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16 New way of second quantization of fermions and bosons

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Abstract. This contribution presents properties of the second quantized not only fermion fields but also boson fields, if the second quantization of both kinds of fields origins in the description of the internal space of fields with the ″basis vectors″ which are the superposition of odd (when describing fermions) or even (when describing bosons) products of the Clifford algebra operators γ^a's. The tensor products of the ″basis vectors″ with the basis in ordinary space forming the creation operators manifest the anticommutativty (of fermions) or commutativity (of bosons) of the ″basis vectors″, explaining the second quantization postulates of both kinds of fields. Creation operators of boson fields have all the properties of the gauge fields of the corresponding fermion fields, offering a new understanding of the fermion and boson fields.

Povzetek: Prispevek razloži drugo kvantizacijo ne le fermionskih ampak tudi bozonskih polj. Notranji prostor fermionov opišejo ″osnovnimi vektorji″, ki so superpozicija produktov lihega števila Cliffordovh operatorjev γ^a, bozonski ″osnovnimi vektorji″ pa so superpozicija produktov sodega števila Cliffordovih operatorjev γ^a. Kreacijski in anihilacijski operatorji, ki so tenzorski produkt končnega števila ″osnovnih vektorjev″ in zvezno neskončnega števila komutirajočih vektorjev v običajnem prostoru, ″podedujejo″ antikomutativnost ali komutativnost od ″osnovnih vektorjev″. Posledično fermionska stanja antikomutirajo in bozonska komutirajo, kar razloži postulate druge kvantizacije za fermionska in bozonska polja in ponudi nov pogled za lastnosti obeh vrst polj.

16.1 Introduction

In a long series of works [19,20,23,25,26,28,29,31] I have found, together with the collaborators ([1,26,32,34,35,37] and the references therein), with H.B. Nielsen and in long discussions with participants during the annual workshops ″What comes beyond the standard models″, the phenomenological success with the model named the *spin-charge-family* theory: The internal space of fermions are in this model described with the Clifford algebra objects of all linear superposition of odd products of γ^a's in $d = (13+1)$. Fermions interact with only gravity — with the vielbeins and the two kinds of the spin connection fields (the gauge fields of $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and $\tilde{S}^{ab} = \frac{1}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$ [1]). Spins from higher

[1] If there are no fermion present the two kinds of the spin connection fields are uniquely described by the vielbeins [36].

dimensions, $d > (3+1)$, described by γ^a's, manifest in $d = (3+1)$ as charges of the *standard model* quarks and leptons and antiquarks and antileptons, appearing in (two times four) families, the quantum numbers of which are determined by the second kind of the Clifford algebra object $\tilde{S}^{ab}$'s. Gravity in higher dimensions manifests as the *standard model* vector gauge fields as well as the scalar Higgs and Yukawa couplings [1, 5, 23, 25, 26, 26–29, 31, 32, 34, 35, 37], predicting new scalar fields, which offer the explanation besides for higgs scalar and Yukawa couplings also for the asymmetry between matter and antimatter in our universe and for the dark matter (represented by the stable of the upper group of four families), predicting a new family — the fourth family to the observed three.

In this contribution I shortly repeat the description of the internal space of the second quantized fermion fields with the odd products of the Clifford operators γ^a's, what leads to the creation operators for fermions without postulating the second quantization requirements of Dirac [7–9]. The creation operators for fermions, which are superposition of tensor products of the ordinary basis and the "basis vectors" describing the internal space of fermions, anticommute, explaining correspondingly the postulates of Dirac, offering also a new understanding of fermion fields ([1] and references therein). The creation operators of fermions appear in families, carrying either left or right handedness, their Hermitian conjugated partners belong to another set of Clifford odd "basis vectors" carrying the opposite handedness.

The main part of this contribution discusses properties of the second quantized boson fields, which are the gauge fields of the corresponding second quantized fermion fields. The internal space of bosons is described by the superposition of even products of γ^a's. The boson fields correspondingly commute. The corresponding creation operators and their Hermitian conjugated partners belong to the same set of "basis vectors", carrying all the quantum numbers in adjoint representations. They interact among themselves and with the corresponding fermion fields.

In Sect. 16.2 the anticommuting Grassmann and Clifford algebras are presenting and the relations among them discussed. The "basis vectors" are defined as the egenvectors of the Cartan subalgebra of the Lorentz algebra for the Grassmann and the two Clifford algebras for odd and for even products of algebras members, and their anticommutation or commutation relations presented, Subsect. 16.2.1. The reduction of the two kinds of the Clifford algebras to only one makes the Clifford odd "basis vectors" anticommuting, giving to different irreducible representations of the Lorentz algebra the family quantum numbers, Subsect. 16.2.2. To make understanding of the properties of the Clifford odd and Clifford even "basis vectors" easier in Subsect. 16.2.3 the case of $d = (5+1)$-dimensional space is chosen and the "basis vectors" of odd, 16.2.3, and even, 16.2.3, Clifford character are presented in details and then generalized to any even d, Subsect. 16.2.4.

In Sect. 16.3 the creation operators of the second quantized fermion and boson fields are discussed, as well as their Hermitian conjugated partners. In Subsect. 16.3.1 the simple action for fermion interacting with bosons and for corresponding bosons, as assumed in the *spin-charge-family* theory is presented.

Sect. 17.5 reviews shortly what one can learn in this contribution.

Both algebras, Grassmann and Clifford, offer "basis vectors" for the description of the internal space of fermions [1, 19, 20] and the corresponding bosons with which fermions interact. The oddness or evenness of "basis vectors", transfered to the creation operators, which are tensor products of the finite number of "basis vectors" and the (continuously) infinite number of momentum (or coordinate) basis, and to their Hermitian conjugated partners annihilation operators, offers the second quantization of fermions and bosons without postulating the second quantized conditions [7–9] for either the half integer spin fermions or integer spin bosons, enabling the explanation of the Dirac's postulates. Further investigations are needed in both case, for the boson case in particular, although promising, the time for this study was too short.

16.2 Grassmann and Clifford algebras

To describe the internal space of fermions and bosons one can use either the Grassmann or the Clifford algebras.
In Grassmann d-dimensional space there are d anticommuting operators θ^a, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, .., d)$, and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, offering together $2 \cdot 2^d$ operators, the half of which are superposition of products of θ^a and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,, (a, b) = (0, 1, 2, 3, 5, \cdots, d)\,. \tag{16.1}$$

Defining [32]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,, \quad \text{leads to} \quad (\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,. \tag{16.2}$$

θ^a and $\frac{\partial}{\partial\theta_a}$ are, up to the sign, Hermitian conjugated to each other. The identity is the self adjoint member of the algebra. We make a choice for the complex properties of θ^a, and correspondingly of $\frac{\partial}{\partial\theta_a}$, as follows

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, ..., -\theta^{d-1}, \theta^d)\,,$$
$$\{\frac{\partial}{\partial\theta_a}\}^* = (\frac{\partial}{\partial\theta_0}, \frac{\partial}{\partial\theta_1}, -\frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}, -\frac{\partial}{\partial\theta_5}, \frac{\partial}{\partial\theta_6}, ..., -\frac{\partial}{\partial\theta_{d-1}}, \frac{\partial}{\partial\theta_d})\,. \tag{16.3}$$

In d-dimensional space of anticommuting Grassmann coordinates and of their Hermitian conjugated partners derivatives, Eqs. (17.3, 16.2), there exist two kinds of the Clifford coordinates (operators) — γ^a and $\tilde{\gamma}^a$ — both expressible in terms of θ^a and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [20].

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \tag{16.4}$$

offering together $2 \cdot 2^d$ operators: 2^d of those which are products of γ^a and 2^d of those which are products of $\tilde{\gamma}^a$. Taking into account Eqs. (16.2, 16.4) it is easy to prove that they form two independent anticommuting Clifford algebras, Refs. ([1] and references therein)

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \\
\{\gamma^a, \tilde{\gamma}^b\}_+ &= 0 \,, \quad (a,b) = (0,1,2,3,5,\cdots,d) \,, \\
(\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a \,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a \,,
\end{aligned} \tag{16.5}$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$.
While the Grassmann algebra can be used to describe the "anticommuting integer spin second quantized fields" and "commuting integer spin second quantized fields" [1,25], the Clifford algebras describe the second quantized fermion fields, if the superposition of odd products of γ^a's or $\tilde{\gamma}^a$'s are used. The superposition of even products of either γ^a's or $\tilde{\gamma}^a$'s describe the commuting second quantized boson fields.
The reduction, Eq. (17.9) of Subsect. (16.2.2), of the two Clifford algebras — γ^a's and $\tilde{\gamma}^a$'s — to only one is needed — γ^a's are chosen — for the correct description of the internal space of fermions. After the decision that only γ^a's are used to describe the internal space of fermions, the remaining ones, $\tilde{\gamma}^a$'s, are used to equip the irreducible representations of the Lorentz group (with the infinitesimal generators $S^{ab} = \frac{i}{4}\{\gamma^a, \gamma^b\}_-$) with the family quantum numbers in the case that the odd Clifford algebra describes the internal space of the second quantized fermions.
It then follows that the even Clifford algebra objects, the superposition of the even products of γ^a's, offer the description of the second quantized boson fields, which are the gauge fields of the second quantized fermion fields, the internal space of which are described by the odd Clifford algebra objects. This will be demonstrated in this contribution.

16.2.1 "Basis vectors" determined by superposition of odd and even products of Clifford objects.

There are $\frac{d}{2}$ members of the Cartan subalgebra of the Lorentz algebra in even dimensional spaces. One can choose

$$\begin{aligned}
&\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d} \,, \\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d} \,, \\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d} \,, \\
&\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab} \,.
\end{aligned} \tag{16.6}$$

Let us look for the "eigenstates" of each of the Cartan subalgebra members, Eq. (16.6), for each of the two kinds of the Clifford algebras separately,

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad \tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) = \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,,
\end{aligned} \tag{16.7}$$

$k^2 = \eta^{aa}\eta^{bb}$. The proof of Eq. (17.7) is presented in Ref. [1], App. (I).
Let us introduce for nilpotents $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$, $(\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b))^2 = 0$ and projectors $\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, $(\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b))^2 = \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$ of both algebras the notation

$$\begin{aligned}
&\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{(k)}^\dagger = \eta^{aa}\stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]}\\
&\stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad \stackrel{ab}{[k]}^\dagger = \stackrel{ab}{[k]}\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,,\\
&\stackrel{ab}{(\tilde{k})}:= \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad \stackrel{ab}{(\tilde{k})}^\dagger = \eta^{aa}\stackrel{ab}{(-\tilde{k})}\,, \quad (\stackrel{ab}{(\tilde{k})})^2 = 0\,,\\
&\stackrel{ab}{[\tilde{k}]}:= \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, \quad \stackrel{ab}{[\tilde{k}]}^\dagger = \stackrel{ab}{[\tilde{k}]}\,, \quad (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,,\\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-\tilde{k}]} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(-\tilde{k})} = 0\,,
\end{aligned} \tag{16.8}$$

Statement 1. *One can define "basis vectors" to be eigenvectors of all the members of the Cartan subalgebras as even or odd products of nilpotents and projectors in any even dimensional space.*

Due to the anticommuting properties of the Clifford algebra objects there are anticommuting and commuting "basis vectors". The anticommuting "basis vectors" contain an odd products of nilpotents, at least one nilpotent, the rest are then projectors. Let us denote the Clifford odd "basis vectors" of the Clifford γ^a kind as $\hat{b}^{m\dagger}_f$, where m and f determine the m^{th} member of the f^{th} irreducible representation. We shall denote by $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$ the Hermitian conjugated partner of the "basis vector" $\hat{b}^{m\dagger}_f$. The "basis vectors" of the Clifford $\tilde{\gamma}^a$ kind would correspondingly be denoted by $\hat{\tilde{b}}^{m\dagger}_f$ and $\hat{\tilde{b}}^m_f$.
It is not difficult to prove the anticommutation relations of the Clifford odd "basis vectors" and their Hermitian conjugated partners for both algebras ([1,19] and references therein). Let us here present only the one of the Clifford algebras — γ^a's.

$$\begin{aligned}
\hat{b}^m_f {}_{*_A}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_f {}_{*_A}|\psi_{oc}> &= |\psi^m_f>\,,\\
\{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_{*_A +}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_f, \hat{b}^{m\dagger}_f\}_{*_A +}|\psi_{oc}> &= |\psi_{oc}>\,,
\end{aligned} \tag{16.9}$$

where $*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_f$ and $\hat{b}^{m'}_{f'}$ among themselves and with the vacuum state $|\psi_{oc}>$ of Eq.(17.10), which takes into account Eq. (17.5),

$$|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^m_f {}_{*_A} \hat{b}^{m\dagger}_f\,|\,1\,>\,, \tag{16.10}$$

for one of the members m, anyone, of the odd irreducible representation f, with $|1>$, which is the vacuum without any structure — the identity. It follows that $\hat{b}^m_f||\psi_{oc}>= 0$. The relations are valid for both kinds of the odd Clifford algebras, we only have to replace $\hat{b}^{m\dagger}_f$ by $\hat{\tilde{b}}^{m\dagger}_f$ and equivalently for the Hermitian conjugated partners.

The Clifford odd "basis vectors" *almost* fulfil the second quantization postulates for fermions. There is, namely, the property, which the second quantized fermions must fulfil in addition to the relations of Eq. (16.9). If the anticommutation relations of "basis vectors" and their Hermitian conjugated partners would fulfil the relation:

$$\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> = \delta^{mm'}\delta_{ff'}|\psi_{oc}>, \tag{16.11}$$

for either γ^a or $\tilde{\gamma^a}$, then the corresponding creation and annihilation operators would fulfil the anticommutation relations for the second quantized fermions, explaining the postulates of Dirac for the second quantized fermion fields. For any $\hat{b}^m_f$ and any $\hat{b}^{m'\dagger}_{f'}$ this is not the case. It turns out that besides $\hat{b}^{m=1}_{f=1} = \stackrel{d-1\,d}{(-)} \cdots \stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{03}{(-i)}$, for example, also $\hat{b}^{m'}_{f'} = \stackrel{d-1\,d}{(-)} \cdots \stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{03}{[+i]}$ and several others give, when applied on $\hat{b}^{m=1\dagger}_{f=1}$, nonzero contributions. There are namely $2^{\frac{d}{2}-1} - 1$ too many annihilation operators for each creation operator which give, applied on the creation operator, nonzero contribution.

The problem is solvable by the reduction of the two Clifford odd algebras to only one [1, 5, 36, 37] as it is presented in subsection 16.2.2: If γ^a's are chosen to determine internal space of fermions, the remaining ones, $\tilde{\gamma}^a$'s, determine then quantum numbers of each family (described by the eigenvalues of $\tilde{S}^{ab}$ of the Cartan subalgebra members). Correspondingly the creation and annihilation operators, expressible as tensor products, $*_T$, of the "basis vectors" and the basis in ordinary (momentum or coordinate) space, fulfil the anticommutation relation for the second quantized fermions.

Let me point out that the Hermitian conjugated partners of the "basis vectors" belong to different irreducible representations of the corresponding Lorentz group than the "basis vectors". This can be understood, since the Clifford odd "basis vectors" have always odd numbers of nilpotents, so that an odd number of $\stackrel{ab}{(k)}$'s transforms under Hermitian conjugation into $\stackrel{ab}{(-k)}$'s, which can not be the member of the "basis vectors", since the even generators of the Lorentz transformations transform always even number of nilpotents, keeping the number of nilpotents always odd. It is different in the case of the Clifford even "basis vectors", since an even number of $\stackrel{ab}{(k)}$'s, transformed with the Hermitian conjugation into en even number of $\stackrel{ab}{(-k)}$'s belongs to the same group of the "basis vectors".

Statement 2. *The Clifford odd $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations of "basis vectors" have their Hermitian conjugated partners in another set of $2^{\frac{d}{2}-1}$ $\cdot 2^{\frac{d}{2}-1}$ "basis vectors". Each of the two sets of the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" has their Hermitian conjugated partners within the same set.*

The Clifford even "basis vectors" commute. Let us denote the Clifford even "basis vectors", described by γ^a's, by $\hat{\mathcal{A}}^{m\dagger}_f$. There is no need to denote their Hermitian conjugated partners by $\hat{\mathcal{A}}^{m}_f$, since in the even Clifford sector the "basis vectors" and their Hermitian conjugated partners appear within the same group. We shall manifest this in the toy model of $d = (5+1)$. In the Clifford even sector m and f are just two indexes: f denotes the subgroups within which the "basis vectors" do not have the Hermitian conjugated partners (Subsect. 16.2.3, Eq. (16.21)).
We shall need also the equivalent "basis vectors" in the Clifford even part of the kind $\tilde{\gamma}^a$'s. Let these "basis vectors" be denoted by $\hat{\tilde{A}}^{m\dagger}_f$.
These commuting even Clifford algebra objects have interesting properties. I shall discuss the properties of even and odd "basis vectors" in Sects. 16.2.3, 16.2.4, first in $d = (5+1)$-dimensional space, then in the general case.

16.2.2 Reduction of the Clifford space

The creation and annihilation operators of an odd Clifford algebra of both kinds, of either γ^a's or $\tilde{\gamma}^a$'s, turn out to obey the anticommutation relations for the second quantized fermions, postulated by Dirac [1], provided that each of the irreducible representations of the corresponding Lorentz group, describing the internal space of fermions, would carry a different quantum number.
But we know that a particular member m has for all the irreducible representations the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either γ^a's or $\tilde{\gamma}^a$'s), Eq. (17.8).
Statement 3. *The only possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us say "family" quantum number, is that we "sacrifice" one of the two vector spaces.*
Let us "sacrifice" $\tilde{\gamma}^a$'s, using $\tilde{\gamma}^a$'s to define the "family" quantum numbers for each irreducible representation of the vector space of "basis vectors of an odd products of γ^a's, while keeping the relations of Eq. (17.5) unchanged: $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$, $(\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a$, $(a,b) = (0,1,2,3,5,\cdots,d)$.

We therefore *postulate*:
Let $\tilde{\gamma}^a$'s operate on γ^a's as follows [20,29,31,32,35]

$$\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}|\psi_{oc}>, \tag{16.12}$$

with $(-)^B = -1$, if B is (a function of) an odd products of γ^a's, otherwise $(-)^B = 1$ [35], $|\psi_{oc}>$ is defined in Eq. (17.10).

Statement 4. *After the postulate of Eq. (17.9) "basis vectors" which are superposition of an odd products of γ^a's obey all the fermions second quantized postulates of Dirac, presented in Eqs. (16.11, 16.9).*
We shall see in Sect. 16.2.3 that the Clifford even "basis vectors" obey the bosons second quantized postulates.

After this postulate the vector space of $\tilde{\gamma}^a$'s is "frozen out". No vector space of $\tilde{\gamma}^a$'s needs to be taken into account any longer for the description of the internal space of fermions or bosons, in agreement with the observed properties of fermions. $\tilde{\gamma}^a$'s obtain the role of operators determining properties of fermion and boson "basis vectors".

Let me add that we shall still use $\tilde{S}^{ab}$ for the description of the internal space of fermion and boson fields, Subsects. 16.2.3, 16.2.3, 16.2.3. $\tilde{S}^{ab}$'s remain as operators. One finds, using Eq. (17.9),

$$\stackrel{ab}{(\tilde{k})}\stackrel{ab}{(k)} = 0\,, \quad \stackrel{ab}{(\tilde{-k})}\stackrel{ab}{(k)} = -i\eta^{aa}\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[k]} = i\stackrel{ab}{(k)}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-k]} = 0\,,$$
$$\stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[\tilde{-k}]}\stackrel{ab}{(k)} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{[\tilde{-k}]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]} \tag{16.13}$$

Taking into account anticommuting properties of both Clifford algebras, γ^a's and $\tilde{\gamma}^a$'s, it is not difficult to prove the relations in Eq. (16.13).

16.2.3 Properties of Clifford odd and even "basis vectors" in $d = (5+1)$

To make discussions easier let us first look for the properties of "basis vectors" in $d = (5+1)$-dimensional space. Let us look at: **i.** internal space of fermions as the superposition of odd products of the Clifford objects γ^a's, **ii.** internal space of the corresponding gauge fields as the superposition of even products of the Clifford objects γ^a's.
Choosing the "basis vectors" to be eigenvectors of all the members of the Cartan subalgebra of the Lorentz algebra and correspondingly the products of nilpotents and projectors (Statement 1.) one finds the "basis vectors" presented in Table 16.1. The table presents the eigenvalues of the "basis vectors" for each member of the Cartan subalgebra for the group $SO(5,1)$.
The *odd I* group (is chosen to) present the "basis vectors" describing the internal space of fermions. Their Hermitian conjugated partners are then the "basis vectors" presented in the group *odd II*.
The *even I* and *even II* represent commuting Clifford even "basis vectors", representing bosons, the gauge fields of fermions.
We shall analyse both kinds of "basis vectors" through the subgroups of the $SO(5,1)$ group. The choices of $SU(2)\times SU(2)\times U(1)$ and $SU(3)\times U(1)$ subgroups of the $SO(5,1)$ group will also be discussed just to see the differences in properties from the properties of the $SO(5,1)$ group.
In Table 16.1 the properties of "basis vectors" are presented as products of nilpotents $\stackrel{ab}{(+i)}$ ($\stackrel{ab}{(+i)}^2 = 0$) and projectors $\stackrel{ab}{[+]}$($\stackrel{ab}{[+]}^2 = \stackrel{ab}{[+]}$). "Basis vectors" for fermions contain an odd number of nilpotents, "basis vectors" for bosons contain an even number of nilpotents. In both cases nilpotents $\stackrel{ab}{(+i)}$ and projectors $\stackrel{ab}{[+]}$ are chosen to be the "eigenvectors" of the Cartan subalgebra. Eq. (16.6), of the Lorentz algebra. The "basis vectors", determining the creation operators for fermions and their Hermitian conjugated partners, $\hat{b}^{m\dagger}_f$ and $\hat{b}^m_f$, respectively, as we shall see in Sub-

sect. 16.2.3 they are superposition of odd products of γ^a, algebraically anticommute, due to the properties of the Clifford algebra

$$\begin{aligned}
&\{\gamma^a,\gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a,\tilde{\gamma}^b\}_+ \,,\\
&\{\gamma^a,\tilde{\gamma}^b\}_+ = 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\
&\quad (\gamma^a)^\dagger = \eta^{aa}\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a\,,\\
&\quad \gamma^a\gamma^a = \eta^{aa}\,, \quad \gamma^a(\gamma^a)^\dagger = I\,, \quad \tilde{\gamma}^a\tilde{\gamma}^a = \eta^{aa}\,, \quad \tilde{\gamma}^a(\tilde{\gamma}^a)^\dagger = I\,, \quad (16.14)
\end{aligned}$$

where I represents the unit operator.

"Basis vectors" of odd products of γ^a's in $d = (5+1)$. Let us see in more details properties of the Clifford odd "basis vectors", analysing them also with respect to two kinds of the subgroups $SO(3,1)\times U(1)$ and $SU(3)\times U(1)$ of the group $SO(5,1)$, with the same number of Cartan subalgebra members in all three cases, $\frac{d}{2} = 3$. We use the expressions for the commuting operators for the subgroup $SO(3,1)\times U(1)$

$$N^3_{\pm}(= N^3_{(L,R)}) := \frac{1}{2}(S^{12} \pm iS^{03})\,, \quad \tilde{N}^3_{\pm}(= \tilde{N}^3_{(L,R)}) := \frac{1}{2}(\tilde{S}^{12} \pm i\tilde{S}^{03}) \quad (16.15)$$

and for the commuting generators for the subgroup $SU(3)$ and $U(1)$

$$\begin{aligned}
&\tau^3 := \frac{1}{2}(-S^{1\,2} - iS^{0\,3})\,, \qquad \tau^8 = \frac{1}{2\sqrt{3}}(-iS^{0\,3} + S^{1\,2} - 2S^{13\;14})\,,\\
&\tau^4 := -\frac{1}{3}(-iS^{0\,3} + S^{1\,2} + S^{5\,6})\,. \qquad (16.16)
\end{aligned}$$

The corresponding relations for $\tilde{\tau}^3, \tilde{\tau}^8$ and $\tilde{\tau}^4$ can be read from Eq. (16.16), if replacing S^{ab} by $\tilde{S}^{ab}$. Recognizing that $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ one reproduces all the relations for the corresponding $\vec{\tau}$ and $\mathcal{N}^3_{\pm}$.

The rest of generators of both kinds of subgroups of the group $SO(5,1)$ can be found in Eqs. (17.26, 17.28) of App. 16.7.

In Table 16.2 the properties of the odd "basis vectors" $\hat{b}^{m\dagger}_f$ are presented with respect to the generators of the group **i.** $SO(5,1)$ (with 15 generators, 3 of them forming the corresponding Commuting among subalgebra), **ii.** $SO(4)\times U(1)$ (with 7 generators and 3 of the corresponding Cartan subalgebra members) and **iii.** $SU(3)\times U(1)$ (with 9 generators and 3 of the corresponding Cartan subalgebra members), together with the eigenvalues of the commuting generators. These "basis vectors" are already presented as a part of Table 16.1. They fulfil together with their Hermitian conjugated partners the anticommutation relations of Eqs. (16.9, 16.11).
The right handed, $\Gamma^{(5+1)} = 1$, fourthplet of the fourth family of Table 16.2 can be found in the first four lines of Table 16.5 if only the $d = (5+1)$ part is taken into account. The left handed fourthplet of the fourth family of Table 16.4 can be found in four lines from line 33 to line 36, again if only the $d = (5+1)$ part is taken into

Table 16.1: $2^d = 64$ "eigenvectors" of the Cartan subalgebra of the Clifford odd and even algebras in $d = (5+1)$-dimensional space are presented, divided into four groups. The first group, odd I, is chosen to represent "basis vectors", named $\hat{b}^{m\dagger}_f$, appearing in $2^{\frac{d}{2}-1} = 4$ "families" ($f = 1, 2, 3, 4$), each "family" with $2^{\frac{d}{2}-1} = 4$ "family" members ($m = 1, 2, 3, 4$). The second group, odd II, contains Hermitian conjugated partners of the first group for each family separately, $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$. The "family" quantum numbers of $\hat{b}^{m\dagger}_f$, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, are written above each "family". The properties of anticommuting "basis vectors" are discussed in Subsects. 16.2.3, 16.2.4. The two groups with the even number of γ^a's, *even I* and *even II*, have their Hermitian conjugated partners within their own group each. The two groups which are products of even number of nilpotents and even or odd number of projectors represent the "basis vectors" for the corresponding boson gauge fields. Their properties are discussed in Subsecs. 16.2.3 and 16.2.4. $\Gamma^{(5+1)}$ and $\Gamma^{(3+1)}$ represent handedness in $d = (3+1)$ and $d = (5+1)$ space calculated as products of γ^a's, App. 16.9.

"basis vectors"	m	f = 1 $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	f = 2 $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	f = 3 $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 stackrel56	f = 4 $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
odd I $\hat{b}^{m\dagger}_f$		03 12 56	03 12 56	03 12 56	03 12 56					
	1	(+i)[+][+]	[+i][+](+)	[+i](+)[+]	(+i)(+)(+)	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	[−i](−)[+]	(−i)(−)(+)	(−i)[−][+]	[−i][−](+)	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	[−i][+](−)	(−i)[+][−]	(−i)(+)(−)	[−i](+)[−]	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	(+i)(−)(−)	[+i](−)[−]	[+i][−](−)	(+i)[−][−]	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1
		03 12 56	03 12 56	03 12 56	03 12 56				$\Gamma^{(5+1)}$	
odd II $\hat{b}^m_f$	1	(−i)[+][+]	[+i][+](−)	[+i](−)[+]	(−i)(−)(−)				−1	
	2	[−i](+)[+]	(+i)(+)(−)	(+i)[−][+]	[−i][−](−)				−1	
	3	[−i][+](+)	(+i)[+][−]	(+i)(−)(+)	[−i](−)[−]				−1	
	4	(−i)(+)(+)	[+i](+)[−]	[+i][−](+)	(−i)[−][−]				−1	
even I	m	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1
even II	m	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1

account.

Statement 5. *In a chosen* d=*dimensional space there is the choice that the "basis vectors" are right handed. Their Hermitian conjugated partners are correspondingly left handed. One could make the opposite choice, like in Table 16.4.*
Then the "basis vectors" of Table 16.2 would be the Hermitian conjugated partners to the left handed "basis vectors" of Table 16.4. For the left handed "basis vectors" the vacuum state $|\psi_{oc}>$, Eq. (17.10), chosen as the $\sum_f \hat{b}^m_f *_A \hat{b}^{m\dagger}_f$, has to be changed, since the vacuum state must have the property that $\hat{b}^m_f|\psi_{oc}>= 0$

Table 16.2: The basic creation operators, "basis vectors" — $\hat{b}^{m=(ch,s)\dagger}_f$ (each is a product of projectors and an odd product of nilpotents, and is the "eigenvector" of all the Cartan subalgebra members, S^{03}, S^{12}, S^{56} and $\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$, Eq. (16.6) (ch (charge), the eigenvalue of S^{56}, and s (spin), the eigenvalues of S^{03} and S^{12}, explain index m, f determines family quantum numbers, the eigenvalues of ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$) — are presented for $d = (5+1)$-dimensional case. This table represents also the eigenvalues of the three commuting operators $N^3_{L,R}$ and S^{56} of the subgroups $SU(2) \times SU(2) \times U(1)$ and the eigenvalues of the three commuting operators τ^3, τ^8 and τ^4 of the subgroups $SU(3) \times U(1)$, in these two last cases index m represents the eigenvalues of the corresponding commuting generators. $\Gamma^{(5+1)} = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6$, $\Gamma^{(3+1)} = i\gamma^0\gamma^1\gamma^2\gamma^3$. Operators $\hat{b}^{m=(ch,s)\dagger}_f$ and $\hat{b}^{m=(ch,s)}_f$ fulfil the anticommutation relations of Eqs. (16.9, 16.11).

f	m	$=(ch,s)$	$\hat{b}^{m=(ch,s)\dagger}_f$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	N^3_L	N^3_R	τ^3	τ^8	τ^4	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
I	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}\,\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
II	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\mid\stackrel{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\mid\stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\mid\stackrel{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\mid\stackrel{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\mid\stackrel{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\mid\stackrel{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
IV	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

and $\hat{b}^{m\dagger}_f\,|\psi_{oc}>= \hat{b}^{m\dagger}_f$.

One can notice that:
i. The family members of "basis vectors" have the same properties in all the families, independently whether one observes the group $SO(d-1,1)$ ($SO(5,1)$ in the case of $d = (5+1)$) or of the subgroups with the same number of commuting operators ($SU(2) \times SU(2) \times U(1)$ or $\times SU(3) \times U(1)$ in $d = (5+1)$case). The families carry different family quantum numbers. This is true for right, ($\Gamma^{(5+1)} = 1$), and for left ($\Gamma^{(5+1)} = -1$), representations.
ii. The sum of all the eigenvalues of all the commuting operators over the $2^{\frac{d}{2}-1}$ family members is equal to zero for each of $2^{\frac{d}{2}-1}$ families, separately for left and separately for right handed representations, independently whether the group $SO(d-1,1)$ ($SO(5,1)$) or the subgroups ($SU(2) \times SU(2) \times U(1)$ or $\times SU(3) \times U(1)$)

are considered.
iii. The sum of the family quantum numbers over the four families is zero as well.
iv. The properties of the left handed family members differ strongly from the right handed ones. It is easy to recognize this in our $d = (5+1)$ case when looking at $SU(3) \times U(1)$ quantum numbers since the right handed realization manifests the "colour" properties of "quarks" and "leptons" and the left handed the "colour" properties of "antiquarks" and "antileptons".
v. For a chosen even d there is a choice for either right or left handed family members. The choice of the handedness of the family members determine also the vacuum state for the chosen "basis vectors".
Let me add that the "basis vectors" and their Hermitian conjugated partners fulfil the anticommutation relations postulated by Dirac for the second quantized fermion fields. When forming tensor products, $*_T$, of these "basis vectors" and the basis of ordinary, momentum or coordinate, space the single fermion creation and annihilation operators fulfil all the requirements of the Dirac's second quantized fermion fields, explaining therefore the postulates of Dirac, Sect. 16.3.

"Basis vectors" of even products of γ^a's in $d = (5+1)$ The Clifford even "basis vectors", they are products of an even number of nilpotents, $\stackrel{ab}{(k)}$, and the rest up to $\frac{d}{2}$ of projectors, $\stackrel{ab}{[k]}$, commute since even products of (anticommuting) γ^a's commute.

Let us see in more details several properties of the Clifford even "basis vectors":

A. The properties of the algebraic, $*_A$, application of the Clifford even "basis vectors" on the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_f$, presented in Table 16.2, teaches us that the Clifford even "basis vectors" describe the internal space of the gauge fields of $\hat{b}^{m\dagger}_f$.

A.i.
Let $\hat{b}^{m`\dagger}_{f`}$ represents the $m`^{th}$ Clifford odd I "basis vector" (the part of the creation operators which determines the internal part of the fermion state) of the $f`^{th}$ family and let $\hat{\mathcal{A}}^{m\dagger}_f$ denotes the m^{th} Clifford *even* II "basis vector" of the f^{th} irreducible representation with respect to S^{ab} — but not with respects to $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, which includes all $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Let us evaluate the algebraic products $\hat{\mathcal{A}}^{m\dagger}_f$ on $\hat{b}^{m`\dagger}_{f`}$ for any (m, m') and (f, f').
Taking into account Eq. (17.8) and Tables (16.1, 16.2) one can easily evaluate the algebraic products $\hat{\mathcal{A}}^{m\dagger}_f$ on $\hat{b}^{m'\dagger}_{f'}$ for any (m, m') and (f, f'). Starting with $\hat{b}^{1\dagger}_1$ one

finds the non zero contributions only if applying $\hat{\mathcal{A}}^{m\dagger}_{3}$, $m=(1,2,3,4)$ on $\hat{b}^{1\dagger}_{1}$

$$
\begin{aligned}
&\hat{\mathcal{A}}^{m\dagger}_{3} *_A \hat{b}^{1\dagger}_{1} (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) : \\
&\hat{\mathcal{A}}^{1\dagger}_{3} (\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}) *_A \hat{b}^{1\dagger}_{1} (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}) \rightarrow \hat{b}^{1\dagger}_{1} \,, \\
&\hat{\mathcal{A}}^{2\dagger}_{3} (\equiv \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[+]}) *_A \hat{b}^{1\dagger}_{1} \rightarrow \hat{b}^{2\dagger}_{1} (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]}) \,, \\
&\hat{\mathcal{A}}^{3\dagger}_{3} (\equiv \stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(-)}) *_A \hat{b}^{1\dagger}_{1} \rightarrow \hat{b}^{3\dagger}_{1} (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}) \,, \\
&\hat{\mathcal{A}}^{4\dagger}_{3} (\equiv \stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}) *_A \hat{b}^{1\dagger}_{1} \rightarrow \hat{b}^{4\dagger}_{1} (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)}) \,.
\end{aligned}
\tag{16.17}
$$

The products of an even number of nilpotents and even or an odd number of projectors, represented by even products of γ^a's, applying on family members of a particular family, obviously transform family members, representing fermions of one particular family, into the same or another family member of the same family. All the rest of $\hat{\mathcal{A}}^{m}_{f}$, $f \neq 3$, applying on $\hat{b}^{1\dagger}_{1}$, give zero for any family f.

Let us comment the above events, concerning only the internal space of fermions and, obviously, bosons: If the fermion, the internal space of which is described by Clifford odd "basis vector" $\hat{b}^{1\dagger}_{1}$, absorbs the boson $\hat{\mathcal{A}}^{1}_{3}$ (with $\mathcal{S}^{03}=0, \mathcal{S}^{12}=0, \mathcal{S}^{56}=0$), its "basis vector" $\hat{b}^{1\dagger}_{1}$ remains unchanged.

The fermion with the "basis vector" $\hat{b}^{1\dagger}_{1}$, if absorbing the boson with $\hat{\mathcal{A}}^{2}_{3}$ (with $\mathcal{S}^{03}=-i, \mathcal{S}^{12}=-1, \mathcal{S}^{56}=0$), changes its internal "basis vector" $\hat{b}^{1\dagger}_{1}$ into the "basis vector" $\hat{b}^{2\dagger}_{1}$ (which carries now $S^{03}=-\frac{i}{2}, S^{12}=-\frac{1}{2}$, and the same $S^{56}=\frac{1}{2}$ as before). The fermion with "basis vector" $\hat{b}^{1\dagger}_{1}$ absorbing the boson with the "basis vector" $\hat{\mathcal{A}}^{3}_{3}$ changes its "basis vector" to $\hat{b}^{3\dagger}_{1}$, while the fermion with the "basis vector" $\hat{b}^{1\dagger}_{1}$ absorbing the boson with the "basis vector" $\hat{\mathcal{A}}^{4}_{3}$ changes its "basis vector" to $\hat{b}^{4\dagger}_{1}$.

Let us see how do the rest of $\hat{\mathcal{A}}^{m}_{f}$, $m=(1,2,3,4)$, $f=(1,2,3,4)$ change the properties of $\hat{b}^{n\dagger}_{1}$, $n=2,3,4$.

It is easy to evaluate if taking into account Eq . (17.8) that

$$
\begin{array}{l}
\hat{\mathcal{A}}^{m\dagger}_4 *_A \hat{b}^{2\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]}) : \\
\hat{\mathcal{A}}^{1\dagger}_4 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[+]}) *_A \hat{b}^{2\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]}) \rightarrow \hat{b}^{1\dagger}_1 \, , \\
\hat{\mathcal{A}}^{2\dagger}_4 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[+]}) *_A \hat{b}^{2\dagger}_1 \rightarrow \hat{b}^{2\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]}) \, , \\
\hat{\mathcal{A}}^{3\dagger}_4 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(-)}) *_A \hat{b}^{2\dagger}_1 \rightarrow \hat{b}^{3\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}) \, , \\
\hat{\mathcal{A}}^{4\dagger}_4 (\equiv \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(-)}) *_A \hat{b}^{2\dagger}_1 \rightarrow \hat{b}^{4\dagger}_1 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)}) \, , \\
\hat{\mathcal{A}}^{m\dagger}_2 *_A \hat{b}^{3\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}) : \\
\hat{\mathcal{A}}^{1\dagger}_2 (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{(+)}) *_A \hat{b}^{3\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}]) \rightarrow \hat{b}^{1\dagger}_1 \, , \\
\hat{\mathcal{A}}^{2\dagger}_2 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{(+)}) *_A \hat{b}^{3\dagger}_1 \rightarrow \hat{b}^{2\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]}) \, , \\
\hat{\mathcal{A}}^{3\dagger}_2 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[-]}) *_A \hat{b}^{3\dagger}_1 \rightarrow \hat{b}^{3\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}) \, , \\
\hat{\mathcal{A}}^{4\dagger}_2 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{[-]}) *_A \hat{b}^{1\dagger}_1 \rightarrow \hat{b}^{4\dagger}_1 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)}) \, ,' \\
\hat{\mathcal{A}}^{m\dagger}_1 *_A \hat{b}^{4\dagger}_1 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)}) : \\
\hat{\mathcal{A}}^{1\dagger}_1 (\equiv \stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(+)}) *_A \hat{b}^{4\dagger}_1 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)}) \rightarrow \hat{b}^{1\dagger}_1 \, , \\
\hat{\mathcal{A}}^{2\dagger}_1 (\equiv \stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{(+)}) *_A \hat{b}^{4\dagger}_1 \rightarrow \hat{b}^{2\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[+]}) \, , \\
\hat{\mathcal{A}}^{3\dagger}_1 (\equiv \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{[-]}) *_A \hat{b}^{4\dagger}_1 \rightarrow \hat{b}^{3\dagger}_1 (\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}) \, , \\
\hat{\mathcal{A}}^{4\dagger}_1 (\equiv \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[-]}) *_A \hat{b}^{4\dagger}_1 \rightarrow \hat{b}^{4\dagger}_1 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{(-)}) \, .
\end{array}
\qquad (16.18)
$$

All the rest of $\hat{\mathcal{A}}^{m}_f$, applying on $\hat{b}^{n\dagger}_1$, give zero for any other f except the one presented in Eqs. (16.17, 16.18).
We can repeat this calculation for all four family members $\hat{b}^{m`\dagger}_{f`}$ of any of families f‘. concluding

$$
\begin{array}{l}
\hat{\mathcal{A}}^{m\dagger}_3 *_A \hat{b}^{1\dagger}_f \rightarrow \hat{b}^{m\dagger}_f \, , \\
\hat{\mathcal{A}}^{m\dagger}_4 *_A \hat{b}^{2\dagger}_f \rightarrow \hat{b}^{m\dagger}_f \, , \\
\hat{\mathcal{A}}^{m\dagger}_2 *_A \hat{b}^{3\dagger}_f \rightarrow \hat{b}^{m\dagger}_f \, , \\
\hat{\mathcal{A}}^{m\dagger}_1 *_A \hat{b}^{4\dagger}_f \rightarrow \hat{b}^{m\dagger}_f \, .
\end{array}
\qquad (16.19)
$$

The recognition of this subsection concerns so far only internal space of fermions, not yet its dynamics in ordinary space. Let us interpret what is noticed:

Statement 6. *A fermion with the "basis vector" $\hat{b}^{m`\dagger}_{f`}$, "absorbing" one of the commuting Clifford even objects, $\hat{\mathcal{A}}^{m\dagger}_f$, transforms into another family member of the same family, to $\hat{b}^{m\dagger}_f$, changing correspondingly the family member quantum numbers and keeping the same family quantum number or remains unchanged.*
The application of the Clifford even "basis vector" $\hat{\mathcal{A}}^{m\dagger}_f$ on the Clifford odd "basis vector" does not cause the change of the family of the Clifford odd "basis vector".

A.ii.

We need to know the quantum numbers of the Clifford even "basis vectors", which obviously manifest properties of the boson fields since they bring to the Clifford odd "basis vectors" — representing the internal space of fermions — the quantum numbers which cause transformation into another fermion with a different Clifford odd "basis vectors" of the same family f. The Clifford even "basis vectors" do not cause the change of the family of fermions.

Let us point out that the Clifford odd "basis vectors" appear in $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ family members in each family, four members in four families in the $d = (5+1)$ case, while the Hermitian conjugated partners belong to another group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford odd "basis vectors", (to $odd\,II$ in Table 16.1), while the Clifford even "basis vectors" have their Hermitian conjugated partners within the same group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members (appearing in our treating case in $even\,II$ in Table 16.1). Since we found in Eqs. (16.19, 16.17, 16.17) that the Clifford even "basis vector" transforms the Clifford odd "basis vector" into another member of the same family, changing the family members quantum numbers for an integer, they must carry the integer quantum numbers.

One can see in Table 16.1 that the members of the group $even\,II$, for example, are Hermitian conjugated to one another in pairs and four of them are self adjoint. Correspondingly † has no special meaning, it is only the decision that all the Clifford even "basis vector" are equipped with †: $\hat{\mathcal{A}}^{m\dagger}_{f}$.

Let us therefore calculate the quantum numbers of $\hat{\mathcal{A}}^{m\dagger}_{f}$, where m and f distinguish among different Clifford even "basis vectors" (with f which does not really denote the family, since $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ defines the whole irreducible representation of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ "basis vectors") with the Cartan subalgebra operators $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, presented in Eqs. (16.6).

In Table 16.3 the eigenvalues of the Cartan subalgebra members of $\mathcal{S}^{ab}$ are presented, as well as the eigenvalues of the commuting operators of subgroups $SU(2)\times SU(2)\times U(1)$, that is the eigenvalues of $(\mathcal{N}^3_L, \mathcal{N}^3_R, \mathcal{S}^{03})$, and of $SU(3)\times U(1)$, that is the eigenvalues of (τ^3, τ^8, τ^4), expressions for which can be found in Eqs. (16.15, 16.16) if one takes into account that $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$. The algebraic application of any member of a group f on the self adjoint operator (denoted in Table 16.3 by $\bigcirc$) of this group f, gives the same member back.

The vacuum state of the Clifford even "basis vectors" is correspondingly the normalized sum of all the self adjoint operators of these Clifford even group $even\,II$. Each of $\hat{\mathcal{A}}^{m\dagger}_{f}$ when applying on such a vacuum state gives the same $\hat{\mathcal{A}}^{m\dagger}_{f}$.

$$|\phi_{oc_{even}}> = \frac{1}{2}\left(\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[-]} + \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]} + \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\right) \quad (16.20)$$

The pairs of "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$, which are Hermitian conjugated to each other, are in Table 16.3 pointed out by the same symbols. This property is independent of the group or subgroups which we choose to observe properties of the "basis vectors". If treating the subgroup $SU(3)\times U(1)$ one finds the 8 members of $\hat{\mathcal{A}}^{m\dagger}_{f}$, which belong to the group $SU(3)$ forming octet which has $\tau^4 = 0$, six of them appear in three pairs Hermitian conjugated to each other, two of them are self adjoint members of the octet, with eigenvalues of all the Cartan subalgebra members equal to zero. There are also two singlets with eigenvalues of all the Cartan subalgebra members equal to zero. And there is the sextet, with three pairs which are mutually Hermitian conjugated. One can notice that the sum of all the eigenvalues of all the

Table 16.3: The "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$, each is the product of projectors and an even number of nilpotents, and is the "eigenvector" of all the Cartan subalgebra members, $\mathcal{S}^{03}$, $\mathcal{S}^{12}$, $\mathcal{S}^{56}$, Eq. (16.6), are presented for $d = (5+1)$-dimensional case. Indexes m and f determine $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ different members $\hat{\mathcal{A}}^{m\dagger}_{f}$. In the third column the "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$ which are Hermitian conjugated partners to each other, and can therefore annihilate each other, are pointed out with the same symbol. For example with $\star$ are equipped the first member with $m = 1$ and $f = 1$ and the last member with $m = 4$ and $f = 3$. The sign $\bigcirc$ denotes the "basis vectors" which are self adjoint $(\hat{\mathcal{A}}^{m\dagger}_{f})^{\dagger} = \hat{\mathcal{A}}^{m\dagger}_{f}$. This table represents also the eigenvalues of the three commuting operators $\mathcal{N}^{3}_{L,R}$ and $\mathcal{S}^{56}$ of the subgroups $SU(2) \times SU(2) \times U(1)$ of the group $SO(5,1)$ and the eigenvalues of the three commuting operators τ^{3}, τ^{8} and τ^{4} of the subgroups $SU(3) \times U(1)$.

f	m	*	$\hat{\mathcal{A}}^{m\dagger}_{f}$	$\mathcal{S}^{03}$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$	$\mathcal{N}^{3}_{L}$	$\mathcal{N}^{3}_{R}$	τ^{3}	τ^{8}	τ^{4}
I	1	$\star\star$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(+)}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\triangle$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{(+)}$	$-i$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\ddagger$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$	$-i$	1	0	1	0	-1	0	0
	4	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[-]}$	0	0	0	0	0	0	0	0
II	1	$\bullet$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{(+)}$	i	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\otimes$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{(+)}$	0	-1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[-]}$	0	0	0	0	0	0	0	0
	4	$\ddagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\stackrel{56}{[-]}$	i	-1	0	-1	0	1	0	0
III	1	$\bigcirc$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}$	0	0	0	0	0	0	0	0
	2	$\odot\odot$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{[+]}$	$-i$	-1	0	0	-1	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
	3	$\bullet$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\stackrel{56}{(-)}$	$-i$	0	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
	4	$\star\star$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}$	0	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
IV	1	$\odot\odot$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[+]}$	i	1	0	0	1	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$
	2	$\bigcirc$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[+]}$	0	0	0	0	0	0	0	0
	3	$\otimes$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(-)}$	0	1	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\triangle$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(-)}$	i	0	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0

Cartan subalgebra members over the 16 members $\hat{\mathcal{A}}^{m\dagger}_{f}$ is equal to zero, independent of whether we treat the group $SO(5,1)$, $SU(2) \times SU(2) \times U(1)$, or $SU(3) \times U(1)$.

A.iii.

In **A.i.** we saw that the application of $\hat{\mathcal{A}}^{m\dagger}_{f}$ on the fermion "basis vectors" $\hat{b}^{m\dagger}_{f}$ transforms the particular member $\hat{b}^{m\dagger}_{f}$ to one of the members of the same family f, changing eigenvalues of the Cartan subalgebra members for an integer. We found in **A.ii** the eigenvalues of the Cartan subalgebra members for each of $2^{\frac{d}{2}-1} \times {}^{\frac{d}{2}-1}$ (equal to 16 in $d = (5+1)$) $\hat{\mathcal{A}}^{m\dagger}_{f}$, recognizing that they do have properties of the boson fields.

It remains to look for the behaviour of these Clifford even "basis vector" when they apply on each other. Let us denote the self adjoint member in each group of "basis vectors" of particular f as $\hat{\mathcal{A}}^{m_0\dagger}_f$. We easily see that

$$\{\hat{\mathcal{A}}^{m\dagger}_f, \hat{\mathcal{A}}^{m'\dagger}_f\}_- = 0\,, \quad \text{if}\,(m, m') \neq m_0 \;\text{or}\; m = m_0 = m'\,, \forall\, f\,,$$
$$\hat{\mathcal{A}}^{m\dagger}_f *_A \hat{\mathcal{A}}^{m_0\dagger}_f = \hat{\mathcal{A}}^{m\dagger}_f\,, \quad \forall\, m\,, \forall\, f\,. \tag{16.21}$$

Two "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_f$ and $\hat{\mathcal{A}}^{m'\dagger}_f$ of the same f and of $(m, m') \neq m_0$ are orthogonal.
The two "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_f$ and $\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $*_A$, of which gives nonzero contribution, like $\hat{\mathcal{A}}^{1\dagger}_1 *_A \hat{\mathcal{A}}^{4\dagger}_2 = \hat{\mathcal{A}}^{1\dagger}_2$, "scatter" into the third one, or annihilate into vacuum $|\phi_{oc_{even}}>$, Eq. (16.20), like $\hat{\mathcal{A}}^{2\dagger}_2 *_A \hat{\mathcal{A}}^{3\dagger}_4 = \hat{\mathcal{A}}^{2\dagger}_4$. [2] To generate creation and annihilation operators the tensor products, $*_T$, of the "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_f$, as well as of the "basis vectors" $\hat{b}^{m\dagger}_f$, with the basis in ordinary, momentum or coordinate, space is needed.

Statement 7. *Two "basis vectors"* $\hat{\mathcal{A}}^{m\dagger}_f$ *and* $\hat{\mathcal{A}}^{m'_0\dagger}_f$ *of the same* f *and of* $(m, m') \neq m_0$ *are orthogonal. The two "basis vectors" with nonzero algebraic product,* $*_A$*, "scatter" into the third one, or annihilate into vacuum.*

B. Let us point out that the choice of the Clifford odd "basis vectors", $odd\,I$, describing the internal space of fermions, and consequently the choice of the Clifford even "basis vectors", $even\,II$, describing the internal space of their gauge fields, is ours. If we choose in Table 16.1 $odd\,II$ to represent the "basis vectors" describing the internal space of fermions, then the corresponding "basis vectors" representing the internal space of bosonic partners are those of $even\,I$.

For a different choice of handedness of the Clifford odd "basis vectors" for describing fermions — making a choice of the left handedness instead of the right handedeness — Table 16.2 should be replaced by Table 16.4 and correspondingly also **A.i.**, **A.ii.**, **A.iii.** should be rewritten.
For an even d *there is a choice for either right or left handed family members.* The choice of the handedness of the family members determine also the vacuum state for the chosen "basis vectors" for either — Clifford odd "basis vectors" of fermions or for the corresponding Clifford even "basis vectors" of the corresponding gauge boson fields.

C. The Clifford even "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_f$, representing the boson gauge fields to the corresponding Clifford odd "basis vectors" $\hat{b}^{m\dagger}_f$, have the properties that they transform Clifford odd "basis vectors" $\hat{b}^{m\dagger}_f$ of each family within the family members. There are the additional Clifford even "basis vectors" $\hat{\tilde{\mathcal{A}}}^{m\dagger}_f$ which transform each family member of particular family into the same family member of some of the rest families.

[2] I use "scatter" in quotation marks since the "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_f$ determine only the internal space of bosons, as also the "basis vectors" $\hat{b}^{m\dagger}_f$ determine only the internal space of fermions.

These Clifford even "basis vectors" $\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}$ are products of an even number of nilpotents and of projectors, which are eigenvectors of the Cartan subalgebra operators $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1\,d}$. The table like Table 16.3 should be prepared and their properties described as in the case of **A.i.**, **A.ii.**, **A.iii.**. A short illustration is to help understanding the role of these Clifford even "basis vectors" $\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}$.

Let us use for the Clifford even "basis vectors" $\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}$ the same arrangement with products of nilpotents and projectors as the one, chosen for the Clifford even "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$ in the case of $d = (5+1)$ in Table 16.3, except that now nilpotents and projectors are eigenvectors of the Cartan subalgebra operators $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$, and are correspondingly written in terms of nilpotents $\overset{ab}{(\tilde{k})}$ and projectors $\overset{ab}{[\tilde{k}]}$. The application of these nilpotents and projectors on nilpotents and projectors appearing in $\hat{b}^{m\dagger}_{f}$ are presented in Eq. (16.13). Making a choice of $\hat{\tilde{\mathcal{A}}}^{1\dagger}_{1} (\equiv \overset{03}{[\tilde{+i}]}\,\overset{12}{(\tilde{+})}\,\overset{56}{(\tilde{+})})$, with quantum numbers $(\mathcal{S}^{03} = 0, \mathcal{S}^{12} = 1, \mathcal{S}^{56} = 1)$, on $\hat{b}^{4\dagger}_{1} (\equiv \overset{03}{(+i)}\,\overset{12}{(-)}\,\overset{56}{(-)})$ with the family members quantum numbers $(S^{03} = \frac{i}{2}, S^{12} = -\frac{1}{2}, S^{56} = -\frac{1}{2})$ and the family quantum numbers $(\tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = -\frac{1}{2}, \tilde{S}^{56} = -\frac{1}{2})$ it follows

$$\hat{\tilde{\mathcal{A}}}^{1\dagger}_{1} (\equiv \overset{03}{[\tilde{+i}]}\,\overset{12}{(\tilde{+})}\,\overset{56}{(\tilde{+})}) *_{A} \hat{b}^{4\dagger}_{1} (\equiv \overset{03}{(+i)}\,\overset{12}{(-)}\,\overset{56}{(-)}) \rightarrow \hat{b}^{4\dagger}_{4} (\equiv \overset{03}{(+i)}\,\overset{12}{[-]}\,\overset{56}{[-]})\,. \tag{16.22}$$

$\hat{b}^{4\dagger}_{4} (\equiv \overset{03}{(+i)}\,\overset{12}{[-]}\,\overset{56}{[-]})$ carry the same family members quantum numbers as $\hat{b}^{4\dagger}_{1}$ $(S^{03} = \frac{i}{2}, S^{12} = -\frac{1}{2}, S^{56} = -\frac{1}{2}))$ but belongs to the different family with the family quantum numbers $(\tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = \frac{1}{2}, \tilde{S}^{56} = \frac{1}{2})$.

The detailed analyse of these last two cases **B.** and **C.** will be studied after this Bled proceedings.

We can conclude that the Clifford even "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$:

a. Have the quantum numbers determined by the Cartan subalgebra members of the Lorentz group of $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$. Applying algebraically, $*_A$, $\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, $\hat{\mathcal{A}}^{m\dagger}_{3}$ transform these "basis vectors" to another ones with the same family quantum numbers, $\hat{b}^{m'\dagger}_{f}$.

b. In any irreducibly representation of $\mathcal{S}^{ab}$ $\hat{\mathcal{A}}^{m\dagger}_{f}$ appear in pairs, which are Hermitian conjugated to each other or they are self adjoint.

c. The self adjoint members $\hat{\mathcal{A}}^{m\dagger}_{f}$ define the vacuum state of the second quantized boson fields.

d. Applying $\hat{\mathcal{A}}^{m\dagger}_{f}$ algebraically to each other these commuting Clifford even "basis vector" forming another Clifford even "basis vector" or annihilate into the vacuum.

e. The choice of the left or the right handedness of the "basis vectors" of an odd Clifford character, describing the internal space of fermions, is ours. The left and the right handed "basis vectors" of an odd Clifford character are namely Hermitian conjugated to each other. With the choice of the handedness of the fermion "basis vectors" also the choice of boson Clifford even "basis vectors" — which are their corresponding gauge fields — are chosen.

f. There exist the Clifford even "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$ (like $\hat{\mathcal{A}}^{1\dagger}_{1}$ $(\equiv \overset{03}{[\tilde{+i}]}\,\overset{12}{(\tilde{+})}\,\overset{56}{(\tilde{1})}))$ which transform the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, representing the internal space of fermions, into the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f'}$ with the same family member m belonging to another family f'.

16.2.4 "Basis vectors" describing internal space of fermions and bosons in any even dimensional space

In Subsect. 16.2.3 the properties of the "basis vectors", describing internal space of fermions and bosons in a toy model with $d = (5+1)$ are presented in order to simplify (to make more illustrative) the discussions on the properties of the Clifford odd "basis vectors" describing the internal space of fermions and the Clifford even "basis vectors" describing the internal space of corresponding bosons, the gauge fields of fermions.
The generalization to any even d is straightforward. For the description of the internal space of fermions I follow here Ref. [1].

a. The "basis vectors" offering the description of the internal space of fermions, $\hat{b}^{m\dagger}_{f}$, must contain an odd product of nilpotents $\overset{ab}{(k)}$, $2n'+1$, in $d = 2(2n+1)$, $n' = (0, 1, 2, \ldots, \frac{1}{2}(\frac{d}{2} - 1)$, and the rest is the product of n'' projectors $\overset{ab}{[k]}$, $n'' = \frac{d}{2} - (2n'+1)$. Nilpotents and projectors are chosen to be "eigenvectors" of the $\frac{d}{2}$ members of the Cartan subalgebra.
After the reduction of the two kinds of the Clifford algebras to only one, γ^a's, the generators S^{ab} of the Lorentz transformations in the internal space of fermions described by γ^a's, determine the $2^{\frac{d}{2}-1}$ family members for each of $2^{\frac{d}{2}-1}$ families, while $\tilde{S}^{ab}$'s determine the $\frac{d}{2}$ numbers (the eigenvalues of the Cartan subalgebra members of the $2^{\frac{d}{2}-1}$ families).
The Cliford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ obey the postulates of Dirac for the second quantized fermion fields

$$
\begin{aligned}
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+} |\psi_{oc}> &= \delta^{mm'}\, \delta_{ff'} |\psi_{oc}>,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*_A+} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>,\\
\hat{b}^{m\dagger}_{f} *_A |\psi_{oc}> &= |\psi^{m}_{f}>,\\
\hat{b}^{m}_{f} *_A |\psi_{oc}> &= 0 \cdot |\psi_{oc}>,
\end{aligned}
\tag{16.23}
$$

with (m, m') denoting the "family" members and (f, f') denoting "families", $*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_{f}$ with their Hermitian conjugated objects $\hat{b}^{m}_{f}$, with the vacuum state $|\psi_{oc}>$, Eq. (17.10), and $\hat{b}^{m\dagger}_{f}$ or $\hat{b}^{m'}_{f'}$ among themselves. It is not difficult to prove the above relations if taking into account Eq. (17.5).
The Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$'s and their Hermitian conjugated partners $\hat{b}^{m}_{f}$'s appear in two independent groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, Hermitian conjugated to each other.

It is our choice which one of these two groups with $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members to take as "basis vectors" $\hat{b}^{\dagger m}_{f}$'s. Making the opposite choice the "basis vectors" change handedness.

b. The "basis vectors" for bosons, $\hat{\mathcal{A}}^{m\dagger}_{f}$, must contain an even number of nilpotents $\stackrel{ab}{(k)}$, $2n'$. In $d = 2(2n+1)$, $n' = (0, 1, 2, \dots, \frac{1}{2}(\frac{d}{2}-1))$, the rest, n'', are projectors $\stackrel{ab}{[k]}$, $n'' = (\frac{d}{2} - (2n'))$.
The "basis vectors" are either self adjoint or have the Hermitian conjugated partners within the same group of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members.
They do not form families, m and f only note a particular "basis vector". One of the members of particular f is self adjoint and participates to the vacuum state which has $2^{\frac{d}{2}-1}$ summands, Eq. (16.20).
The Clifford even "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$ commute, $\{\hat{\mathcal{A}}^{m\dagger}_{f}, \hat{\mathcal{A}}^{m'\dagger}_{f}\}_{-} = 0$, if both have the same index f and none of them or both of them are self adjoint operators.

$$\{\hat{\mathcal{A}}^{m\dagger}_{f}, \hat{\mathcal{A}}^{m'\dagger}_{f}\}_{-} = 0\,, \quad \text{if}\,(m, m') \neq m_0 \,\text{or}\, m = m_0 = m'\,, \forall f\,,$$
$$\hat{\mathcal{A}}^{m\dagger}_{f} *_A \hat{\mathcal{A}}^{m_0\dagger}_{f} = \hat{\mathcal{A}}^{m\dagger}_{f}\,, \quad \forall m\,, \forall f\,. \qquad (16.24)$$

The two "basis vectors", $\hat{\mathcal{A}}^{m\dagger}_{f}$ and $\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $*_A$, of which gives nonzero contribution, "scatter" into the third one, or annihilate into the vacuum $|\phi_{oc_{even}}>$.
Quantum numbers of $\hat{\mathcal{A}}^{m\dagger}_{f}$ are determined by the Cartan subalgebra members of the Lorentz group $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.
If a fermion with the "basis vector" $\hat{b}^{m\dagger}_{f}$ "absorbs" one of the commuting Clifford even objects, $\hat{\mathcal{A}}^{m`\dagger}_{f'}$, it transforms into another family member of the same family, to $\hat{b}^{m'\dagger}_{f}$, changing correspondingly the family member quantum numbers, keeping the family quantum number the same, or remains unchanged.
The remaining group of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors", presented in Table 16.1 do not influence the chosen Clifford odd "basic vectors", but rather their Hermitian conjugated partners $\hat{b}^{m}_{f}$.
There are the even "basis vectors" $\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}$, the nilpotents and projectors of which are $\stackrel{ab}{(\tilde{k})}$, $\stackrel{ab}{[\tilde{k}]}$, respectively. These "basis vectors" $\hat{\tilde{\mathcal{A}}}^{m\dagger}_{f}$, if applying on the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, transform these "basis vectors" into "basis vectors" $\hat{b}^{m\dagger}_{f'}$ belonging to different family f, while the family member quantum number m remains unchanged.
Exchanging the role of the Clifford odd "basis vector" $\hat{b}^{m\dagger}_{f}$ and their Hermitian conjugated partners $\hat{b}^{m}_{f}$ (what means in the case of $d = (5+1)$ the exchange of $odd\,I$, which is right handed, with $odd\,II$, which is lefthanded, in Table 16.1), not only causes the change of the handedness of the new $\hat{b}^{n\dagger}_{f}$, but also the change of the role of the Clifford even "basis vectors" (what means in the case of $d = (5+1)$ the exchange of *even* II with *even* I).

16.3 Second quantized fermion and boson fields with internal space described by Clifford algebra

After the **reduction** of the Clifford space to only the part determined by γ^a's, the "basis vectors", which are superposition of odd products of γ^a's, determine the internal space of fermions. The "basis vectors" are orthogonal and appear in even dimensional spaces in $2^{\frac{d}{2}-1}$ families, each with $2^{\frac{d}{2}-1}$ family members. Quantum numbers of family members are determined by S^{ab}, quantum numbers of families are determined by $\tilde{\gamma}^a$'s, or better by $\tilde{S}^{ab}$'s. $\tilde{\gamma}^a$'s anticommute among themselves and with γ^a's, as they did before the reduction of the Clifford space, Eq. (17.11). "Basis vectors" $\hat{b}^{m\dagger}_f$, determining internal space of fermions, are in even dimensional spaces products of an odd number of nilpotents and an even number of projectors, chosen to be eigenvectors of the $\frac{d}{2}$ Cartan subalgebra members of the Lorentz algebra S^{ab}, Table 16.2. There are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Hermitian conjugated partners of "basis vectors", denoted by $\hat{b}^m_f (= (\hat{b}^{m\dagger}_f)^\dagger$. It is our choice which one of these two groups of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members are "basis vectors" and which one are their Hermitian conjugated partners. These two groups differ in handedness as can be seen in Table 16.1, if observing $odd\,I$ and $odd\,II$, as well as if we compare Table 16.2 and Table 16.4.

The Clifford odd anticommuting "basis vectors", describing the internal space of fermions, obey together with their Hermitian conjugated partners the postulates of Dirac for the second quantized fermion fields, Eq. (17.11).
The Clifford even products of γ^a's (with the even number of nilpotents) form twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ "basis vectors", $\hat{\mathcal{A}}^{m\dagger}_f$, describing properties of bosons, Table 16.3. *Each of the two groups are commuting objects due to the fact that even number of γ^a's commute.*

Also the Clifford even "basis vectors" are chosen to be the eigenvectors of the Cartan subalgebra of the Lorentz group, this time determined by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eqs. (16.19, 16.21). While the Clifford odd "basis vectors" and their Hermitian conjugated partners form two independent groups, the Clifford even "basis vectors" have their Hermitian conjugated partners within each of the two groups.
The choice of the "basis vectors" among the two groups of the Clifford odd products of nilpotents and projectors for the description of the internal space of fermions, distinguishing also in handedness and other properties (Table 16.2 and Table 16.4) made as well the choice foe the Clifford even "basis vectors" describing the corresponding boson fields. We notice in Table 16.1 that the choice of $odd\,I$ for the description of the internal space of fermions makes $even\,II$ to be the corresponding boson field.
The remaining group of the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors", presented in Table 16.1 as $even\,I$ are not the boson partners to the chosen Clifford odd $odd\,I$ "basic vectors", but rather to their Hermitian conjugated partners $\hat{b}^m_f$, presented as $odd\,II$ in the same Table 16.1.

The creation operators, either for creating fermions or for creating bosons, must have besides the "basis vectors" defining the internal space of fermions and bosons also the *basis in ordinary space in momentum or coordinate representation*. I follow here shortly Ref. [1].

Let us briefly present the relations concerning the momentum or coordinate part of the single particle states. The longer version is presented in Ref. ([1] in Subsect. 3.3 and in App. J)

$$
\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p\,>\,, \quad <\vec{p}\,| =< 0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}') =< 0_p\,|\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{p}'}|\,0_p\,>\,,\\
&\text{leading to}\\
\hat{b}_{\vec{p'}}, \hat{b}^{\dagger}_{\vec{p}} &= \delta(\vec{p'}-\vec{p})\,,
\end{aligned}
\tag{16.25}
$$

where the normalization $< 0_p\,|\,0_p\, >= 1$ to identity is assumed. While the quantized operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ commute $\{\hat{p}^i\,, \hat{p}^j\}_- = 0$ and $\{\hat{x}^k\,, \hat{x}^l\}_- = 0$, this is not the case for $\{\hat{p}^i\,, \hat{x}^j\}_- = i\eta^{ij}$. It therefore follows

$$
\begin{aligned}
&<\vec{p}\,|\,\vec{x}> = < 0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}\,>= (< 0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\;\hat{b}^{\dagger}_{\vec{p}}\,|0_{\vec{p}}\,>)^{\dagger}\\
&\{\hat{b}^{\dagger}_{\vec{p}}\,,\,\hat{b}^{\dagger}_{\vec{p}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}_{\vec{p}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{p}\,'}\}_- = 0\,,\\
&\{\hat{b}^{\dagger}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}_{\vec{x}\,'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}\,'}\}_- = 0\,,\\
&\quad\text{while}\\
&\{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{x}}\}_- = e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, \qquad ,\{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{p}}\}_- = e^{-i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,,
\end{aligned}
\tag{16.26}
$$

Statement 8. *While the internal space of either fermions or bosons has the finite degrees of freedom —* $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ *— the momentum basis has obviously continuously infinite degrees of freedom.*

Let us use the common symbol $\hat{a}^m_f$ for both "basis vectors" $\hat{b}^{m\dagger}_f$ and $\hat{\mathcal{A}}^{m\dagger}_f$. And let be taken into account that either fermion or boson second quantized states are solving equations of motion, which relate p^0 and $\vec{p}$: $p^0 = |\vec{p}|$. Then the solution of the equations of motion can be written as the superposition of the tensor products, $*_T$, of a finite number of "basis vectors" describing the internal space of a second quantized single particle state, $\hat{a}^m_f$, and the continuously infinite momentum basis

$$
\{\hat{\mathbf{a}}^{s\dagger}_f(\vec{p}) = \sum_m c^{sm}{}_f\,(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}}\;*_T\;\hat{a}^{m\dagger}_f\}|vac_c > *_T|0_{\vec{p}}>\,, \tag{16.27}
$$

where $\vec{p}$ determines the momentum in ordinary space and s determines all the rest of quantum numbers. The state written here as $|vac_o > *_T|0_{\vec{p}}>$ is considered as the vacuum for a starting single particle state from which one obtains the other single particle states by the operators, like $\hat{b}^{\dagger}_{\vec{p}}$, which pushes the momentum by an amount $\vec{p}$ and the vacuum for either fermions $|\psi_{oc}>$, Eq. (17.10), or bosons $|\phi_{oc_{even}}>$, Eq. (16.20).
The creation operators for fermions can be therefore written as

$$
\{\hat{\mathbf{b}}^{s\dagger}_f(\vec{p}) = \sum_m c^{sm}{}_f\,(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}}\;*_T\;\hat{b}^{m\dagger}_f\}|\psi_{oc}> *_T|0_{\vec{p}}>\,, \tag{16.28}
$$

while for the corresponding gauge bosons it follows

$$\{\hat{\mathcal{A}}^{s\dagger}_{f}(\vec{p}) = \sum_{m} \mathcal{C}^{sm}{}_{f}(\vec{p})\, \hat{b}^{\dagger}_{\vec{p}} *_{T} \hat{\mathcal{A}}^{m\dagger}_{f}\}|\phi_{oc_{even}}> *_{T} |0_{\vec{p}}> . \qquad (16.29)$$

Since the "basis vectors" $\hat{b}^{m\dagger}_{f}$, describing the internal space of fermion, and their Hermitian conjugated partners do fulfil the anticommuting properties of Eq. (17.11), then also $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and $(\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}))^{\dagger}$, Eq. (17.12), fulfil the anticommutation relations of Eq. (17.11) due the commutativity of operators $\hat{b}^{\dagger}_{\vec{p}} = (\hat{b}^{\dagger}_{-\vec{p}})^{\dagger} = \hat{b}_{-\vec{p}}$ and anticommutativity of "basis vectors".
The "basis vectors" for fermions bring to the second quantized fermions, that is to the creation and correspondingly to the annihilation operators operating on the vacuum state, the *anticomutativity* and $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ quantum numbers of family members and of families for each of continuously ∞ many $\vec{p}$. The fermion single particle states therefore already anticommute.

The $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" $\hat{\mathcal{A}}^{m\dagger}_{f}$, appearing in pairs which are Hermitian conjugated to each other, fulfil the commuting properties of Eq. (16.21), transfering these commuting properties also to $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of $\hat{\mathcal{A}}^{s\dagger}_{f}(\vec{p})$, Eq. (17.12), for any of continuously ∞ $\vec{p}$, so that $\hat{\mathcal{A}}^{s\dagger}_{f}(\vec{p})$ fulfil the commutation relations of Eq. (16.21) according to commutativity properties of operators $\hat{\mathcal{A}}^{m\dagger}_{\vec{p}}$.
Statement 9. *The odd products of the Clifford objects γ^{a}'s offer the "basis vectors" to describe the internal space of the second quantized fermion fields. The even products of the Clifford objects γ^{a}'s offer the "basis vectors" to describe the internal space of the second quantized boson fields. They are the gauge fields of the fermion fields described by the odd Clifford objects.*

Statement 9.a *The description of the internal space of fermions with the odd Clifford algebra explains the second quantization postulates of Dirac. The quantized single fermion states anticommute.*

The $\hat{\mathcal{A}}^{s\dagger}_{f}(\vec{p})$ "basis vectors" bring to the second quantized bosons, that is to the creation operators and annihilation operators, appearing in pairs or as self adjoint operators, operating on the vacuum state, the *commutativity* properties and $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ quantum numbers, explaining properties of boson particles. The ordinary basis, $\hat{b}^{\dagger}_{\vec{p}}$, brings to the creation operators the continuously infinite degrees of freedom.

Statement 9.b *The description of the internal space of bosons with the even Clifford algebra explains the second quantization postulates for gauge fields. The quantized single boson states commute.*

Let us represent here the anticommutation relations for the creation and annihilation operators of the second quantized fermion fields $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ by taking into account Eq. (17.11)

$$
\begin{aligned}
\{\hat{\mathbf{b}}^{s'}_{f`}(\vec{p'})\,,\,\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+}\,|\psi_{oc}>|0_{\vec{p}}> &= \delta^{ss'}\delta_{ff'}\,\delta(\vec{p}'-\vec{p})\,|\psi_{oc}>|0_{\vec{p}}>\,,\\
\{\hat{\mathbf{b}}^{s'}_{f`}(\vec{p'})\,,\,\hat{\mathbf{b}}^{s}_{f}(\vec{p})\}_{+}\,|\psi_{oc}>|0_{\vec{p}}> &= 0\,|\psi_{oc}>|0_{\vec{p}}>\,,\\
\{\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p'})\,,\,\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+}\,|\psi_{oc}>|0_{\vec{p}}> &= 0\,|\psi_{oc}>|0_{\vec{p}}>\,,\\
\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\,|\psi_{oc}>|0_{\vec{p}}> &= |\psi^{s}_{f}(\vec{p})>\\
\hat{\mathbf{b}}^{s}_{f}(\vec{p})\,|\psi_{oc}>|0_{\vec{p}}> &= 0\,|\psi_{oc}>|0_{\vec{p}}>\\
|p^{0}| &= |\vec{p}|\,.
\end{aligned}
\tag{16.30}
$$

The creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}, p^0))$ and their Hermitian conjugated partners annihilation operators $\hat{\mathbf{b}}^{s}_{f}(\vec{p}, p^0))$, creating and annihilating the single fermion state, respectively, fulfil when applying on the vacuum state, $|\psi_{oc}>|0_{\vec{p}}>$, the anticommutation relations for the second quantized fermions, postulated by Dirac (Ref. [1], Subsect. 3.3.1, Sect. 5).
The anticommutation relations of Eq. (17.14) are valid also if we replace the vacuum state, $|\psi_{oc}>|0_{\vec{p}}>$, by the Hilbert space of Clifford fermions generated by the tensor product multiplication, $*_{T_H}$, of any number of the Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{\mathbf{b}}^{s}_{f}{}^{\dagger}(\vec{p})$ of any $(s, f, \vec{p})$), Ref. ([1], Sect. 5.).

The commutation relations among boson creation operators $\hat{\mathcal{A}}^{s\dagger}_{f}(\vec{p})$ can be written as

$$
\{\hat{\mathcal{A}}^{s\dagger}_{f}(\vec{p})\,,\hat{\mathcal{A}}^{s'\dagger}_{f}(\vec{p}')\}_{-} = f^{ss's''ff`f''}\,\hat{\mathcal{A}}^{s''\dagger}_{f''}\,\delta(\vec{p}-\vec{p}')\,. \tag{16.31}
$$

Let us present an example with $\vec{p} = (0, 0, p^3, 0, 0)$ and the choice $\hat{\mathcal{A}}^{3\dagger}_{1}(\vec{p})$ and $\hat{\mathcal{A}}^{2\dagger}_{2}(\vec{p}')$, taken from Table 16.3, one finds

$$
\{\hat{\mathcal{A}}^{3\dagger}_{1}(\vec{p})\,,\hat{\mathcal{A}}^{1\dagger}_{2}(\vec{p}')\}_{-} = -\delta(\vec{p}-\vec{p}')\,\hat{\mathcal{A}}^{2\dagger}_{1}(\vec{p})\,. \tag{16.32}
$$

One can notice that the sums over each of the quantum numbers ($\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \mathcal{N}^3_L, \mathcal{N}^3_R, \tau^3, \tau^8,$ of the left hand side are equal to the corresponding quantum numbers on the right hand side.

The study of properties of the second quantized bosons with the internal space of which is described by the Clifford even algebra has just started and needs further consideration.
Let us point out that when breaking symmetries, like in the case of $d = (5+1)$ into $SU(2) \times SU(2) \times U(1)$, one easily sees that the same, either the right or the left representations appear within the same, only the right, Table 16.2, or only the left, Table 16.4, representation, manifesting the right (left) hand fermions and the left (right) handed antifermions [24]. The same observation demonstrates also Table 16.5, in which in each octet of u-quarks and d-quarks of any colour and in the octet of colourless leptons the left and the right members of fermions and antifermions appear.

16.3.1 Simple action for fermion and boson fields

Let the space be $d = 2(2n+1)$-dimensional. The *spin-charge-family* theory proposes $d = (13+1)$-dimensional space, or larger, so that the "basis vectors". describing the internal space of fermions and bosons, offers the properties of the observed quarks and leptons and their antiquarks and antileptons, as well as the corresponding boson fields, as we learn in thic contribution.
The action for the second quantized massless fermion and antifermion fields, and the corresponding massless boson fields in $d = 2(2n+1)$-dimensional space is therefore

$$
\begin{aligned}
\mathcal{A} &= \int d^d x\, E\, \frac{1}{2}\,(\bar{\psi}\,\gamma^a p_{0a}\psi) + h.c. + \\
&\quad \int d^d x\, E\,(\alpha\, R + \tilde{\alpha}\,\tilde{R})\,, \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E}\{p_\alpha, E f^{\alpha}{}_a\}_-\,, \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}\,, \\
R &= \frac{1}{2}\{f^{\alpha[a}f^{\beta b]}\;(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^{c}{}_{b\beta})\} + h.c.\,, \\
\tilde{R} &= \frac{1}{2}\{f^{\alpha[a}f^{\beta b]}\;(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\,\tilde{\omega}^{c}{}_{b\beta})\} + h.c.\,.
\end{aligned}
\tag{16.33}
$$

Here [3] $f^{\alpha[a}f^{\beta b]} = f^{\alpha a}f^{\beta b} - f^{\alpha b}f^{\beta a}$.
It is proven in Refs. [26, 38] that the spin connection gauge fields manifest in $d = (3+1)$ as the ordinary gravity, the known vector gauge fields and the scalar gauge fields, offering the (simple) explanation for the origin of higgs assumed by the *standard model*, explaining as well the Yukawa couplings.

16.4 Conclusions

In the *spin-charge-family* theory the Clifford algebra is used to describe the internal space of fermion fields, what brings new insights, new recognitions about properties of fermion and boson fields ([1] and references therein):
The use of the odd Clifford algebra elements γ^a's to describe the internal space of fermions offers not only the explanation for all the assumptions of the *standard model*, with the appearance of the families of quarks and leptons and antiquarks and antileptons included, but also for the appearance of the dark matter in the universe, for the explanation of the second quantized postulates for fermions of

[3] $f^{\alpha}{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^{\alpha}{}_b = \delta^a{}_b,\; e^a{}_\alpha f^{\beta}{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

Dirac, for the matter/antimatter asymmetry in the universe, and for several other observed phenomena, making several predictions.
This article is the first trial to describe the internal space of bosons while using the even products of Clifford algebra objects γ^a's.
Although this study of the internal space of boson fields with the even Clifford algebra objects needs further considerations, yet the properties demonstrated in this paper are at least very promising.

Let me repeat briefly what I hope that we have learned.
i. There are two kinds of the anticommuting algebras, the Grassmann algebra, offering in d-dimensional space $2 \cdot 2^d$ operators, and the two Clifford algebras, each with 2^d operators. The Grassmann algebra operators are expressible with the operators of the two Clifford algebras and opposite, Eq. (16.4), and opposite. The two Clifford algebras are independent of each other, Eq. (16.5), forming two independent spaces.
ii. Either the Grassmann algebra or the two Clifford algebras can be used to describe the internal space of anticommuting objects, if the odd products of operators are used to describe the internal space of these objects, and of commuting objects, if the even products of operators are used to describe the internal space of these objects.
iii. The "basis vectors" can be found in each of these algebras, which are eigenvectors of the Cartan subalgebras, Eq. (16.6), of the corresponding Lorentz algebras $\mathcal{S}^{ab}$, S^{ab} and $\tilde{S}^{ab}$, Eq. (17.7).
iv. After the reduction of the two Clifford algebras to only one — γ^{ab}'s — assuming how does $\tilde{\gamma}^a$ apply on γ^a: $\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}|\psi_{oc}>$, with $(-)^B = -1$, if B is (a function of) an odd products of γ^a's, otherwise $(-)^B = 1$, there remain twice $2^{\frac{d}{2}-1}$ iredducible representations of S^{ab}, each with the $2^{\frac{d}{2}-1}$ members. $\tilde{\gamma}^a$'s operate on superposition of products of γ^a's.
v. The "basis vectors", which are superposition of odd products of γ^a's, can be arranged to fulfil the anticommutation relations, postulated by Dirac, explaining correspondingly the anticommutation postulates of Dirac, Eqs. (16.9, 16.11).
v.a. The Clifford odd $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations of "basis vectors" have their Hermitian conjugated partners in another set of $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ "basis vectors", Tables (16.1, 16.2). The two sets of "basis vectors" differ in handedness, Tables (16.2, 16.4).
v.b. It is our choice which set we use to describe the creation operators and which one to describe the annihilation operators. Correspondingly we have either left or right handed creation operators.
v.c. The family members of "basis vectors" have the same properties in all the families. The sum of all the eigenvalues of all the commuting operators over the $2^{\frac{d}{2}-1}$ family members is equal to zero for each of $2^{\frac{d}{2}-1}$ families, separately for left and separately for right handed representations. The sum of the family quantum numbers over the four families is zero.
vi. The Clifford even "basis vectors", which are superposition of even products of γ^a's, commute.
vi.a. The Clifford even "basis vectors" have their Hermitian conjugated partners

within the same group of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members, Table 16.3, or are self adjoint.
vi.b. Each of the two groups of the Clifford even $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ "basis vectors" applies algebraically on only one of the two Clifford odd "basis vectors", (in Table 16.1 Clifford *even* II "basis vectors" apply on Clifford odd I "basis vectors"), conserving the quantum numbers of the internal space.
vi.c. The Clifford even "basis vectors", applying algebraically on the Clifford odd "basis vectors", transform the Clifford odd "basis vector" into another member of the same family, Eqs. (16.17, 16.18, 16.19).
vi.d. The Clifford even "basis vectors" have obviously the quantum numbers of the adjoint representations with respect to the fundamental representation of the Clifford odd partners of the Clifford even "basis vectors", Table 16.3.
vi.e. The sum of all the eigenvalues of all the Cartan subalgebra members over the members of Clifford even "basis vectors" is equal to zero, independent of the choice of the subgroups (with the same number of the Cartan subalgeba), Table 16.3.
vi.f. Two Clifford even "basis vectors" ($\hat{\mathcal{A}}^{m\dagger}_{f}$ and $\hat{\mathcal{A}}^{m'\dagger}_{f}$) of the same f and of $(m, m') \neq m_0$ are orthogonal. The two "basis vectors" with non zero algebraic product, $*_A$, "scatter" into the third one, or annihilate into the vacuum,.
vi.g. The superposition of products of even number of $\tilde{\gamma}^a$'s transform the member of the Clifford odd "basis vector" of particular family into the same family member of another family.
vii. The creation and annihilation operators for either the Clifford odd or the Clifford even fields, contain besides the corresponding "basis vectors" also the basis in ordinary, coordinate or momentum, space, Eqs. (17.12, 16.28, 16.29).
vii.a. The tensor products, $*_T$, of the "basis vectors" describing the internal space of fermions or bosons and the basis in ordinary space have the properties of creation and annihilation operators for either fermion or boson fields, defining the states when applying on the corresponding vacuum states, Eqs. (17.10, 16.20).
vii.b. While the internal space of either fermions or bosons has the finite degrees of freedom — $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ — the momentum basis has obviously continuously infinite degrees of freedom. Correspondingly the single particle states have continuously infinite degrees of freedom.
vii.c. There are the "basis vectors" describing the internal spaces of either fermions or bosons, which bring commutativity or anticommutativity to creation and annihilation operators.
vii.d. The single particle states described by applying the Clifford odd creation operators on the vacuum state, anticommute, while the single particle states described by applying the Clifford even creation operators on the vacuum state commute. The same rules are valid also when applying creation operators on the corresponding Hilbert spaces, Ref. (nh2021RPPNP), Sect. 5.
vii.e. Fermion fields described by using the Clifford odd creation operators interact with exchange of the corresponding boson fields described by the Clifford even creation operators, Eq. (16.19). Bosons fields interacts on both ways, with boson fields (if the corresponding two "basis vectors" have non zero algebraic product, $*_A$), as well as with fermions.
vii.f. The application of the creation operators with the Clifford even "basis vec-

tors", in which all the γ^a's are replaced by $\tilde{\gamma}^a$'s, on the fermion creation operators, transform the fermion creation operator to another one, belonging to different family with the unchanged family members of the "basis vectors", Subsect. (16.2.4, part **b.**).

Let me conclude this contribution by saying that so far the description of the internal space of the second quantized fermions with the Clifford odd "basis vectors" offers a new insight into the Hilbert space of the second quantized fermions (although there are still open questions waiting to be discussed, like it is the appearance of the Dirac sea in the usual approaches), the equivalent description of the internal space of the second quantized boson fields with the Clifford even "basis vectors" needs, although to my opinion very promising, a lot of further study.

16.5 Eigenstates of Cartan subalgebra of Lorentz algebra

The eigenvectors of S^{ab} and $\tilde{S}^{ab}$ in the space determined by γ^a's is as follows

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= -\frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned} \tag{16.34}$$

with $k^2 = \eta^{aa}\eta^{bb}$.
The proof of the first two equations of Eq.(16.34) goes as follows, $a \neq b$ is assumed:
$\frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{i}{2}\frac{1}{2}(-\eta^{aa}\gamma^b + \frac{\eta^{aa}\eta^{bb}}{ik}\gamma^a) = \frac{k}{2}\frac{1}{2}(\gamma^a - \eta^{aa}\frac{i}{k}\gamma^b)$.
$\frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{i}{2}\frac{1}{2}(\gamma^a\gamma^b - \frac{i}{k}\eta^{aa}\eta^{bb}) = \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$.

For proving the second two equations it must be recognized that after the reduction of the Clifford space to only the part spent by γ^a's, that is after requiring
$\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}|\psi_{oc}>$,
with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B =$ 1 [35], the relations of Eq. (16.5) remain unchanged.
One can see this as follows (I follow here Ref. [1], Statement 3a. of App.I)
$\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab} = \tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a = \tilde{\gamma}^a i\gamma^b + \tilde{\gamma}^b i\gamma^a = i\gamma^b(-i)\gamma^a + i\gamma^a(-i)\gamma^b =$ $2\eta^{ab}$.
$\{\tilde{\gamma}^a, \gamma^b\}_+ = 0 = \tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a = \gamma^b(-i)\gamma^a + \gamma^b i\gamma^a = 0$.
Taking this into account it follows
$\tilde{S}^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{i}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\gamma^b\gamma^a = \frac{i}{2}\frac{1}{2}(-\eta^{aa}\gamma^b +$ $\frac{\eta^{aa}\eta^{bb}}{ik}\gamma^a) = \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$,
$\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{i}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\gamma^b\gamma^a = \frac{i}{2}\frac{1}{2}(-\gamma^a\gamma^b + \frac{i}{k}\eta^{aa}\eta^{bb}) = -\frac{k}{2}\frac{1}{2}(1 +$ $\frac{i}{k}\gamma^a\gamma^b)$,
where it is taken into account that $k^2 = \eta^{aa}\eta^{bb}$.

16.6 Clifford odd and even "basis vectors" continue

In Table 16.2 the Clifford odd "basis vectors" of the right handedness were chosen for the description of the internal space of fermions in $d = (5+1)$-dimensional space, noted in Table 16.1 as $odd\,I$.

If we make a choice of $odd\,II$ for the Clifford odd "basis vectors" in Table 16.1, and take the $odd\,I$ as their Hermitian conjugated partners, then these "basis vectors" are left (not right) handed and have properties presented in Table 16.4. We can compare their properties by the properties of the right handed "basis vectors" appearing in Table 16.2. The two groups $odd\,I$ and $odd\,II$ are Hermitian conjugated to each other. We clearly see if comparing both tables, Table 16.2 and

Table 16.4: The "basis vectors", this time left handed — $\hat{b}^{m=(ch,s)\dagger}_{f}$ (each is a product of projectors and an odd number of nilpotents, and is the "eigenstate" of all the Cartan subalgebra members, S^{03}, S^{12}, S^{56} and $\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$, Eq. (16.6) (ch (charge), the eigenvalue of S^{56}, and s (spin), the eigenvalues of S^{03} and S^{12}, explain index m, f determines family quantum numbers, the eigenvalues of ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$) — are presented for $d = (5+1)$-dimensional case. Their Hermitian conjugated partners — $\hat{b}^{m=(ch,s)}_{f}$ — can be found in Table 16.2 as "basis vectors". This table represents also the eigenvalues of the three commuting operators $N^3_{L,R}$ and S^{56} of the subgroups $SU(2) \times SU(2) \times U(1)$ and the eigenvalues of the three commuting operators τ^3, τ^8 and τ^4 of the subgroups $SU(3) \times U(1)$, in these two last cases index m represents the eigenvalues of the corresponding commuting generators. $\Gamma^{(5+1)} = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6 = -1$, $\Gamma^{(3+1)} = i\gamma^0\gamma^1\gamma^2\gamma^3$. Operators $\hat{b}^{m=(ch,s)\dagger}_{f}$ and $\hat{b}^{m=(ch,s)}_{f}$ fulfil the anticommutation relations of Eqs. (16.9, 16.11).

f	m	$=(ch,s)$	$\hat{b}^{m=(ch,s)\dagger}_{f}$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	N^3_L	N^3_R	τ^3	τ^8	τ^4	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
I	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\,|\stackrel{56}{(+)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\,|\stackrel{56}{(+)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\,|\stackrel{56}{[-]}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	1	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\,|\stackrel{56}{[-]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
II	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\,|\stackrel{56}{(+)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\,|\stackrel{56}{(+)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\,|\stackrel{56}{[-]}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\,|\stackrel{56}{[-]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
III	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\,|\stackrel{56}{[+]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\,|\stackrel{56}{[+]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\,|\stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\,|\stackrel{56}{(-)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
IV	1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\,|\stackrel{56}{[+]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\,|\stackrel{56}{[+]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\,|\stackrel{56}{(-)}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\,|\stackrel{56}{(-)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 16.4, that they do differ in properties. In particular the difference among these two kinds of "basis vectors" is easily seen in the $SU(3) \times U(1)$ subgroup, that is in (τ^3, τ^8, τ^4) values.
In Table 16.5 one finds the left and the right handed content of one of the families, the fourth ones, presented in Ref. [1], Table 5, if $d = (5+1)$ is taken as the subspace of the space $d = (13+1)$.

16.7 Some useful relations in Grassmann and Clifford space, needed also in App. 16.8

The generator of the Lorentz transformation in Grassmann space is defined as follows [20]

$$\begin{aligned} \mathbf{S}^{ab} &= (\theta^a p^{\theta b} - \theta^b p^{\theta a}) \\ &= S^{ab} + \tilde{S}^{ab} , \quad \{S^{ab}, \tilde{S}^{cd}\}_- = 0 , \end{aligned} \tag{16.35}$$

where S^{ab} and $\tilde{S}^{ab}$ are the corresponding two generators of the Lorentz transformations in the Clifford space, forming orthogonal representations with respect to each other.
We make a choice of the Cartan subalgebra of the Lorentz algebra as follows

$$\begin{aligned} &\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\; d} , \\ &S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d} , \\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d} , \\ &\text{if} \quad d = 2n . \end{aligned} \tag{16.36}$$

We find the infinitesimal generators of the Lorentz transformations in Clifford space

$$\begin{aligned} S^{ab} &= \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) , \quad S^{ab\dagger} = \eta^{aa}\eta^{bb} S^{ab} , \\ \tilde{S}^{ab} &= \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) , \quad \tilde{S}^{ab\dagger} = \eta^{aa}\eta^{bb} \tilde{S}^{ab} , \end{aligned} \tag{16.37}$$

where γ^a and $\tilde{\gamma}^a$ are defined in Eqs. (16.4, 16.5). The commutation relations for either $\mathbf{S}^{ab}$ or S^{ab} or $\tilde{S}^{ab}$, $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, are

$$\begin{aligned} &\{S^{ab}, \tilde{S}^{cd}\}_- = 0 , \\ &\{S^{ab}, S^{cd}\}_- = i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac}) , \\ &\{\tilde{S}^{ab}, \tilde{S}^{cd}\}_- = i(\eta^{ad} \tilde{S}^{bc} + \eta^{bc} \tilde{S}^{ad} - \eta^{ac} \tilde{S}^{bd} - \eta^{bd} \tilde{S}^{ac}) . \end{aligned} \tag{16.38}$$

The infinitesimal generators of the two invariant subgroups of the group $SO(3,1)$ can be expressed as follows

$$\vec{N}_{\pm}(= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm i S^{01}, S^{31} \pm i S^{02}, S^{12} \pm i S^{03}) . \tag{16.39}$$

The infinitesimal generators of the two invariant subgroups of the group $SO(4)$ are expressible with $S^{ab}, (a,b) = (5,6,7,8)$ as follows

$$\begin{aligned}
\vec{\tau}^{1} &:= \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78})\,,\\
\vec{\tau}^{2} &:= \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78})\,,
\end{aligned} \tag{16.40}$$

while the generators of the $SU(3)$ and $U(1)$ subgroups of the group $SO(6)$ can be expressed by $S^{ab}, (a,b) = (9,10,11,12,13,14)$

$$\begin{aligned}
\vec{\tau}^{3} :=\; & \frac{1}{2}\{S^{9\,12} - S^{10\,11}\,, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12},\\
& S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}\,, S^{11\,14} - S^{12\,13}\,,\\
& S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\}\,,\\
\tau^{4} :=\; & -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})\,.
\end{aligned} \tag{16.41}$$

The group $SO(6)$ has $\frac{d(d-1)}{2} = 15$ generators and $\frac{d}{2} = 3$ commuting operators. The subgroups $SU(3) \times U(1)$ have the same number of commuting operators, expressed with τ^{33}, τ^{38} and τ^{4}, and 9 generators, 8 of $SU(3)$ and one of $U(1)$. The rest of 6 generators, not included in $SU(3) \times U(1)$, can be expressed as $\frac{1}{2}\{S^{9\,12} + S^{10\,11}, S^{9\,11} - S^{10\,12}, S^{9\,14} + S^{10\,13}, S^{9\,13} - S^{10\,14}, S^{11\,14} + S^{12\,13}, S^{11\,13} - S^{12\,14}$.

The hyper charge Y can be defined as $Y = \tau^{23} + \tau^{4}$.
The equivalent expressions for the "family" charges, expressed by $\tilde{S}^{ab}$, follow if in Eqs. (17.26 - 17.28) S^{ab} are replaced by $\tilde{S}^{ab}$.
Let us present some useful relations from Ref. [23].

$$\begin{array}{llll}
\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0\,, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa}\stackrel{ab}{[-k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(-k)} = 0\,,\\
\stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[k]} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[-k]} = \stackrel{ab}{[-k]}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[-k]} = 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(k)} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(-k)} = \stackrel{ab}{(-k)}\;.
\end{array} \tag{16.42}$$

16.8 One family representation in $d = (13+1)$-dimensional space with $2^{\frac{d}{2}-1}$ members representing quarks and leptons and antiquarks and antileptons in the *spin-charge-family* theory

In Table Table so13+1. the "basis vectors" of one irreducible representation, one family, of the Clifford odd basis vectors of left handedness, $\Gamma^{(13+1)}$, is presented, including all the quarks and the leptons and the antiquarks and the antileptons of

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
1	u_R^{c1}	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	u_R^{c1}	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	d_R^{c1}	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	d_R^{c1}	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	d_L^{c1}	$\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	d_L^{c1}	$-\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	u_L^{c1}	$-\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	u_L^{c1}	$\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	u_R^{c2}	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	u_R^{c2}	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	d_R^{c2}	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	d_R^{c2}	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	d_L^{c2}	$\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	d_L^{c2}	$-\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	u_L^{c2}	$-\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	u_L^{c2}	$\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	u_R^{c3}	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	u_R^{c3}	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	d_R^{c3}	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	d_R^{c3}	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	d_L^{c3}	$\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	d_L^{c3}	$-\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	u_L^{c3}	$-\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	u_L^{c3}	$\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{[-]}\;\overset{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	ν_R	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	e_R	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	e_L	$\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	e_L	$-\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	ν_L	$-\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	$\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\bar{d}_L^{\bar{c}1}$	$\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}_L^{\bar{c}1}$	$\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}_L^{\bar{c}1}$	$-\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}_L^{\bar{c}1}$	$-\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}_R^{\bar{c}1}$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}_R^{\bar{c}1}$	$-\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}_R^{\bar{c}1}$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}_R^{\bar{c}1}$	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\;\overset{11\,12}{(+)}\;\overset{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
42	$\bar{d}^{\bar{c}2}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}^{\bar{c}2}_{L}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\bar{u}^{\bar{c}2}_{L}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\bar{d}^{\bar{c}2}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\bar{d}^{\bar{c}2}_{R}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\bar{u}^{\bar{c}2}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\bar{u}^{\bar{c}2}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\bar{d}^{\bar{c}3}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\bar{d}^{\bar{c}3}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\bar{u}^{\bar{c}3}_{L}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\bar{u}^{\bar{c}3}_{L}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\bar{d}^{\bar{c}3}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\bar{d}^{\bar{c}3}_{R}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\bar{u}^{\bar{c}3}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\bar{u}^{\bar{c}3}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\bar{e}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\bar{e}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\bar{\nu}_{L}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\bar{\nu}_{L}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\bar{\nu}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\bar{\nu}_{R}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\bar{e}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\bar{e}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 16.5: The left handed ($\Gamma^{(13,1)} = -1$ [23]) multiplet of spinors — the members of the fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented in the massless basis using the technique presented in Refs. [23,31,34,35]. It contains the left handed ($\Gamma^{(3,1)} = -1$) weak ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$, Eq. (17.27)), and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, Eq. (17.27)) quarks and leptons and the right handed ($\Gamma^{(3,1)} = 1$) weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $SU(3)\times U(1)$ part: Quarks are triplets of three colours ($c^i = (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, Eq. (17.28)) carrying the "fermion charge" ($\tau^4 = \frac{1}{6}$, Eq. (17.28)). The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). The same multiplet contains also the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons. Antiquarks distinguish from antileptons again only in the $SU(3)\times U(1)$ part: Antiquarks are antitriplets, carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anticolourless antileptons carry the "fermion charge" ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The vacuum state, on which the nilpotents and projectors operate, is presented in Eq. (17.10). The reader can find this Weyl representation also in Refs. [25], [26], [31] and the references therein.

the *standard model*. The needed definitions of the quantum numbers are presented in App. 16.7.

In Tables 16.1, 16.2, 16.3 a simple toy model for $d = (5+1)$-dimensional space is discussed, and the properties of fermions (appearing in families) and their gauge boson fields (the vielbeins and the two kinds of the spin connection fields) analysed. The manifold $(5+1)$ was suggested to break either into $SU(2)\times SU(2)\times U(1)$ or to $SU(3)\times U(1)$ to study properties of the fermion and boson second quantized fields, with second quantization origining in the anticommutativity or commutativity of "basis vectors".

Here only one family of "basis vectors" is presented to see that while the starting "basis vectors" can be either left or right handed, the subgroups,of the starting

group contain left and right handed members, as it is $SU(2) \times SU(2) \times U(1)$ of $SO(5+1)$ in the toy model.
The breaks of the symmetries, manifesting in Eqs. (17.26, 17.27, 17.28), are in the *spin-charge-family* theory caused by the condensate and the nonzero vacuum expectation values (constant values) of the scalar fields carrying the space index $(7, 8)$ (Refs. [23, 31] and the references therein), all originating in the vielbeins and the two kinds of the spin connection fields. The space breaks first to $SO(7,1)$ $\times SU(3) \times U(1)_{II}$ and then further to $SO(3,1) \times SU(2)_I \times U(1)_I \times SU(3) \times U(1)_{II}$, what explains the connections between the weak and the hyper charges and the handedness of spinors.

16.9 Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} = \alpha\, \varepsilon_{a_1 a_2 \dots a_{d-1} a_d}\, S^{a_1 a_2} \cdot S^{a_3 a_4} \cdots S^{a_{d-1} a_d}\,, \qquad (16.43)$$

with α, which is chosen so that $\Gamma^{(d)} = \pm 1$.
In the Grassmann case S^{ab} is defined in Eq. (16.6), while in the Clifford case Eq. (16.43) simplifies, if we take into account that $S^{ab}|_{a\neq b} = \frac{i}{2}\gamma^a\gamma^b$ and $\tilde{S}^{ab}|_{a\neq b} = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n\,. \qquad (16.44)$$

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