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On z-monodromies in embedded graphs

Adam Tyc

Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54, 10-710 Olsztyn, Poland

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Abstract

We characterize all permutations which occur as the z-monodromies of faces in connected simple finite graphs embedded in surfaces whose duals are also simple.

Keywords: Central circuit, chess coloring, embedded graph, zigzag, z-monodromy.

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1 Introduction

Zigzags are closed walks in embedded graphs which generalize the concept of *Petrie polygons* in regular polyhedra [2]. They were used in computer graphics [6] and in enumerating all combinatorial possibilities for fullerenes in mathematical chemistry [1, 4]. Zigzags are also closely related to *Gauss code problem*: if an embedded graph contains a single zigzag, then this zigzag is a geometrical realization of a certain Gauss code (see [5, Section 17.7] for the planar case and [3, 9] for the case when a graph is embedded in an arbitrary surface). More results on zigzags can be found in [8, 10, 13, 15].

We will consider zigzags in connected simple finite graphs embedded in surfaces whose duals are also simple. The latter condition guarantees that for any two consecutive edges on a face there is a unique zigzag containing them. This property is the crucial tool in the concept of *z-monodromy*. For a face F, the z-monodromy M_F is a permutation on the set of all oriented edges obtained by orienting each edge of F in the two possible ways. If e_0, e is a pair of consecutive edges in F, then $M_F(e)$ is the first oriented edge of F that occurs in the zigzag containing e_0, e after e.

Such z-monodromies were introduced in [12] and exploited to prove that any triangulation of an arbitrary (not necessarily oriented) closed surface can be shredded to a triangulation with a single zigzag. There are precisely 7 types of z-monodromies for triangle faces and each of them is realized. The properties and some applications of z-monodromies of

E-mail address: adam.tyc@matman.uwm.edu.pl (Adam Tyc)

triangle faces can be found in [11, 16]. See also [14] for a generalization of z-monodromies on pairs of edges.

Faces of embedded graph under consideration contains at least three edges. We characterize permutations that occur as z-monodromies of k-gonal faces for any $k \geq 3$. More precisely, a permutation σ on the set

$$[k]_{\pm} = \{1, \dots, k, -k, \dots, -1\}$$

occurs as the z-monodromy if and only if it satisfies the following conditions:

- if $\sigma(i) = j$, then $\sigma(-j) = -i$;
- $\sigma(i) \neq -i$.

In the plane case, our construction is based on the *chess coloring* of 4-regular plane graphs. For every permutation σ satisfying the above conditions there is a plane graph with a face F whose z-monodromy is σ; furthermore, this graph contains a unbounded triangle face T such that every zigzag passing through F does not pass through T. To extend the construction on the general case, we take any graph embedded in a surface with a triangle face and replace this face by the above plane graph.

We consider the case when an embedded graph and its dual both are simple. In the general case, zigzags cannot be reconstructed from pairs of consecutive edges. This shows that the concept of z-monodromy cannot be generalized in a direct way.

2 Zigzags in embedded graphs

Let S be a connected closed 2-dimensional (not necessarily orientable) surface. Let Γ be a 2-cell embedding of a connected finite graph in S, in other words, a *map* [7, Definition 1.3.6]. The difference $S \setminus \Gamma$ is a disjoint union of open disks and the closures of these disks are the *faces*. We say that a face is *k-gonal* if it contains precisely k edges. We will always assume that the following condition is satisfied:

(SS) Γ and the dual map Γ^* (see [7, p.52]) both are embeddings of simple graphs.

The fact that one of the graphs is simple does not implies that the same holds for the other graph. For example, Γ^* is not simple if Γ contains a vertex of degree 2 or two distinct faces with intersection containing more than one edge. The condition (SS) implies that each face in our graphs is k-gonal with $k \geq 3$.

A *zigzag* in Γ is a *sequence* of edges $\{e_i\}_{i \in \mathbb{N}}$ satisfying the following conditions for every $i \in \mathbb{N}$:

- e_i and e_{i+1} are distinct, they have a common vertex and belong to the same face,
- the faces containing e_i, e_{i+1} and e_{i+1}, e_{i+2} are distinct and the edges e_i and e_{i+2} are non-intersecting.

Since Γ is finite, there is a natural number $n > 1$ such that $e_{i+n} = e_i$ for every natural i. Thus, every zigzag will be represented as a cyclic sequence $e_1, \dots, e_n$, where n is the smallest number satisfying this condition.

Any zigzag is completely determined by every pair of consecutive edges contained in this zigzag. Conversely, for every pair of distinct edges e, e' which have a common vertex and belong to the same face there is a unique zigzag containing the sequence e, e'. This property will be used in the next section.

If $Z = \{e_1, \dots, e_n\}$ is a zigzag, then the reversed sequence $Z^{-1} = \{e_n, \dots, e_1\}$ also is a zigzag. A zigzag cannot contain a sequence $e, e', \dots, e', e$ which implies that $Z \neq Z^{-1}$ for any zigzag Z. In other words, a zigzag cannot be self-reversed (see [12] for the proof for triangulations; in our case the proof is similar).

Example 2.1. Consider the cube Q_3 whose vertices are $1, \dots, 8$, see Fig. 1.

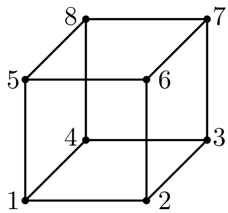


Figure 1: The cube Q_3

It contains precisely 4 zigzags up to reversing:

$12, 23, 37, 78, 85, 51;\quad 12, 26, 67, 78, 84, 41;\quad 14, 43, 37, 76, 65, 51;\quad 23, 34, 48, 85, 56, 62.$

Let BP_n be the n-gonal bipyramid, where $1, \dots, n$ are the consecutive vertices of the base and the remaining two vertices are a, b.

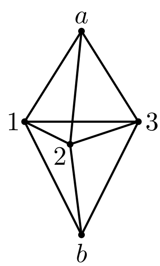


Figure 2: The bipyramid BP_3

If $n = 3$ (see Fig. 2), then it contains a single zigzag (up to reversing):

$$a1, 12, 2b, b3, 31, 1a, a2, 23, 3b, b1, 12, 2a, a3, 31, 1b, b2, 23, 3a.$$

The same holds for BP_n if n is odd. If n is even, then BP_n contains 2 or 4 zigzags up to reversing.

Every zigzag in Γ induces in a natural way a zigzag in Γ^* and vice versa.

Remark 2.2. Zigzags can be defined in maps of non-simple graphs [8]. In this case, there are simple examples showing that a zigzag cannot be determined by any pair of its consecutive edges.

3 Main result

Let Γ be as in the previous section and let F be a k-gonal face of Γ. Denote by $v_0, \dots, v_{k-1}$ the consecutive vertices of F in a fixed orientation on the boundary of this face (it is possible that $v_i = v_j$ if $|i - j| \geq 3$). Consider the set of all oriented edges of F

$$\Omega(F) = \{e_1, \dots, e_k, -e_k, \dots, -e_1\},$$

where $e_i = v_{i-1}v_i$ and $-e_i = v_iv_{i-1}$ are mutually reversed oriented edges of F (the indices are taken modulo k); it is clear that $\Omega(F)$ consists of $2k$ mutually distinct elements. Let D_F be the following permutation on $\Omega(F)$

$$D_F = (e_1, e_2, \dots, e_k)(-e_k, \dots, -e_2, -e_1).$$

In other words, D_F transfers every oriented edge of F to the next oriented edge in the corresponding orientation on the boundary.

The *z-monodromy* of F is the mapping $M_F : \Omega(F) \to \Omega(F)$ defined as follows. For any $e \in \Omega(F)$ we take $e_0 \in \Omega(F)$ such that $D_F(e_0) = e$. There is a unique zigzag, where e_0, e are consecutive edges. The first element of $\Omega(F)$ contained in this zigzag after e_0, e is denoted by $M_F(e)$.

Remark 3.1. The z-monodromy is defined when (SS) is satisfied. This concept cannot be carried out on the general case immediately.

Lemma 3.2. *The following assertions are fulfilled:*

(1) If $M_F(e) = e'$ for some $e, e' \in \Omega(F)$, then $M_F(-e') = -e$.

(2) M_F is bijective.

(3) $M_F(e) \neq -e$ for every $e \in \Omega(F)$.

Proof. (1). Let $e \in \Omega(F)$. Consider $e_0 \in \Omega(F)$ satisfying $D_F(e_0) = e$. If Z is the zigzag containing the pair e_0, e, then

$$e' = M_F(e) \text{ and } e'_0 = D_F M_F(e)$$

are the next two elements of $\Omega(F)$ in Z. Observe that $D_F(-e'_0) = -e'$. The reversed zigzag Z^{-1} contains the sequence $-e'_0, -e'$ and $-e$ is the first element of $\Omega(F)$ contained in Z^{-1} after this pair. This means that $M_F(-e') = -e$.

(2). It is sufficient to show that M_F is injective. Suppose that $M_F(e) = M_F(e') = e''$. By (1), we have $-e = M_F(-e'') = -e'$ which implies that $e = e'$.

(3). Let e and e_0 be as in the proof of (1). If $M_F(e) = -e$, then there is a zigzag Z containing the sequences e_0, e and $-e, D_F(-e)$. Since $D_F(-e) = -e_0$, Z passes through both pairs e_0, e and $-e, -e_0$. This implies that $Z = Z^{-1}$ which is impossible. □

The set $\Omega(F)$ is naturally identified with

$$[k]_\pm = [k]_+ \cup [k]_-$$

where

$$[k]_+ = \{1, \dots, k\}, \text{ and } [k]_- = \{-k, \dots, -1\}$$

(e_i and $-e_i$ correspond to i and $-i$, respectively). Then, by Lemma 3.2, the z-monodromy M_F is a permutation σ of $[k]_\pm$ satisfying the following conditions:

(M1) if $\sigma(i) = j$, then $\sigma(-j) = -i$;

(M2) $\sigma(i) \neq -i$.

Our main result is the following.

Theorem 3.3. *Let S be a connected closed 2-dimensional (not necessarily orientable) surface and let $k \geq 3$. Let also σ be a permutation of $[k]_{\pm}$ satisfying* (M1) *and* (M2). *There is a connected finite graph Γ embedded in S and satisfying* (SS) *which contains a k-gonal face F whose z-monodromy is σ.*

In Section 4, we prove Theorem 3.3 for plane graphs (graphs embedded in a sphere). Graphs on surfaces different from a sphere will be considered in Section 5.

4 The plane case

4.1 Preliminary

Let G be a 4-regular plane graph. The dual graph G^* is bipartite and there exists a *chess coloring* of faces of G in two colors b and w. For $c \in \{b, w\}$ we take a vertex inside every face of G assigned with the color c and join two such vertices by an edge if the corresponding faces have a common vertex at theirs boundaries. The obtained plane graph will be denoted by $\mathcal{R}_c(G)$. The graphs $\mathcal{R}_b(G)$ and $\mathcal{R}_w(G)$ are dual (see Fig. 3).

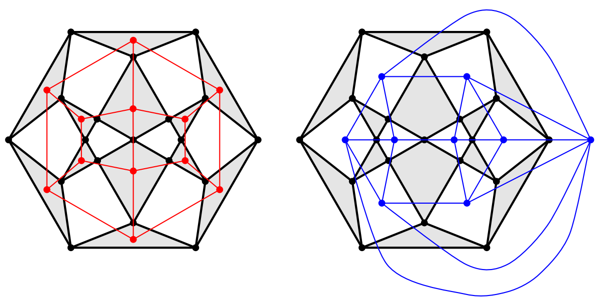


Figure 3: The chess coloring and the related graphs

Consider a plane graph Γ. The *medial graph* of Γ is the graph $\mathcal{M}(\Gamma)$ whose vertex set is the edge set of Γ and two vertices of $\mathcal{M}(\Gamma)$ are joined by an edge if they have a common vertex and belong to the same face in Γ. The graph $\mathcal{M}(\Gamma)$ is also plane. This graph is 4-regular and its face set is the union of the vertex set and the face set of Γ. Thus, $\mathcal{M}(\Gamma)$ is chess colored. Let b be the color used to coloring the faces of $\mathcal{M}(\Gamma)$ corresponding to the vertices of Γ. The remaining faces of $\mathcal{M}(\Gamma)$ (corresponding to the faces of Γ) are colored in w. Then

$$\mathcal{R}_b(\mathcal{M}(\Gamma)) = \Gamma \qquad \text{and} \qquad \mathcal{R}_w(\mathcal{M}(\Gamma)) = \Gamma^*.$$

For example, the graph marked in black in Fig. 3 is the medial graph of the graphs marked in red and blue.

A *central circuit* is a circuit in the medial graph which is obtained by starting with an edge and continuing at each vertex by the edge opposite to the entering one [4, p.5]. We will consider central circuits as cyclic sequences of vertices and distinguish each central circuit from the reversed. If Γ and Γ^* both are simple, then there is a one-to-one correspondence between zigzags in Γ and central circuits in $\mathcal{M}(\Gamma)$: a sequence formed by edges of Γ is a zigzag if and only if this sequence is a central circuit in $\mathcal{M}(\Gamma)$. In Fig. 4. the part of the zigzag is marked by the bold red line and the corresponding part of the central circuit is marked by the bold black line.

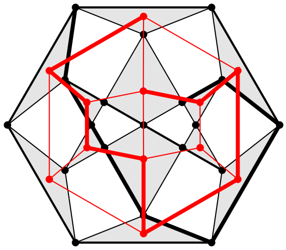


Figure 4: A zigzag and the corresponding central circuit

4.2 Main construction

Let σ be a permutation on $[k]_\pm$ satisfying (M1) and (M2). We construct a 4-regular plane graph G which is the medial graph of a plane graph, where σ occurs as the z-monodromy of a face F (this plane graph contains a k-gonal face F such that σ is M_F).

Consider a circle C embedded in the plane. We take mutually distinct points $p_1, \dots, p_k$ from C such that these points occur along C in the clockwise order; these points will be the edges of the mentioned above face F. Next, denote by C' a circle inside the part of the plane bounded by C and take mutually distinct points $a_{12}, a_{23}, \dots, a_{(k-1)k}, a_{k1}$ occurring on C' in the clockwise order. Similarly, let C'' be a circle inside the part of the plane bounded by C' and let $r_1, l_k, r_2, l_1, \dots, r_k, l_{k-1}$ be mutually distinct points that occur along C'' in the clockwise order. For every a_{ij} we take two segments that intersect precisely in a_{ij} and join p_i with l_i and p_j with r_j, respectively. Note that all such segments intersect each of C, C', C'' in precisely one point and the interiors of any two of these segments are disjoint if they contain distinct points a_{ij}. Denote by S_i the union of the segment joining r_i with p_i, the arc of C between p_i and p_{i+1} and the segment joining p_{i+1} with l_{i+1} if $i < k$; for $i = k$ we replace every index $i + 1$ by 1. See Fig. 5 for the case $k = 6$.

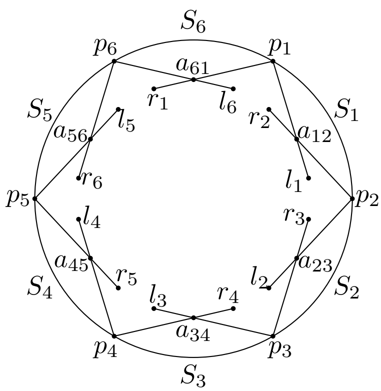


Figure 5: The beginning of construction for $k = 6$

We will work with the relation $\sim$ on the set

$$\mathcal{O}=\bigcup_{i=1}^{k}\{l_i,r_i\}$$

such that for any $i,j\in[k]_\pm$ satisfying $\sigma(i)=j$ one of the following possibilities is realized:

(1) $l_i\sim r_j$ if $i,j\in[k]_+$,

(2) $r_{-i}\sim l_{-j}$ if $i,j\in[k]_-$,

(3) $l_i\sim l_{-j}$ if $i\in[k]_+$ and $j\in[k]_-$,

(4) $r_{-i}\sim r_j$ if $i\in[k]_-$ and $j\in[k]_+$.

The relation $\sim$ is irreflexive and symmetric. Indeed, if $l_i\sim l_i$ (the case (3)) or $r_i\sim r_i$ (the case (4)), then we get $\sigma(i)=-i$ which contradicts (M2). Thus, $\sim$ is irreflexive. Now, we show that if $l_i\sim l_j$, then $l_j\sim l_i$ (the remaining three cases are similar). If $l_i\sim l_j$ with $i,j\in[k]_+$ (the case (3)), then $\sigma(i)=-j$ and, by (M1), $\sigma(j)=-i$ and $j\in[k]_+,-i\in[k]_-$; i.e. $l_j\sim l_i$. Note that for each $x\in\mathcal{O}$ there is a unique $x'\in\mathcal{O}$ such that $x\sim x'$.

If a pair of points from $\mathcal{O}$ is in the relation $\sim$, then we join them by a curve homeomorphic to the segment $[0,1]$ inside the part of the plane bounded by C''. The following conditions must be satisfied:

• the curves have no more than finitely many intersections and self-intersections,

• for every such intersection point either there are precisely two distinct curves passing once through this point or there is a single curve passing twice through it,

• all intersections are transversal.

Let $L_1,\dots,L_k$ be the curves described above (we take an arbitrary numeration that does not depend on the endpoints from $\mathcal{O}$).

Note that each of $l_1,r_1,\dots,l_k,r_k$ is a common point of a unique S_i and a unique L_j. Thus, we obtain a family $\mathcal{C}$ of closed curves

$$S_{i_1}\cup L_{i_1}\cup S_{i_2}\cup L_{i_2}\cup\dots$$

such that every S_i and L_j is contained in precisely one of these curves. Let V be the set of all intersection and self-intersection points of curves from $\mathcal{C}$. In particular, all p_i and all a_{ij} belong to V.

Example 4.1. Let $k=6$ and

$$\sigma=(1,-6,-4,2)(3,-5)(5,-3)(-2,4,6,-1).$$

The relation $\sim$ on the set $\mathcal{O}=\{l_1\dots,l_6,r_1,\dots,r_6\}$ is as follows

$$l_1\sim l_6,\;\; r_6\sim l_4,\;\; r_4\sim r_2,\;\; l_2\sim r_1,\;\; l_3\sim l_5,\;\; r_5\sim r_3.$$

One of suitable connections between points from $\mathcal{O}$ is presented in Fig. 6.

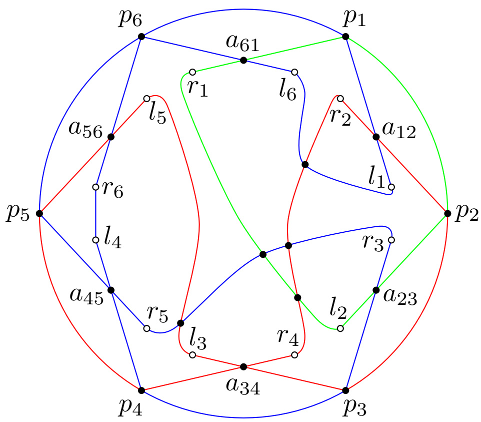


Figure 6: A suitable connection between points of $\mathcal{O}$

In this case, $\mathcal{C}$ consists of precisely 3 closed curves with 17 intersection points, i.e. $|V| = 17$.

Let $G = G(\mathcal{C})$ be the graph whose vertex set is V and two vertices are joined by an edge if they are two consecutive points on one of curves from $\mathcal{C}$. In fact, we consider G as a graph whose vertices are points on the plane and edges are parts of curves from $\mathcal{C}$ joining these points. It is easy to see that G is a 4-regular and the curves from $\mathcal{C}$ correspond to pairs of mutually reversed central circuits from G. We make the chess coloring of G and split the set of its faces into the two sets corresponding to the colors. Let b be the color of faces whose boundaries are cycles with vertices p_i, p_j, a_{ij}; as above, w is the other color.

As in Subsection 4.1, we obtain the dual graphs $\mathcal{R}_b(G)$ and $\mathcal{R}_w(G)$. These graphs are not necessarily simple; they may contain the following fragments:

(A) loops,

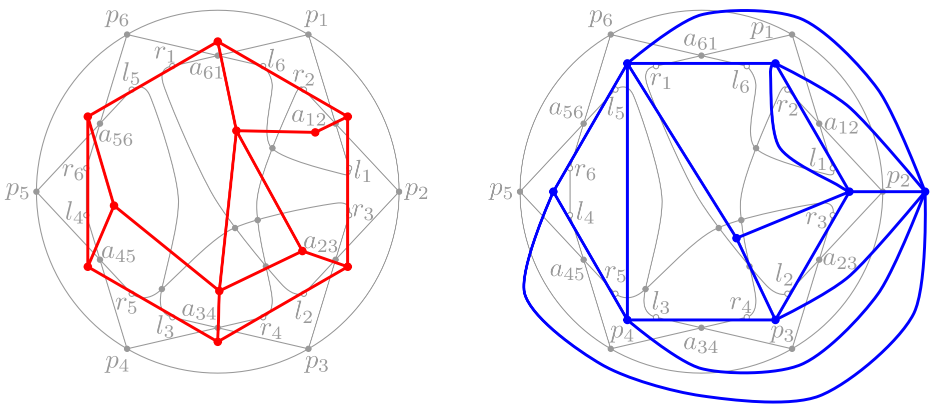
(B) multiple edges,

(C) edges that belong to the boundary of one face only (in particular, edges with vertices of degree 1),

(D) pairs of faces whose intersection of boundaries contains more than one edge.

If one of the cases (A) or (B) occurs in one of the graphs $\mathcal{R}_b(G), \mathcal{R}_w(G)$, then the case (C) or (D), respectively, occurs in the dual graph.

Example 4.2. Let $\mathcal{C}$ be as in Example 4.1. The graphs $\mathcal{R}_b(G)$ and $\mathcal{R}_w(G)$ are presented in Fig. 7a and Fig. 7b, respectively.



(a) (b)

Figure 7: The dual graphs $\mathcal{R}_b(G)$ and $\mathcal{R}_w(G)$

The graph $\mathcal{R}_b(G)$ contains a pair of faces with two common edges (the case (D)) which corresponds to a double edge in $\mathcal{R}_w(G)$ (the case (B)).

Now, we show how modify the graph G such that the connections between the points from $\mathcal{O}$ induced by the relation $\sim$ do not change and the graphs $\mathcal{R}_b(G), \mathcal{R}_w(G)$ become simple.

Suppose that e is an edge joining the vertices v' and v'' in $\mathcal{R}_b(G)$ or $\mathcal{R}_w(G)$ (both the cases are similar). We consider separately the cases $v' \neq v''$ and $v' = v''$ (see Fig. 8a and 8b, respectively). If $v' \neq v''$, then we consider the following edges in the same graph ($\mathcal{R}_b(G)$ or $\mathcal{R}_w(G)$):

• e'_+ and e'_- which occur directly after e in the clockwise and the anticlockwise order on edges incident to v', respectively;

• e''_+ and e''_- which occur directly after e in the clockwise and the anticlockwise order on edges incident to v'', respectively.

If $v' = v''$, then we exclude the case when the loop e is the boundary of a face (this case will be considered separately). The edge e splits the plane into two parts and we consider the following edges:

• e'_+ and e'_- are the edges contained in one of these parts which occur directly after e in the clockwise and the anticlockwise order on edges incident to v', respectively;

• e''_+ and e''_- are the edges contained in the other part of the plane which occur directly after e in the clockwise and the anticlockwise order on edges incident to v'', respectively.

There are precisely two parts of central circuits in G (up to reversing) that pass through e (since e, e'_δ, e''_δ are vertices of G, where $\delta \in \{+,-\}$):

$$\ldots, e'_+, e, e''_+, \ldots \qquad \text{and} \qquad \ldots, e'_-, e, e''_-, \ldots$$

which are marked in blue and red in Fig. 8.

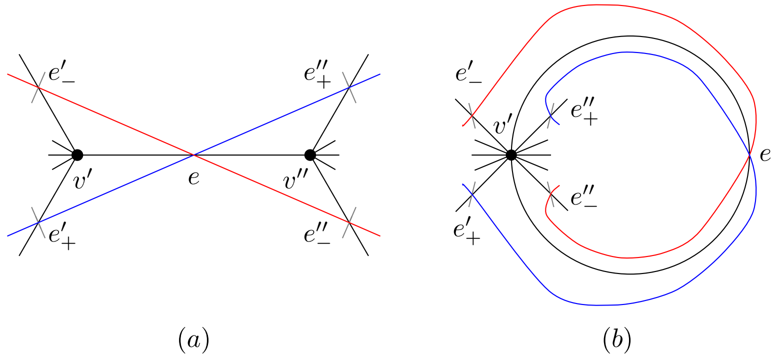


Figure 8: Parts of central circuits

For each of these cases we replace e by the graphs presented in Fig. 9a and Fig. 9b, respectively.

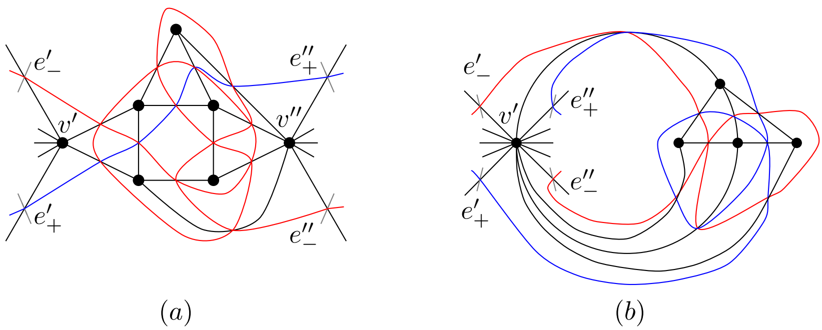


Figure 9: Two types of expansion

This operation will be called the *expansion* of e. It replaces the edge e by an intersecting trail E_δ, $\delta \in \{+,-\}$, joining e'_δ and e''_δ. So, the mentioned above parts of central circuits will be replaced by

$$\ldots, e'_+, E_+, e''_+, \ldots \qquad \text{and} \qquad \ldots, e'_-, E_-, e''_-, \ldots,$$

respectively (they are marked in blue and red in Fig. 9). Thus, central circuits do not change in a significant way.

Remark 4.3. We can obtain the same result using other pairs of graphs instead of the graphs from Fig. 9. This pair is the first which we found.

Now, we explain how transform $\mathcal{R}_b(G), \mathcal{R}_w(G)$ to simple graphs if at least one of the possibilities (A)–(D) occurs. Without loss of generality we can consider $\mathcal{R}_b(G)$. Furthermore, we restrict ourselves to the cases (A) and (B) (since (C) and (D) correspond to (A) and (B), respectively, in the dual graph). The case (A) will be decomposed in two subcases.

(A1). Suppose that $\mathcal{R}_b(G)$ contains a face whose boundary is a loop. The corresponding parts of mutually reversed central circuits from G are also loops. The loops can be removed from these graphs without changing the central circuits in a significant way (see Fig. 10).

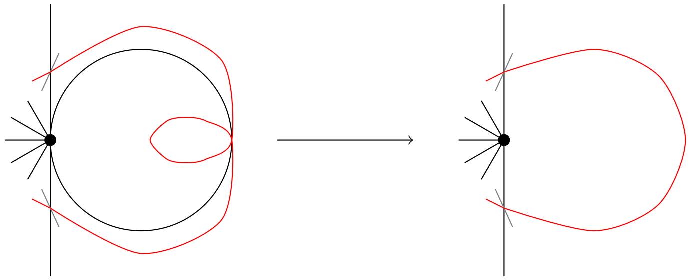


Figure 10: Removing a loop

(A2). If e is a loop in $\mathcal{R}_b(G)$ which is not a boundary of a face, then we use the expansion to e.

(B). If two distinct vertices are connected by $m \geq 2$ edges, then we expand any $m-1$ of them.

Example 4.4. Since $\mathcal{R}_w(G)$ from Example 4.2 contains two edges connecting the same pair of vertices (the case (C)), we expand one of these edges, see Fig. 11.

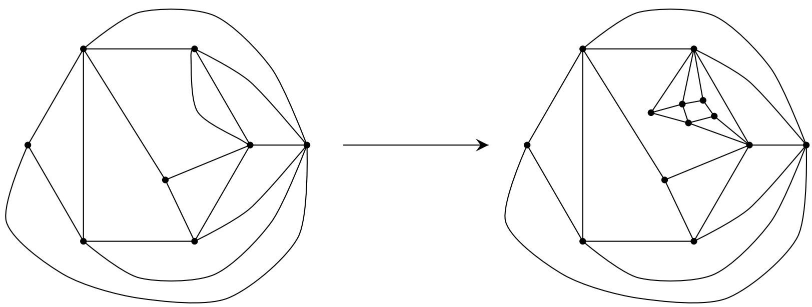


Figure 11: The expansion of an edge in $\mathcal{R}_w(G)$

This simultaneously modify $\mathcal{R}_b(G)$ and we obtain a graph without the possibility (A), see Fig. 12.

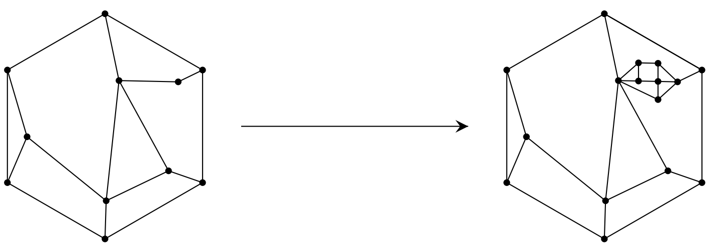


Figure 12: The corresponding modification of $\mathcal{R}_b(G)$

So, we can simultaneously transform $\mathcal{R}_b(G)$ and $\mathcal{R}_w(G)$ to simple graphs such that the relation $\sim$ on elements of $\mathcal{O}$ is not changed. In particular, we come to a new (4-regular plane) graph G and assert that $\mathcal{R}_b(G)$ contains a face for which σ occurs as the z-monodromy of one of faces.

Recall that C is a cycle in G with the vertices $p_1, \dots, p_k$ and it is the boundary of the outer face of G. This face has a common edge only with faces whose boundaries contain the vertices p_i, p_j, a_{ij}. By the definition of $\mathcal{R}_b(G)$, we have the following:

• every face with the boundary containing p_i, p_j, a_{ij} in G is a vertex in $\mathcal{R}_b(G)$ which we denote by v_{ij};

• the face bounded by C in G is a face in $\mathcal{R}_b(G)$ which will be denoted by F;

• every p_i corresponds to an edge of F.

Consider the oriented edges $e_j = v_{ij}v_{jl}$ and $-e_j = v_{jl}v_{ij}$ in $\mathcal{R}_b(G)$, where i, j, l are three consecutive elements in the cyclic sequence $1, \dots, k$. The pair of mutually reversed oriented edges $e_j, -e_j$ corresponds to the vertex p_j in G. Thus,

$$\Omega(F) = \{e_1, \dots, e_k, -e_k, \dots, -e_1\}.$$

Let $e_0, e \in \Omega(F)$ be such that $D_F(e_0) = e$. There is a unique zigzag Z in $\mathcal{R}_b(G)$ containing the pair e_0, e. The element e' which occurs in Z directly after this pair does not belong to $\Omega(F)$. The edges e_0, e, e' are three consecutive vertices in the central circuit in G corresponding to Z such that e_0, e are two consecutive vertices from the cycle C and e' is one of elements a_{ij}. Let x be the first element from $\mathcal{O}$ such that the central circuit containing e_0, e, e' passes through x (as a curve on the plane) directly after this triple (x is a point on the plane, but not a vertex of the graph). There is a unique $x' \in \mathcal{O}$ such that $x \sim x'$ and the central circuit passes through x'. Since there is no elements of $\Omega(F)$ between x and x' in the central circuit, the first element of $\Omega(F)$ that occurs after x' corresponds to $M_F(e)$. Therefore, σ occurs as M_F.

Example 4.5. Let F be the outer face of $\mathcal{R}_b(G)$ from Example 4.4 and let e_i be the oriented edge of F corresponding to p_i whose direction is defined by the clockwise orientation on the boundary of F (see Fig. 13).

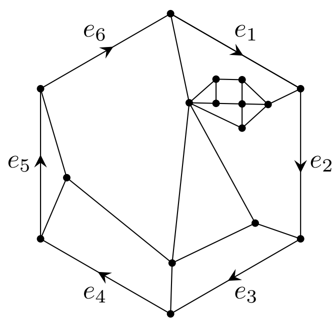


Figure 13: The new graph $\mathcal{R}_b(G)$

A direct verification shows that

$$M_F = (e_1, -e_6, -e_4, e_2)(e_3, -e_5)(e_5, -e_3)(-e_2, e_4, e_6, -e_1),$$

i.e. the permutation

$$\sigma = (1, -6, -4, 2)(3, -5)(5, -3)(-2, 4, 6, -1)$$

from Example 4.1 occurs as M_F in $\mathcal{R}_b(G)$.

5 The non-plane case

In this section, we consider an arbitrary connected closed 2-dimensional (not necessarily orientable) surface S different from a sphere. We show that any permutation σ on $[k]_\pm$ satisfying (M1) and (M2) occurs as the z-monodromy of k-gonal face in a graph embedded in S. Let Γ be a graph embedded in a sphere (a plane graph) such that σ occurs as the z-monodromy of a face F of Γ. We assume that $\Gamma = \mathcal{R}_b(G)$, where G is the 4-regular graph from Section 4.

Let e be an edge in G. It is contained in the boundaries of precisely two faces F_1, F_2 in G. We assume that F_1 and F_2 correspond to a face distinct from F and a vertex of Γ, respectively. Let us take three circles B_1, B_2, B_3 that intersect like the Borromean rings. Consider the graph G' obtained from G by adding B_1, B_2, B_3 as in Fig. 14.

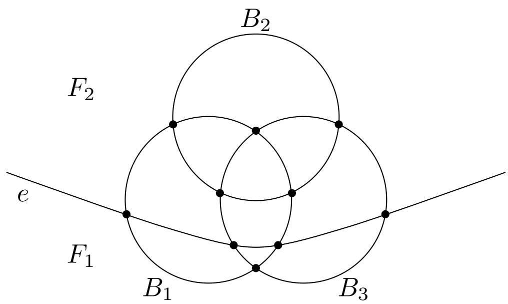


Figure 14: Constructing of G'

It must be pointed out that the circles B_1, B_2, B_3 do not intersect the remaining edges of G. The graph G' is 4-regular and $\mathcal{R}_b(G')$ is obtained from $\Gamma = \mathcal{R}_b(G)$ by adding the graph $\tilde{G}$ marked in red in Fig. 15 to the vertex v corresponding to the face F_2.

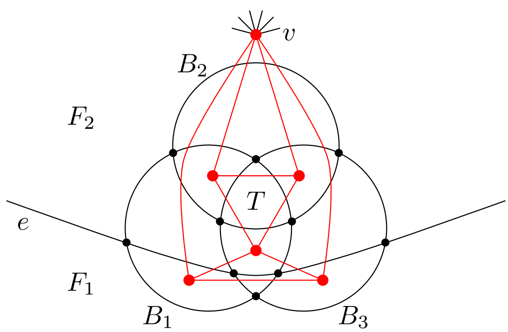


Figure 15: The graph $\tilde{G}$

It is clear that $\mathcal{R}_b(G')$ and $\mathcal{R}_w(G')$ are simple. Denote by T the face of $\tilde{G}$ which is contained in F_2 and does not contain v. Note that B_1, B_2, B_3 induce central circuits of G'. Each zigzag of $\mathcal{R}_b(G')$ passing through T corresponds to one of B_i. Observe that F is the face of $\mathcal{R}_b(G')$ and the zigzags corresponding to B_1, B_2, B_3 do not contain edges of this face. This means that the z-monodromy of F in $\mathcal{R}_b(G')$ is also σ.

Consider any graph Γ' embedded in S that contains a triangle face T'. We take the connected sum of the sphere containing $\mathcal{R}_b(G')$ and S by removing the interiors of faces

T and T' and identifying theirs boundaries by a homeomorphism that sends vertices to vertices. We come to a new graph embedded in S containing F as a face. Since every zigzag of $\mathcal{R}_b(G')$ containing an edge of F does not pass through any edge of T, the z-monodromy of F in the new graph is the same as in $\mathcal{R}_b(G')$ and, consequently, as in $\mathcal{R}_b(G)$.

ORCID iDs

Adam Tyc https://orcid.org/0000-0003-4870-5731

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