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The Schwinger model in point form*

D. Kupelwieser[a], W. Schweiger[a], and W. H. Klink[b]

[a] Institut für Physik, Universität Graz, A-8010 Graz, Austria
[b] Dept. Physics and Astronomy, The University of Iowa, Iowa City, IA 52242-1479, U.S.

Abstract. We attempt to solve the Schwinger model, i.e. massless QED in 1+1 dimensions, by quantizing it on a space-time hyperboloid $x_\mu x^\mu = \tau^2$. The Fock-space representation of the 2-momentum operator is derived and its algebraic structure is analyzed. We briefly outline a solution strategy.

1 Introduction

The Schwinger model is quantum electrodynamics of massless fermions in 1 space and 1 time dimension [1] and serves as a popular testing ground for non-perturbative methods in quantum field theory (QFT). It is an exactly solvable, super-renormalizable gauge theory that exhibits various interesting phenomena [2], such as confinement, which one would like to understand better in 1+3-dimensional QFTs. Originally it was solved by means of functional methods [1]. Later on also operator solutions were found [3] and spectrum and eigenstates of the theory were calculated by quantizing it at equal time $x^0 = \text{const.}$ [4, 5] or at equal light-cone time $x^+ = x^0 + x^1 = \text{const.}$ [6]. We rather attempt to solve the Schwinger model by means of canonical quantization on the space-time hyperboloid $x_0^2 - x_1^2 = \tau^2$. Each of these quantization hypersurfaces is associated with a particular form of relativistic Hamiltonian dynamics [7], namely the instant form, the front form and the point form, respectively.

The quantization surface in point form is a space-time hyperboloid which is invariant under the action of the Lorentz group. The kinematic (interaction independent) generators of the Poincaré group are therefore those of the Lorentz subgroup. All the interactions go into the components of the 2-momentum P^μ, i.e. the generators of space-time translations, which provide the dynamics of the system. One of the main virtues of point-form dynamics is obviously a simple behavior of wave functions and operators under Lorentz transformations. This has already been exploited in applications to relativistic few-body systems [8], but corresponding studies of interacting quantum field theories are still very sparse. The best-known paper is that of Fubini et al. [9], who deal with point-form QFT in 2-dimensional Euclidean space-time. We rather want to extend equal-τ quantization in Minkowski space-time, as it was worked out in Ref. [10] for free field theories, to the interacting case. The solution being known, the Schwinger model

* Talk delivered by D. Kupelwieser

would be an interesting example to test the point-form approach against other methods. The hope is then that point-form quantum field theory will eventually represent a useful alternative in the study of 4-dimensional quantum field theories.

The Lagrangian of the Schwinger model is

$$\mathcal{L} = \mathcal{L}_\gamma + \mathcal{L}_e + \mathcal{L}_{\text{int}} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{photon part}} + \underbrace{\frac{i}{2}\bar{\psi}\overset{\leftrightarrow}{\not\partial}\psi}_{\text{fermion part}} + \underbrace{\frac{1}{2}e\bar{\psi}\not{A}\psi}_{\text{interaction part}} \tag{1}$$

with the 2×2 Dirac matrices being represented, as usual, in the Weyl basis, i.e. $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$ and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_3$.

2 The 2-Momentum Operator

2.1 The free part

This exposition follows closely Ref. [10] to which we refer for further details.

Fermions: In order to obtain the Fock-space representation of the free fermion 2-momentum operator, we Fourier-expand the Dirac field $\psi(x)$ in terms of plane waves using the fermion and antifermion annihilation (creation) operators $c^{(\dagger)}(p)$ and $d^{(\dagger)}(p)$ and the spinor basis $\{u(p), v(p)\}$. In the massless case, the spinors are $(p^0 = |p^1|)$:

$$u(p) = \frac{1}{\sqrt{2p^0}}\begin{pmatrix} p^0 - p^1 \\ p^0 + p^1 \end{pmatrix} \quad \text{and} \quad v(p) = \frac{1}{\sqrt{2p^0}}\begin{pmatrix} p^1 - p^0 \\ p^1 + p^0 \end{pmatrix} . \tag{2}$$

The free fermion 2-momentum operator in point-form is then obtained from the stress-energy tensor $\Theta_e^{\mu\nu}$ by integrating over the space-time hyperboloid $x_\mu x^\mu = \tau^2$:

$$P_e^\mu = \int_{\mathbb{R}^2} \underbrace{2d^2x\,\delta\left(x^2 - \tau^2\right)\,\theta(x^0)\,x_\nu}_{\text{point-form “surface” element}}\,\Theta_e^{\nu\mu} , \quad \text{with} \quad \Theta_e^{\nu\mu} = \frac{i}{2}\bar{\psi}\gamma^\nu \overset{\leftrightarrow}{\partial}{}^\mu\psi . \tag{3}$$

Inserting now the plain-wave expansion for the fields and interchanging momentum and x integrations we are left with the covariant distribution

$$\begin{aligned} W_\nu(q) &= 2\int_{\mathbb{R}^2} d^2x\,\delta(x^2 - \tau^2)\,\theta(x^0)\,x_\nu\,e^{-iqx} \\ &= 2\pi\delta(q^2)\epsilon(q^0)q_\nu + 2\pi\theta(q^2)\delta(q^0)J_0(\tau\sqrt{q^2})g_{\nu 0} \\ &\quad - \frac{\pi\tau}{\sqrt{q^2}}\theta(q^2)\left[iY_1(\tau\sqrt{q^2}) + \epsilon(q^0)J_1(\tau\sqrt{q^2})\right] q_\nu \\ &\quad - \frac{2i\tau}{\sqrt{-q^2}}\theta(-q^2)K_1(\tau\sqrt{-q^2})\,q_\nu . \end{aligned} \tag{4}$$

When evaluating equation (3) for the free parts of the Lagrangian (1), W_ν is contracted with spinor products of the form $\bar{u}\gamma^\nu u$, $\bar{u}\gamma^\nu v$, etc. All the contractions with q_ν vanish and only the term $\propto \theta(q^2)\delta(q^0)\, g_{\nu 0}$ survives. The result, as already shown by Biernat et al. [10] using a different trick to evaluate W_ν, is (after normal ordering)

$$P^\mu_e = \int \frac{dp^1}{2p^0}\, p^\mu \left(c^\dagger(p)\, c(p) + d^\dagger(p)\, d(p)\right), \tag{5}$$

i.e. the same as in instant form.

Photons: For the free photon 2-momentum operator we proceed in an analogous way. [1] The Fourier expansion of the vector potential $A^\mu(x)$ in terms of plane waves gives rise to the photon creation- and annihilation operators $a^\dagger_\kappa(k)$ and $a_\kappa(k)$ and to polarization vectors $\epsilon^\mu_\kappa(k)$, with $\kappa = 0, 1$ labeling the polarization. The polarization vectors are orthonormalized according to $\epsilon^\mu_{\kappa'}(k)\epsilon_{\kappa\mu}(k) = g_{\kappa'\kappa}$. In order to preserve the nice covariance properties of the point form, we work within the Lorenz gauge and use the Gupta-Bleuler quantization procedure. As a consequence there are no physical photons left. The 0- and the 1-component of the photon field are pure gauge degrees of freedom. Proceeding in analogy to the fermion part we find for the Fock-space representation of the free photon 2-momentum operator again the same result as for equal-time quantization, i.e.

$$P^\mu_\gamma = \sum_{\kappa=0}^{1} \int \frac{dk^1}{2k^0}\, k^\mu\, g^{\kappa\kappa} a^\dagger_\kappa(k)\, a_\kappa(k)\,. \tag{6}$$

2.2 The interaction part

Since there is no derivative in the interaction part of the Lagrangian (1), the interaction part of the stress-energy tensor is simply given by $\Theta^{\mu\nu}_{\text{int}} = -g^{\mu\nu}\,\mathcal{L}_{\text{int}}$. The interaction part of the 2-momentum operator is then

$$P^\mu_{\text{int}} = -\int_{\mathbb{R}^2} 2\, d^2x\, \delta(x^2 - \tau^2)\, \theta(x^0)\, x^\mu\, \mathcal{L}_{\text{int}}(x)\,. \tag{7}$$

One can check explicitly that the corresponding integral for the interaction part of the boost generator vanishes as expected [10].

To obtain the Fock-space representation of P^μ_{int} we proceed as before. The only difference is now that $W_\nu(q)$ does not provide a momentum conserving δ function. But this is not surprising. Both components of the momentum operator are interaction dependent so that one cannot expect momentum conservation at interaction vertices. But what one can do is to analyze the algebraic structure of P^μ_{int}. By appropriately collecting terms it can be cast into the form

$$P^\mu_{\text{int}} = -e \sum_{\kappa=0}^{1} \int \frac{dk^1}{2k^0} \left(\mathcal{A}(X^\mu_\kappa)(k)\, a_\kappa(k) + \mathcal{A}^\dagger(X^\mu_\kappa)(k)\, a^\dagger_\kappa(k)\right) \tag{8}$$

[1] See also Ref. [11] for a detailed derivation of the gluon 2-momentum operator.

with

$$\mathcal{A}(X^{\mu}_{\kappa})(k) = \int \frac{dp^1}{2p^0} \int \frac{dp'^1}{2p'^0} \left(c^{\dagger}(p'), d(p)\right) X^{\mu}_{(\kappa)}(k, p', p) \begin{pmatrix} c(p) \\ d^{\dagger}(p) \end{pmatrix} \quad (9)$$

The distribution W^{μ} for different combinations of the momenta p, p' and k together with the different spinor products determines essentially the elements of the 2×2 matrix $X^{\mu}_{(\kappa)}(k, p', p)$.

3 The Eigenvalue Problem

Putting all the pieces together we finally end up with the eigenvalue problem

$$\left(P^{\mu}_{e} + P^{\mu}_{\gamma} + P^{\mu}_{\text{int}}\right) |\Psi\rangle = \mathcal{A}(E^{\mu})|\Psi\rangle + \sum_{\kappa=0}^{1} \int \frac{dk^1}{2k^0} \Big(k^{\mu} g^{\kappa\kappa} a^{\dagger}_{\kappa}(k)\, a_{\kappa}(k) - e\mathcal{A}(X^{\mu}_{\kappa})(k)\, a_{\kappa}(k) - e\mathcal{A}^{\dagger}(X^{\mu}_{\kappa})(k)\, a^{\dagger}_{\kappa}(k)\Big)|\Psi\rangle = p^{\mu}|\Psi\rangle \quad (10)$$

which we want to solve non-perturbatively. Here we have also expressed the fermion kinetic energy in terms of the $\mathcal{A}$s to emphasize that the fermion creation and annihilation operators occur only in bilinear combinations. The argument E^{μ} is essentially a diagonal matrix containing $\pm\delta(p^{1\prime} - p^1)$.

A possible strategy to solve this eigenvalue problem was proposed in Ref. [12]. The first step is to keep the number of modes finite. This could, e.g., be done without spoiling Lorentz-transformation properties by compactifying the x^1 direction such that one ends up with a deSitter space. A finite number of modes means also that only a finite number of fermion-antifermion pairs can be created. In order to keep the number of bosons finite the boson algebra is then considered as a contraction limit of another unitary algebra that restricts the number of bosons in any mode. In this way one ends up with a solvable algebraic problem that involves only a finite number of modes and a finite number of particles. The interesting question will be whether the well known results for Schwinger model are recovered upon performing the necessary contractions that restore the original theory.

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