

Distinguishing colorings, proper colorings, and covering properties without AC*

Amitayu Banerjee [†] 

Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053, Budapest, Hungary

Zalán Molnár [‡] , Alexa Gopalsingh 

Eötvös Loránd University, Department of Logic, Múzeum krt. 4, 1088, Budapest, Hungary

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Abstract

We work with simple graphs in ZF (i.e., the Zermelo–Fraenkel set theory without the Axiom of Choice (AC)) and assume that the sets of colors can be either well-orderable or non-well-orderable, to prove that the following statements are equivalent to König’s Lemma:

- (a) Any infinite locally finite connected graph G such that the minimum degree of G is greater than k , has a chromatic number for any fixed integer k greater than or equal to 2.
- (b) Any infinite locally finite connected graph has a chromatic index.
- (c) Any infinite locally finite connected graph has a distinguishing number.
- (d) Any infinite locally finite connected graph has a distinguishing index.

The above results strengthen some recent results of Stawiski since he assumed that the sets of colors can be well-ordered. We formulate new conditions for the existence of irreducible proper coloring, minimal edge cover, maximal matching, and minimal dominating set in connected bipartite graphs and locally finite connected graphs, which are either equivalent to AC or König’s Lemma. Moreover, we show that if the Axiom of Choice for families of 2-element sets holds, then the Shelah–Soifer graph has a minimal dominating set.

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[†]Corresponding author.

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Razlikovalna barvanja, pravilna barvanja in lastnosti pokritij brez aksioma izbire*

Amitayu Banerjee[†] 

Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053, Budapest, Hungary

Zalán Molnár[‡] , Alexa Gopaulsingh 

Eötvös Loránd University, Department of Logic, Múzeum krt. 4, 1088, Budapest, Hungary

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Povzetek

Obravnavamo enostavne grafe v okviru ZF (tj. Zermelo–Fraenklove teorije množic brez aksioma izbire) in predpostavljamo, da se da množico barv bodisi dobro urediti bodisi ne, zato da dokažemo, da so naslednje trditve, ekvivalentne Kőnigovi lemi:

- (a) Vsak neskončen lokalno končen povezan graf G , za katerega je minimalna stopnja njegovih točk večja od k , ima kromatsko število za katerokoli fiksno celo število k večje ali enako 2.
- (b) Vsak neskončen lokalno končen povezan graf ima kromatski indeks.
- (c) Vsak neskončen lokalno končen povezan graf ima razlikovalno število.
- (d) Vsak neskončen lokalno končen povezan graf ima razlikovalni indeks.

Zgornji rezultati izboljšujejo nekaj nedavnih rezultatov Stawiskega, saj je on predpostavil, da se da množico barv dobro urediti. Oblikujemo nove pogoje za obstoj nereducibilnega pravilnega barvanja, minimalnega pokritja povezav, maksimalnega prirejanja in minimalne dominacijske množice v povezanih dvodelnih grafih in lokalno končnih povezanih grafih, ki so ekvivalentni bodisi aksiomu izbire bodisi Kőnigovi lemi. Pokažemo tudi, da če aksiom izbire velja za družine 2-elementnih množic, potem ima Shelah–Soiferjev graf minimalno dominacijsko množico.

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Ključne besede: Aksiom izbire, pravilna barvanja, razlikovalna barvanja, minimalno pokritje povezav, maksimalno prirejanje, minimalna dominacijska množica.

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