, \beta_2, \gamma_3)\} \cup \{(\beta_0, \gamma_1, 0, \beta_3)\} \subseteq I.$$

Assume next $d = 0$. Then comparing $(a, b, c, 0)$ with $(\gamma_0, 0, \gamma_2, \beta_3)$ we get $a = \gamma_0$. Since $(\gamma_0, b, c, 0)$ and $(\beta_0, \beta_1, \gamma_2, 0) \in C_1$ already coincide in one position and differ in two positions, $b \neq \beta_1$. Hence we can introduce $b = \gamma_1$. We next find that $c = \beta_2$ by comparing $(\gamma_0, \gamma_1, c, 0)$ with $(\beta_0, 0, \beta_2, \gamma_3)$. So we have another possibility for a subset of I:

$$C_{1.2} = C_1 \cup \{(\gamma_0, 0, \beta_2, \gamma_3)\} \cup \{(\gamma_0, \gamma_1, \beta_2, 0)\} \subseteq I.$$

Suppose $C_{1.1} \subseteq I$. Then we need to prove that $(\gamma_0, \gamma_1, \beta_2, 0)$ and $(0, \gamma_1, \gamma_2, \gamma_3)$ belong to I. Assume on the contrary that $(\gamma_0, \gamma_1, \beta_2, 0) \notin I$. Then there is a vertex $(a, b, c, d) \in I$ such that $a \neq \gamma_0, b \neq \gamma_1, c \neq \beta_2$, and $d \neq 0$. Since $e((a, b, c, d), (0, 0, 0, 0)) = 1$ and as $d \neq 0$, exactly one of a, b, c equals 0. In the first case compare $(0, b, c, d)$ with $(\beta_0, \gamma_1, 0, \beta_3) \in C_{1.1}$. It follows that $d = \beta_3$. Then, since $e((0, b, c, \beta_3), (0, \beta_1, \beta_2, \beta_3)) \geq 2$, these two vertices must be the same. But this means that $c = \beta_2$, a contradiction. Assume next $b = 0$. Then comparing $(a, 0, c, d)$ with $(\gamma_0, \beta_1, 0, \gamma_3) \in C_{1.1}$ we find that $d = \gamma_3$. It follows that $(a, 0, c, \gamma_3) = (\beta_0, 0, \beta_2, \gamma_3)$, another contradiction because $c \neq \beta_2$. In the last case, $c = 0$, compare $(a, b, 0, d)$ with $(\gamma_0, 0, \gamma_2, \beta_3) \in C_{1.1}$ to see that $d = \beta_3$. Therefore $(a, b, 0, \beta_3) = (\beta_0, \gamma_1, 0, \beta_3)$, the final contradiction, since $b \neq \gamma_1$. We conclude that $(\gamma_0, \gamma_1, \beta_2, 0) \in I$.

We proceed similarly for the vertex $(0, \gamma_1, \gamma_2, \gamma_3)$ and assume that it does not belong to I. Then there exists $(a, b, c, d) \in I$, such that $a \neq 0, b \neq \gamma_1, c \neq \gamma_2$, and $d \neq \gamma_3$. Since $e((a, b, c, d), (0, 0, 0, 0)) = 1$, exactly one of b, c, d equals 0. In the first case compare $(a, 0, c, d)$ with $(\beta_0, \beta_1, \gamma_2, 0) \in C_{1.1}$ to get $a = \beta_0$. But then $(\beta_0, 0, c, d) = (\beta_0, 0, \beta_2, \gamma_3)$, which is not possible since $d \neq \gamma_3$. In the second case we have $a = \beta_0$ by considering $(a, b, 0, d)$ and $(\beta_0, 0, \beta_2, \gamma_3) \in C_{1.1}$. Then $(\beta_0, b, 0, d) = (\beta_0, \gamma_1, 0, \beta_3)$, which is not possible since $b \neq \gamma_1$. Finally, if $d = 0$, we have $a = \gamma_0$ (compare $(a, b, c, 0)$ with $(\gamma_0, 0, \gamma_2, \beta_3) \in C_{1.1}$), therefore $(\gamma_0, b, c, 0) = (\gamma_0, \gamma_1, \beta_2, 0)$. Another contradiction, since $b \neq \gamma_1$. We conclude that $(0, \gamma_1, \gamma_2, \gamma_3) \in I$ and hence I is as required.

Assume $C_{1.2} \subseteq I$. Then we need to show that $(\beta_0, \gamma_1, 0, \beta_3)$ and $(0, \gamma_1, \gamma_2, \gamma_3)$ belong to I. Suppose $(\beta_0, \gamma_1, 0, \beta_3) \notin I$. Then there exists $(a, b, c, d) \in I$, such that $a \neq \beta_0, b \neq \gamma_1, c \neq 0$, $d \neq \beta_3$, and exactly one of a, b, d equals 0. If $a = 0$ then $c = \beta_2$ by considering $(0, b, c, d)$ and $(\gamma_0, \gamma_1, \beta_2, 0)$. But then $(0, b, \beta_2, d) = (0, \beta_1, \beta_2, \beta_3)$, contradicting $d \neq \beta_3$. If $b = 0$, then consider $(a, 0, c, d)$ and $(0, \beta_1, \beta_2, \beta_3)$ to get $c = \beta_2$. Now $(a, 0, \beta_2, d) = (\beta_0, 0, \beta_2, \gamma_3)$, contradicting $a \neq \beta_0$. Finally, if $d = 0$, then $(a, b, c, 0)$ and $(\beta_0, 0, \beta_2, \gamma_3)$ give $c = \beta_2$. But then $(a, b, \beta_2, 0) = (\gamma_0, \gamma_1, \beta_2, 0)$, which contradicts $b \neq \gamma_1$. Therefore, $(\beta_0, \gamma_1, 0, \beta_3) \in I$.

Suppose $(0, \gamma_1, \gamma_2, \gamma_3) \notin I$. Then we have $(a, b, c, d) \in I$, where $a \neq 0, b \neq \gamma_1, c \neq \gamma_2$, $d \neq \gamma_3$, and one of b, c, d is 0. If $b = 0$ then $a = \gamma_0$ (consider $(a, 0, c, d)$ and $(\gamma_0, \beta_1, 0, \gamma_3)$), and therefore $(\gamma_0, 0, c, d) = (\gamma_0, 0, \gamma_2, \beta_3)$, contradicting $c \neq \gamma_2$. If $c = 0$ we get $a = \beta_0$ (consider $(a, b, 0, d)$ and $(\beta_0, 0, \beta_2, \gamma_3)$), and so $(\beta_0, b, 0, d) = (\beta_0, \gamma_1, 0, \beta_3)$, contradicting $b \neq \gamma_1$. Finally, for $d = 0$ we have $a = \gamma_0$ (consider $(a, b, c, 0)$ and $(\gamma_0, 0, \gamma_2, \beta_3)$). Now $(\gamma_0, b, c, 0) = (\gamma_0, \gamma_1, \beta_2, 0)$, contradicting $b \neq \gamma_1$. We conclude that $(0, \gamma_1, \gamma_2, \gamma_3) \in I$ and hence also in this case I is as claimed.

The proofs for the remaining 7 cases, that is, for the sets C_i, $i = 2, 3, ..., 8$, go along the same lines as the above proof for C_1. In each of these sets we miss four vertices and exactly one of the integers $\gamma_0, \gamma_1, \gamma_2$, and γ_3. (In the above case, γ_1 was missing.) Now, exactly one among the four missing vertices does not contain the missing integer (above such a vertex is $(\gamma_0, 0, \gamma_2, \beta_3)$). Then we prove for this vertex that belongs to I. We continue by selecting a vertex not in I that coincides with the previous six vertices in at least one position, and coincides in at least one position also with one of the three missing vertices. (Above the vertex $(0, 0, \gamma_2, \gamma_3)$ played this role.) Now we proceed as above and detect the remaining two vertices. We note that the order in which we prove the inclusion of these two vertices is essential because the inclusion of one of them is needed to prove the inclusion of the other.

Conversely, assume that I contains 9 vertices as described in the statement. We need to prove that an arbitrary vertex $x = (\delta_0, \delta_1, \delta_2, \delta_3) \notin I$ is dominated by at least one vertex from I.

Note first that for each i, each of the α_i, β_i, and γ_i, appears in exactly three vertices from I. If, say, u, v, and w contain α_i, then $e(u, v) = 1$, $e(u, w) = 1$, $e(v, w) = 1$, and, moreover, u, v, and w coincide in the position where α_i stands. We now distinguish four cases.

Suppose $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 0$. Then $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ dominates x.

Let $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 3$ and let $\delta_k \neq \alpha_k$, where $k \in \{0, 1, 2, 3\}$. Then x is dominated by the two vertices that contain α_k and do not contain α_i for $i \neq k$.

The next case is $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 2$. Let $\delta_k \neq \alpha_k$ and $\delta_\ell \neq \alpha_\ell$, where $k, \ell \in \{0, 1, 2, 3\}$. Consider the two vertices that contain α_k and none of the remaining α_i's. These two vertices can coincide with x only in δ_ℓ. Since they differ in the corresponding position (which is because they agree in α_k), one of them dominates x.

The last case to consider is when $e(x, (\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 1$. Let $\delta_k = \alpha_k$. Then $\delta_i \neq \alpha_i$ for $i \neq k$. Assume that no vertex from I dominates x. Let I_1 be the set of the three vertices from I that contain β_k and let I_2 be the set of the three vertices that contain γ_k. Since x is dominated by no vertex from $I_1 \cup I_2$, we have three positions to agree with each of them. For such a fixed position, x can agree with at most one vertex from I_1 and with at most one vertex from I_2. Hence only in the optimal case we can have $e(x, u) \geq 1$ for each $u \in I_1 \cup I_2$. This in particular implies that $\delta_i \in \{\beta_i, \gamma_i\}$ for any $i \neq k$. No two of the δ_i's, where

$i \neq k$, appear in the same vertex from $I_1 \cup I_2$, for otherwise we would have two vertices from this set that would agree on two positions. Note further that for any two integers r and s that appear as coordinates of vertices from I, there is exactly one vertex in I that contains r and s. There are three pairs of integers of the form $\{\delta_i, \delta_j\}$, where $i, j \neq k$. None of these two such integers appear in the same vertex from $I_1 \cup I_2$. Hence these pairs must appear on the two vertices from $I \setminus (I_1 \cup I_2 \cup \{(0,0,0,0)\})$. Therefore, two of these pairs appear on the same vertex, but this means that x is equal to one them, the final contradiction which completes the first step of the proof.

It remains to prove that if $2 \in \{n_i, 0 \leq i \leq 3\}$, then G contains no independent dominating T_1-set. Suppose I is such a set and assume $(0,0,0,0) \in I$. We may also without loss of generality assume $n_3 = 2$. Since $(1,0,0,0) \notin I$, there exists a vertex $(a,b,c,d) \in I$ such that $a \neq 1$ and $b, c, d \neq 0$. Hence $d = 1$ and because $e((a,b,c,1),(0,0,0,0)) = 1$ we have $a = 0$. Thus

$$\{(0,0,0,0),(0,b,c,1)\} \subseteq I.$$

Consider next $(0,1,0,0) \notin I$. Then there exists $(e,f,g,h) \in I$ that dominates $(0,1,0,0)$. Similarly as above we get $h = 1$ and $f = 0$. Since $e((e,0,g,1),(0,0,0,0)) = 1$ and $e((e,0,g,1),(0,b,c,1)) = 1$ we also have $g \neq 0$ and $g \neq c$. If $n_2 = 2$ this is a contradiction because no vertex of I dominates $(0,1,0,0)$. So let $n_2 > 2$. Then

$$\{(0,0,0,0),(0,b,c,1),(e,0,g,1)\} \subseteq I.$$

Now $(0,0,1,0) \notin I$, hence there is a vertex $(k,l,m,n) \in I$, such that $k, l, n \neq 0$ and $m \neq 1$. Similarly as above we find that

$$\{(0,0,0,0),(0,b,c,1),(e,0,g,1),(k,l,0,1)\} \subseteq I.$$

If $n_0 = 2$ or $n_1 = 2$ we have a contradiction because $0, e, k$ are pairwise different as well as are $0, b, l$. So let $n_0 \geq 3$ and $n_1 \geq 3$. Since $(0,l,g,1) \notin I$, there is a vertex $(x,y,z,w) \in I$ with $x \neq 0$, $y \neq l$, $z \neq g$, and $w = 0$. The set I is T_1 hence $(x,y,z,0) = (e,l,c,0)$ or $(x,y,z,0) = (k,b,g,0)$. But both possibilities are impossible because $y \neq l$ and $z \neq g$, respectively. □

4 Idomatic partitions into T_1-sets

We next turn our attention to idomatic partitions of products of four complete graphs and characterize the products which admit such partitions into T_1-sets as follows:

Theorem 4.1 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$, $n_i \geq 2$. Then G has an idomatic partition into T_1-sets if and only if $n_i \geq 3$ and there exist indices $j, k \in [4]$, $j \neq k$, such that $3|n_j$ and $3|n_k$.*

In the rest of the section we prove this theorem, where the arguments are in part parallel to those from [6].

Note first that by Theorem 3.1, $n_i \geq 3$ is a necessary condition for the existence of an idomatic partition into T_1-sets. Hence assume in the rest that this is the case.

Suppose that G has an idomatic partition into T_1-sets. By Theorem 3.1, each part (every T_1-set) in an idomatic partition into T_1-sets has nine vertices, and thus 9 is a divisor of $n_0n_1n_2n_3$, so there exists at least one $j \in [4]$ such that $3|n_j$. By the commutativity of the direct product, we can assume that $j = 3$. Let I_ℓ be a T_1-set of our idomatic partition. By Theorem 3.1, I_ℓ is of the form

$$\begin{aligned}\{&(\alpha_0, \alpha_1, \alpha_2, \alpha_3), (\alpha_0, \beta_1, \beta_2, \beta_3), (\alpha_0, \gamma_1, \gamma_2, \gamma_3),\\ &(\beta_0, \alpha_1, \beta_2, \gamma_3), (\beta_0, \gamma_1, \alpha_2, \beta_3), (\beta_0, \beta_1, \gamma_2, \alpha_3),\\ &(\gamma_0, \alpha_1, \gamma_2, \beta_3), (\gamma_0, \beta_1, \alpha_2, \gamma_3), (\gamma_0, \gamma_1, \beta_2, \alpha_3)\}\end{aligned}$$

for some pairwise different $\alpha_i, \beta_i, \gamma_i \in [n_i]$, $0 \leq i \leq 3$. The number of vertices of the form (x, y, z, α_3) with fixed $\alpha_3 \in [n_3]$ in G is exactly $n_0n_1n_2$. In every T_1-set of the partition in which α_3 occurs, there are exactly three vertices $(x, y, z, \alpha_3), (x', y', z', \alpha_3)$ and $(x'', y'', z'', \alpha_3)$, where x, x', and x'' are pairwise different and so are y, y', and y''. Hence we must be able to partition the $n_0n_1n_2$ vertices containing α_3 into triples of vertices described above. Therefore, $3|n_0n_1n_2$.

Conversely, assume that there exist indices $j, k \in [4]$, $j \neq k$, such that $3|n_j$ and $3|n_k$. The graph $G = K_{n_0} \times K_{n_1} \times K_{n_2} \times K_{n_3}$ can be seen as the Cayley graph associated with the direct product group $\mathcal{G} = Z_{n_0} \times Z_{n_1} \times Z_{n_2} \times Z_{n_3}$ with connector set $\big([n_0]\backslash\{0\}\big) \times \big([n_1]\backslash\{0\}\big) \times \big([n_2]\backslash\{0\}\big) \times \big([n_3]\backslash\{0\}\big)$, where Z_{n_i} denotes the additive cyclic group of the integers modulo n_i. Using the commutativity of the direct product again, we can assume that $j = 2$ and $k = 3$. Let a_j be the element of order $\frac{n_j}{3}$ in the group Z_{n_j} for $j \in \{2, 3\}$. Let $H_0 = \langle(1, 0, 0, 0)\rangle$ denote the cyclic subgroup of $\mathcal{G}$ generated by the element $(1, 0, 0, 0)$. Similarly, let $H_1 = \langle(0, 1, 0, 0)\rangle$, $H_2 = \langle(0, 0, a_2, 0)\rangle$ and $H_3 = \langle(0, 0, 0, a_3)\rangle$. It is obvious that $H_i \cap H_j = \{(0, 0, 0, 0)\}$ for $i, j \in [4]$, $i \neq j$. As $\mathcal{G}$ is abelian it follows that $H_0H_1H_2H_3 = \{h_0 + h_1 + h_2 + h_3 \mid h_i \in H_i \text{ for } i \in [4]\}$ is a subgroup of order $\frac{n_0n_1n_2n_3}{9}$ in $\mathcal{G}$. Let us use the notation $r = \frac{n_0n_1n_2n_3}{9}$ and $P = H_0H_1H_2H_3 = \{p_1, p_2, ..., p_r\}$, where we may without loss of generality assume that $p_1 = (0, 0, 0, 0)$. Let β_k and γ_k be any elements in $Z_{n_k} \setminus \{0\}$, with $\beta_k \neq \gamma_k$ for $k \in \{0, 1\}$, and let β_j and γ_j be any elements in $Z_{n_j} \setminus \{0\}$, with $\beta_j \neq \gamma_j$; $\beta_j, \gamma_j \notin \langle a_j\rangle$; and if $a_j \neq 0$ then

$\beta_j \not\equiv \gamma_j \bmod a_j$, $j \in \{2, 3\}$. By standard group theory arguments,

$$\begin{gathered} P, (0, \beta_1, \beta_2, \beta_3) + P, (0, \gamma_1, \gamma_2, \gamma_3) + P, \\ (\beta_0, 0, \beta_2, \gamma_3) + P, (\beta_0, \gamma_1, 0, \beta_3) + P, (\beta_0, \beta_1, \gamma_2, 0) + P, \\ (\gamma_0, 0, \gamma_2, \beta_3) + P, (\gamma_0, \beta_1, 0, \gamma_3) + P, (\gamma_0, \gamma_1, \beta_2, 0) + P \end{gathered}$$

is a partition of $\mathcal{G}$ into left cosets of P. If we denote

$$\begin{aligned} D = \{&(0, \beta_1, \beta_2, \beta_3), (0, \gamma_1, \gamma_2, \gamma_3), (\beta_0, 0, \beta_2, \gamma_3), (\beta_0, \gamma_1, 0, \beta_3), \\ &(\beta_0, \beta_1, \gamma_2, 0), (\gamma_0, 0, \gamma_2, \beta_3), (\gamma_0, \beta_1, 0, \gamma_3), (\gamma_0, \gamma_1, \beta_2, 0)\}, \end{aligned}$$

then no element from D belongs to the subgroup P due to the construction of D. Moreover, there exists no element $z \in P$ such that $x + z = y$ for some pairwise different elements $x, y \in D$. Otherwise, $z = (z_0, z_1, z_2, z_3)$ is such that $z_2 \in \{\pm\beta_2, \pm\gamma_2, \pm(\beta_2 - \gamma_2)\}$ or $z_3 \in \{\pm\beta_3, \pm\gamma_3, \pm(\beta_3 - \gamma_3)\}$. All these possibilities lead to a contradiction with the conditions of choosing β_j and γ_j, $j \in \{2, 3\}$. Hence our statement about the partition of $\mathcal{G}$ into left cosets of P holds. For instance, in the above construction we could have chosen $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 1$ and $\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 2$.

Now we will construct an idomatic partition of G into T_1-sets. For every $p_i \in P$, $1 \le i \le r$, we introduce

$$\begin{aligned} C_i = \{&p_i, p_i + (0, \beta_1, \beta_2, \beta_3), p_i + (0, \gamma_1, \gamma_2, \gamma_3), \\ &p_i + (\beta_0, 0, \beta_2, \gamma_3), p_i + (\beta_0, \gamma_1, 0, \beta_3), p_i + (\beta_0, \beta_1, \gamma_2, 0), \\ &p_i + (\gamma_0, 0, \gamma_2, \beta_3), p_i + (\gamma_0, \beta_1, 0, \gamma_3), p_i + (\gamma_0, \gamma_1, \beta_2, 0)\}, \end{aligned}$$

where

$$\begin{aligned} &p_i + x = (p_{i_0}, p_{i_1}, p_{i_2}, p_{i_3}) + (x_0, x_1, x_2, x_3) \\ &= (p_{i_0} + x_0 \bmod n_0, p_{i_1} + x_1 \bmod n_1, p_{i_2} + x_2 \bmod n_2, p_{i_3} + x_3 \bmod n_3). \end{aligned}$$

It is clear that for arbitrary pairwise different $x, y \in D \cup \{(0, 0, 0, 0)\}$ and $i \in \{1, 2, ..., r\}$ the vertices $p_i + x$ and $p_i + y$ are non-adjacent because of non-adjacency of x and y. That is, all C_i are independent. By Theorem 3.1, all C_i are T_1-sets and by our statement above we get $\bigcup_{i=1}^{r} C_i = G$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Hence $C_1, C_2, ..., C_r$ is an idomatic partition of the graph G into T_1-sets. $\square$

5 Concluding remarks

We have started the paper with fall colorings and noted that they coincide with idomatic partitions. The *fall chromatic number* $\chi_f(G)$ of a graph G is defined as the minimum order of a fall coloring of G. Clearly, $\chi_f(G) \ge \chi(G)$.

Since Hedetniemi's conjecture holds for complete graphs, see [5, 8], we have $\chi(\times_{i=1}^k K_{n_i}) = \min\{n_i \mid 1 \le i \le k\}$. Let $n_\ell = \min\{n_i \mid 1 \le i \le k\}$. In Section 2 we have noticed (for four factors) that

$$I_j = [n_1] \times \cdots \times [n_{\ell-1}] \times \{j\} \times [n_{\ell+1}] \times \cdots \times [n_k]\,,$$

where $j \in [n_\ell]$, is an independent dominating set. Consequently, $\{I_j \mid j \in [n_\ell]\}$ is a fall coloring. Hence $\chi_f(\times_{i=1}^k K_{n_i}) \le n_\ell = \chi(\times_{i=1}^k K_{n_i}) \le \chi_f(\times_{i=1}^k K_{n_i})$ and so $\chi_f(\times_{i=1}^k K_{n_i}) = \chi(\times_{i=1}^k K_{n_i})$.

An alternative argument for the above conclusion could use the independence number of the direct product of complete graphs, see [7, Corollary 1].

Acknowledgements

We wish to thank Mario Valencia-Pabon for several useful discussions. S. Klavžar was supported in part by the Slovenian Research Agency, program P1-0297.

References

[1] E.J. Cockayne, S.T. Hedetniemi, Disjoint independent dominating sets in graphs, Discrete Math. 15 (1976) 213–222.

[2] J.E. Dunbar, S.M. Hedetniemi, S.T. Hedetniemi, D.P. Jacobs, J. Knisely, R.C. Laskar, D.F. Rall, Fall colorings of graphs, J. Comb. Math. Comb. Comput. 33 (2000) 257–273.

[3] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.

[4] J. Lyle, N. Drake, R. Laskar, Independent domatic partitioning or fall coloring of strongly chordal graphs, Congr. Numer. 172 (2005) 149–159.

[5] N. Sauer, Hedetniemi's conjecture—a survey, Discrete Math. 229 (2001) 261–292.

[6] M. Valencia-Pabon, Idomatic partitions of direct products of complete graphs, manuscript, 2008.

[7] M. Valencia-Pabon, J. Vera, Independence and coloring properties of direct products of some vertex-transitive graphs, Discrete Math. 306 (2006) 2275–2281.

[8] X. Zhu, A survey on Hedetniemi's conjecture, Taiwanese J. Math. 2 (1998) 1–24.