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UNIVERZA NA PRIMORSKEM

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

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Editoral

The present issue of *Ars Mathematica Contemporanea* contains a selection of articles related to the topics presented at the International Conference on Graph Theory and Combinatorics, held form 1 to 3 May 2013, in Koper. The conference was organised in honour of Professor Dragan Marušič on the occasion of his 60th birthday. It brought together many of the researchers with whom Dragan's discussions on mathematics have been (and still are) the most enjoyable and fruitful.

There were 14 invited talks at the conference, covering some of Dragan's favourite topics related to symmetries of graphs and other combinatorial structures, such as the hamiltonicity problem in vertex-transitive graphs and the polycirculant conjecture.

The 15 articles in this special issue of *Ars Mathematica Contemporanea* have been selected for publication after the same thorough refereeing process that papers go through for regular issues of this journal. We are convinced that these high quality articles will have a great impact, and will positively influence future research on the topics concerned.

Klavdija Kutnar, Štefko Miklavič, Tomaž Pisanski and Primož Šparl

ARS MATHEMATICA CONTEMPORANEA

DM=60 – A Surprise Conference

On 1 May 2013, a prominent Slovenian mathematician Dragan Marušič turned 60. His former students and colleagues decided to mark this event in a way that scientists know best: by organising a special scientific conference and publishing a *Festschrift*, in honour of Dragan. Following the tradition of similar scientific festivities, and in order to reduce costs, the conference was organised by invitation only. A list of speakers was very carefully hand-picked from eminent mathematicians, Dragan's long-time collaborators and his students. Not all were able to attend, but there was a twist in the organisation that makes DM=60 unique. The conference was a surprise conference. In 2013 the University of Primorska's FAMNIT organised 14 international conferences, and Dragan was led to believe that his 60th birthday would be commemorated later that year at the annual summer school at Rogla.

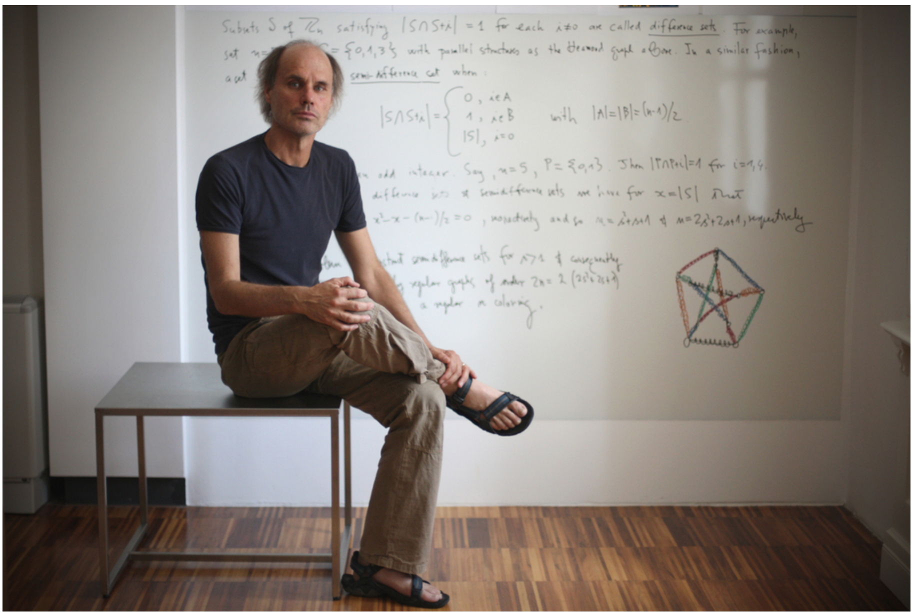

Figure 1: *Professor Dragan Marušič with his favorite CFSG-free proof that* S_5 *and* A_5 *are the only simple primitive groups of degree* $2p$

I will never forget the expression on Dragan's face when he entered the big lecture room that morning, on his birthday. The auditorium that was supposed to be empty (due to a national holiday) was full of mathematicians. The conference started promptly and ran for three days. What a joy for Mathematics! Students were able to meet first class mathematicians, and colleagues were able to work with each other on various unsolved problems, and discuss future projects together. All the lectures at the conference touched upon the work that Dragan has done in the past, and the influence he has had on his colleagues and students for years. Reprints of Dragan's published papers were bound into a volume and

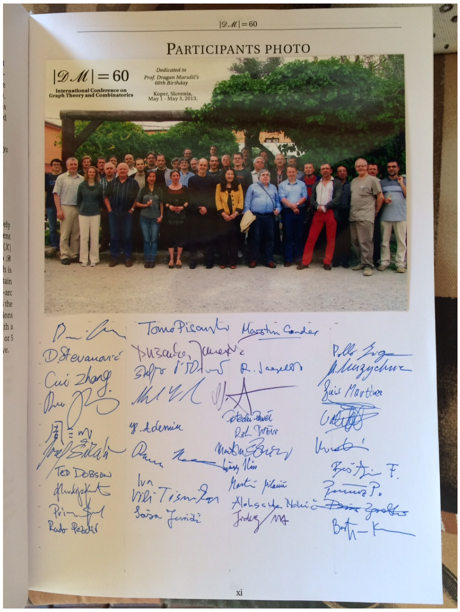

Figure 2: *The DM-60 conference photo, with signatures of the speakers*

presented to him, together with the signatures of all invited speakers. (See Figure 2.) This volume comprises a remarkable 1729 pages of original mathematics – about a page for every week of Dragan's life after his PhD! (See Figure 3.)

Dragan is Rector of the University of Primorska, which is based in the city of Koper, on the Adriatic coast. Currently among its students of mathematics about one third are from abroad. And although the University of Primorska is ten times smaller than the University of Ljubljana, it attracted four times as many new mathematics PhD students in 2014. These things highlight Dragan's successful leadership of this young University.

It has also been very interesting to watch the unfolding of the career of Professor Dragan Marušič. His talents were clearly visible already in high school, but it took a long time before his work was recognised as mainstream. He submitted his first paper as un undergraduate, and completed a very good PhD at the University of Reading, UK, under the guidance of the legendary Crispin Nash-Williams. But this was not sufficient to get him position in the main Mathematics department in Slovenia. As a free thinker, a follower of Grateful Dead, an anti-nuclear activist, and long-haired vegetarian, Dragan did not fit he traditional mould of Slovenian society. Even after returning home from the USA, where he began his postdoctoral academic career, the doors to the Mathematics department in Ljubljana remained closed for Dragan. What a waste for Slovenian mathematics!

Dragan had started dreaming about democracy and the University of Primorska a long time earlier, and at the first free elections in Slovenia, he ran as a candidate for parliament, for the opposition alliance DEMOS. Creating the University of Primorska was his the main subject on his agenda. (See Figure 4.) Although he won in his district, the votes

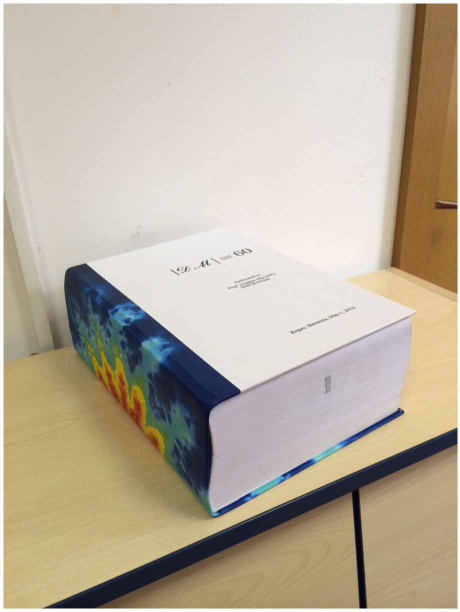

Figure 3: *Collection of the 1729 pages of original mathematics produced by Dragan Marušič over a 30 year period*

he gained helped a politician who was higher in the party list to enter the parliament. But his campaign to create a new university was taken up and realised by others who were less passionate about it but in a better position politically to succeed.

Dragan is the third rector of the University, but the first critical intellectual to hold this position. (The previous two rectors were government ministers before wearing their University insignia.) His primary goal is to promote and ensure the quality and visibility of his University. He will not compromise his integrity, and has no fear of exposing corruption among officials. Sadly, a lot of people in Slovenia are envious of Dragan's scientific achievements and the flourishing of mathematics at the University of Primorska under his leadership. A few months after the conference took place, an anonymous pamphlet was distributed to the media, ministries, and prosecutors, claiming that the DM=60 conference never took place, and that Dragan spent public money on his own birthday party. I am glad that this special volume gives proof of the quality of the DM=60 conference, and that its funding was well-spent.

Tomaž Pisanski

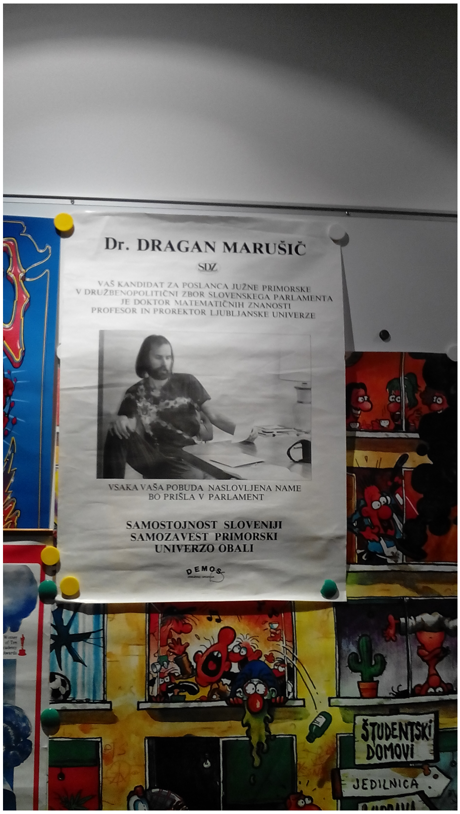

Figure 4: *Poster of Dragan Marušič running for the Slovenian Parliament in 1990 – including a slogan about creating the University of Primorska*

Dragan Marušič, Bibliography: 1980–2013

[1] B. Alspach, M. D. E. Conder, D. Marušič and M.-Y. Xu, A classification of 2-arc-transitive circulants, *J. Algebraic Combin.* **5** (1996), 83–86.

[2] B. Alspach, D. Marušič and L. Nowitz, Constructing graphs which are $1/2$-transitive, *J. Austral. Math. Soc. Ser. A* **56** (1994), 391–402.

[3] A. Araluze, I. Kovács, K. Kutnar, L. Martínez and D. Marušič, Partial sum quadruples and bi-Abelian digraphs, *J. Combin. Theory Ser. A* **119** (2012), 1811–1831.

[4] A. Araluze, K. Kutnar, L. Martínez and D. Marušič, Edge connectivity in difference graphs and some new constructions of partial sum families, *European J. Combin.* **32** (2011), 352–360.

[5] A. Blokhuis, G. Kiss, I. Kovács, A. Malnič, D. Marušič and J. Ruff, Semiovals contained in the union of three concurrent lines, *J. Combin. Des.* **15** (2007), 491–501.

[6] P. J. Cameron, M. Giudici, G. A. Jones, W. M. Kantor, M. H. Klin, D. Marušič and L. A. Nowitz, Transitive permutation groups without semiregular subgroups, *J. London Math. Soc. (2)* **66** (2002), 325–333.

[7] V. Carli, N. Jovanović, A. Podlesek, A. Roy, Z. Rihmer, S. Maggi, D. Marušič, C. Cesaro, A. Marušič and M. Sarchiapone, The role of impulsivity in self-mutilators, suicide ideators and suicide attempters: a study of 1265 male incarcerated individuals, *J. Affective Disord.* **123** (2010), 116–122.

[8] M. D. E. Conder, A. Malnič, D. Marušič, T. Pisanski and P. Potočnik, The edge-transitive but not vertex-transitive cubic graph on 112 vertices, *J. Graph Theory* **50** (2005), 25–42.

[9] M. D. E. Conder, A. Malnič, D. Marušič and P. Potočnik, A census of semisymmetric cubic graphs on up to 768 vertices, *J. Algebraic Combin.* **23** (2006), 255–294.

[10] M. D. E. Conder and D. Marušič, A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer, *J. Combin. Theory Ser. B* **88** (2003), 67–76.

[11] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Minimal normal subgroups of transitive permutation groups of square-free degree, *Discrete Math.* **307** (2007), 373–385.

[12] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, *J. Combin. Theory Ser. B* **97** (2007), 371–380.

[13] E. Dobson and D. Marušič, An unusual decomposition of a complete 7-partite graph of order 28, *Discrete Math.* **308** (2008), 4595–4598.

[14] E. Dobson and D. Marušič, On semiregular elements of solvable groups, *Comm. Algebra* **39** (2011), 1413–1426.

[15] S. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, *J. Combin. Theory Ser. B* **98** (2008), 1349–1372.

[16] S. Du and D. Marušič, Biprimitive graphs of smallest order, *J. Algebraic Combin.* **9** (1999), 151–156.

[17] S. Du and D. Marušič, An infinite family of biprimitive semisymmetric graphs, *J. Graph Theory* **32** (1999), 217–228.

[18] S. Du, D. Marušič and A. O. Waller, On 2-arc-transitive covers of complete graphs, *J. Combin. Theory Ser. B* **74** (1998), 276–290.

[19] T. Durkee, D. Marušič and V. Poštuvan, Prevalence of pathological internet use among adolescents in europe: demographic and social factors, *Addiction* **107** (2012), 2210–2222.

[20] Y.-Q. Feng, K. Kutnar, A. Malnič and D. Marušič, On 2-fold covers of graphs, *J. Combin. Theory Ser. B* **98** (2008), 324–341.

[21] H. H. Glover, K. Kutnar, A. Malnič and D. Marušič, Hamilton cycles in $(2, \mathrm{odd}, 3)$-Cayley graphs, *Proc. Lond. Math. Soc. (3)* **104** (2012), 1171–1197.

[22] H. H. Glover, K. Kutnar and D. Marušič, Hamiltonian cycles in cubic Cayley graphs: the $\langle 2, 4k, 3\rangle$ case, *J. Algebraic Combin.* **30** (2009), 447–475.

[23] H. H. Glover and D. Marušič, Hamiltonicity of cubic Cayley graphs, *J. Eur. Math. Soc. (JEMS)* **9** (2007), 775–787.

[24] M. Hladnik, D. Marušič and T. Pisanski, Cyclic Haar graphs, *Discrete Math.* **244** (2002), 137–152, algebraic and topological methods in graph theory (Lake Bled, 1999).

[25] R. Jajcay, A. Malnič and D. Marušič, On the number of closed walks in vertex-transitive graphs, *Discrete Math.* **307** (2007), 484–493.

[26] G. Kiss, A. Malnič and D. Marušič, A new approach to arcs, *Acta Math. Hungar.* **84** (1999), 181–188.

[27] S. Klavžar, D. Marušič, B. Mohar and T. Pisanski, Preface [Algebraic and topological methods in graph theory], *Discrete Math.* **307** (2007), 299.

[28] S. Klavžar, D. Marušič, B. Mohar and T. Pisanski, Preface: Algebraic and topological graph theory (Bled'07), *Discrete Math.* **310** (2010), 1651–1652, held at Lake Bled, June 2007.

[29] I. Kovács, K. Kutnar and D. Marušič, Classification of edge-transitive rose window graphs, *J. Graph Theory* **65** (2010), 216–231.

[30] I. Kovács, K. Kutnar, D. Marušič and S. Wilson, Classification of cubic symmetric tricirculants, *Electron. J. Combin.* **19** (2012), Paper 24, 14.

[31] I. Kovács, A. Malnič, D. Marušič and Š. Miklavič, One-matching bi-Cayley graphs over abelian groups, *European J. Combin.* **30** (2009), 602–616.

[32] I. Kovács, D. Marušič and M. E. Muzychuk, Primitive bicirculant association schemes and a generalization of Wielandt's theorem, *Trans. Amer. Math. Soc.* **362** (2010), 3203–3221.

[33] I. Kovács, D. Marušič and M. E. Muzychuk, On dihedrants admitting arc-regular group actions, *J. Algebraic Combin.* **33** (2011), 409–426.

[34] I. Kovács, D. Marušič and M. E. Muzychuk, On G-arc-regular dihedrants and regular dihedral maps, *J. Algebraic Combin.* **38** (2013), 437–455.

[35] K. Kutnar, U. Borštnik, D. Marušič and D. Janežič, Interconnection networks for parallel molecular dynamics simulation based on hamiltonian cubic symmetric topology, *J. Math. Chem.* **45** (2009), 372–385.

[36] K. Kutnar, A. Malnič and D. Marušič, Chirality of toroidal molecular graphs, *J. Chem. Inf. Mod.* **45** (2005), 1527–1535.

[37] K. Kutnar, A. Malnič, D. Marušič and Š. Miklavič, Distance-balanced graphs: symmetry conditions, *Discrete Math.* **306** (2006), 1881–1894.

[38] K. Kutnar, A. Malnič, D. Marušič and Š. Miklavič, The strongly distance-balanced property of the generalized Petersen graphs, *Ars Math. Contemp.* **2** (2009), 41–47.

[39] K. Kutnar and D. Marušič, Hamiltonicity of vertex-transitive graphs of order $4p$, *European J. Combin.* **29** (2008), 423–438.

[40] K. Kutnar and D. Marušič, On cyclic edge-connectivity of fullerenes, *Discrete Appl. Math.* **156** (2008), 1661–1669.

[41] K. Kutnar and D. Marušič, Recent trends and future directions in vertex-transitive graphs, *Ars Math. Contemp.* **1** (2008), 112–125.

[42] K. Kutnar and D. Marušič, A complete classification of cubic symmetric graphs of girth 6, *J. Combin. Theory Ser. B* **99** (2009), 162–184.

[43] K. Kutnar and D. Marušič, Hamilton cycles and paths in vertex-transitive graphs—current directions, *Discrete Math.* **309** (2009), 5491–5500.

[44] K. Kutnar and D. Marušič, On certain graph theory application, *Lecture notes in economics and mathematical systems* **613** (2009), 283–291.

[45] K. Kutnar and D. Marušič, Some topics in graph theory, *Lecture notes in economics and mathematical systems* **613** (2009), 3–22.

[46] K. Kutnar, D. Marušič and D. Janežič, Fullerenes via their automorphism groups, *MATCH Commun. Math. Comput. Chem.* **63** (2010), 267–282.

[47] K. Kutnar, D. Marušič, Š. Miklavič and P. Šparl, Strongly regular tri-Cayley graphs, *European J. Combin.* **30** (2009), 822–832.

[48] K. Kutnar, D. Marušič, D. W. Morris, J. Morris and P. Šparl, Hamiltonian cycles in Cayley graphs whose order has few prime factors, *Ars Math. Contemp.* **5** (2012), 27–71.

[49] K. Kutnar, D. Marušič and P. Šparl, An infinite family of half-arc-transitive graphs with universal reachability relation, *European J. Combin.* **31** (2010), 1725–1734.

[50] K. Kutnar, D. Marušič and D. Vukičević, On decompositions of leapfrog fullerenes, *J. Math. Chem.* **45** (2009), 406–416.

[51] K. Kutnar, D. Marušič and C. Zhang, Hamilton paths in vertex-transitive graphs of order $10p$, *European J. Combin.* **33** (2012), 1043–1077.

[52] K. Kutnar, D. Marušič and C. Zhang, On cubic non-Cayley vertex-transitive graphs, *J. Graph Theory* **69** (2012), 77–95.

[53] C. Li, D. Marušič and J. Morris, Classifying arc-transitive circulants of square-free order, *J. Algebraic Combin.* **14** (2001), 145–151.

[54] C. H. Li, Z. P. Lu and D. Marušič, On primitive permutation groups with small suborbits and their orbital graphs, *J. Algebra* **279** (2004), 749–770.

[55] M. Lovrečič Saražin and D. Marušič, Vertex-transitive expansions of $(1, 3)$-trees, *Discrete Math.* **310** (2010), 1772–1782.

[56] A. Malnič and D. Marušič, Imprimitive groups and graph coverings, in: *Coding theory, design theory, group theory (Burlington, VT, 1990)*, Wiley, New York, Wiley-Intersci. Publ., pp. 227–235, 1993.

[57] A. Malnič and D. Marušič, Constructing 4-valent $\frac{1}{2}$-transitive graphs with a non-solvable automorphism group, *J. Combin. Theory Ser. B* **75** (1999), 46–55.

[58] A. Malnič and D. Marušič, Constructing $\frac{1}{2}$-arc-transitive graphs of valency 4 and vertex stabilizer $Z_2 \times Z_2$, *Discrete Math.* **245** (2002), 203–216.

[59] A. Malnič, D. Marušič, Š. Miklavič and P. Potočnik, Semisymmetric elementary abelian covers of the Möbius-Kantor graph, *Discrete Math.* **307** (2007), 2156–2175.

[60] A. Malnič, D. Marušič, R. G. Möller, N. Seifter, V. Trofimov and B. Zgrablič, Highly arc transitive digraphs: reachability, topological groups, *European J. Combin.* **26** (2005), 19–28.

[61] A. Malnič, D. Marušič and P. Potočnik, Elementary abelian covers of graphs, *J. Algebraic Combin.* **20** (2004), 71–97.

[62] A. Malnič, D. Marušič and P. Potočnik, On cubic graphs admitting an edge-transitive solvable group, *J. Algebraic Combin.* **20** (2004), 99–113.

[63] A. Malnič, D. Marušič, P. Potočnik and C. Wang, An infinite family of cubic edge-but not vertex-transitive graphs, *Discrete Math.* **280** (2004), 133–148.

[64] A. Malnič, D. Marušič and N. Seifter, Constructing infinite one-regular graphs, *European J. Combin.* **20** (1999), 845–853.

[65] A. Malnič, D. Marušič, N. Seifter, P. Šparl and B. Zgrablič, Reachability relations in digraphs, *European J. Combin.* **29** (2008), 1566–1581.

[66] A. Malnič, D. Marušič, N. Seifter and B. Zgrablić, Highly arc-transitive digraphs with no homomorphism onto $\mathbb{Z}$, *Combinatorica* **22** (2002), 435–443.

[67] A. Malnič, D. Marušič and P. Šparl, On strongly regular bicirculants, *European J. Combin.* **28** (2007), 891–900.

[68] A. Malnič, D. Marušič, P. Šparl and B. Frelih, Symmetry structure of bicirculants, *Discrete Math.* **307** (2007), 409–414.

[69] A. Malnič, D. Marušič and C. Wang, Cubic edge-transitive graphs of order $2p^3$, *Discrete Math.* **274** (2004), 187–198.

[70] A. Marušič, K. Belšak and D. Marušič, Sensitization of emotions as a risk factor for ischemic heart disease, *Psychiatria Danub.* **20** (2008), 30–35.

[71] A. Marušič and D. Marušič, Suicidology - expanding its research tools, *Crisis* **23** (2002), 95–97.

[72] D. Marušič, Vertex minimal planar cyclic graphs, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **1981** (1980), 161–169 (1981).

[73] D. Marušič, On vertex symmetric digraphs, *Discrete Math.* **36** (1981), 69–81.

[74] D. Marušič, Cayley properties of vertex symmetric graphs, *Ars Combin.* **16** (1983), 297–302.

[75] D. Marušič, Hamiltonian circuits in Cayley graphs, *Discrete Math.* **46** (1983), 49–54.

[76] D. Marušič, Some problems in vertex symmetric graphs, in: *Finite and infinite sets, Vol. I, II (Eger, 1981)*, North-Holland, Amsterdam, volume 37 of *Colloq. Math. Soc. János Bolyai*, pp. 581–589, 1984.

[77] D. Marušič, Vertex transitive graphs and digraphs of order p^k, in: *Cycles in graphs (Burnaby, B.C., 1982)*, North-Holland, Amsterdam, volume 115 of *North-Holland Math. Stud.*, pp. 115–128, 1985.

[78] D. Marušič, Hamiltonian cycles in vertex symmetric graphs of order $2p^2$, *Discrete Math.* **66** (1987), 169–174.

[79] D. Marušič, On vertex-transitive graphs of order qp, *J. Combin. Math. Combin. Comput.* **4** (1988), 97–114.

[80] D. Marušič, Research problems: Problem 92, problem 93, *Discrete Math.* **70** (1988), 215.

[81] D. Marušič, Strongly regular bicirculants and tricirculants, *Ars Combin.* **25** (1988), 11–15, eleventh British Combinatorial Conference (London, 1987).

[82] D. Marušič, Strong regularity and circulant graphs, *Discrete Math.* **78** (1989), 119–125.

[83] D. Marušič, Hamiltonicity of tree-like graphs, *Discrete Math.* **80** (1990), 167–173.

[84] D. Marušič, On tree-like cubic vertex-transitive graphs, *Ars Combin.* **29** (1990), 91–96, twelfth British Combinatorial Conference (Norwich, 1989).

[85] D. Marušič, The seven "weirdos", *Obzornik Mat. Fiz.* **37** (1990), 105–108.

[86] D. Marušič, Hamiltonicity of vertex-transitive pq-graphs, in: *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity (Prachatice, 1990)*, North-Holland, Amsterdam, volume 51 of *Ann. Discrete Math.*, pp. 209–212, 1992.

[87] D. Marušič, Research problems: Problems 211-212, *Discrete Math.* **131** (1994), 397–398.

[88] D. Marušič, A family of one-regular graphs of valency 4, *European J. Combin.* **18** (1997), 59–64.

[89] D. Marušič, Half-transitive group actions on finite graphs of valency 4, *J. Combin. Theory Ser. B* **73** (1998), 41–76.

[90] D. Marušič, Recent developments in half-transitive graphs, *Discrete Math.* **182** (1998), 219–231, graph theory (Lake Bled, 1995).

[91] D. Marušič, Constructing cubic edge- but not vertex-transitive graphs, *J. Graph Theory* **35** (2000), 152–160.

[92] D. Marušič, In memoriam: some personal impressions of Crispin Nash-Williams, *Discrete Math.* **244** (2002), 2–4, algebraic and topological methods in graph theory (Lake Bled, 1999).

[93] D. Marušič, On 2-arc-transitivity of Cayley graphs, *J. Combin. Theory Ser. B* **87** (2003), 162–196, dedicated to Crispin St. J. A. Nash-Williams.

[94] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.* **299** (2005), 180–193.

[95] D. Marušič, Corrigendum to: "On 2-arc-transitivity of Cayley graphs" [J. Combin. Theory Ser. B **87** (2003), no. 1, 162–196],(2006), 761–764.

[96] D. Marušič, Hamilton cycles and paths in fullerenes, *J. Chem. Inf. Mod.* **47** (2007), 732–736.

[97] D. Marušič, Preface: Special issue of discrete mathematics on Hamiltonicity problem for vertex-transitive (Cayley) graphs, *Discrete Math.* **309** (2009), 5425.

[98] D. Marušič and J. Morris, Normal circulant graphs with noncyclic regular subgroups, *J. Graph Theory* **50** (2005), 13–24.

[99] D. Marušič and R. Nedela, Maps and half-transitive graphs of valency 4, *European J. Combin.* **19** (1998), 345–354.

[100] D. Marušič and R. Nedela, On the point stabilizers of transitive groups with non-self-paired suborbits of length 2, *J. Group Theory* **4** (2001), 19–43.

[101] D. Marušič and R. Nedela, Partial line graph operator and half-arc-transitive group actions, *Math. Slovaca* **51** (2001), 241–257.

[102] D. Marušič and R. Nedela, Finite graphs of valency 4 and girth 4 admitting half-transitive group actions, *J. Aust. Math. Soc.* **73** (2002), 155–170.

[103] D. Marušič and T. D. Parsons, Hamiltonian paths in vertex-symmetric graphs of order $5p$, *Discrete Math.* **42** (1982), 227–242.

[104] D. Marušič and T. D. Parsons, Hamiltonian paths in vertex-symmetric graphs of order $4p$, *Discrete Math.* **43** (1983), 91–96.

[105] D. Marušič and T. Pisanski, Weakly flag-transitive configurations and half-arc-transitive graphs, *European J. Combin.* **20** (1999), 559–570.

[106] D. Marušič and T. Pisanski, The Gray graph revisited, *J. Graph Theory* **35** (2000), 1–7.

[107] D. Marušič and T. Pisanski, The remarkable generalized Petersen graph $G(8,3)$, *Math. Slovaca* **50** (2000), 117–121.

[108] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croat. Chem. Acta* **73** (2000), 969–981.

[109] D. Marušič, T. Pisanski and S. Wilson, The genus of the GRAY graph is 7, *European J. Combin.* **26** (2005), 377–385.

[110] D. Marušič and P. Potočnik, Semisymmetry of generalized Folkman graphs, *European J. Combin.* **22** (2001), 333–349.

[111] D. Marušič and P. Potočnik, Bridging semisymmetric and half-arc-transitive actions on graphs, *European J. Combin.* **23** (2002), 719–732.

[112] D. Marušič and P. Potočnik, Classifying 2-arc-transitive graphs of order a product of two primes, *Discrete Math.* **244** (2002), 331–338, algebraic and topological methods in graph theory (Lake Bled, 1999).

[113] D. Marušič and C. E. Praeger, Tetravalent graphs admitting half-transitive group actions: alternating cycles, *J. Combin. Theory Ser. B* **75** (1999), 188–205.

[114] D. Marušič and R. Scapellato, Characterizing vertex-transitive pq-graphs with an imprimitive automorphism subgroup, *J. Graph Theory* **16** (1992), 375–387.

[115] D. Marušič and R. Scapellato, A class of non-Cayley vertex-transitive graphs associated with $\mathrm{PSL}(2,p)$, *Discrete Math.* **109** (1992), 161–170, algebraic graph theory (Leibnitz, 1989).

[116] D. Marušič and R. Scapellato, Imprimitive representations of $\mathrm{SL}(2, 2^k)$, *J. Combin. Theory Ser. B* **58** (1993), 46–57.

[117] D. Marušič and R. Scapellato, A class of graphs arising from the action of $\mathrm{PSL}(2, q^2)$ on cosets of $\mathrm{PGL}(2, q)$, *Discrete Math.* **134** (1994), 99–110, algebraic and topological methods in graph theory (Lake Bled, 1991).

[118] D. Marušič and R. Scapellato, Classification of vertex-transitive pq-digraphs, *Istit. Lombardo Accad. Sci. Lett. Rend. A* **128** (1994), 31–36 (1995).

[119] D. Marušič and R. Scapellato, Classifying vertex-transitive graphs whose order is a product of two primes, *Combinatorica* **14** (1994), 187–201.

[120] D. Marušič and R. Scapellato, Permutation groups with conjugacy complete stabilizers, *Discrete Math.* **134** (1994), 93–98, algebraic and topological methods in graph theory (Lake Bled, 1991).

[121] D. Marušič and R. Scapellato, Permutation groups, vertex-transitive digraphs and semiregular automorphisms, *European J. Combin.* **19** (1998), 707–712.

[122] D. Marušič, R. Scapellato and N. Zagaglia Salvi, A characterization of particular symmetric $(0, 1)$ matrices, *Linear Algebra Appl.* **119** (1989), 153–162.

[123] D. Marušič, R. Scapellato and N. Zagaglia Salvi, Generalized Cayley graphs, *Discrete Math.* **102** (1992), 279–285.

[124] D. Marušič, R. Scapellato and B. Zgrablić, On quasiprimitive pqr-graphs, *Algebra Colloq.* **2** (1995), 295–314.

[125] D. Marušič and P. Šparl, On quartic half-arc-transitive metacirculants, *J. Algebraic Combin.* **28** (2008), 365–395.

[126] D. Marušič and A. O. Waller, Half-transitive graphs of valency 4 with prescribed attachment numbers, *J. Graph Theory* **34** (2000), 89–99.

[127] D. Marušič and M.-Y. Xu, A $\frac{1}{2}$-transitive graph of valency 4 with a nonsolvable group of automorphisms, *J. Graph Theory* **25** (1997), 133–138.

[128] T. Pisanski, M. Boben, D. Marušič, A. Orbanić and A. Graovac, The 10-cages and derived configurations, *Discrete Math.* **275** (2004), 265–276.

[129] R. Šćepanović, G. Ringel, D. Marušič, G. L. Chia and B. Alspach, Nonseparable graphs with a given number of cycles, *J. Graph Theory* **18** (1994), 777–789.

[130] D. Wasserman, V. Carli, C. Wasserman, A. Apter, J. Balazs, R. Bracale, R. Brunner, C. Bursztein-Lipsicas, P. Corcoran, D. Cosman, T. Durkee, D. Feldman, J. Gadoros, F. Guillemin, C. Haring, J.-P. Kahn, M. Kaess, H. Keeley, D. Marušič, B. Nemes, V. Poštuvan, S. Reiter-Theil, F. Resch, P. Sáiz, M. Sarchiapone, M. Sisask, A. Varnik and C. W. Hoven, Saving and empowering young lives in europe (seyle): a randomized controlled trial, *BMC Public Health* **10** (2010), 1–14.

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Odd-order Cayley graphs with commutator subgroup of order pq are hamiltonian*

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Abstract

We show that if G is a nontrivial, finite group of odd order, whose commutator subgroup $[G, G]$ is cyclic of order $p^\mu q^\nu$, where p and q are prime, then every connected Cayley graph on G has a hamiltonian cycle.

Keywords: Cayley graph, hamiltonian cycle, commutator subgroup.

Math. Subj. Class.: 05C25, 05C45

1 Introduction

It has been conjectured that there is a hamiltonian cycle in every connected Cayley graph on any finite group, but all known results on this problem have very restrictive hypotheses (see [2, 13, 15] for surveys). One approach is to assume that the group is close to being abelian, in the sense that its commutator subgroup is small. This is illustrated by the following theorem that was proved in a series of papers by Marušič [12], Durnberger [3, 4], and Keating-Witte [10]:

Theorem 1.1 (D. Marušič, E. Durnberger, K. Keating, and D. Witte, 1985)**.** If G is a nontrivial, finite group, whose commutator subgroup $[G, G]$ is cyclic of order p^μ, where p prime and $\mu \in \mathbb{N}$, then every connected Cayley graph on G has a hamiltonian cycle.

Under the additional assumption that G has odd order, we extend this theorem, by allowing the order of $[G, G]$ to be the product of two prime-powers:

Theorem 1.2. If G is a nontrivial, finite group of odd order, whose commutator subgroup $[G, G]$ is cyclic of order $p^\mu q^\nu$, where p and q are prime, and $\mu, \nu \in \mathbb{N}$, then every connected Cayley graph on G has a hamiltonian cycle.

*To Dragan Marušič on his 60th birthday.

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Remark 1.3. Of course, we would like to prove the conclusion of Theorem 1.2 without the assumption that $|G|$ is odd, or with a weaker assumption on the order of $[G, G]$.

If $\mu, \nu \leq 1$, then there is no need to assume that $[G, G]$ is cyclic:

Corollary 1.4. If G is a nontrivial, finite group of odd order, whose commutator subgroup $[G, G]$ has order pq, where p and q are distinct primes, then every connected Cayley graph on G has a hamiltonian cycle.

This yields the following contribution to the ongoing search [11] for hamiltonian cycles in Cayley graphs on groups whose order has few prime factors:

Corollary 1.5. If p and q are distinct primes, then every connected Cayley graph of order $9pq$ has a hamiltonian cycle.

Here is an outline of the paper. Miscellaneous definitions and preliminary results are collected in Section 2. (Also, Corollaries 1.4 and 1.5 are derived from Theorem 1.2 in §2E.) The paper's main tool is a technique known as "Marušič's Method." A straightforward application of this method is given in Section 3, and some other consequences are in Section 4. The proof of Theorem 1.2 is in Section 5, except that one troublesome case is postponed to Section 6.

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2 Preliminaries

2A Assumptions, definitions, and notation

Assumption 2.1.

(1) G is always a finite group.

(2) S is a generating set for G.

Definition 2.2. The *Cayley graph* $\mathrm{Cay}(G; S)$ is the graph whose vertex set is G, with an edge from g to gs and an edge from g to gs^{-1}, for every $g \in G$ and $s \in S$.

Notation 2.3.

- We let $G' = [G, G]$ and $\overline{G} = G/G'$. Also, for $g \in G$, we let $\overline{g} = gG'$ be the image of g in $\overline{G}$.
- For $g, h \in G$, we let $g^h = h^{-1}gh$ and $[g, h] = g^{-1}h^{-1}gh$.
- If H is an abelian subgroup of G and $k \in \mathbb{Z}$, we let

$$H^k = \{\, h^k \mid h \in H \,\}.$$

This is a subgroup of H (because H is abelian).

Notation 2.4. For $g \in G$ and $s_1, \ldots, s_n \in S \cup S^{-1}$, we use $[g](s_1, \ldots, s_n)$ to denote the walk in $\mathrm{Cay}(G; S)$ that visits (in order), the vertices

$$g,\ gs_1,\ gs_1s_2,\ gs_1s_2s_3,\ \ldots,\ gs_1s_2\cdots s_n.$$

We often write $(s_1, \ldots, s_n)$ for $[e](s_1, \ldots, s_n)$.

Definition 2.5. Suppose

- N is a normal subgroup of G, and
- $C = (s_i)_{i=1}^n$ is a hamiltonian cycle in $\mathrm{Cay}(G/N; S)$.

The *voltage* of C is $\prod_{i=1}^n s_i$. This is an element of N, and it may be denoted ΠC.

Remark 2.6. If $C = [g](s_1, \ldots, s_n)$, and N is abelian, then $\prod_{i=1}^n s_i = (\Pi C)^g$.

Proof. There is some ℓ with $(s_1s_2\cdots s_\ell)g \in N$. Then

$$C = (s_{\ell+1}, s_{\ell+2}, \ldots, s_n, s_1, s_2, \ldots, s_\ell),$$

so

$$\begin{aligned}(\Pi C)^g &= g^{-1}(s_{\ell+1}s_{\ell+2}\cdots s_n\, s_1s_2\cdots s_\ell)g \\ &= \big[(s_1s_2\cdots s_\ell)g\big]^{-1}(s_1s_2\cdots s_\ell)(s_{\ell+1}s_{\ell+2}\cdots s_n)\,\big[(s_1s_2\cdots s_\ell)g\big] \\ &= (s_1s_2\cdots s_n)^{(s_1s_2\cdots s_\ell)g} \\ &= s_1s_2\cdots s_n.\end{aligned}$$

□

2B Factor Group Lemma and Marušič's Method

Lemma 2.7 ("Factor Group Lemma" [15, §2.2])**.** Suppose

- N is a cyclic, normal subgroup of G,
- $(s_i)_{i=1}^m$ is a hamiltonian cycle in $\mathrm{Cay}(G/N; S)$, and
- the product $s_1s_2\cdots s_m$ generates N.

Then $(s_1, s_2, \ldots, s_m)^{|N|}$ is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

The following simple observation allows us to assume $|N|$ is square-free whenever we apply the Factor Group Lemma (2.7).

Lemma 2.8 ([10, Lem. 3.2])**.** Suppose

- N is a cyclic, normal subgroup of G,
- $\underline{N} = N/\Phi$ is the maximal quotient of N that has square-free order,
- $\underline{G} = G/\Phi$,
- $(s_1, s_2, \ldots, s_m)$ is a hamiltonian cycle in $\mathrm{Cay}(\underline{G}/\underline{N}; S)$, and
- the product $\underline{s_1}\,\underline{s_2}\cdots\underline{s_m}$ generates $\underline{N}$.

Then $s_1s_2\cdots s_m$ generates N, so $(s_1, s_2, \ldots, s_m)^{|N|}$ is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Remark 2.9 (cf. [7, Thm. 5.1.1])**.** When applying Lemma 2.8, it is sometimes helpful to know that if

- N, $\underline{N} = N/\Phi$, and $\underline{G} = G/\Phi$ are as in Lemma 2.8, and
- S is a minimal generating set of G.

Then $\underline{S}$ is a minimal generating set of $\underline{G}$.

Lemma 2.10 ("Marušič's Method" [12], cf. [10, Lem. 3.1])**.** Suppose

- $S_0 \subseteq S$,
- $\langle S_0 \rangle$ contains G',
- there are hamiltonian cycles $C_1, \ldots, C_r$ in $\mathrm{Cay}(\langle S_0\rangle/G'; S_0)$ that all have an oriented edge in common, and

$*$ for every $\gamma \in G'$, there is some i, such that $\big\langle \gamma \cdot \Pi C_i \big\rangle = G'$.

Then there is a hamiltonian cycle in $\mathrm{Cay}(G/G'; S)$ whose voltage generates G'. Hence, the Factor Group Lemma (2.7) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Corollary 2.11. Assume $G' = \mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are distinct primes. Then, in the situation of Marušič's Method (2.10), the final condition ($*$) can be replaced with either of the following:

(1) $r = 3$, and $\big\langle (\Pi C_i)^{-1}(\Pi C_j) \big\rangle = G'$ whenever $1 \le i < j \le 3$.

(2) $r = 4$, and

- $\big\langle (\Pi C_1)^{-1}(\Pi C_2) \big\rangle$ contains $\mathbb{Z}_p$, and
- $\big\langle (\Pi C_1)^{-1}(\Pi C_3) \big\rangle = \big\langle (\Pi C_2)^{-1}(\Pi C_4) \big\rangle = \mathbb{Z}_q$.

Proof. Let $\gamma \in G'$.

(1) Consider the three elements $\gamma \cdot \Pi C_1$, $\gamma \cdot \Pi C_2$, and $\gamma \cdot \Pi C_3$ of $\mathbb{Z}_p \times \mathbb{Z}_q$. By assumption, no two have the same projection to $\mathbb{Z}_p$, so only one of them can have trivial projection. Similarly for the projection to $\mathbb{Z}_q$. Therefore, there is some i, such that $\gamma \cdot \Pi C_i$ projects nontrivially to both $\mathbb{Z}_p$ and $\mathbb{Z}_q$. Therefore $\langle \gamma \cdot \Pi C_i \rangle = G'$.

(2) There is some $i \in \{1, 2\}$, such that $\gamma \cdot \Pi C_i$ projects nontrivially to $\mathbb{Z}_p$. We may assume the projection of $\gamma \cdot \Pi C_i$ to $\mathbb{Z}_q$ is trivial (otherwise, we have $\langle \gamma \cdot \Pi C_i \rangle = G'$, as desired). Then $\gamma \cdot \Pi C_{i+2}$ has the same (nontrivial) projection to $\mathbb{Z}_p$, but has a different (hence, nontrivial) projection to $\mathbb{Z}_q$. So $\langle \gamma \cdot \Pi C_{i+2} \rangle = G'$. □

2C Some known results

We recall a few results that provide hamiltonian cycles in $\mathrm{Cay}(G; S)$ under certain assumptions.

Theorem 2.12 (Witte [14])**.** If $|G| = p^\mu$, where p is prime and $\mu > 0$, then every connected Cayley digraph on G has a directed hamiltonian cycle.

Theorem 2.13 (Ghaderpour-Morris [6])**.** If G is a nontrivial, nilpotent, finite group, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.

Theorem 2.14 (Ghaderpour-Morris [5])**.** If $|G| = 27p$, where p is prime, then every connected Cayley graph on G has a hamiltonian cycle.

The proof of the preceding theorem has the following consequence.

Corollary 2.15. If G is a finite group, such that $|G/G'| = 9$ and G' is cyclic of order $p^\mu \cdot 3^\nu$, where $p \geq 5$ is prime, then every connected Cayley graph on G has a hamiltonian cycle.

Proof. Let $\underline{G} = G/(G')^{3p}$. Then $|\underline{G}| = 27p$ and $|\underline{G}'| = 3p$, so the proof of [5, Props. 3.4 and 3.6] provides a hamiltonian cycle in $\operatorname{Cay}\big(\underline{G}/\underline{G}'; S\big)$ whose voltage generates $\underline{G}'$. Then Lemma 2.8 provides a hamiltonian cycle in $\operatorname{Cay}(G; S)$. □

Theorem 2.16 (Alspach [1, Thm. 3.7])**.** Suppose

- $s \in S$,
- $\langle s \rangle \triangleleft G$,
- $|G/\langle s \rangle|$ is odd, and
- there is a hamiltonian cycle in $\operatorname{Cay}\big(G/\langle s \rangle; S\big)$.

Then there is a hamiltonian cycle in $\operatorname{Cay}(G; S)$.

This has the following immediate consequence, since every subgroup of a cyclic, normal subgroup is normal:

Corollary 2.17. Suppose

- G' is cyclic,
- $s \in S \cap G'$,
- $|G/\langle s \rangle|$ is odd, and
- there is a hamiltonian cycle in $\operatorname{Cay}(G/\langle s \rangle; S)$.

Then there is a hamiltonian cycle in $\operatorname{Cay}(G; S)$.

2D Group theoretic preliminaries

We recall a few elementary facts about finite groups.

Lemma 2.18 ([6, Lem. 3.4])**.** Suppose

- $\langle a, b \rangle = G$,
- G' is cyclic of square-free order, and
- $G' \subseteq Z(G)$.

Then $|[a, b]|$ is a divisor of both $\langle \overline{a} \rangle$ and $|\overline{G}/\langle \overline{a} \rangle|$.

Lemma 2.19 ([6, Lem. 3.5])**.** If $G = \langle a, b \rangle$, and G' is cyclic, then $G' = \langle [a, b] \rangle$.

Corollary 2.20. Suppose

- $\langle a, G' \rangle = G$, and
- G' is cyclic of square-free order.

Then a does not centralize any nontrivial subgroup of G'.

Proof. Let γ be a generator of the cyclic group G', and let $\underline{G} = G/\langle [a,\gamma] \rangle$, so $\underline{a}$ centralizes $\underline{\gamma}$. Then $\underline{G}' = \langle \underline{\gamma} \rangle \subseteq Z(\underline{G})$, so Lemma 2.18 tells us that $|\underline{G}'| = |[\underline{a}, \underline{\gamma}]|$ is a divisor of $|\overline{G}/\langle \overline{a} \rangle| = 1$. This means $\underline{G}$ is abelian, so $\langle [a,\gamma] \rangle = G' = \langle \gamma \rangle$. This implies that a does not centralize any nontrivial power of γ. In other words, a does not centralize any nontrivial subgroup of G'. □

Lemma 2.21. Suppose

- $G' = \mathbb{Z}_{3^\mu}$ is cyclic of order 3^μ, for some $\mu \in \mathbb{N}$, and
- $G/(G')^3$ is a nonabelian group of order 27.

Then

(1) $\mu = 1$ (so $|G| = 27$),

(2) $(ab)^3 = a^3 b^3$ for all $a, b \in G$, and

(3) the elements of order 3 (together with e) form a subgroup of G.

Proof. Note that $|G| = 3^{\mu+2}$, so G is a 3-group. Since G' is cyclic (and 3 is odd), it is not difficult to show

$$(ab)^3 \in a^3 b^3 (G')^3, \text{ for all } a, b \in G. \tag{2.21A}$$

(This is a special case of [9, Satz III.10.2(c), p. 322].)

(1) Since $G/G' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, there is a 2-element generating set $\{a, b\}$ of G. (In fact, every minimal generating set has exactly two elements [9, 3.15, p. 273].) Since $a^3, b^3 \in G'$, we see from (2.21A) that we may assume $b^3 \in (G')^3$ (by replacing b with ba or ba^{-1}, if necessary). Furthermore, by modding out $(G')^9$, there is no harm in assuming $\mu \leq 2$, so $(G')^3 \subseteq Z(G)$. Therefore $[a, b^3] = e$, so [9, Satz 10.6(b), p. 326] tells us that $[a,b]^3 = e$. Since $\langle [a,b] \rangle = G'$ (see Lemma 2.19), this implies $\mu = 1$.

(2) Since $\mu = 1$, we have $(G')^3 = \{e\}$, so this is immediate from (2.21A).

(3) This is immediate from (2). (Also, it is a special case of [9, Satz III.10.6(a), p. 326].) □

2E Proofs of Corollaries 1.4 and 1.5

Proof of Corollary 1.4. Assume, without loss of generality, that $p < q$. Then Sylow's Theorem implies that G' has a unique Sylow q-subgroup Q, so $Q \triangleleft G$. Therefore G acts on Q by conjugation. Since $Q \cong \mathbb{Z}_q$, we know that the automorphism group of Q is abelian (more precisely, it is cyclic of order $q-1$), so this implies that G' centralizes Q. So $Q \subseteq Z(G')$. Since G'/Q is cyclic (indeed, it is of prime order, namely, p), this implies that G' is abelian. Since $p \neq q$, we know that every abelian group of order pq is cyclic, so we conclude that G' is cyclic. Therefore Theorem 1.2 applies. □

Proof of Corollary 1.5. Assume $|G| = 9pq$. We may assume p and q are odd, for otherwise $|G|$ is of the form $18p$, so [11, Prop. 9.1] applies. Therefore $|G|$ is odd, so it suffices to show $|G'|$ is a divisor of pq, for then Corollary 1.4 (or Theorem 1.1) applies.

Note that we may assume $3 \notin \{p, q\}$, for otherwise $|G|$ is of the form $27p$, so Theorem 2.14 applies. Therefore, neither $|\mathrm{Aut}(\mathbb{Z}_9)| = 6$ nor $|\mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 48$ is divisible by either p or q, so Burnside's Transfer Theorem [7, Thm. 7.4.3, p. 252] implies that G has a normal subgroup N of order pq. Since $|G/N| = 9$, and every group of order 9 is abelian, we know that $G' \subseteq N$, so $|G'|$ is a divisor of $|N| = pq$, as desired. □

Let us also record the fact that almost all cases of Theorem 1.2 will be proved by using Marušič's Method (2.10):

Theorem 2.22. Assume

- S is a minimal generating set for a nontrivial, finite group G of odd order,
- G' is cyclic of order $p^\mu q^\nu$, where p and q are prime, and $\mu, \nu \in \mathbb{N}$,
- for all $s \in S$, we have $s \notin G'$ and $G' \not\subseteq \langle s \rangle$,
- $G/(G')^3$ is **not** the nonabelian group of order 27 and exponent 3, and
- either $G/G' \not\cong \mathbb{Z}_3 \times \mathbb{Z}_3$, or $\#S \neq 2$.

Then, for every $\gamma \in G'$, there exists a hamiltonian cycle C in $\mathrm{Cay}\big(G/G'; S\big)$, such that $\gamma \Pi C$ generates G'.

3 The usual application of Marušič's Method

Applying Marušič's Method (2.10) requires the existence of more than one hamiltonian cycle in a quotient of $\mathrm{Cay}(G; S)$. In practice, one usually starts with a single hamiltonian cycle and modifies it in various ways to obtain the others that are needed. The following result describes a modification that will be used repeatedly in the proof of Theorem 1.2.

Lemma 3.1 (cf. Durnberger [3] and Marušič [12])**.** Assume:

- C_0 is an oriented hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$,
- $a, b \in S^{\pm 1}$, $g \in G$, and $m \in \mathbb{Z}^+$,
- C_0 contains:
  - the oriented path $[ga^{-(m+1)}](a^m, b, a^{-m})$, and
  - either the oriented edge $[g](b)$ or the oriented edge $[gb](b^{-1})$.

Then there are hamiltonian cycles $C_0, C_1, \ldots, C_m$ in $\mathrm{Cay}(\overline{G}; S)$, such that

$$\Big(\big(\Pi C_0\big)^{-1} \big(\Pi C_k\big) \Big)^g = \begin{cases} [a^k, b^{-1}]\,[a^k, b^{-1}]^a & \text{if } C_0 \text{ contains } [g](b), \\ [b^{-1}, a^k]\,[a^k, b^{-1}]^a & \text{if } C_0 \text{ contains } [gb](b^{-1}). \end{cases}$$

Proof. Note that $[ga^{-(m+1)}](a^m, b, a^{-m})$ contains the subpath $[ga^{-(k+1)}](a^k, b, a^{-k})$ for $0 \le k \le m$.

Case 1. Assume that C_0 contains $[g](b)$. Construct C_k by:

- replacing the oriented edge $[g](b)$ with the oriented path $[g](a^{-k}, b, a^k)$, and
- replacing the oriented path $[ga^{-(k+1)}](a^k, b, a^{-k})$ with the oriented edge $[ga^{-(k+1)}](b)$

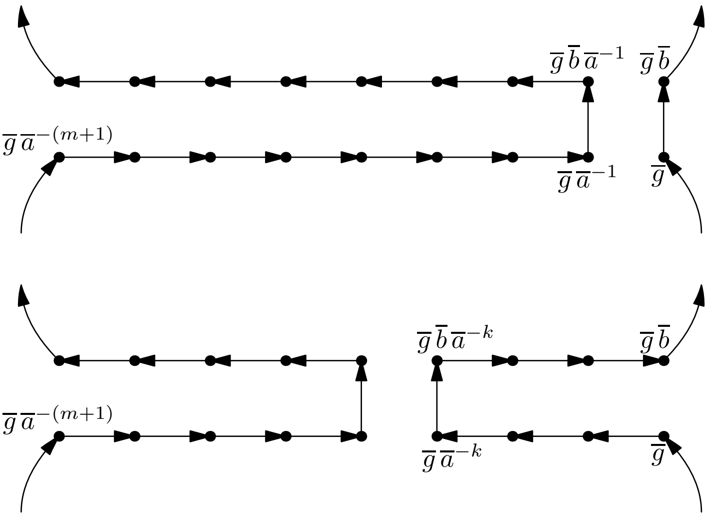

Figure 1: A portion of the hamiltonian cycles C_0 (top) and C_k (bottom).

(see Figure 1).

To calculate the voltage of C_k, write $C_0 = [g](s_1, \ldots, s_n)$. There is some ℓ with $\overline{s_1} \cdots \overline{s_\ell} = \overline{a}^{-1}$, so

$$C_k = [g]\big(a^{-k}, b, a^k,\, (s_i)_{i=2}^{\ell-k},\, b, (s_i)_{i=\ell+k+2}^{n}\big).$$

For convenience, let

$$h = \prod_{i=\ell+1}^{n} s_i \equiv \left(\prod_{i=1}^{\ell} s_i\right)^{-1} \equiv a \ (\operatorname{mod} G').$$

Then, from Remark 2.6 (and the fact that G' is commutative), we have

$$\begin{aligned}
\big(\Pi C_k\big)^g &= (a^{-k} b a^k) \left(\prod_{i=2}^{\ell-k} s_i\right) b \left(\prod_{i=\ell+k+2}^{n} s_i\right) \\
&= (a^{-k} b a^k b^{-1}) \left(\prod_{i=1}^{\ell} s_i\right) a^{-k} b a^k b^{-1} \left(\prod_{i=\ell+1}^{n} s_i\right) \\
&= [a^k, b^{-1}] \cdot \left(\prod_{i=1}^{\ell} s_i\right) \left(\prod_{i=\ell+1}^{n} s_i\right) \cdot [a^k, b^{-1}]^h \\
&= [a^k, b^{-1}] \cdot \big(\Pi C_0\big)^g \cdot [a^k, b^{-1}]^a \\
&= \big(\Pi C_0\big)^g \cdot [a^k, b^{-1}]\, [a^k, b^{-1}]^a.
\end{aligned}$$

Case 2. Assume that C_0 contains $[gb](b^{-1})$. This is similar. Construct C_k by:

- replacing the oriented edge $[gb](b^{-1})$ with the oriented path $[gb](a^{-k}, b^{-1}, a^k)$, and

- replacing the oriented path $[ga^{-(k+1)}](a^k, b, a^{-k})$ with the oriented edge $[ga^{-(k+1)}](b)$.

(See Figure 1, but reverse the orientation of the paths in the right half of the figure.)

To calculate the voltage of C_k, write $C_0 = [gb](s_1, \ldots, s_n)$. There is some ℓ with $\overline{s_1} \cdots \overline{s_\ell} = \overline{ab}^{-1}$, so

$$C_k = [gb]\big(a^{-k}, b^{-1}, a^k,\ (s_i)_{i=2}^{\ell-k},\ b, (s_i)_{i=\ell+k+2}^{n}\big).$$

For convenience, let

$$h = \prod_{i=\ell+1}^{n} s_i \equiv \left(\prod_{i=1}^{\ell} s_i\right)^{-1} \equiv ab \pmod{G'}.$$

Then

$$\begin{aligned}
\big(\Pi C_k\big)^{gb} &= (a^{-k}b^{-1}a^k)\left(\prod_{i=2}^{\ell-k} s_i\right) b \left(\prod_{i=\ell+k+2}^{n} s_i\right) \\
&= (a^{-k}b^{-1}a^k b)\left(\prod_{i=1}^{\ell} s_i\right) a^{-k}ba^kb^{-1}\left(\prod_{i=\ell+1}^{n} s_i\right) \\
&= b^{-1}(ba^{-k}b^{-1}a^k)b \cdot \left(\prod_{i=1}^{\ell} s_i\right)\left(\prod_{i=\ell+1}^{n} s_i\right) \cdot [a^k, b^{-1}]^h \\
&= [b^{-1}, a^k]^b \cdot \big(\Pi C_0\big)^{gb} \cdot [a^k, b^{-1}]^{ab} \\
&= \big(\Pi C_0\big)^{gb} \cdot [b^{-1}, a^k]^b\, [a^k, b^{-1}]^{ab}.
\end{aligned}$$

□

Remark 3.2. In the situation of Lemma 3.1, we have $\Big\langle \big(\Pi C_0\big)^{-1}\big(\Pi C_k\big)\Big\rangle = \langle [a^k, b^{-1}]\rangle$ if either

1. C_0 contains $[g](b)$ and a does not invert any nontrivial element of $\langle [a^k, b^{-1}]\rangle$, or
2. C_0 contains $[gb](b^{-1})$ and a does not centralize any nontrivial element of $\langle [a^k, b^{-1}]\rangle$.

Note that if $|G|$ is odd, then the hypothesis on a in (1) is automatically satisfied (because no element of odd order can ever invert a nontrivial element).

Corollary 3.3 (cf. [4, Case iv] and [10, Case 4.3])**.** Assume

- $a \in S$ with $\langle \overline{a} \rangle \neq \overline{G}$,
- $(s_i)_{i=1}^{d}$ is a hamiltonian cycle in $\operatorname{Cay}\big(\overline{G}/\langle \overline{a}\rangle; S\big)$,
- $a^r \prod_{i=1}^{d} s_i \in G'$, with $0 \le r \le |\overline{a}| - 2$, and
- $0 \le k \le |\overline{a}| - 3$.

Then the walk

$$\begin{aligned}
C_k = \big(a^k, s_1, &a^{-(k+1)}, (s_{2i}, a^{|\overline{a}|-2}, s_{2i+1}, a^{-(|\overline{a}|-2)})_{i=1}^{(d-3)/2}, \\
& s_{d-1}, a^r, s_d, a^{-(|\overline{a}|-k-2)}, s_1, a^{|\overline{a}|-k-3}, (s_i)_{i=2}^{d-1}, a^{-(|\overline{a}|-r-2)}, s_d\big)
\end{aligned}$$

is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$ (see Figure 2), and we have

$$\Pi C_k = (\Pi C_0)[a^{-k}, s_1^{-1}][a^{-k}, s_1^{-1}]^{a^{-1}}.$$

Proof. C_0 contains the oriented edge (s_1) and the oriented path $[a^{|\overline{a}|-2}](a^{-(|\overline{a}|-3)}, s_1, a^{|\overline{a}|-3})$, so we may apply Lemma 3.1 with $g = e$, $b = s_1$, and a^{-1} in the role of a. □

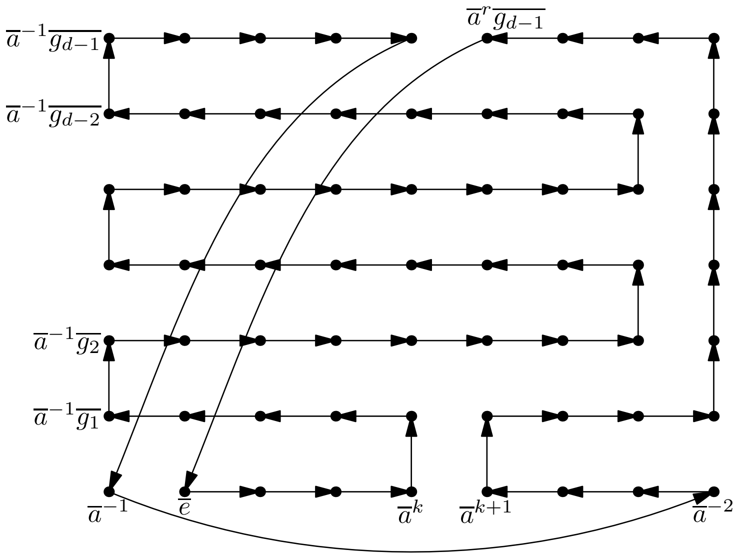

Figure 2: A hamiltonian cycle C_k in $\mathrm{Cay}(\overline{G}; S)$, where $g_j = \prod_{i=1}^{j} s_i$.

4 Other applications of Marušič's Method

Here are some other situations in which we can apply Marušič's Method (2.10).

Theorem 4.1 ([10, §4 and §5])**.** Suppose

- $|G|$ is odd,
- $G' = \mathbb{Z}_{p^\mu}$ is cyclic of prime-power order,
- S is a generating set of G,
- $S \cap G' = \emptyset$, and
- G is **not** the nonabelian group of order 27 with exponent 3.

Then there exist hamiltonian cycles C_1 and C_2 in $\mathrm{Cay}(G/G'; S)$ that have an oriented edge in common, such that $(\Pi C_1)^{-1}(\Pi C_2)$ generates G'.

Proof. Lemma 2.8 allows us to assume $|G'| = p$. Then the desired conclusion is implicit in [10, §4 and §5] unless $|G/G'| \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $p = 3$.

Therefore $G/(G')^3$ is a nonabelian group of order 27, so Lemma 2.21(1) tells us $|G| = 27$. By assumption, the exponent of G is greater than 3, so we conclude from

Lemma 2.21(3) that S contains an element b with $|b| \geq 9$. We may assume S is minimal, so $\#S = 2$; write $S = \{a, b\}$. Then we have the following two hamiltonian cycles in $\mathrm{Cay}(\overline{G}; S)$:

$$C_1 = (a^2, b)^3 \text{ and } C_2 = (a^2, b^{-1})^3.$$

Since Lemma 2.21(2) tells us $(xy)^3 = x^3y^3$ for all $x, y \in G$, and we have $x^3 \in G' = Z(G)$ for all $x \in G$, we see that

$$(\Pi C_1)^{-1}(\Pi C_2) = \big((a^2b)^3\big)^{-1}\big(a^2b^{-1}\big)^3 = \big((a^2)^3b^3\big)^{-1}\big((a^2)^3(b^{-1})^3\big) = b^{-6} \neq e,$$

since $|b| \geq 9$. □

We will use the following version of this result in Subcase ii of Case 5.12.

Proposition 4.2. Suppose

- $|G|$ is odd,
- $G' = \mathbb{Z}_p$ has prime order,
- Z is a subgroup of $Z(G)$,
- $S \cap G'Z = \emptyset$, and
- G is not nilpotent.

Then there exist hamiltonian cycles C_1 and C_2 in $\mathrm{Cay}\big(G/(G'Z); S\big)$ that have an oriented edge in common, such that $\big\langle(\Pi C_1)^{-1}(\Pi C_2)\big\rangle = G'$.

Proof. Choose $a, b \in S$ with $[a, b] \neq e$. Since G is not nilpotent, we may assume a does not centralize G'. Furthermore, since we are using Marušič's Method (2.10), there is no harm in assuming $S = \{a, b\}$.

If $b \notin \langle a, G', Z\rangle$, then the proof of [10, Case 5.3] provides two hamiltonian cycles $C_1 = (s_i)_{i=1}^n$ and $C_2 = (t_i)_{i=1}^n$ in $\mathrm{Cay}\big(G/(G'Z); a, b\big)$, such that $\Pi C_1 \neq \Pi C_2$ (and the two cycles have an oriented edge in common). From the construction, it is clear that $(s_i)_{i=1}^n$ is a permutation of $(t_i)_{i=1}^n$, so $(\Pi C_1)^{-1}(\Pi C_2) \in G'$.

We may now assume $b \in \langle a, G', Z\rangle$. Then, letting $n = |G : \langle a, G', Z\rangle|$, there is some i, such that $b^i \in a^iG'Z$ and $0 < i < n$. Therefore, we have the following two hamiltonian cycles in $\mathrm{Cay}\big(G/(G'Z); S\big)$ that both contain the oriented edge (b):

$$C_1 = (b, a^{-(i-1)}, b, a^{n-i-1}),$$
$$C_2 = (b, a^{n-i-1}, b, a^{-(i-1)}) = [a^{-1}]C_1.$$

The sequence of edges in C_2 is a permutation of the sequence of edges in C_1, therefore $(\Pi C_1)^{-1}(\Pi C_2) \in G'$. Also, since a does not centralize G', it is not difficult to see that $(\Pi C_1)^{-1}(\Pi C_2)$ is nontrivial, and therefore generates G'. □

Lemma 4.3. Assume

- $G' = \mathbb{Z}_{p^\mu} \times \mathbb{Z}_{q^\nu}$, where p and q are prime,
- $S \cap G' = \emptyset$,
- there exist $a, b \in S \cup S^{-1}$, with $a \neq b$, such that $aG' = bG'$,

- the generating set S is minimal, and
- $|G|$ is odd.

Then there is a hamiltonian cycle in $\mathrm{Cay}(G;S)$.

Proof. Write $b = a\gamma$, with $\gamma \in G'$.

Case 1. Assume $\langle\gamma\rangle = G'$. We apply Marušič's Method (2.10), so Lemma 2.8 allows us to assume $G' = \mathbb{Z}_p \times \mathbb{Z}_q$. Since $|\overline{a}| \geq 3$, it is easy to find an oriented hamiltonian cycle C_0 in $\mathrm{Cay}(\overline{G};S)$ that has (at least) 2 oriented edges α_1 and α_2 that are labeled a. We construct two more hamiltonian cycles C_1 and C_2 by replacing one or both of α_1 and α_2 with a b-edge. (Replace one a-edge to obtain C_1; replace both to obtain C_2.) Then there are conjugates γ_1 and γ_2 of γ, such that

$$(\Pi C_0)^{-1}(\Pi C_1) = \gamma_1, \quad (\Pi C_1)^{-1}(\Pi C_2) = \gamma_2, \quad (\Pi C_0)^{-1}(\Pi C_2) = \gamma_1\gamma_2.$$

By the assumption of this case, we know that γ_1 and γ_2 generate G'. Also, since $|G|$ is odd, we know that no element of G inverts any nontrivial element of G', so $\gamma_1\gamma_2$ also generates G'. Therefore, Marušič's Method 2.11(1) applies.

Case 2. Assume $\langle\gamma\rangle \neq G'$. Since S is minimal, we know $\langle\gamma\rangle$ contains either $\mathbb{Z}_{p^\mu}$ or $\mathbb{Z}_{q^\nu}$. By the assumption of this case, we know it does not contain both. So let us assume $\langle\gamma\rangle = N \times \mathbb{Z}_{q^\nu}$, where N is a proper subgroup of $\mathbb{Z}_{p^\mu}$.

Assume, for the moment, that $G/(G')^p$ is not the nonabelian group of order 27 and exponent 3. We use Marušič's Method (2.10), so Lemma 2.8 allows us to assume $G' = \mathbb{Z}_p \times \mathbb{Z}_q$. Applying Theorem 4.1 to $G/\mathbb{Z}_q$ provides us with hamiltonian cycles C_1 and C_2 in $\mathrm{Cay}(G/G'; S \smallsetminus \{b\})$, such that $\langle(\Pi C_1)^{-1}(\Pi C_2)\rangle$ contains $\mathbb{Z}_p$. (Furthermore, the two cycles have an oriented edge in common.) Since S is a minimal generating set, we know that C_i contains an edge labelled $a^{\pm 1}$. (In fact, more than one, so we can take one that is not the edge in common with the other cycle.) Assume, without loss of generality, that it is labelled a. Replacing this edge with b results in a hamiltonian cycle C_i', such that $\langle(\Pi C_i)^{-1}(\Pi C_i')\rangle = \langle\gamma\rangle = \mathbb{Z}_q$. Then Marušič's Method 2.11(2) applies.

We may now assume that $G/(G')^p$ is the nonabelian group of order 27 and exponent 3. Then $G/\langle\gamma\rangle$ is a 3-group, so Theorem 2.12 tells us there is a directed hamiltonian cycle C_0 in the Cayley digraph $\overrightarrow{\mathrm{Cay}}(G/\langle\gamma\rangle; S \smallsetminus \{b\})$. Since $S \smallsetminus \{b\}$ is a minimal generating set of $G/\langle\gamma\rangle$, there must be at least two edges α_1 and α_2 that are labeled a in C. Now the proof of Case 1 applies (but with $\langle\gamma\rangle$ in the place of G'). □

5 Proof of Theorem 1.2

Assumption 5.1. We always assume:

(1) The generating set S is minimal.

(2) $S \cap G' = \emptyset$ (see Corollary 2.17).

(3) p and q are distinct (see Theorem 1.1).

(4) G is not nilpotent (see Theorem 2.13). This implies $G/(G')^{pq}$ is not nilpotent [9, Satz 3.5, p. 270].

(5) There do not exist $a, b \in S \cup S^{-1}$ with $a \neq b$ and $aG' = bG'$ (see Lemma 4.3).

(6) There does not exist $s \in S$, such that $G' \subseteq \langle s\rangle$ (see Theorem 2.16).

Remark 5.2. We consider several cases that are exhaustive up to permutations of the variables a, b, and c, and interchanging p and q. Here is an outline of the cases:

- There exist $a, b \in S$, such that $\langle [a, b] \rangle = G'$.
  - (5.3) $\overline{b} \in \langle \overline{a} \rangle$.
  - (5.4) $\overline{b} \notin \langle \overline{a} \rangle$ and $|\overline{a}| \geq 5$.
  - (5.5) $|\overline{a}| = |\overline{b}| = 3$ and $\langle \overline{a} \rangle \neq \langle \overline{b} \rangle$.
- There exist $a, b, c \in S$, such that $\mathbb{Z}_{p^\mu} \subseteq \langle [a, b] \rangle$ and $\mathbb{Z}_{q^\nu} \subseteq \langle [a, c] \rangle$.
  - (5.7) $\overline{b}, \overline{c} \in \langle \overline{a} \rangle$.
  - (5.8) $\langle \overline{a} \rangle \subsetneq \langle \overline{a}, \overline{b} \rangle \subsetneq \langle \overline{a}, \overline{b}, \overline{c} \rangle$.
  - (5.9) a centralizes $G'/(G')^{pq}$.
  - (5.10) $\overline{b}, \overline{c} \notin \langle \overline{a} \rangle$.
  - (5.11) $\overline{c} \in \langle \overline{a} \rangle$ and $\overline{b} \notin \langle \overline{a} \rangle$.
- There do not exist $a, b, c \in S$, such that $\langle [a, b], [a, c] \rangle = G'$. (5.12)

Case 5.3. Assume there exist $a, b \in S$, such that $\langle [a, b] \rangle = G'$ and $\overline{b} \in \langle \overline{a} \rangle$.

Proof. We use Marušič's Method (2.11), so there is no harm in assuming $S = \{a, b\}$. Then $\langle \overline{a} \rangle = \langle \overline{a}, \overline{b} \rangle = \overline{G}$. Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$. Let $n = |\overline{a}| = |\overline{G}|$, fix k with $\overline{b} = \overline{a}^k$, and choose $\gamma \in G'$, such that $b = a^k \gamma$. Note that

- $a^n = e$ (since Corollary 2.20 implies that a cannot centralize a nontrivial subgroup of G'), and
- $\langle \gamma \rangle = G'$ (since $\langle a \rangle \ltimes \langle \gamma \rangle = \langle a, b \rangle = G$).

We may assume $1 \leq k < n/2$, by replacing b with its inverse if necessary. We may also assume $n \geq 5$ (otherwise, we must have $k = 1$, contrary to Assumption 5.1(5)). Therefore $n - k - 2 > 0$.

We have the following three hamiltonian cycles in $\mathrm{Cay}(\overline{G}; a, b)$:

$$C_1 = (a^n), \qquad C_2 = (a^{n-k-1}, b, a^{-(k-1)}, b), \qquad C_3 = (a^{n-k-2}, b, a^{-(k-1)}, b, a).$$

Their voltages are

$$\begin{aligned}
\Pi C_1 &= a^n = e, \\
\Pi C_2 &= a^{n-k-1} b a^{-(k-1)} b = a^{n-k-1} (a^k \gamma) a^{-(k-1)} (a^k \gamma) = a^n \cdot a^{-1} \gamma a \gamma = \gamma^a \gamma, \\
\Pi C_3 &= a^{n-k-2} b a^{-(k-1)} b a = a^{-1} (a^{n-k-1} b a^{-(k-1)} b) a = (\Pi C_2)^a.
\end{aligned}$$

Since $|G|$ is odd, we know that a does not invert $\mathbb{Z}_p$ or $\mathbb{Z}_q$. Therefore ΠC_2 generates G'. Hence, the conjugate ΠC_3 must also generate G'. Furthermore, as was mentioned above, we know that a does not centralize any nontrivial element of G', so $(\Pi C_2)(\Pi C_3)^{-1}$ also generates G'. (Also note that all three hamiltonian cycles contain the oriented edge (a).) Hence, Marušič's Method 2.11(1) applies. □

Case 5.4. Assume there exist $a, b \in S$, such that $\langle [a, b] \rangle = G'$ and $\overline{b} \notin \langle \overline{a} \rangle$. Also assume $|\overline{a}| \geq 5$.

Proof (cf. proof of [10, Case 4.3]). We use Marušič's Method (2.11), so there is no harm in assuming $S = \{a, b\}$. Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$. Let $d = |\overline{G}/\langle\overline{a}\rangle|$, so there is some r with $\overline{b}^d\overline{a}^r = \overline{e}$ and $0 \leq r < |\overline{a}|$. We may assume $r \leq |\overline{a}| - 2$, by replacing b with its inverse if necessary.

Applying Corollary 3.3 to the hamiltonian cycle (b^{-d}) yields hamiltonian cycles C_0, C_1, and C_2 (since $2 = 5 - 3 \leq |\overline{a}| - 3$). Note that all of these contain the oriented edge $\overline{b}(b^{-1})$. Furthermore, the voltage of C_k is

$$\Pi C_k = \pi[a^{-k}, b]\,[a^{-k}, b]^{a^{-1}},$$

where $\pi = \Pi C_0$ is independent of k.

Since $[a^{-1}, b]$ generates G', and a does not invert any nontrivial element of G' (recall that $|G|$ is odd), it is easy to see that G' is generated by the difference of any two of

$$e,\ [a^{-1}, b],\ \text{and}\ [a^{-2}, b] = [a^{-1}, b][a^{-1}, b]^{a^{-1}}.$$

Using again the fact that a does not invert any element of G', this implies that G' is generated by the difference of any two of the three voltages, so Marušič's Method 2.11(1) applies. □

Case 5.5. Assume there exist $a, b \in S$, such that $\langle[a, b]\rangle = G'$, $|\overline{a}| = |\overline{b}| = 3$ and $\langle\overline{a}\rangle \neq \langle\overline{b}\rangle$.

Proof. This proof is rather lengthy. It can be found in Section 6. □

Assumption 5.6. Henceforth, we assume there do not exist $a, b \in S \cup S^{-1}$, such that $\langle[a, b]\rangle = G'$.

Case 5.7. Assume $\mathbb{Z}_{p^\mu} \subseteq \langle[a, b]\rangle$, $\mathbb{Z}_{q^\nu} \subseteq \langle[a, c]\rangle$, and $\langle\overline{b}, \overline{c}\rangle \subseteq \langle\overline{a}\rangle$.

Proof. We use Marušič's Method (2.11), so there is no harm in assuming $S = \{a, b, c\}$. (Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, so $\langle[a, b]\rangle = \mathbb{Z}_p$ and $\langle[a, c]\rangle = \mathbb{Z}_q$.) Then, since $\overline{b}, \overline{c} \in \langle\overline{a}\rangle$, we must have $\langle\overline{a}\rangle = \overline{G}$. Therefore, Corollary 2.20 tells us that a does not centralize any nonidentity element of G'. Fix k and ℓ with $\overline{b} = \overline{a}^k$ and $\overline{c} = \overline{a}^\ell$. We may write $b = a^k\gamma_1$ and $c = a^\ell\gamma_2$, for some $\gamma_1 \in \mathbb{Z}_p$ and $\gamma_2 \in \gamma_q$.

Since 1, k, and ℓ are distinct (see Assumption 5.1(5)), we may assume $1 < k < \ell < n/2$, by interchanging b and c and/or replacing b and/or c with its inverse if necessary. Therefore $\ell \geq 3$ and $k + \ell \leq n - 2$, so we have the following three hamiltonian cycles in $\mathrm{Cay}(\overline{G}; a, b, c)$:

$$\begin{aligned}
C_1 &= (a^{-n}) \\
C_2 &= (a^{-(\ell-1)}, c,\ b, a^{-(k-1)}, b,\ a^{n-k-\ell-2},\ c) \\
C_3 &= (a^{-(\ell-2)}, c,\ b, a^{-(k-1)}, b,\ a^{n-k-\ell-2},\ c, a^{-1}).
\end{aligned}$$

Note that each of these contains the oriented edge (a^{-1}).

Since a does not centralize any nonidentity element of G', we know $\Pi C_1 = e$. A straightforward calculation shows

$$\Pi C_2 = (\gamma_1\gamma_1^{a^{-1}})^{a^{-k-1}}(\gamma_2^{a^{-1}}\gamma_2),$$

which generates G'. Therefore, $\Pi C_3 = (\Pi C_2)^{a^{-1}}$ and $(\Pi C_2)^{-1}(\Pi C_3)$ also generate G'. (For the latter, note that a^{-1} does not centralize any nonidentity element of G'.) Therefore Marušič's Method 2.11(1) applies. □

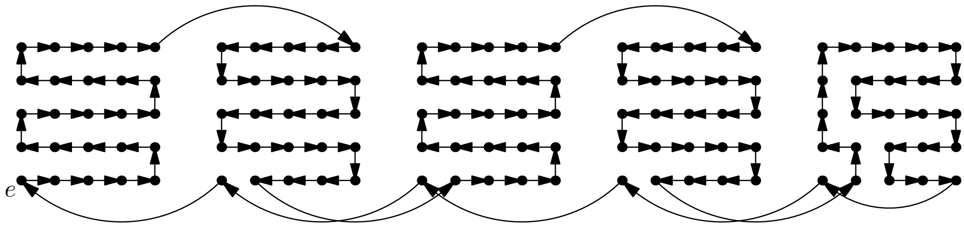

Figure 3: A hamiltonian cycle X.

Case 5.8. Assume $\mathbb{Z}_{p^\mu} \subseteq \langle [a,b] \rangle$, $\mathbb{Z}_{q^\nu} \subseteq \langle [a,c] \rangle$, and there exists $s \in \{a, b\}$, such that $\langle \overline{a} \rangle \subsetneq \langle \overline{a}, \overline{s} \rangle \subsetneq \langle \overline{a}, \overline{b}, \overline{c} \rangle$.

Proof. We use Marušič's Method (2.11), so there is no harm in assuming $S = \{a, b, c\}$. Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, so $\langle [a,b] \rangle = \mathbb{Z}_p$ and $\langle [a,c] \rangle = \mathbb{Z}_q$. Choose $A, B, C \geq 3$, such that $\overline{a}^A = \overline{e}$, and every element of $\overline{G}$ can be written uniquely in the form

$$\overline{a}^x \overline{b}^y \overline{c}^z \text{ with } \begin{array}{l} 0 \leq x < A, \\ 0 \leq y < B, \\ 0 \leq z < C. \end{array}$$

More precisely, we may let

$$\begin{cases} A = |\overline{a}|,\, B = |\langle \overline{a}, \overline{b} \rangle : \langle \overline{a} \rangle|,\, C = |\overline{G} : \langle \overline{a}, \overline{b} \rangle| & \text{if } s = b, \\ A = |\overline{a}|,\, C = |\langle \overline{a}, \overline{c} \rangle : \langle \overline{a} \rangle|,\, B = |\overline{G} : \langle \overline{a}, \overline{c} \rangle| & \text{if } s = c. \end{cases}$$

Then we have the following hamiltonian cycle X in $\operatorname{Cay}(\overline{G}; a, b, c)$ (see Figure 3):

$$X = \Big(a, \Big(a^{A-2}, (b, a^{-(A-1)}, b, a^{A-1})^{(B-1)/2}, c, \\ (a^{-(A-1)}, b^{-1}, a^{A-1}, b^{-1})^{(B-1)/2}, a^{-(A-2)}, c \Big)^{(C-1)/2}, \\ b, a^{-1}, b^{B-2}, a, (a^{A-2}, b^{-1}, a^{-(A-2)}, b^{-1})^{(B-3)/2}, \\ a^{A-2}, b^{-1}, a^{-(A-3)}, b^{-1}, a^{A-2}, c^{-(C-1)} \Big).$$

We obtain a new hamiltonian cycle X^p by replacing a subpath of the form $[g]\big(a^{A-1}, b, a^{-(A-1)}\big)$ with $[g]\big(a^{-(A-1)}, b, a^{A-1}\big)$. Then $(\Pi X)^{-1}(\Pi X^p)$ is a conjugate of

$$\big(a^{A-1} b a^{-(A-1)}\big)^{-1} \big(a^{-(A-1)} b a^{A-1}\big) = [b, a^{A-1}]^a [b, a^{A-1}].$$

Similarly, replacing a subpath of the form $[g]\big(a^{A-1}, c, a^{-(A-1)}\big)$ with $[g]\big(a^{-(A-1)}, c, a^{A-1}\big)$ results in a hamiltonian cycle X_q, such that $(\Pi X)^{-1}(\Pi X_q)$ is a conjugate of $[c, a^{A-1}]^a [c, a^{A-1}]$. Furthermore, doing both replacements results in a hamiltonian cycle X_q^p, such that $(\Pi X^p)^{-1}(\Pi X_q^p)$ is also a conjugate of $[c, a^{A-1}]^a [c, a^{A-1}]$. Note that all four of these hamiltonian cycles contain the oriented edge $c(c^{-1})$.

Since $G' \not\subseteq \langle a \rangle$ (see Assumption 5.1(6)), we may assume $a^A \in \mathbb{Z}_p$ (by interchanging p and q if necessary). Since $[c, a] \in \mathbb{Z}_q$, this implies that c centralizes a^A, so $[c, a^{A-1}] =$

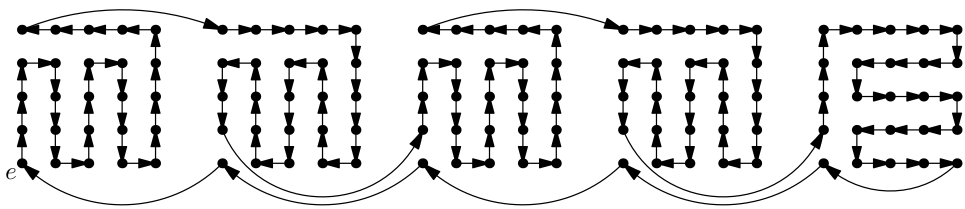

Figure 4: A hamiltonian cycle Y_1.

$[c, a^{-1}]$ generates $\mathbb{Z}_q$. Since a does not invert any nontrivial element of $\mathbb{Z}$ (recall that G has odd order), this implies that $[c, a^{A-1}]^a[c, a^{A-1}]$ generates $\mathbb{Z}_q$.

Assume, for the moment, that $[b, a^{A-1}]$ generates $\mathbb{Z}_p$. Since a does not invert any nontrivial element of $\mathbb{Z}_p$, this implies that $[b, a^{A-1}]^a[b, a^{A-1}]$ generates $\mathbb{Z}_p$. Therefore, Marušič's Method 2.11(2) applies.

We may now assume $[b, a^{A-1}]$ does not generate $\mathbb{Z}_p$. This means $[b, a^{A-1}] = e$. Since $[b, a^{-1}] \neq e$, we conclude that $[b, a^A] \neq e$, so

$$b \text{ does not centralize } \mathbb{Z}_p.$$

We have the following hamiltonian cycle Y_1 in $\mathrm{Cay}(\overline{G}; a, b, c)$ (see Figure 4):

$$Y_1 = \Big(b, \Big(b^{B-3}, (a, b^{-(B-2)}, a, b^{B-2})^{(A-1)/2}, b, a^{-(A-1)}, c, \\ a^{A-1}, b^{-1}, (b^{-(B-2)}, a^{-1}, b^{B-2}, a^{-1})^{(A-1)/2}, b^{-(B-3)}, c\Big)^{(C-1)/2}, \\ b^{B-2}, a, (a^{A-2}, b^{-1}, a^{-(A-2)}, b^{-1})^{(B-1)/2}, a^{A-1}, c^{-(C-1)}\Big).$$

We create a new hamiltonian cycle Y_2 by replacing a subpath of the form $[g]\big(a^{-(A-1)}, c, a^{A-1}\big)$ with $[g]\big(a^{A-1}, c, a^{-(A-1)}\big)$. This is the same as the construction of X_q from X, but with a and a^{-1} interchanged, so the same calculation shows

$$(\Pi Y_1)^{-1}(\Pi Y_2) \text{ is a conjugate of } [c, a^{-(A-1)}]^{a^{-1}}[c, a^{-(A-1)}], \text{ which generates } \mathbb{Z}_q.$$

Furthermore, since Y_1 and Y_2 both contain the oriented path $[b^{B-3}](b, a, b^{-1})$, and either the oriented edge $[b^{B-2}](a)$ or the oriented edge $[b^{B-2}a](a^{-1})$, Remark 3.2 provides hamiltonian cycles Y_1' and Y_2', such that $(\Pi Y_i)^{-1}(\Pi Y_i')$ generates $\mathbb{Z}_p$. Since all four hamiltonian cycles contain the oriented edge $[c](c^{-1})$, Marušič's Method 2.11(2) applies. □

Case 5.9. Assume $\mathbb{Z}_{p^\mu} \subseteq \langle [a, b] \rangle$, $\mathbb{Z}_{q^\nu} \subseteq \langle [a, c] \rangle$, and a centralizes $G'/(G')^{pq}$.

Proof. We use Marušič's Method (2.11), so there is no harm in assuming $S = \{a, b, c\}$. Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, so $\langle [a, b] \rangle = \mathbb{Z}_p$ and $\langle [a, c] \rangle = \mathbb{Z}_q$.

Note that $[a, b^{-1}, c] \in \mathbb{Z}_p$, $[c, a^{-1}, b] \in \mathbb{Z}_q$, and $[b, c^{-1}, a] = e$ (because a centralizes G'). Since $\mathbb{Z}_p \cap \mathbb{Z}_q = \{e\}$, and the Three-Subgroup Lemma [7, Thm. 2.3, p. 19] tells us

$$[a, b^{-1}, c]^b[b, c^{-1}, a]^c[c, a^{-1}, b]^a = e,$$

we conclude that $[a, b^{-1}, c] = [c, a^{-1}, b] = e$, so

$$c \text{ centralizes } \mathbb{Z}_p \text{ and } b \text{ centralizes } \mathbb{Z}_q.$$

We know $G' \not\subseteq Z(G)$, because G is not nilpotent (see Assumption 5.1(4)). Since a centralizes G', this implies we may assume c does not centralize G' (by interchanging b and c if necessary). So c does not centralize $\mathbb{Z}_q$. Since a, b, and G' all centralize $\mathbb{Z}_q$, this implies $c \notin \langle a, b, G' \rangle$. In other words, $\overline{c} \notin \langle \overline{a}, \overline{b} \rangle$. Furthermore, applying Corollary 2.20 to the group $\langle a, b \rangle$ tells us that $\langle \overline{a} \rangle \neq \langle \overline{a}, \overline{b} \rangle$. Therefore $\langle \overline{a} \rangle \subsetneq \langle \overline{a}, \overline{b} \rangle \subsetneq \langle \overline{a}, \overline{b}, \overline{c} \rangle$, so Case 5.8 applies. □

Case 5.10. Assume $\mathbb{Z}_{p^\mu} \subseteq \langle [a, b] \rangle$, $\mathbb{Z}_{q^\nu} \subseteq \langle [a, c] \rangle$, and $\overline{b}, \overline{c} \notin \langle \overline{a} \rangle$.

Proof. We use Marušič's Method (2.11), so there is no harm in assuming $S = \{a, b, c\}$. Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, so $\langle [a, b] \rangle = \mathbb{Z}_p$ and $\langle [a, c] \rangle = \mathbb{Z}_q$. We may assume $\langle \overline{a}, \overline{b} \rangle = \langle \overline{a}, \overline{c} \rangle = \overline{G}$, for otherwise Case 5.8 applies.

Let us begin by showing that a does not centralize any nontrivial element of G'. Suppose not. Then we may assume that a centralizes $\mathbb{Z}_p$. Let $\underline{G} = G/\mathbb{Z}_q = G/\langle [a, c] \rangle$. Since $\langle a, c, G' \rangle = G$, we know that $\langle \underline{a}, \underline{c}, \underline{\mathbb{Z}_p} \rangle = \underline{G}$, so $\underline{a}$ is in the center of $\underline{G}$. This contradicts the fact that $\langle [\underline{a}, \underline{b}] \rangle = \underline{\mathbb{Z}_p}$ is nontrivial.

Since $\overline{G}$ is abelian (and because $\overline{b}, \overline{c} \notin \langle \overline{a} \rangle$), it is easy to choose a hamiltonian cycle $(s_i)_{i=1}^d$ in $\operatorname{Cay}(\overline{G}/\langle \overline{a} \rangle; S)$ that contains both an edge labeled b (or b^{-1}) and an edge labeled c (or c^{-1}). Note that

$$C_0 = \left((s_i)_{i=1}^{d-1}, a^{|\overline{a}|-1}, (s_{d-2i+1}^{-1}, a^{-(|\overline{a}|-2)}, s_{d-2i}^{-1}, a^{|\overline{a}|-2})_{i=1}^{(d-1)/2}, a \right)$$

is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$.

Subcase i. *Assume* $|\overline{a}| > 3$. We may assume $s_1 = b^{-1}$ and $s_2 = c^{-1}$. Then C_0 contains the four subpaths

$$(b^{-1}), \quad [b^{-1}a^2](a^{-1}, b, a), \quad [b^{-1}](c^{-1}), \quad [b^{-1}c^{-1}a^{-2}](a, c, a^{-1}).$$

Therefore, we may let g be either b^{-1} or $b^{-1}c^{-1}$ in Lemma 3.1, so Remark 3.2(2) tells us we have hamiltonian cycles C^b and C^c, such that $(\Pi C_0)^{-1}(\Pi C^b)$ is a generator of $\mathbb{Z}_p$, and $(\Pi C_0)^{-1}(\Pi C^c)$ is a generator of $\mathbb{Z}_q$. Since $|\overline{a}| > 3$, we see that

$$C^b, \text{ like } C_0, \text{ contains } [b^{-1}](c^{-1}) \text{ and } [b^{-1}c^{-1}a^{-2}](a, c, a^{-1}),$$

so Remark 3.2(2) provides a hamiltonian cycle C^b_c, such that $(\Pi C^b)^{-1}(\Pi C^b_c)$ is a generator of $\mathbb{Z}_q$. Therefore, Marušič's Method 2.11(2) applies (since each of these four hamiltonian cycles contains the oriented edge $[a^{-1}](a)$).

Subcase ii. *Assume* $d > 3$. We may assume $s_1 = b^{-1}$ and $s_3 = c^{-1}$. Then C_0 contains the four subpaths

$$(b^{-1}), \ [b^{-1}a^2](a^{-1}, b, a), \quad [s_1 s_2](c^{-1}), \ [s_1 s_2 c^{-1} a^2](a^{-1}, c, a).$$

Therefore, we may let g be either b^{-1} or $s_1 s_2 c^{-1}$ in Lemma 3.1, so Remark 3.2(2) tells us we have hamiltonian cycles C^b and C^c, such that $(\Pi C_0)^{-1}(\Pi C^b)$ is a generator of $\mathbb{Z}_p$, and $(\Pi C_0)^{-1}(\Pi C^c)$ is a generator of $\mathbb{Z}_q$. It is clear that C^b, like C_0, contains $[s_1 s_2](c^{-1})$

and $[s_1 s_2 c^{-1} a^2](a^{-1}, c, a)$, so Remark 3.2(2) provides a hamiltonian cycle C^b_c, such that $(\Pi C^b)^{-1}(\Pi C^b_c)$ is a generator of $\mathbb{Z}_q$. Therefore, Marušič's Method 2.11(2) applies (since each of these four hamiltonian cycles contains the oriented edge $[a^{-1}](a)$).

Subcase iii. *Assume* $|\overline{a}| = 3$ *and* $d = 3$. Since $d = 3$, we may assume $\overline{b} \equiv \overline{c} \pmod{\langle \overline{a} \rangle}$ (by replacing c with its inverse if necessary). Let

$$C_0 = (b^{-1}, c^{-1}, a^2, c, a^{-1}, b, a^2),$$

so C_0 is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$. Then C_0 contains the four subpaths

$$(b^{-1}),\ [b^{-1}a^2](a^{-1}, b, a), \quad [b^{-1}](c^{-1}),\ [b^{-1}c^{-1}a^{-2}](a, c, a^{-1}).$$

Therefore, we may let g be either b^{-1} or $b^{-1}c^{-1}$ in Lemma 3.1, so Remark 3.2(2) tells us we have hamiltonian cycles

$$C^b = (a, b^{-1}, a^{-1}, c^{-1}, a^2, c, b, a)$$

and

$$C^c = (b^{-1}, a^{-1}, c^{-1}, a^2, c, b, a^2),$$

such that $(\Pi C_0)^{-1}(\Pi C^b)$ is a generator of $\mathbb{Z}_p$, and $(\Pi C_0)^{-1}(\Pi C^c)$ is a generator of $\mathbb{Z}_q$. Furthermore, C^c contains the oriented paths $[ab^{-1}](b)$ and $[a^{-1}](a, b^{-1}, a^{-1})$, so, by letting $g = a$ in Lemma 3.1 (and replacing b with b^{-1}), Remark 3.2(2) tells us we have a hamiltonian cycle

$$C^c_b = (a^2, b^{-1}, c^{-1}, a^2, c, a^{-1}, b),$$

such that $(\Pi C^c)^{-1}(\Pi C^c_b)$ is a generator of $\mathbb{Z}_p$. Therefore Marušič's Method 2.11(2) applies (since all four of these hamiltonian cycles contain the oriented edge $[b^{-1}c^{-1}](a)$). □

Case 5.11. Assume $\mathbb{Z}_{p^\mu} \subseteq \langle [a, b] \rangle$, $\mathbb{Z}_{q^\nu} \subseteq \langle [a, c] \rangle$, $\overline{c} \in \langle \overline{a} \rangle$, and $\overline{b} \notin \langle \overline{a} \rangle$.

Proof. We use Marušič's Method (2.10), so there is no harm in assuming $S = \{a, b, c\}$. Furthermore, Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, so $\langle [a, b] \rangle = \mathbb{Z}_p$ and $\langle [a, c] \rangle = \mathbb{Z}_q$. Also note that, from Assumption 5.1(5), we know $\overline{c} \notin \{\overline{a}^{\pm 1}\}$, so we must have $|\overline{a}| > 3$.

Let $d = |\overline{G}/\langle \overline{a} \rangle|$. Since $\overline{c} \in \langle \overline{a} \rangle$, we have $\langle \overline{a}, \overline{b} \rangle = \overline{G}$, so (b^d) is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}/\langle \overline{a} \rangle; S)$. Choose r such that $a^r b^d \in G'$ and $0 \leq r \leq |\overline{a}| - 1$. Assume $r < |\overline{a}|/2$ (so $r \leq |\overline{a}| - 3$), by replacing b with its inverse if necessary. Then letting $k = |\overline{a}| - 3$ in Corollary 3.3 provides us with a hamiltonian cycle $C^0 = C_{|\overline{a}|-3}$.

Choose ℓ with $\overline{c} = \overline{a}^\ell$, and write $c = a^\ell \gamma$, where $\mathbb{Z}_q \subseteq \langle \gamma \rangle$. We may assume $0 \leq \ell < |\overline{a}|/2$ (by replacing c with its inverse, if necessary). Then $\ell \leq |\overline{a}| - 3$, so we see from Figure 2 that $C_{|\overline{a}|-3}$ contains the path $[a^\ell b](a^{-(\ell+1)})$. Replacing this with the path $[a^\ell b](c^{-1}, a^{\ell-1}, c^{-1})$ results in a hamiltonian cycle C^1, such that $(\Pi C^0)^{-1}(\Pi C^1)$ is a conjugate of

$$c^{-1} a^{\ell-1} c^{-1} \cdot a^{\ell+1} = (a^\ell \gamma)^{-1} a^{\ell-1} (a^\ell \gamma)^{-1} \cdot a^{\ell+1} = \gamma^{-1} (\gamma^{-1})^a.$$

Since $|G|$ is odd, we know that a does not invert any nontrivial element of G', so this is a generator of $\langle \gamma \rangle$, which contains $\langle [a, c] \rangle = \mathbb{Z}_q$.

Furthermore, from Figure 2, we see that $C_{|\overline{a}|-3}$ contains both the oriented edge $[b^{-1}a^{-1}](b)$ and the oriented path $[b^{-1}a](a^{-1}, b, a)$. Then, by construction, C^1 also contains these paths. Therefore, we may apply Lemma 3.1 with $g = b^{-1}a^{-1}$, so Remark 3.2(1) tells us we have hamiltonian cycles $\widehat{C}^0$ and $\widehat{C}^1$, such that $(\Pi C^i)^{-1}(\Pi \widehat{C}^i)$ is a generator of $\mathbb{Z}_p$. Therefore Marušič's Method 2.11(2) applies (since there are many oriented edges, such as $[a^{-1}](a^{-1})$, that are in all four hamiltonian cycles). □

Case 5.12. Assume there do not exist $a, b, c \in S$, such that $\langle [a, b], [a, c] \rangle = G'$.

Proof. Let $\underline{G} = G/(G')^{pq}$, so $\underline{G}' = \mathbb{Z}_{pq}$. The assumption of this case implies that we may partition S into two nonempty sets S_p and S_q, such that

- $\underline{S_p}$ centralizes $\underline{S_q}$ in $\underline{G}$, and
- for $r \in \{p, q\}$, and $a, b \in S_r$, we have $[\underline{a}, \underline{b}] \in \underline{\mathbb{Z}_r}$.

Let $G_p = \langle S_p \rangle$, $G_q = \langle S_q \rangle$, and $Z = \underline{G_p} \cap \underline{G_q} \subseteq Z(\underline{G})$.

Since $\underline{G}$ is not nilpotent (see Assumption 5.1(4)), we know that $\underline{G}' \not\subseteq Z(\underline{G})$. Therefore, we may assume $\mathbb{Z}_q \not\subseteq Z(\underline{G})$ (by interchanging p and q if necessary). Since $\underline{G_p} \cap \underline{G_q} \subseteq Z(\underline{G})$, this implies $\mathbb{Z}_q \not\subseteq \underline{G_p}$.

Subcase i. *Assume there exist $a_p, b_p, a_q, b_q \in S$, such that $\langle [\underline{a_p}, \underline{b_p}] \rangle = \underline{\mathbb{Z}_p}$, $\langle [\underline{a_q}, \underline{b_q}] \rangle = \underline{\mathbb{Z}_q}$, and $\{b_p, b_q\}$ is a minimal generating set of $\langle \overline{a_p}, \overline{b_p}, \overline{a_q}, \overline{b_q} \rangle / \langle \overline{a_p}, \overline{a_q} \rangle$.* We use Marušič's Method (2.10) with $S_0 = \{a_p, b_p, a_q, b_q\}$. Assume, for simplicity, that $S = S_0$. Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, so $G = \underline{G}$.

After perhaps replacing some generators with their inverses, it is easy to find:

- a hamiltonian cycle $(s_i)_{i=1}^m$ in $\mathrm{Cay}\big(\langle \overline{a_p}, \overline{a_q} \rangle; a_p, a_q\big)$, such that $s_{m-2} = a_p$ and $s_{m-1} = a_q$, and
- a hamiltonian cycle $(t_j)_{j=1}^n$ in $\mathrm{Cay}\big(\overline{G}/\langle \overline{a_p}, \overline{a_q} \rangle; b_p, b_q\big)$, such that $t_1 = b_p$ and $t_3 = b_q$.

We have the following hamiltonian cycle C_0 in $\mathrm{Cay}(G; S)$:

$$C_0 = \Big(\big((s_i)_{i=1}^{n-2}, t_{2j-1}, (s_{n-1-i}^{-1})_{i-1}^{n-2}, t_{2j} \big)_{j=1}^{(m-1)/2}, (s_i)_{i=1}^{n-1}, (t_{m-j}^{-1})_{j=1}^{m-1}, s_n \Big).$$

Much as in the proof of Lemma 3.1, we construct a hamiltonian cycle C_1 by

- replacing the oriented edge $[s_m^{-1} b_p](b_p^{-1})$ with the path $[s_m^{-1} b_p](a_q^{-1}, b_p^{-1}, a_q)$, and
- the oriented path $[s_m^{-1} a_q^{-1} a_p^{-1}](a_p, b_p, a_p^{-1})$ with $[s_m^{-1} a_q^{-1} a_p^{-1}](b_p)$.

Then there exist $g, h \in G$, such that

$$(\Pi C_0)^{-1}(\Pi C_1) = [b_p^{-1}, a_q]^g \, [a_p^{-1}, b_p]^h = e^g \cdot [a_p^{-1}, b_p]^h = [a_p^{-1}, b_p]^h,$$

which generates $\mathbb{Z}_p$.

Similarly, we may construct hamiltonian cycles C_0' and C_1' from C_0 and C_1 by

- replacing the oriented edge $[s_m^{-1} t_1 t_2 b_q](b_q^{-1})$ with the path $[s_m^{-1} t_1 t_2 b_q](a_q^{-1}, b_q^{-1}, a_q)$, and
- the oriented path $[s_m^{-1} a_q^{-1} a_p^{-1} t_1 t_2](a_p, b_q, a_p^{-1})$ with $[s_m^{-1} a_q^{-1} a_p^{-1} t_1 t_2](b_q)$.

Then, for $k \in \{0, 1\}$, essentially the same calculation shows there exist $g', h' \in G$, such that

$$(\Pi C_k)^{-1}(\Pi C'_k) = [b_q^{-1}, a_q]^{g'}[a_p^{-1}, b_q]^{h'} = [b_q^{-1}, a_q]^{g'} \cdot e^{h'} = [b_q^{-1}, a_q]^{g'},$$

which generates $\mathbb{Z}_q$.

All four hamiltonian cycles contain the oriented edge (s_1), so Marušič's Method 2.11(2) applies.

Subcase ii. *Assume $\underline{G_p}$ is not the nonabelian group of order* 27 *and exponent* 3. We will apply Marušič's Method (2.11), so Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$, which means $\underline{G} = G$.

Claim. We may assume $S_q \cap (G'Z) = \emptyset$. Suppose $a_q \in S_q \cap (G'Z)$. By the minimality of S, we know $a_q \notin G_p$. Since Z and $\mathbb{Z}_p$ are contained in G_p, this implies $G' \subseteq \langle G_p, a_q\rangle$. Therefore, the minimality of S implies that $S_q \setminus \{a_q\}$ is a minimal generating set of $\overline{G}/\langle\overline{G_p}, \overline{a_q}\rangle$. So Subcase i applies. This completes the proof of the claim.

Now, applying Proposition 4.2 to G_q tells us there exist hamiltonian cycles C_q and C'_q in $\mathrm{Cay}\big(\overline{G_q}/\overline{Z}; S_q\big)$, such that C_q and C'_q have an oriented edge in common, and $\langle(\Pi C_q)^{-1}$ $(\Pi C'_q)\rangle = \mathbb{Z}_q$.

Also, Theorem 4.1 provides hamiltonian cycles C_p and C'_p in $\mathrm{Cay}\big(\overline{G_p}; S_p\big)$, such that C_p and C'_p have an oriented edge in common, and $\langle(\Pi C_p)^{-1}(\Pi C'_p)\rangle = \mathbb{Z}_p$.

For $r \in \{p, q\}$, write $C_r = (s_{r,i})_{i=1}^{n_r}$ and $C'_r = (t_{r,i})_{i=1}^{n_r}$. Since C_r and C'_r have an edge in common, we may assume $s_{r,n_r} = t_{r,n_r}$.

Let

$$C = \Big((s_{p,i})_{i=1}^{n_p-1}, (s_{q,i})_{i=1}^{n_q-1}, \\ \big(s_{p,n_p-2i+1}^{-1}, (s_{q,n_q-j}^{-1})_{j=1}^{n_q-2}, s_{p,n_p-2i}^{-1}, (s_{q,j})_{j=2}^{n_q-1} \big)_{i=1}^{(n_p-1)/2}, s_{q,n_q} \Big). \tag{5.12A}$$

Then C is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$.

For $r \in \{p, q\}$, a path of the form $[g](s_{r,i})_{i=1}^{n_r-1}$ appears near the start of C. We obtain a new hamiltonian cycle C^r in $\mathrm{Cay}\big(\overline{G}; S\big)$ by replacing this with $[g](t_{r,i})_{i=1}^{n_r-1}$. We can also construct a hamiltonian cycle $C^{p,q}$ by making both replacements. Then

$$\langle(\Pi C)^{-1}(\Pi C^r)\rangle = \langle(\Pi C_r)^{-1}(\Pi C'_r)\rangle = \mathbb{Z}_r,$$

and

$$\langle(\Pi C^q)^{-1}(\Pi C^{p,q})\rangle = \langle(\Pi C_p)^{-1}(\Pi C'_p)\rangle = \mathbb{Z}_p,$$

so Marušič's Method 2.11(2) applies (since all four hamiltonian cycles contain the oriented edge $[s_{q,n_q}^{-1}](s_{q,n_q})$).

Subcase iii. *Assume $\underline{G_p}$ is the nonabelian group of order* 27 *and exponent* 3. We have $p = 3$, and Lemma 2.21(1) tells us $\mu = 1$; i.e., $G' = \mathbb{Z}_3 \times \mathbb{Z}_{q^\nu}$. Therefore $\underline{G} = G/(G')^q$.

Let $C_p = (s_{p,i})_{i=1}^{27}$ be a hamiltonian cycle in $\mathrm{Cay}\big(\underline{G_p}; S_p\big)$. Also, for $r = q$, Theorem 4.1 provides hamiltonian cycles $C_q = (s_{q,i})_{i=1}^{n_q}$ and $C'_q = (t_{q,i})_{i=1}^{n_q}$ in $\mathrm{Cay}(\overline{G_q}; S_q)$, such that $s_{q,n_q} = t_{q,n_q}$ and $(\Pi C_q)^{-1}(\Pi C'_q)$ generates $\mathbb{Z}_{q^\nu}$. Define the hamiltonian cycle C as in (5.12A) (with $n_p = 27$). We obtain a new hamiltonian cycle C^q in $\mathrm{Cay}\big(\overline{G}; S\big)$ by

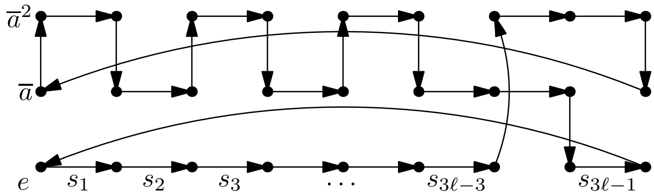

Figure 5: A hamiltonian cycle C_0.

replacing an occurrence of $(s_{q,i})_{i=1}^{n_q-1}$ with the path $(t_{q,i})_{i=1}^{n_q-1}$. Much as in Subcase ii, we have

$$\langle (\underline{\Pi C})^{-1}(\underline{\Pi C^q}) \rangle = \langle (\underline{\Pi C_q})^{-1}(\underline{\Pi C'_q}) \rangle = \underline{\mathbb{Z}_q},$$

so $\underline{\Pi C}$ and $\underline{\Pi C^q}$ cannot both be trivial. Therefore, applying the Factor Group Lemma (2.7) with $N = \mathbb{Z}_q$ provides a hamiltonian cycle in $\mathrm{Cay}(\underline{G}; S)$, and then Lemma 2.8 tells us there is a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

6 Proof of Case 5.5

In this section, we prove Case 5.5. Therefore, the following assumption is always in effect:

Assumption 6.1. Assume there exist $a, b \in S$, such that $\langle [a,b] \rangle = G'$, $|\overline{a}| = |\overline{b}| = 3$, and $\langle \overline{a} \rangle \neq \langle \overline{b} \rangle$.

The proof will consider two cases.

Case I. Assume $\#S > 2$.

Proof. Let c be a third element of S, and let $\ell = |\overline{G} : \langle \overline{a}, \overline{b} \rangle|$. (Since S is a minimal generating set, and $G' = \langle [a,b] \rangle \subseteq \langle a, b \rangle$, we must have $\ell > 1$.) We use Marušič's Method (2.10) with $S_0 = \{a, b, c\}$; assume, for simplicity, that $S = S_0$. Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$. Let

$$(s_i)_{i=1}^{3\ell} = \big((b, c, b^{-1}, c)^{(\ell-1)/2}, b^2, c^{-(\ell-1)}, b \big),$$

so $(s_i)_{i=1}^{3\ell}$ is a hamiltonian cycle in $\mathrm{Cay}\big(\overline{G}/\langle \overline{a} \rangle; b, c\big)$. Note that

$$s_1 = s_5 = b.$$

From the definition of $(s_i)_{i=1}^{3\ell}$, it is easy to see that $\overline{\prod_{i=1}^{3\ell} s_i} = \overline{b}^3 = \overline{e}$, so we have the following hamiltonian cycle C_0 in $\mathrm{Cay}(\overline{G}; a, b, c)$ (see Figure 5):

$$C_0 = \big((s_j)_{j=1}^{3\ell-3}, a^{-1}, s_{3\ell-2}, s_{3\ell-1}, a^{-1}, s_{3\ell}, \\ (a, s_{2j-1}, a^{-1}, s_{2j})_{j=1}^{3(\ell-1)/2}, s_{3\ell-2}, a^{-1}, s_{3\ell-1}, s_{3\ell} \big).$$

Since $s_1 = b$, we see that C_0 contains the oriented edge (b), and it also contains the oriented path $[a^{-2}](a, b, a^{-1})$, so Lemma 3.1 provides a hamiltonian cycle C_1, such that

$$(\Pi C_0)^{-1}(\Pi C_1) \text{ is a conjugate of } [a, b^{-1}][a, b^{-1}]^a.$$

Similarly, since $s_5 = b$ and $s_1 s_2 s_3 s_4 = c^2$, we see that C_1 contains both the oriented edge $[c^2](b)$ and the oriented path $[c^2 a^{-2}](a, b, a^{-1})$, so Lemma 3.1 provides a hamiltonian cycle C_2, such that

$$(\Pi C_1)^{-1}(\Pi C_2) \text{ is also a conjugate of } [a, b^{-1}][a, b^{-1}]^a.$$

Since no element of G inverts any nontrivial element of G' (recall that $|G|$ is odd), this implies that $(\Pi C_i)^{-1}(\Pi C_j)$ generates G' whenever $i \neq j$. So Marušič's Method 2.11(1) applies (since all three hamiltonian cycles contain the oriented edge $[s_1](s_2)$. □

Case II. Assume $\#S = 2$.

Proof. We have $S = \{a, b\}$, so $|G| = 9p^\mu q^\nu$. We may assume $p, q > 3$, for otherwise Corollary 2.15 applies (perhaps after interchanging p and q).

One very special case with a lengthy proof will be covered separately:

Assumption 6.2. Assume Proposition 6.4 below does not provide a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Under this assumption, we will always use the Factor Group Lemma (2.7) with $N = G'$, so Lemma 2.8 allows us to assume $G' = \mathbb{Z}_{pq}$.

Let

$$C = (a^{-2}, b^{-1}, a, b^{-1}, a^{-2}, b^2),$$

so C is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; a, b)$. We have

$$\Pi C = a^{-2}b^{-1}ab^{-1}a^{-2}b^2 = [a, b]^a[a, b][a, b]^b(a^{-3})^{b^2}. \tag{6.2A}$$

Let $\underline{G} = G/\mathbb{Z}_p$, so $\underline{G}' = \mathbb{Z}_q$. Since $p, q > 3$, we know $\gcd\big(|\overline{G}|, |G'|\big) = 1$, so $G \cong \overline{G} \ltimes G'$ [7, Thm. 6.2.1(i)]. Therefore $G' \cap Z(G)$ is trivial, so we may

$$\text{assume that } a \text{ does not centralize } \mathbb{Z}_q$$

(perhaps after interchanging a with b). Therefore a acts on $\mathbb{Z}_q$ via a nontrivial cube root of unity. Since the nontrivial cube roots of unity are the roots of the polynomial $x^2 + x + 1$, this implies that $\underline{[a, b]^{a^2}[a, b]^a[a, b]} = \underline{e}$, so

$$\underline{[a, b]^a[a, b]} = (\underline{[a, b]^{a^2}})^{-1} = (\underline{[a, b]^{a^{-1}}})^{-1}$$

(since $|\overline{a}| = 3$). Furthermore, $\underline{a^{-3}} = \underline{e}$ (since a has trivial centralizer in $\mathbb{Z}_q$). Hence,

$$\begin{aligned} \underline{\Pi C} &= \underline{[a, b]^a}\ \underline{[a, b]}\ \underline{[a, b]^b}(\underline{a}^{-3})^{b^2} \\ &= (\underline{[a, b]}^{a^{-1}})^{-1}\ \underline{[a, b]}^b\ \underline{e} \\ &= (\underline{[a, b]}^{a^{-1}})^{-1}\ \underline{[a, b]}^b. \end{aligned}$$

Therefore

$$\underline{\Pi C} \neq \underline{e} \text{ unless } y^b = y^{a^{-1}} \text{ for all } y \in \mathbb{Z}_q. \tag{6.2B}$$

Hence, we may assume $\langle \Pi C \rangle$ contains $\mathbb{Z}_q$ (by replacing b with its inverse if necessary).

Subcase i. *Assume a centralizes $\mathbb{Z}_p$.* Since $G' \cap Z(G)$ is trivial, we know that b does not centralize $\mathbb{Z}_p$. Also, we may assume $\langle \Pi C \rangle \neq G'$, for otherwise the Factor Group Lemma (2.7) applies. Therefore ΠC must project trivially to $\mathbb{Z}_p$. Fixing $r, k \in \mathbb{Z}$ with

$$[a,b]^b = [a,b]^r \text{ and } a^{-3} = [a,b]^k$$

(and using the fact that $r^2 + r + 1 \equiv 0 \pmod p$), we see from (6.2A) that this means

$$0 \equiv 1 + 1 + r + kr^2 \equiv 1 - r^2 + kr^2 \equiv r^2(r - 1 + k) \pmod p,$$

so

$$k \equiv 1 - r \pmod p.$$

Therefore $k \not\equiv 0 \pmod p$ (since r is a primitive cube root of unity). Also, since a centralizes $\mathbb{Z}_p$, we have

$$\begin{aligned}[a^{-1}, b^{-1}]^{-kr} \equiv \left([a, b^{-1}]^{-1}\right)^{-kr} = \left([a,b]^{b^{-1}}\right)^{-kr} = [a,b]^{-k} \\ = a^3 = (a^{-1})^{-3} \pmod{\mathbb{Z}_q}.\end{aligned}$$

Therefore, replacing a and b with their inverses replaces k with $-kr$ (modulo p), and it obviously replaces r with r^2. Hence, we may assume that we also have

$$-kr \equiv 1 - r^2 \equiv r^3 - r^2 = -(1-r)r^2 \equiv -kr^2 \pmod p,$$

so $r \equiv 1 \pmod p$. This contradicts the fact that b does not centralize $\mathbb{Z}_p$.

Subcase ii. *Assume a does not centralize $\mathbb{Z}_p$.* We may assume that the preceding subcase does not apply when a and b are interchanged (and perhaps p and q are also interchanged). Therefore, we may assume that either

- b centralizes both $\mathbb{Z}_p$ and $\mathbb{Z}_q$, in which case, interchanging p and q in (6.2B) tells us that ΠC projects nontrivially to both $\mathbb{Z}_p$ and $\mathbb{Z}_q$, so the Factor Group Lemma (2.7) applies, or
- b has trivial centralizer in G'.

Henceforth, we assume a and b both have trivial centralizer in G'.

We may assume $y^b = y^a$ for $y \in \mathbb{Z}_q$, by replacing b with its inverse if necessary. We may also assume $\langle \Pi C \rangle \neq G'$ (for otherwise the Factor Group Lemma (2.7) applies). Since $\langle \Pi C \rangle$ contains $\mathbb{Z}_q$, this means that $\langle \Pi C \rangle$ does not contain $\mathbb{Z}_p$. By interchanging p and q in (6.2B), we conclude that $x^b = x^{a^{-1}}$ for $x \in \mathbb{Z}_p$. We are now in the situation where a hamiltonian cycle in $\mathrm{Cay}(G; a, b)$ is provided by Proposition 6.4 below. □

The remainder of this section proves Proposition 6.4, by applying the Factor Group Lemma (2.7) with $N = \mathbb{Z}_{q^\nu}$. To this end, the following lemma provides a hamiltonian cycle in $\mathrm{Cay}\big(G/\mathbb{Z}_{q^\nu}; S\big)$.

Lemma 6.3. Assume

- $G = \mathbb{Z}_{p^\mu} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle x \rangle \rtimes \big(\langle a \rangle \times \langle b_0 \rangle\big)$, with $p > 3$,
- $b = x b_0$,
- $x^b = x^{a^{-1}} = x^r$, where r is a primitive cube root of unity in $\mathbb{Z}_{p^\mu}$,

- $k \in \mathbb{Z}$, such that
  - $k \equiv 1 \pmod{3}$,
  - $k \equiv r \pmod{p^\mu}$, and
  - $0 \le k < 3p^\mu$,
- ℓ is the multiplicative inverse of k, modulo $3p^\mu$ (and $0 \le \ell < 3p^\mu$),
- $C = \big(a, b^{-2}, (a^{-1}, b^2)^{k-1}, a^{-2}, b^2, (a, b^{-2})^{\ell-k-1}, a^{-2}, (b^{-2}, a)^{3p^\mu-\ell-1}\big)$, and
- $\widetilde{C}$ is the walk obtained from C by interchanging a and b, and also interchanging k and ℓ.

Then either C or $\widetilde{C}$ is a hamiltonian cycle in $\mathrm{Cay}(G; a, b)$.

Proof. Define

$$v_{2i+\epsilon} = (ba)^i b^\epsilon \qquad \text{for } \epsilon \in \{0,1\},$$
$$w_j = (ba)^j b^{-1},$$

and let $V = \{v_i\}$ and $W = \{w_j\}$. Note that, since $x^{ab} = x$, we have $|ab| = 3p^\mu$, so $\#V = 6p^\mu$ and $\#W = 3p^\mu$, so G is the disjoint union of V and W. With this in mind, it is easy to see that $C_1 = (b^{-2}, a)^{3p^\mu}$ is a hamiltonian cycle in $\mathrm{Cay}(G; a, b)$.

Removing the edges of the subpaths (b^{-2}) and $[(ba)^k](b^{-2}, a, b^{-2})$ from C_1 results in two paths:

- path P_1 from $b^{-2} = b$ to $(ba)^k$, and
- path P_2 from $(ba)^{k+1}b$ to e (since $(ba)^k(b^{-2}ab^{-2}) = (ba)^k(bab) = (ba)^{k+1}b$).

The union of P_1 and P_2 covers all the vertices of G *except* the interior vertices of the removed subpaths, namely,

all vertices except b^{-1}, $(ba)^k b^{-1}$, $(ba)^k b$, $(ba)^{k+1}$, and $(ba)^{k+1}b^{-1}$.

By ignoring y in calculation (6.4A) below, we see that $b^{-1}a^{-1} = (a^{-1}b^{-1})^k$, which means

$$ab = (ba)^k.$$

Since $b^{-2} = b$, this implies

$$ab^{-2} = (ba)^k.$$

Also, since $a^{-1} = a^2$, we have

$$ba^{-1}b^2 = ba^2b^2 = (ba)(ab)b = (ba)\big((ba)^k\big)b = (ba)^{k+1}b.$$

Therefore

$$Q_1 = (a, b^{-2}) \text{ is a path from the end of } P_2 \text{ to the end of } P_1,$$

and

$$Q_2 = [b](a^{-1}, b^2) \text{ is a path from the start of } P_1 \text{ to the start of } P_2.$$

So, letting $-P_1$ be the reverse of the walk P_1, we see that

$$C_2 = Q_1 \cup -P_1 \cup Q_2 \cup P_2$$

is a closed walk.

Note that the interior vertices of Q_1 are

$$a = (ab)b^{-1} = (ba)^k b^{-1}$$

and

$$ab^{-1} = (ab)b = (ba)^k b,$$

and the interior vertices of Q_2 are

$$ba^{-1} = ba^2 = (ba)(ab)b^{-1} = (ba)(ba)^k b^{-1} = (ba)^{k+1} b^{-1}$$

and

$$ba^{-1}b = \big((ba)^{k+1}b^{-1}\big)b = (ba)^{k+1}.$$

These are all but one of the vertices that are not in the union of P_1 and P_2, so

$$C_2 \text{ is a cycle that covers every vertex except } b^{-1}.$$

Notice that the only a-edge removed from C_1 is $[(ba)^k b^{-2}](a) = [(ba)^k b](a)$. Since

$$k^2 \equiv (r^2)^2 = r^4 \equiv r \not\equiv 1 \pmod{p^\mu},$$

and ℓ is the multiplicative inverse of k, modulo $3p^\mu$, we know $k \neq \ell$, so this removed edge is not equal to $[(ba)^\ell b](a)$. Therefore $[(ba)^\ell b](a)$ is an edge of C_2. Now, we create a walk C^* by removing this edge from C_2, and replacing it with the path $[(ba)^\ell b](a^{-2})$. Since

$$(ab)^\ell = \big((ba)^k\big)^\ell = (ba)^{k\ell} = ba,$$

we see that the interior vertex of this path is

$$[(ba)^\ell b]a^{-1} = [b(ab)^\ell]a^{-1} = [b(ba)]a^{-1} = b^2 = b^{-1}.$$

Therefore C^* covers every vertex, so it is a hamiltonian cycle.

Since $ab = (ba)^k$ and $ba = (ab)^\ell$, it is obvious that interchanging a and b will also interchange k and ℓ. Therefore, we may assume $k < \ell$, by interchanging a and b if necessary. Then the edge $[(ba)^\ell b](a)$ is in P_2, rather than being in P_1. If we let P_2' be the path obtained by removing this edge from P_2, and replacing it with $[(ba)^\ell b](a^{-2})$, then we have

$$\begin{aligned} C &= \big((a, b^{-2}),\ \ (a^{-1}, b^2)^{k-1}, a^{-1},\ \ (a^{-1}, b^2),\ \ (a, b^{-2})^{\ell-k-1}, a^{-2}, (b^{-2}, a)^{3p^\mu - \ell - 1} \big) \\ &= Q_1 \cup -P_1 \cup Q_2 \cup P_2' \\ &= C^* \end{aligned}$$

is a hamiltonian cycle in $\mathrm{Cay}(G; a, b)$. □

Proposition 6.4. Assume

- $\overline{G} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$,
- $G' = \mathbb{Z}_{p^\mu} \times \mathbb{Z}_{q^\nu}$, with $p \neq q$ and $p, q > 3$,
- $S = \{a, b\}$ has only two elements,
- a and b have trivial centralizer in G', and

- ab centralizes $\mathbb{Z}_{p^\mu}$ and ab^{-1} centralizes $\mathbb{Z}_{q^\nu}$.

Then $\operatorname{Cay}(G; a, b)$ has a hamiltonian cycle.

Proof. Since $\gcd\bigl(|\overline{G}|, |G'|\bigr) = 1$, we have

$$G \cong G' \rtimes \overline{G} \cong (\mathbb{Z}_{p^\mu} \times \mathbb{Z}_{q^\nu}) \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_3).$$

Write $\mathbb{Z}_{p^\mu} = \langle x \rangle$ and $\mathbb{Z}_{q^\nu} = \langle y \rangle$. Since a does not centralize any nontrivial element of G', we may assume $a \in \mathbb{Z}_3 \times \mathbb{Z}_3$ (after replacing it by a conjugate). Write $b = \gamma b_0$, with $\gamma \in G'$ and $b_0 \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\langle a, b \rangle = G$, we must have $\langle \gamma \rangle = G'$, so we may assume $\gamma = xy$; therefore $b = xyb_0$.

Choose $r \in \mathbb{Z}$ with $x^{a^{-1}} = x^r$. Since $|a| = 3$ and a does not centralize any nontrivial element of $\mathbb{Z}_{p^\mu}$, we know that r is a primitive cube root of unity, modulo p^μ. Also, since ab centralizes $\mathbb{Z}_{p^\mu}$, we have $x^b = x^r$.

Define k and ℓ as in Lemma 6.3. Then, letting $\underline{G} = G/\mathbb{Z}_{q^\nu}$ (and perhaps interchanging a with b), Lemma 6.3 tells us that

$$C = \bigl(a, b^{-2}, (a^{-1}, b^2)^{k-1}, a^{-2}, b^2, (a, b^{-2})^{\ell-k-1}, a^{-2}, (b^{-2}, a)^{3p^\mu-\ell-1}\bigr)$$

is a hamiltonian cycle in $\operatorname{Cay}\bigl(\underline{G}; a, b\bigr)$.

To calculate the voltage of C, choose $s \in \mathbb{Z}$ with $y^a = y^s$, and let

$$y_1 = y^{s^2-(1+s+s^2+\cdots+s^{k-1})} = y^{s^2-1}$$

(since $1 + s + s^2 \equiv 0 \pmod{q}$ and $k \equiv 1 \pmod{3}$), and note that

$$\begin{aligned}
(a^{-1}b^{-1})^k &= \bigl(a^{-1}(xyb_0)^{-1}\bigr)^k && \text{(6.4A)}\\
&= \bigl(a^{-1}b_0^{-1}y^{-1}x^{-1}\bigr)^k \\
&= x^{-k}\bigl(a^{-1}b_0^{-1}y^{-1}\bigr)^k && (x \text{ commutes with } a^{-1}b_0^{-1} \text{ and } y)\\
&= x^{-r}\bigl(a^{-1}b_0^{-1}\bigr)^k y^{-(1+s+s^2+\cdots+s^{k-1})} && \left(\begin{matrix} k \equiv r \pmod{p^\mu} \text{ and} \\ y^{a^{-1}b_0^{-1}} = y^{a^2b_0^2} = y^{s^4} = y^s \end{matrix}\right)\\
&= x^{-r}b_0^{-1}a^{-1}y^{-s^2}y_1 && \left(\begin{matrix} a \text{ and } b_0 \text{ commute, } k \equiv 1 \\ \pmod{3}, \text{ and definition of } y_1 \end{matrix}\right)\\
&= b_0^{-1}x^{-1}y^{-1}a^{-1}y_1 && (x^r = x^{b_0} \text{ and } y^{s^2} = y^{a^2} = y^{a^{-1}})\\
&= b^{-1}a^{-1}y^{s^2-1} && (b = xyb_0 \text{ and } y_1 = y^{s^2-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Pi C &= ab^{-2}(a^{-1}b^2)^{k-1}a^{-2}b^2(ab^{-2})^{\ell-k-1}a^{-2}(b^{-2}a)^{3p^\mu-\ell-1} \\
&= ab(a^{-1}b^{-1})^{k-1}ab\big(b(ab)^{\ell-k-1}a\big)(ba)^{3p^\mu-\ell-1} && (|a|=|b|=3) \\
&= ab(a^{-1}b^{-1})^{k}(a^{-1}b^{-1})^{-1}ab\big(ba\big)^{\ell-k}(ba)^{-\ell-1} && (|ba|=3p^\mu) \\
&= ab(a^{-1}b^{-1})^{k}(ba)ab(ba)^{-k}(ba)^{-1} \\
&= ab\big(b^{-1}a^{-1}y^{s^2-1}\big)ba^2b\big(b^{-1}a^{-1}y^{s^2-1}\big)(a^{-1}b^{-1}) && \left(\begin{aligned}(ba)^{-k} &= (a^{-1}b^{-1})^k \\ &= b^{-1}a^{-1}y^{s^2-1}\end{aligned}\right) \\
&= y^{s^2-1}bay^{s^2-1}a^{-1}b^{-1} \\
&= y^{s^2-1}y^{(s^2-1)s} && \left(\begin{aligned}y^{a^{-1}b^{-1}} &= y^{a^2b^2} \\ &= y^{s^4} = y^s\end{aligned}\right) \\
&= y^{(s^2-1)(1+s)}.
\end{aligned}$$

Since s is a primitive cube root of unity modulo q^ν, we know $s \not\equiv \pm 1 \pmod{q}$. Therefore, the exponent of y is not divisible by q, which means $\Pi C \notin \langle y^q \rangle$, so ΠC generates $\mathbb{Z}_{q^\nu}$. Hence, the Factor Group Lemma (2.7) provides the desired hamiltonian cycle in $\mathrm{Cay}(G; a, b)$. □

References

[1] B. Alspach, Lifting Hamilton cycles of quotient graphs, *Discrete Math.* **78** (1989), 25–36. MR 1020643

[2] S. J. Curran and J. A. Gallian, Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey, *Discrete Math.* **156** (1996), 1–18. MR 1405010

[3] E. Durnberger, Connected Cayley graphs of semidirect products of cyclic groups of prime order by abelian groups are Hamiltonian, *Discrete Math.* **46** (1983), 55–68. MR 0708162

[4] E. Durnberger, Every connected Cayley graph of a group with prime order commutator group has a Hamilton cycle, in: B. Alspach, C. Godsil (eds.), *Cycles in Graphs (Burnaby, B.C., 1982),* North-Holland, Amsterdam, 1985, 75–80. MR 0821506

[5] E. Ghaderpour and D. W. Morris, Cayley graphs of order $27p$ are hamiltonian, *Internat. J. Comb.* **2011** (2011), Article ID 206930, 16 pages. MR 2822405

[6] E. Ghaderpour and D. W. Morris, Cayley graphs on nilpotent groups with cyclic commutator subgroup are hamiltonian, *Ars Contemp. Math.* **7** (2014), 55–72. MR 3029452

[7] D. Gorenstein, *Finite Groups,* Chelsea, New York, 1980. MR 0569209

[8] M. J. Hall, *The Theory of Groups,* Macmillan, New York, 1959. MR 0103215

[9] B. Huppert, *Endliche Gruppen I,* Springer, Berlin, 1967. MR 0224703

[10] K. Keating and D. Witte, On Hamilton cycles in Cayley graphs with cyclic commutator subgroup, in: B. R. Alspach, C. D. Godsil (eds.), *Cycles in Graphs (Burnaby, B.C., 1982)*, North-Holland, Amsterdam, 1985, 89–102. MR 0821508

[11] K. Kutnar, D. Marušič, J. Morris, D. W. Morris, and P. Šparl, Hamiltonian cycles in Cayley graphs whose order has few prime factors, *Ars Math. Contemp.* **5** (2012), 27–71. MR 2853700

[12] D. Marušič, Hamiltonian circuits in Cayley graphs, *Discrete Math.* **46** (1983), 49–54. MR 0708161

[13] I. Pak and R. Radoičić, Hamiltonian paths in Cayley graphs, *Discrete Math.* **309** (2009), 5501–5508. MR 2548568

[14] D. Witte, Cayley digraphs of prime-power order are Hamiltonian, *J. Combin. Theory Ser. B* **40** (1986), 107–112. MR 0830597

[15] D. Witte and J. A. Gallian, A survey: Hamiltonian cycles in Cayley graphs, *Discrete Math.* **51** (1984), 293–304. MR 0762322

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Arc-transitive graphs of valency 8 have a semiregular automorphism*

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Abstract

One version of the polycirculant conjecture states that every vertex-transitive graph has a non-identity semiregular automorphism that is, a non-identity automorphism whose cycles all have the same length. We give a proof of the conjecture in the arc-transitive case for graphs of valency 8, which was the smallest open valency.

Keywords: Arc-transitive graphs, polycirculant conjecture, semiregular automorphism.

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1 Introduction

All graphs considered in this paper are finite and simple. An automorphism of a graph Γ is a permutation of the vertex-set $\mathrm{V}(\Gamma)$ which preserves the adjacency relation. The set of automorphisms of Γ forms a group, denoted $\mathrm{Aut}(\Gamma)$. A graph Γ is said to be *G-vertex-transitive* if G is a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on $\mathrm{V}(\Gamma)$. Similarly, Γ is said to be *G-arc-transitive* if G acts transitively on the arcs of Γ. (An *arc* is an ordered pair of adjacent vertices). When $G = \mathrm{Aut}(\Gamma)$, the prefix G in the above notation is sometimes omitted.

A group G acting on a set Ω is *semiregular* on Ω if the stabiliser G_ω is trivial for every $\omega \in \Omega$; note that this implies that the action is faithful. An element $g \in G$ is called *semiregular* provided that it generates a semiregular group. Equivalently, an element $g \in G$ is semiregular if $\langle g \rangle$ acts faithfully on Ω and that all of the cycles of the permutation induced by g have the same length.

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In 1981, Marušič conjectured [7] that every vertex-transitive graph with at least two vertices has a non-identity semiregular automorphism. This is sometimes called the *polycirculant conjecture*. While there has been a lot of work on this conjecture and some of its variants, it is still wide open. There has been progress in some directions (see [6] for a survey). For example, the conjecture has been settled for graphs of certain orders [2, 7, 8] and for graphs of valency at most four [1, 8].

In the arc-transitive case, slightly more can be said, but we first need a few definitions. If Γ is a graph and $G \leq \mathrm{Aut}(\Gamma)$ then $G_v^{\Gamma(v)}$ denotes the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ of the vertex v. A permutation group is called *quasiprimitive* if each of its nontrivial normal subgroup is transitive, while a vertex-transitive graph Γ is called *locally-quasiprimitive* if $\mathrm{Aut}(\Gamma)_v^{\Gamma(v)}$ is quasiprimitive. Using this terminology, we have the following theorem due to Giudici and Xu.

Theorem 1.1. *[5, Theorem 1.1] A locally-quasiprimitive vertex-transitive graph has a non-identity semiregular automorphism.*

A transitive group of prime degree is clearly quasiprimitive (in fact primitive) and hence it follows from Theorem 1.1 that arc-transitive graphs of prime valency have a non-identity semiregular automorphism. In an upcoming paper [4], the author and Giudici deal with the case of arc-transitive graphs of valency twice a prime. The main result of this paper is to prove the polycirculant conjecture for arc-transitive graphs of valency 8, the smallest open valency.

Theorem 1.2. *An arc-transitive graph of valency* 8 *has a non-identity semiregular automorphism.*

The prime 2 plays a special role in our proof. In some sense, we use the fact that any proper factorisation of 8 must include 2 as one of the factors. In particular, if Γ is a G-arc-transitive graph of valency 8 and N is a normal subgroup of G that is not semiregular and that has at least three orbits then the quotient graph Γ/N is an arc-transitive graph of valency either 2 or 4. In the former case, we can view Γ/N as an asymmetric arc-transitive digraph of out-valency 4, which turns out to be very useful (see Theorem 2.7). In the latter case, the valency of Γ is exactly twice the valency of Γ/N which gives us very precious information about the kernel of the action of G on N-orbits (see the proof of Theorem 1.2).

This special role of the prime 2 is also evident in the proof of the case of valency twice a prime [4]. In the more general case of valency a product of two primes, Xu [13] has made significant progress and it seems the bottleneck is the case when G is solvable. (See [3] for some results in this case.)

2 Preliminaries

2.1 Permutation groups

We start with a few very basic lemmas about permutation groups. Since the proofs are short, we include them for the sake of completeness.

Lemma 2.1. *Let p be a prime and let G be a transitive permutation group of degree a power of p. Then G contains a non-identity semiregular element.*

Proof. Let S be a Sylow p-subgroup of G. By [12, Theorem 3.4'], S is transitive. Let z be a non-identity element of the centre of S. If z fixes a point then it must fix all of them, which is a contradiction. It follows that z is semiregular. □

Lemma 2.2. *If G is a transitive permutation group with a non-trivial abelian normal subgroup N that has at most two orbits, then N contains a non-identity semiregular element.*

Proof. If N is semiregular then we are done. We may thus assume that there exists a non-identity element $n \in N$ fixing some point v. Since N is abelian, n fixes the N-orbit v^N pointwise. It follows that N has exactly two orbits. Let u be a point not in v^N and let $g \in G$ such that $v^g = u$. Then n acts semiregularly on u^N, while n^g fixes u^N pointwise and acts semiregularly on v^N. It follows that nn^g is a non-identity semiregular element of N. □

Lemma 2.3. *Let G be a permutation group and let K be a normal subgroup of G such that G/K acts faithfully on the K-orbits. If G/K contains a semiregular element gK of order r coprime to $|K|$ then G contains a semiregular element of order r.*

Proof. There exists $k \in K$ such that $g^r = k$. Let $h = g^{|k|}$. Since $|k|$ and r are coprime, it follows that $|h| = r$ and hK is a non-identity power of gK hence it is semiregular. We now show that h is semiregular. Suppose h^i fixes a point v for some integer i. Then $h^iK = (hK)^i$ fixes v^K and, since hK is semiregular, $h^iK = K$. This implies that $h^i \in K$. Since h has order coprime to $|K|$, it follows that $h^i = 1$. □

Lemma 2.4. *A transitive permutation group of degree 8 is either primitive or it is a $\{2, 3\}$-group.*

Proof. Let G be a transitive permutation group of degree 8 that is not primitive. Then G has a non-trivial system of imprimitivity, with blocks of size either 2 or 4. It follows that G is isomorphic to a subgroup of $\mathrm{Sym}(4) \wr \mathrm{Sym}(2)$ or $\mathrm{Sym}(2) \wr \mathrm{Sym}(4)$ and hence it is a $\{2, 3\}$-group. □

We also need the following folklore lemma

Lemma 2.5. *Let p be a prime, let Γ be a connected graph and let $G \le \mathrm{Aut}(\Gamma)$. If, for every $v \in \mathrm{V}(\Gamma)$, $|G_v^{\Gamma(v)}|$ is coprime to p, then so is $|G_u|$ for every $u \in \mathrm{V}(\Gamma)$.*

Proof. We prove this by contradiction. Let p be a prime dividing $|G_u|$ for some $u \in \mathrm{V}(\Gamma)$ and let $g \in G_u$ have order p. Since g is non-trivial, it must move some vertex. Let w be a vertex of Γ moved by g at minimal distance from u. By the connectivity of Γ, there is a path $u, u_1, \ldots, u_t, w$ such that g fixes each u_i. Then $g \in G_{u_t}$ and g acts nontrivially on $\Gamma(u_t)$. Thus p divides $|G_{u_t}^{\Gamma(u_t)}|$ and the result follows. □

2.2 Quotient graphs

We will need some basic facts about quotient graphs, which we now collect. (For a reference, see [11] for example.)

Let Γ be a graph and let $N \le \mathrm{Aut}(\Gamma)$. The *quotient graph* Γ/N is the graph whose vertices are the N-orbits with two such N-orbits v^N and u^N adjacent whenever there is a pair of vertices $v' \in v^N$ and $u' \in u^N$ that are adjacent in Γ. Clearly, if Γ is connected then so is Γ/N.

Let Γ be a G-arc-transitive graph, let $N \trianglelefteq G$ and let K be the kernel of the action of G on N-orbits. Then $K = NK_v \trianglelefteq G$, $G/K \le \mathrm{Aut}(\Gamma/N)$ and Γ/N is G/K-arc-transitive. If N has at least 3 orbits then the valency of Γ/N is at least 2 and divides the valency of Γ. If Γ/N has the same valency as Γ then $N = K$ and K is semiregular.

Some of these facts are used in the proof of the following lemma.

Lemma 2.6. *Let p be an odd prime and let Γ be a connected 4-valent G-arc-transitive graph such that p divides $|\mathrm{V}(\Gamma)|$. If G is solvable then G contains a semiregular element of order p.*

Proof. The proof goes by induction on $|\mathrm{V}(\Gamma)|$. Let $v \in \mathrm{V}(\Gamma)$. If $G_v^{\Gamma(v)}$ is a 2-group then it follows from Lemma 2.5 that G_v is a 2-group and every element of G of order p is semiregular. We thus assume that $G_v^{\Gamma(v)}$ is not a 2-group and, since it is a transitive group of degree 4, it is 2-transitive. Since G is solvable, it contains a non-trivial normal elementary abelian q-group N. If N has at most two orbits then $p = q$ and the result follows from Lemma 2.2.

From now on, we assume that N has at least three orbits. Since $G_v^{\Gamma(v)}$ is 2-transitive, this implies that Γ/N is 4-valent and hence N is semiregular and G/N acts faithfully on Γ/N. If $p = q$ then N contains a semiregular element of order p and the conclusion holds. Suppose now that $p \neq q$. Then Γ/N is a connected, 4-valent G/N-arc-transitive graph. Since p is coprime to $|N|$, it divides $|\mathrm{V}(\Gamma/N)|$. By the induction hypothesis, G/N contains a semiregular element of order p. The result then follows from Lemma 2.3. □

2.3 Digraphs

Finally, we will need a few notions about digraphs. We follow the terminology of [10] closely. A *digraph* $\vec{\Gamma}$ consists of a finite non-empty set of *vertices* $\mathrm{V}(\vec{\Gamma})$ and a set of *arcs* $\mathrm{A}(\vec{\Gamma}) \subseteq V \times V$, which is an arbitrary binary relation on V. A digraph $\vec{\Gamma}$ is called *asymmetric* provided that the relation $\mathrm{A}(\vec{\Gamma})$ is asymmetric.

If (u, v) is an arc of $\vec{\Gamma}$ then we say that v is an *out-neighbour* of u and that u is an *in-neighbour* of v. The symbols $\vec{\Gamma}^+(v)$ and $\vec{\Gamma}^-(v)$ will denote the set of out-neighbors of v and the set of in-neighbors of v, respectively. We also say that u is the *tail* and v the *head* of (u, v), respectively. The digraph $\vec{\Gamma}$ is said to be of out-valence k if $|\vec{\Gamma}^+(v)| = k$ for every $v \in \mathrm{V}(\vec{\Gamma})$.

An automorphism of $\vec{\Gamma}$ is a permutation of $\mathrm{V}(\vec{\Gamma})$ which preserves the relation $\mathrm{A}(\vec{\Gamma})$. The set of automorphisms of $\vec{\Gamma}$ forms a group, denoted $\mathrm{Aut}(\vec{\Gamma})$. We say that $\vec{\Gamma}$ is arc-transitive provided that $\mathrm{Aut}(\vec{\Gamma})$ acts transitively on $\mathrm{A}(\vec{\Gamma})$.

We say that two arcs a and b of $\vec{\Gamma}$ are *related* if they have a common tail or a common head. Let R denote the transitive closure of this relation. The *alternet* of $\vec{\Gamma}$ (with respect to a) is the subdigraph of $\vec{\Gamma}$ induced by the R-equivalence class $R(a)$ of the arc a. (i.e. the digraph with vertex-set consisting of all heads and tails of arcs in $R(a)$ and whose arc-set is $R(a)$). If the alternet with respect to (u, v) contains an arc of the form (v, w) then this alternet is called *degenerate*.

If the alternet of the arc (u, v) is non-degenerate then it is a connected bipartite digraph where the first bipartition set consists only of sources while the second bipartition set contains only sinks. An important case occurs when this alternet is in fact a complete bipartite digraph in which case we will simply say that the alternet is *complete bipartite*. We say that $\vec{\Gamma}$ is *loosely attached* if $\vec{\Gamma}$ has no degenerate alternets and the intersection of the set of sinks of one alternet intersects the set of sources of another alternet in at most one vertex.

We define the *digraph of alternets* $\mathrm{Al}(\vec{\Gamma})$ of $\vec{\Gamma}$ as the digraph the vertices of which are the alternets of $\vec{\Gamma}$ and with two alternets A and B forming an arc (A, B) of $\mathrm{Al}(\vec{\Gamma})$ whenever the intersection of the set of sinks of A with the set of sources of B is non-empty.

We are now ready to prove the following theorem.

Theorem 2.7. *Let $\vec{\Gamma}$ be a connected asymmetric arc-transitive digraph of out-valence 4. Then $\vec{\Gamma}$ has a non-identity semiregular automorphism.*

Proof. The proof goes by induction on $|\mathrm{V}(\vec{\Gamma})|$. Since $\vec{\Gamma}$ is connected and arc-transitive (and finite), it follows that it is vertex-transitive and strongly connected (see for example [9, Lemma 2]).

Let $v \in \mathrm{V}(\vec{\Gamma})$ and let $G = \mathrm{Aut}(\vec{\Gamma})$. Without loss of generality, we may assume that every prime that divides $|G|$ also divides $|G_v|$. If G_v is a 2-group then the conclusion follows from Lemma 2.1. We may thus assume that G_v is not a 2-group and hence neither is $G_v^{\Gamma^+(v)}$. Since $G_v^{\Gamma^+(v)}$ is a transitive permutation group of degree 4, it is 2-transitive.

Since $\vec{\Gamma}$ has out-valence 4, $G_v^{\Gamma^+(v)}$ is a $\{2,3\}$-group, hence so are G_v and G and therefore G is solvable by Burnside's Theorem. It follows that G has an abelian minimal normal subgroup N. If N is semiregular then the conclusion holds. We may thus assume that N is not semiregular and hence $N_v^{\Gamma^+(v)}$ is a non-trivial normal subgroup of $G_v^{\Gamma^+(v)}$ and therefore is transitive. The same argument yields that $N_v^{\Gamma^-(v)}$ is also transitive.

Let (u,v) be an arc of $\vec{\Gamma}$. We have just seen that $N_u^{\Gamma^+(u)}$ and $N_v^{\Gamma^-(v)}$ are both transitive. This implies that $\Gamma^+(u) \subseteq v^N$ and $\Gamma^-(v) \subseteq u^N$. On the other hand, N is abelian and hence N_u fixes u^N pointwise and N_v fixes v^N pointwise. It follows that the alternet of $\vec{\Gamma}$ with respect to (u,v) is not degenerate and is complete bipartite.

If $\vec{\Gamma}$ is not loosely attached then it follows that, for every vertex x, there exists at least one other vertex y such that $\vec{\Gamma}^-(x) = \vec{\Gamma}^-(y)$ and $\vec{\Gamma}^+(x) = \vec{\Gamma}^+(y)$. It follows easily that $\vec{\Gamma}$ has a non-identity semiregular automorphism in this case.

We may thus assume that $\vec{\Gamma}$ is loosely attached. Let $\vec{\Gamma}' = \mathrm{Al}(\vec{\Gamma})$ be the digraph of alternet of $\vec{\Gamma}$. It follows from [10, Lemmas 3.1-3.3] that $\vec{\Gamma}'$ is a connected asymmetric digraph of out-valence 4 and that $G = \mathrm{Aut}(\vec{\Gamma}')$. It also follows easily that an automorphism which is semiregular on $\vec{\Gamma}'$ is also semiregular on $\vec{\Gamma}$.

Note that $|\mathrm{V}(\vec{\Gamma}')| = |\mathrm{V}(\vec{\Gamma})|/4$ and hence we may apply the induction hypothesis to $\vec{\Gamma}'$ to conclude that it has a non-identity semiregular automorphism g. Together with the observation in the previous sentence, this concludes the proof. □

3 Proof of Theorem 1.2

Let Γ be an arc-transitive graph of valency 8. We must show that Γ has a non-identity semiregular automorphism.

Clearly, we may assume that Γ is connected. Let $G = \mathrm{Aut}(\Gamma)$. If Γ is locally-quasiprimitive then the result follows from Theorem 1.1. We therefore assume that $G_v^{\Gamma(v)}$ is not quasiprimitive. By Lemma 2.4, $G_v^{\Gamma(v)}$ is a $\{2,3\}$-group and hence, by Lemma 2.5, so is G_v. We may also assume that G itself is a $\{2,3\}$-group and hence it is solvable by Burnside's Theorem.

It follows that G has an elementary abelian minimal normal subgroup N. If N has at most two orbits then the result follows from Lemma 2.2. We may thus assume that N has at least three orbits and, in particular, $N_v^{\Gamma(v)}$ is intransitive. If N is semiregular then the conclusion follows. We may thus assume that $N_v \neq 1$ and hence $N_v^{\Gamma(v)} \neq 1$. In particular, Γ/N has valency strictly less than 8 and $N_v^{\Gamma(v)}$ is a non-trivial, intransitive normal subgroup of $G_v^{\Gamma(v)}$. Since the orbits of $N_v^{\Gamma(v)}$ are blocks of $G_v^{\Gamma(v)}$ it follows that $N_v^{\Gamma(v)}$ is a 2-group and hence N is an elementary abelian 2-group. Let K be the kernel of the action of G on N-orbits.

If Γ/N is 4-valent then v is adjacent to at most two vertices from any N-orbit. It follows that $K_v^{\Gamma(v)}$ is a 2-group and hence so are K_v and K. If $|\mathrm{V}(\Gamma/N)|$ is a power of 2 then so is $|\mathrm{V}(\Gamma)|$ and the result follows from Lemma 2.1. We may thus assume that $|\mathrm{V}(\Gamma/N)|$ is not a power of 2 and, by Lemma 2.6, G/K contains a semiregular element of odd order. It then follows from Lemma 2.3 that G contains a semiregular element of odd order.

It remains to deal with the case when Γ/N is 2-valent. In this case, there is a natural orientation of Γ as a connected asymmetric 4-valent digraph $\vec{\Gamma}$ and $\mathrm{Aut}(\vec{\Gamma})$ is a subgroup of index 2 in G. By Theorem 2.7, $\mathrm{Aut}(\vec{\Gamma})$ has a semiregular element. This concludes the proof.

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References

[1] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, *J. Combin. Theory, Ser. B* **97** (2007), 371–380.

[2] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Minimal normal subgroups of transitive permutation groups of square-free degree, *Discrete Math.* **307** (2007), 373–385.

[3] E. Dobson and D. Marušič, On semiregular elements of solvable groups, *Comm. Algebra* **39** (2011), 1413–1426.

[4] M. Giudici and G. Verret, Semiregular automorphisms in arc-transitive graphs of valency $2p$, in preparation.

[5] M. Giudici and J. Xu, All vertex-transitive locally-quasiprimitive graphs have a semiregular automorphism, *J. Algebraic Combin.* **25** (2007), 217–232.

[6] K. Kutnar and D. Marušič, Recent Trends and Future Directions in Vertex-Transitive Graphs, *Ars. Math. Contemp.* **1** (2008), 112–125.

[7] D. Marušič, On vertex symmetric digraphs, *Discrete Math.* **36** (1981), 69–81.

[8] D. Marušič and R. Scapellato, Permutation groups, vertex-transitive digraphs and semiregular automorphisms, *European J. Combin.* **19** (1998), 707–712.

[9] P. M. Neumann, Finite Permutation Groups, edge-coloured graphs and matrices, in: M. P. J. Curran (ed.), *Topics in groups theory and computations*, Academic Press, London (1977).

[10] P. Potočnik and G. Verret, On the vertex-stabiliser in arc-transitive digraphs, *J. Combin. Theory, Ser. B* **100** (2010), 497–509.

[11] C. E. Praeger, Imprimitive symmetric graphs, *Ars Combin.* **19** (1985), 149–163.

[12] H. Wielandt, *Finite permutation groups*, translated from German by R. Bercov, Academic Press, New York (1964).

[13] J. Xu, Semiregular automorphisms of arc-transitive graphs with valency pq, *European J. Combin.* **29** (2008), 622–629.

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Hamilton paths in Cayley graphs on Coxeter groups: I*

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Abstract

We consider several families of Cayley graphs on the finite Coxeter groups A_n, B_n, and D_n with regard to the problem of whether they are Hamilton-laceable or Hamilton-connected. It is known that every connected bipartite Cayley graph on A_n, $n \geq 2$, whose connection set contains only transpositions and has valency at least three is Hamilton-laceable. We obtain analogous results for connected bipartite Cayley graphs on B_n, and for connected Cayley graphs on D_n. Non-bipartite examples arise for the latter family.

Keywords: Hamilton path, Cayley graph, Coxeter group, Hamilton-connected, Hamilton-laceable.

Math. Subj. Class.: 05C25, 05C70

1 Introduction

The motivational stream for this paper is a confluence of many rivulets varying in age and intrigue. We now explore this history and do so in spite of postponing definitions until completing the brief excursion.

The oldest is Lovász's 1969 question [15] asking whether every connected vertex-transitive graph has a Hamilton path. A closely related thread arose more or less simultaneously, namely, the question of whether every connected Cayley graph has a Hamilton cycle. The latter question has attracted considerable attention for more than forty years. There have been three survey papers of which I am aware [2, 9, 17] and many, many individual papers dealing with the question. References [10, 11, 14] are examples of some recent papers on the topic.

Altshuler [6] studied Hamilton cycles in certain embeddings of trivalent graphs on the torus where all faces are hexagons. He was unable to completely settle the problem of

*Dedicated to Dragan Marušič on the occasion of his sixtieth birthday.

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whether all such graphs are hamiltonian. Many of these graphs (but not all) are Cayley graphs on dihedral groups. This was the first specific instance of looking for Hamilton cycles in Cayley graphs on dihedral groups.

The author and Zhang [5] proved that all connected trivalent Cayley graphs on dihedral groups are hamiltonian. Unfortunately, we were unable to extend the result to any larger valency. So the general problem of determining whether all connected Cayley graphs on dihedral groups are hamiltonian remains unresolved. There has been some progress. It has been known for almost thirty years that the problem has an affirmative answer if it could be proved that the answer is yes when all the elements in the connection set are reflections. As a corollary of a general result in a recent paper [4], we now know that all connected Cayley graphs on dihedral groups are hamiltonian whenever the order of the group is a multiple of 4.

Two apparently unrelated threads come from computer science. The older of the two is related to algorithms for generating all the permutations of an n-set [13, 16]. Several of the algorithms correspond to a Hamilton path in a Cayley graph on the symmetric group S_n with a connection set composed of transpositions. Recently, there has been considerable interest in other Cayley graphs on symmetric groups, where the connection set contains only transpositions. The most celebrated graph of this type is the *star graph of dimension* n [1] (as it frequently is called).

The penultimate thread is fairly new and arises in computational biology. Analysis of genomes evolving by inversions leads to a graph theoretic interpretation that involves signed permutations [12]. This involves the Cayley graphs we study in Sections 3 and 4.

An older paper [8] actually ties these threads together (although it may not yet be apparent). In that paper the following theorem appears.

Theorem 1.1. *Let G be a finite group generated by reflections $R_1, \ldots, R_n$. Then there is a hamiltonian circuit in the Cayley diagram for G corresponding to these generators.*

Upon reading Theorem 1.1, we might think this settles the problem for connected Cayley graphs on dihedral groups because, as mentioned above, it suffices to settle that problem when all the members of the connection set are reflections. However, a careful reading of [8] leads to the discovery that they prove there is a presentation for every finite Coxeter group so that the corresponding Cayley graph has a Hamilton cycle. It is highly likely that Theorem 1.1 is true, but it is yet to be proven.

The final thread arises out of a strong generalization of Theorem 1.1 for symmetric groups (see Theorem 2.3 in the next section). The purpose of this paper is to start extending Theorem 2.3 to Cayley graphs on other Coxeter groups. The two main results are Theorems 4.3 and 5.2.

We are ready to start useful background information. The *Cayley graph* $\mathrm{Cay}(G; S)$ on the group G with *connection set* S is the graph whose vertex set is the set of elements of G, with an edge joining g and h if and only if $h = gs$ for some $s \in S$. There is a restriction on the connection set S, namely, $1 \notin S$ and S is *inverse-closed*, that is, $s \in S$ if and only if $s^{-1} \in S$.

We shall use two notations for permutations because each of them is convenient for certain contexts. One common notation is *cyclic notation* in which a permutation is written as a product of disjoint cycles. The cycles are enclosed in parentheses and upon observing a cycle $(\cdots i\ j \cdots)$, this is to be interpreted as meaning the permutation maps i to j. There are no commas in this notation, instead, the elements in the cycles are separated by extra space. We also adopt the convention that fixed points, that is, 1-cycles, are not written down. Thus, the transposition interchanging i and j is written $(i\ j)$.

The other notation for permutations we use is *range notation*. That is, if we write a permutation as $a_1a_2 \dots a_n$, we mean the permutation that maps i to a_i for $i = 1, 2, \dots, n$. Thus, the identity permutation in the symmetric group S_5 is written as 12345.

We shall employ the Orbit-Stabilizer Theorem and state it now for convenience.

Theorem 1.2. *If G is a permutation group acting on a finite set A, then the order of G is given by*

$$|G| = |\mathcal{O}(x)||G_x|,$$

where $x \in A$, $\mathcal{O}(x)$ denotes the orbit of G containing x, and G_x is the stabilizer of x.

A graph X is *Hamilton-connected* whenever one can find a Hamilton path joining any two arbitrarily chosen vertices. Similarly, a bipartite graph X, for which both parts have the same cardinality, is *Hamilton-laceable* whenever one can find a Hamilton path joining any two arbitrarily chosen vertices lying in different parts.

2 The symmetric groups

A *Coxeter group* is a group generated by reflections $R_1, R_2, \dots, R_n$ such that the only other relations are of the form $(R_iR_j)^k = 1$. Given a Coxeter group G, we associate a graph with it, called a *Coxeter diagram*, where there is a vertex associated with each of the generating reflections.

It is easy to see that R_i and R_j commute if and only if $(R_iR_j)^2 = 1$. So we do not place an edge in the Coxeter diagram if and only if $(R_iR_j)^2 = 1$. If $(R_iR_j)^3 = 1$, then we join R_i and R_j by an edge and do not label the edge. Finally, if $(R_iR_j)^k = 1$ and $k > 3$, then we join R_i and R_j by an edge and label the edge with k.

The symmetric group S_n is a Coxeter group corresponding to the Coxeter diagram given in Figure 1 with the last edge removed. (Thus, we see that S_n is the Coxeter group A_{n-1}.) The generator R_i, $1 \leq i \leq n-1$, is the reflection of E^n, n-dimensional euclidean space, through the orthogonal complement of the vector with -1 in coordinate i, 1 in coordinate $i+1$ and zeros in all other coordinates. We present the known results for the symmetric groups for completeness and because it takes little space.

Definition 2.1. Let S be a collection of transpositions in S_n. We define an *auxiliary graph* $\mathrm{aux}(S)$ by letting the vertices be labelled $1, 2, \dots, n$ and joining i and j with an edge if and only if $(i\ j) \in S$.

The following proposition is easily proved by induction using Theorem 1.2.

Proposition 2.2. *If $X = \mathrm{Cay}(S_n; S)$, where S consists of transpositions only, then X is connected if and only if $\mathrm{aux}(S)$ is connected.*

The proof given in [8] for Theorem 1.1 applies to the connection set consisting of the transpositions $(1\ 2), (2\ 3), \dots, (n-1\ n)$. The next theorem is a strong generalization in that it tells us that all of the connected Cayley graphs on the symmetric group, whose connection sets contain only transpositions, are Hamilton-laceable. This vastly extends the connection sets involved, and strongly extends the conclusion as well. Of course, Hamilton-laceable is the best we can hope for because the graphs are bipartite.

Theorem 2.3. (Araki [7]) *If $X = \mathrm{Cay}(S_n; S)$ is connected, S consists of transpositions only, and $n \geq 4$, then X is bipartite and Hamilton-laceable.*

3 Path extension and Johnson graphs

The proofs of the main results to follow are variations on a single theme. Namely, choose a single vertex u and a *target vertex* v with the object of finding a Hamilton path joining u to v. We proceed by building longer and longer paths from u until we have a path from u that spans all the vertices and terminates at v. We call this technique *path extension*.

The following lemma is the path extension lemma and is used many times. We employ the notion of a quotient graph arising from a partition of the vertex set. It is defined as follows. Given a graph X and a partition $A_1, A_2, \dots, A_t$ of its vertex set, we define the *quotient graph* with respect to the partition to be the graph of order t whose vertices correspond to the parts, where two vertices are adjacent if and only if there was at least one edge in X between the corresponding parts.

We remind the reader that a k-matching is a set of k vertex-disjoint edges.

Lemma 3.1. *Let X be a graph whose vertex set is partitioned into parts $A_1, A_2, \dots, A_t$, let Y_i denote the subgraph induced on A_i, $i = 1, 2, \dots, t$, and let $X/\mathcal{A}$ denote the quotient graph with respect to the partition. We are interested in two scenarios.*

(i) If each Y_i is Hamilton-connected, then we assume that whenever two parts are joined by an edge, there is in fact a 3-matching between the parts. In this case, if there is a Hamilton path in $X/\mathcal{A}$ from A_i to A_j, then there is a Hamilton path in X joining any vertex in A_i to any vertex in A_j.

(ii) If each Y_i is Hamilton-laceable, let B_i, C_i denote the parts of the bipartition of Y_i. We now assume that whenever two parts A_i and A_j are joined by an edge, then there is a 2-matching between A_i and A_j so that the four end vertices of the two edges intersect each of the sets B_i, B_j, C_i, C_j. In this case, if there is a Hamilton path in $X/\mathcal{A}$ from A_i to A_j, then there is a Hamilton path joining any vertex of C_i to any vertex of one of B_j or C_j, and any vertex of B_i to any vertex of the other one of B_j or C_j.

Proof. In scenario (i), choose an arbitrary vertex u in A_i. Let v be the target vertex in A_j. There is a Hamilton path P' joining A_i and A_j in $X/\mathcal{A}$. Let A_k be the second vertex of P'.

There must be a vertex $u' \in A_i$ distinct from u such that u' has a neighbor $w \in A_k$ because there is a 3-matching between A_i and A_k. Thus, take a path from u to u' that spans the vertices of A_i. Then add on the edge from u' to w.

There now must be a vertex $w' \in A_k$, distinct from w, with a neighbor in the next part. We then extend the path by adding on a path from w to w' that spans the vertices of A_k.

We continue in the obvious way noting that we may enter A_j, the last part in P', at a vertex distinct from v because there is a 3-matching between A_j and the preceding part on P'. We then complete the path to a Hamilton path from u to v by adding a path spanning A_j that terminates at v.

The proof when each Y_i is bipartite is essentially the same outside of respecting the bipartition of the subgraphs. □

Recall that the *Johnson graph* $J(n, r)$ has all the r-subsets of an n-set as its vertices, where two vertices are adjacent if and only if their corresponding subsets have exactly $r - 1$ elements in common. We need to define another graph. Let $C = \{a_1, a_2, \dots, a_m\}$ be a non-empty subset of $\{0, 1, 2, \dots, n\}$ such that the elements are listed in the order $a_1 < a_2 < \cdots < a_m$. We define the graph $QJ(n, C)$ in the following way. For each

$a_i \in C$, we include a copy of the Johnson graph $J(n, a_i)$. Thus far the Johnson graphs are vertex-disjoint with no edges between them. We then insert edges between $J(n, a_i)$ and $J(n, a_{i+1})$, for each i, using set inclusion, that is, we join an a_i-subset S_1 and an a_{i+1}-subset S_2 if S_1 is contained in S_2.

The graph $QJ(n, C)$ can be pictured as having levels made up of Johnson graphs with edges between successive levels based on set inclusion. The following theorem is proved in [3].

Theorem 3.2. *The graph $QJ(n, C)$ is Hamilton-connected for every non-empty C.*

4 Wreath products

Because we shall be working with signed permutations throughout the rest of the paper, we adopt a convention that simplifies notation. Instead of writing $-k$ for a positive integer k, we write $\overline{k}$. We extend this in the obvious way in that $k = \overline{\overline{k}}$, and use $\overline{x}$ for $-x$.

Consider the Coxeter diagram shown in Figure 1. The generator R_i, $1 \leq i \leq n-1$, is the reflection of E^n through the orthogonal complement of the vector with $\overline{1}$ in coordinate i, 1 in coordinate $i+1$ and zeros in all other coordinates. The generator R_n is the reflection of E^n through the orthogonal complement of the vector with $\overline{1}$ in coordinate n and zeros in all other coordinates. This is the Coxeter group B_n and it is easy to see that

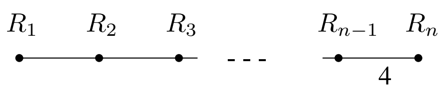

Figure 1

it is isomorphic to the wreath product $S_n \wr S_2$. This group may be visualized as the set of all permutations acting on the set $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$ such that if $f(i) = y$, then $f(\overline{i}) = \overline{y}$. This then gives us a compact notation for the elements of $S_n \wr S_2$, namely, we write

$$a_1 a_2 \ldots a_n$$

to be the permutation mapping i to a_i and $\overline{i}$ to $\overline{a_i}$ for $i = 1, 2, \ldots, n$. That is, the elements are all the signed permutations of $1, 2, \ldots, n$.

Note that $S_n \wr S_2$ is imprimitive with the complete block system composed of the blocks $\{i, \overline{i}\}$ for $i = 1, 2, \ldots, n$. Thus, there is a natural homomorphism

$$\varphi : S_n \wr S_2 \to S_n,$$

with kernel isomorphic to S_2^n, representing the action on the block system. If $\varphi(f) = (i\ j)$ is a transposition in S_n, then either $f = (i\ j)(\overline{i}\ \overline{j})g$ or $f = (i\ \overline{j})(\overline{i}\ j)g$, where g is in the kernel. When $g = 1$, we call such an element of $S_n \wr S_2$ a *double transposition*. If $f \in S_n \wr S_2$ is a transposition, it is easy to see that $f \in \ker(\varphi)$, that is, $f = (i\ \overline{i})$ for some i in cyclic notation.

Let $X = \mathrm{Cay}(S_n \wr S_2; S)$, where S contains only transpositions and double transpositions. We define an auxiliary graph $\mathrm{aux}(S)$ in this case similar to what we did for Cayley graphs on S_n. The vertices are again the integers $1, 2, 3, \ldots, n$. We join i and j by an edge if and only if there is a double transposition $f \in S$ for which the homomorphic image $\varphi(f) = (i\ j)$.

The next result provides some useful structural information.

Lemma 4.1. *If S is a collection of $n-1$ double transpositions in $S_n \wr S_2$ such that* $\mathrm{aux}(S)$ *is a tree, then the subgroup $\langle S \rangle$ generated by S is isomorphic to S_n, has two orbits such that i and $\bar{i}$ belong to different orbits for $1 \le i \le n$, has index 2^n in $S_n \wr S_2$, and the graphs induced on the left cosets of $\langle S \rangle$ are precisely the components of* $\mathrm{Cay}(S_n \wr S_2; S)$.

Proof. We induct on n. When $n = 2$, S contains a single double transposition. We can check easily that the conclusions follow as there are only two possibilities for the double transposition. Let S be a collection of double transpositions satisfying the hypotheses for some $n > 2$, and assume the result holds for $n - 1$.

We may assume the elements on which $S_n \wr S_2$ is acting are labelled so that n is a leaf of the tree $\mathrm{aux}(S)$. Remove from S the double transposition τ involving the element n and let S' denote the set of $n-1$ transpositions left over. The subgroup $\langle S' \rangle$ fixes n and $\overline{n}$, and by induction satisfies the conclusions of the theorem when restricted to $\{1, \overline{1}, 2, \overline{2}, \ldots, n-1, \overline{n-1}\}$.

Thus, $\langle S \rangle_n$, the stabilizer of n, has order $(n-1)!$ and is isomorphic to S_{n-1}. If $\tau = (n\ y)(\overline{n}\ \overline{y})$, then let the orbit of $\langle S' \rangle$ containing y be $\mathcal{O}_1$. Because τ maps n to y and the other $n-1$ double transpositions map elements of $\mathcal{O}_1$ to elements of $\mathcal{O}_1$, the orbit of $\langle S \rangle$ containing n is $\{n\} \cup \mathcal{O}_1$. So the stabilizer of n has order $(n-1)!$ and the orbit containing n has cardinality n. We then know that $|\langle S \rangle| = n!$ by Theorem 1.2. Hence, $\langle S \rangle$ is isomorphic to S_n.

Clearly, the orbit containing $\overline{n}$ is $\{\overline{n}\} \cup \mathcal{O}_2$, where $\mathcal{O}_2$ is the other orbit of the restriction of S_{n-1} to $\{1, \overline{1}, 2, \overline{2}, \ldots, n-1, \overline{n-1}\}$. Hence, $\langle S \rangle$ has two orbits such that $\{i, \overline{i}\}$ intersects both orbits for $1 \le i \le n$.

Because $|S_n \wr S_2| = 2^n n!$ and $|\langle S \rangle| = n!$, it certainly is the case that $\langle S \rangle$ has index 2^n in the $S_n \wr s_2$. So that property holds.

Examining the Cayley graph $\mathrm{Cay}(S_n \wr S_2; S)$, we know that we have a component consisting of the vertices corresponding to the elements of $\langle S \rangle$. Because left-multiplication is an automorphism of a Cayley graph, the components of this Cayley graph are induced on the left cosets of $\langle S \rangle$. □

Lemma 4.2. *If $X = \mathrm{Cay}(S_n \wr S_2; S)$, where S contains only transpositions and double transpositions, then X is connected if and only if S contains at least one transposition and* $\mathrm{aux}(S)$ *is connected.*

Proof. If $\mathrm{aux}(S)$ is not connected, then it is clear that X is not connected. Thus, if X is connected, then $\mathrm{aux}(S)$ is connected.

Multiplying on the right by a double transposition either switches two positions, or switches two positions and negates both entries. Thus, if S contains only double transpositions, the signed permutation $123 \ldots n$ is not in the same component as a signed permutation with a single negative entry. Hence, if X is connected, then S must contain a transposition. This completes the proof of one direction.

We now assume that $\mathrm{aux}(S)$ is connected and S contains a transposition $(i\ \bar{i})$. Let T be a spanning tree of $\mathrm{aux}(S)$. Let $Y = \mathrm{Cay}(S_n \wr S_2; T)$. We know from Lemma 4.1 that Y has 2^n components so let Y' be the component containing $123\ldots n$.

Let $\mathcal{I}(A)$ denote the involution consisting of the product of the transpositions $(k\ \overline{k})$ as k runs through elements of A, where A is a subset of $\{1, 2, \ldots, n\}$. Left multiplication by $\mathcal{I}(A)$ is an automorphism of Y mapping Y' to the component containing $\mathcal{I}(A)$.

Consider the component containing $\mathcal{I}(k)$, where we are writing k rather than $\{k\}$. If we choose any element $a_1 a_2 \ldots a_n$ of Y' with $a_i \in \{k, \overline{k}\}$, then the corresponding element of the component containing $\mathcal{I}(k)$ has all entries the same except that in coordinate i it has $\overline{a_i}$. Thus, there is an edge joining these two vertices via $(i\ \bar{i})$. In a similar way, there is an edge from Y' to any component containing $\mathcal{I}(A)$, where A is a singleton.

In a similar manner, we can find an edge from the component containing $\mathcal{I}(k)$ to any component containing $\mathcal{I}(k, \ell)$, where $\ell \neq k$ and we do not include the set brackets around k and ℓ. It now is obvious that we can use edges generated by $(i\ \bar{i})$ to connect all the components of Y into a single component of X. □

Theorem 4.3. *If $X = \mathrm{Cay}(S_n \wr S_2; S)$ is connected, has valency at least three, and S contains only double transpositions and transpositions, then X is bipartite and Hamilton-laceable.*

Proof. First we show that X is bipartite. Let A consist of the signed permutations $f = a_1 a_2 \cdots a_n$ such that $\varphi(f)$ is an even permutation and f has an even number of negative terms, or $\varphi(f)$ is an odd permutation and f has an odd number of negative terms. Let B be the remaining elements of $S_n \wr S_2$. It is easy to see that if we multiply any element of A on the right by an element of S, we obtain an element of B and vice versa. We conclude that X is bipartite.

Small values of n produce some anomalous situations and we investigate them separately. When $n = 2$, all of the possibilities giving valency 3 are isomorphic to the cartesian product of a 4-cycle and K_2. This is known to be Hamilton-laceable. When the valency is 4, the graph is isomorphic to $K_{4,4}$ which is Hamilton-laceable. Hence, the result is true for $n = 2$.

We cannot apply induction for the $n = 3$ case because the valency may be 3 and upon deleting an element from the connection set, we obtain a subgraph whose components are even length cycles. Even cycles are not Hamilton-laceable so that we must do this case separately.

Let X satisfy the hypotheses and $n = 3$. Because X is connected, $\mathrm{aux}(S)$ is connected and contains a spanning tree. The spanning tree must be a path of length 2 because $n = 3$. By relabelling the elements on which the group $S_n \wr S_2$ acts, if necessary, we may assume the spanning tree is 123. Note that the spanning tree does not uniquely determine the connection set for X. For example, the edge 12 arises from at least one of the double transpositions $(1\ 2)(\bar{1}\ \bar{2})$ and $(1\ \bar{2})(\bar{1}\ 2)$ belonging to S. (Of course, both of these double transpositions could belong to S.)

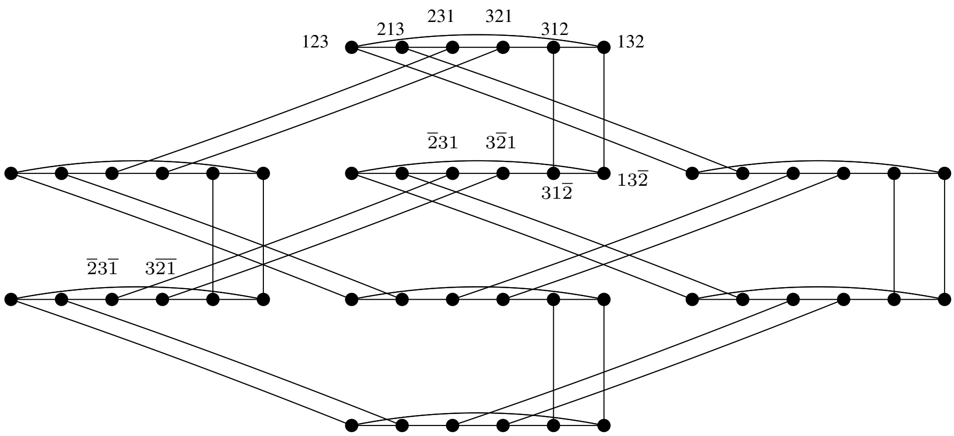

Figure 2

If the double transpositions $(1\ 2)(\overline{1}\ \overline{2})$ and $(2\ 3)(\overline{2}\ \overline{3})$, and the transposition $(3\ \overline{3})$ belong to S, then they generate a spanning subgraph of X isomorphic to the graph shown in Figure 2. In fact, no matter which double transpositions are chosen corresponding to the spanning tree together with either of the transposition $(1\ \overline{1})$ or $(3\ \overline{3})$, we obtain a spanning subgraph of X isomorphic to the graph in Figure 2. If we have the transposition $(2\ \overline{2})$, we obtain edges between the eight 6-cycles such that the two edges joining two fixed 6-cycles are incident with diametrically opposed vertices on the 6-cycles instead of neighboring vertices as in the graph shown in Figure 2.

The essential point is that we have two trivalent bipartite graphs of order 48 that need to be directly checked whether they are Hamilton-laceable. This may seem to be a daunting task, but as a matter of fact it is fairly straightforward.

Consider the graph in Figure 2. It suffices to find a Hamilton path from the vertex 123 to any vertex in the other part of the bipartition because X is vertex-transitive. For example, suppose you want a Hamilton path terminating at the vertex $31\overline{2}$ in the figure. Construct a path starting with 123 that spans the 6-cycle containing 123 and terminates at 132. Continue by taking the edge to $13\overline{2}$ followed by using all the vertices of this 6-cycle and terminating at the vertex $31\overline{2}$. We now have a path starting at 123 and terminating at $31\overline{2}$.

It is easy to see how to transform this starting path into a Hamilton path terminating at $31\overline{2}$. Remove the edge joining $\overline{2}31$ and $3\overline{2}1$ from the path and take the two edges down to the vertices $\overline{2}3\overline{1}$ and $3\overline{2}\overline{1}$ in another 6-cycle. Join $3\overline{2}\overline{1}$ and $\overline{2}3\overline{1}$ using all the vertices of that 6-cycle. It is easy to see that we can continue to delete an edge from the expanding path and move to another 6-cycle and pick up all of its vertices until reaching a Hamilton path.

The preceding technique together with Posa exchanges establishes that the graph of Figure 2 is Hamilton-laceable. The other possible isomorph is a little harder to work with, but it is still fairly easy to establish that it is Hamilton-laceable. Hence, the theorem is true for $n = 3$.

We continue the proof by induction on n. Let $n \geq 4$ and assume the theorem holds for $n-1$. Because $\mathrm{aux}(S)$ is connected, it has a spanning tree T. Moreover, because T has at least two leaves, T has a leaf j such that the connection set S contains a transposition $(i\ \bar{i})$ for which $i \neq j$. Relabel the elements $1, 2, \ldots, n$, if necessary, so that a double transposition g satisfying $\varphi(g) = (n-1\ n)$ belongs to S, where j is relabelled as n.

Now let S' denote all the elements of S that fix n. The group $\langle S' \rangle$ generated by S' is isomorphic to $S_{n-1} \wr S_2$ by Lemma 4.2, and the Cayley graph $Y = \mathrm{Cay}(S_{n-1} \wr S_2; S')$ is connected and bipartite. We know that Y is Hamilton-laceable by induction.

The Cayley graph $X' = \mathrm{Cay}(S_n \wr S_2; S')$ is disconnected with components isomorphic to Y. This follows because $\langle S' \rangle$ generates all signed permutations of $1, 2, 3, \ldots, n$ with n fixed and this is the component of X' containing the identity. If we left multiply $\langle S' \rangle$ by any element of $h \in S_n \wr S_2$, we get all the signed permutations for which n is mapped to $h(n)$, that is, the last coordinate is $h(n)$. Left multiplication is an automorphism of both X and X' so that all the left cosets of $\langle S' \rangle$ induce isomorphs of Y. This sets the stage for the induction proof via Lemma 3.1.

We use the components of X' to give us the partition of $V(X)$. Let $\mathcal{C}(z)$ denote the component consisting of the signed permutations ending with z. If there is an edge from one part of the component $\mathcal{C}(x)$ to one part of the component $\mathcal{C}(y)$, $x \neq y$, then left multiplication by a double transposition from S' gives an edge joining the other two parts of the same two components. Thus, a crucial hypothesis of Lemma 3.1 is satisfied.

It suffices to show that there are Hamilton paths in X from $123 \cdots n$ to every vertex of B because X is vertex-transitive. The double transposition g satisfying $\varphi(g) = (n-1\ n)$ is either $(n-1\ n)(\overline{n-1}\ \overline{n})$ or $(n-1\ \overline{n})(\overline{n-1}\ n)$. We first consider the case that $g = (n-1\ n)(\overline{n-1}\ \overline{n})$.

Let $x \neq y$ such that $x \neq \overline{y}$. There is a signed permutation in $\mathcal{C}(x)$ ending yx. Right multiplication by g gives an edge from $\mathcal{C}(x)$ to $\mathcal{C}(y)$.

Let v be an arbitrary vertex in B in a component $\mathcal{C}(x)$ such that $x \neq n$. By Lemma 3.1 it suffices to find a Hamilton path in the quotient graph $X/\mathcal{A}$ from $\mathcal{C}(n)$ to $\mathcal{C}(x)$. We claim there is a sequence $y_1, y_2, \ldots, y_{2n}$ composed of the elements $1, \overline{1}, \ldots, n, \overline{n}$ such that $y_1 = n$, $y_{2n} = x$ and $j, \overline{j}$ are never consecutive.

Letting $i < n$, use the sequence

$$n, n-1, \ldots, i+1, \overline{1}, \overline{2}, \ldots, \overline{n}, 1, 2, \ldots, i$$

when $x = i$. When $x = \overline{i}$, we negate every term in this sequence other than the first.

When $x = \overline{n}$, use

$$n, n-1, \ldots, 1, \overline{2}, \overline{1}, \overline{3}, \ldots, \overline{n}.$$

From the above remark, there are edges joining consecutive components corresponding to the sequence and the desired Hamilton path exists.

We have to modify the approach somewhat when $x = n$, that is, the target vertex $v = a_1 a_2 \cdots a_{n-1} n$ also lies in $\mathcal{C}(n)$. We start with a path P from $123 \cdots n$ to v that spans the vertices of $\mathcal{C}(n)$. We examine a_{n-1}.

As we traverse P backwards, find the first vertex w for which the $n-1$ entry is different from a_{n-1}. In other words, the subpath $P[123 \cdots n, w]$ terminates in a vertex w whose $n-1$ entry is not a_{n-1}, but every vertex of the subpath $P(w, a_1 a_2 \cdots n]$ has a_{n-1} in coordinate $n-1$. Let w' be the successor of w on P.

Remove the edge ww' from P. The vertices wg and $w'g$ lie in different components because they differ in coordinate n. It is easy to see how to slightly modify the preceding argument so that we obtain a path from w to w' spanning all the vertices of the remaining components. This yields a Hamilton path joining $123\cdots n$ and v.

The other case is $g = (n\ \overline{n-1})(n-1\ \overline{n})$. It is different because multiplying on the right by g not only switches the elements in coordinates $n-1$ and n, it also changes the signs of both. However, it takes only a small modification of the above procedure to handle this case. Use the same sequences $y_1, y_2, \ldots, y_{2n}$, but when terminating the path spanning a left coset, stop at a vertex whose last two coordinates are $\overline{y_{i+1}}y_i$. Multiplying on the right by g then takes you to a vertex in the correct left coset. This completes the proof. □

5 An Index 2 Subgroup Of $S_n \wr S_2$

The groups we considered in the preceding section were the signed permutations of length n. A natural subgroup for each of these groups is the collection of signed permutations with an even number of negative terms. This group is the Coxeter group D_n. It is easy to see that D_n has index 2 in $S_n \wr S_2$.

The Coxeter diagram for the group D_n is shown in Figure 3. The generator R_i, $1 \leq i \leq n-1$, is the reflection through the orthogonal complement of the vector with $\overline{1}$ in coordinate i, 1 in coordinate $i+1$, and zeros elsewhere. The generator R_n is the reflection through the orthogonal complement of the vector with $\overline{1}$ in the last two coordinates and zeros elsewhere.

The first step is to decide which connection sets we are going to allow for the Cayley graphs on D_n. We shall use double transpositions as we have done for $S_n \wr S_2$, but now we require a product of two transpositions that negates two coordinates, that is, a permutation $f \in S_n \wr S_2$ such that $f(i) = \overline{i}$, $f(j) = \overline{j}$, and f fixes all other elements k, where $i \neq j$ and $k \notin \{i, j\}$. We call such a permutation a *double negator*.

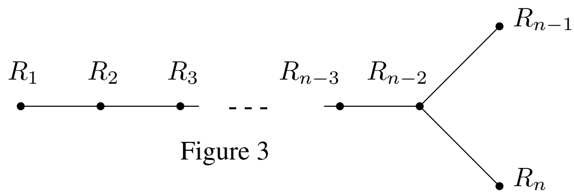

Figure 3

In order to set the stage for what follows, we examine $n = 2, 3$ ahead of time. The case of $n = 2$ is particularly simple, and not particularly edifying, because $|D_2| = 4$. So the only Cayley graph on D_n of valency 3 is K_4. It is not bipartite and certainly is Hamilton-connected.

In general, we let $X = \mathrm{Cay}(D_n; S)$ be a Cayley graph on D_n such that S contains only double transpositions and double negators. We again define $\mathrm{aux}(S)$ by letting the vertices be $1, 2, \ldots, n$, and joining i and j with an edge if and only if there is a double transposition $f \in S$ such that $\varphi(f) = (i\ j)$.

We return to our consideration of the two smallest values of n. There is considerably more complexity when $n = 3$. Note that $|D_3| = 24$.

If $\mathrm{aux}(S)$ is not connected, then it must have a singleton component. Without loss of generality we may assume vertex 3 is a singleton. This means that every double transposition in the connection set S fixes both 3 and $\overline{3}$, and double negators fix all the blocks. Thus, the block $\{3, \overline{3}\}$ is fixed by the group $\langle S \rangle$ generated by S. Hence, $\langle S \rangle$ is a proper subgroup of D_3 which implies that X is not connected. Therefore, we see that if X is connected, then $\mathrm{aux}(S)$ is connected.

We are assuming that X is connected so that $\mathrm{aux}(S)$ contains a spanning tree. The spanning tree must be a path of length 2 because $\mathrm{aux}(S)$ has order 3. Without loss of generality we may assume the set on which the group D_3 is acting is labelled so that the path forming the spanning tree consists of the edges 12 and 23.

We now consider possible special subgraphs of X. First, suppose the double transpositions generating the edges 12 and 23 are $(1\ 2)(\overline{1}\ \overline{2})$ and $(2\ 3)(\overline{2}\ \overline{3})$. These two double transpositions generate the subgraph shown in Figure 4.

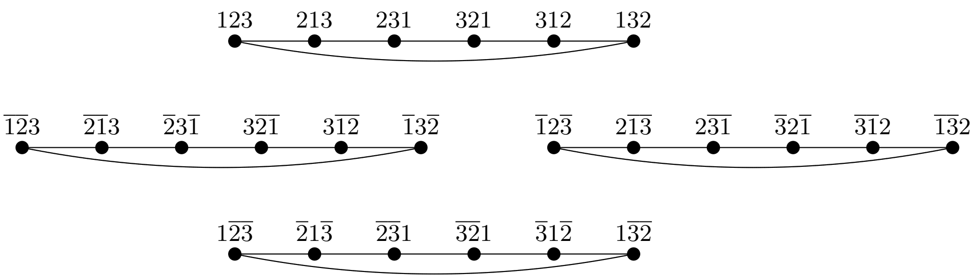

Figure 4

In order for X to be connected, we need either a double negator or a negative double transposition in S (where a double transposition is *negative* when it has the form $(i\ \overline{j})(\overline{i}\ j)$). If we have the double negator $(1\ \overline{1})(2\ \overline{2})$ in S, we obtain the trivalent spanning subgraph Y_1 shown in Figure 5. The graph Y_1 is not bipartite and it can be verified directly that it is Hamilton-connected.

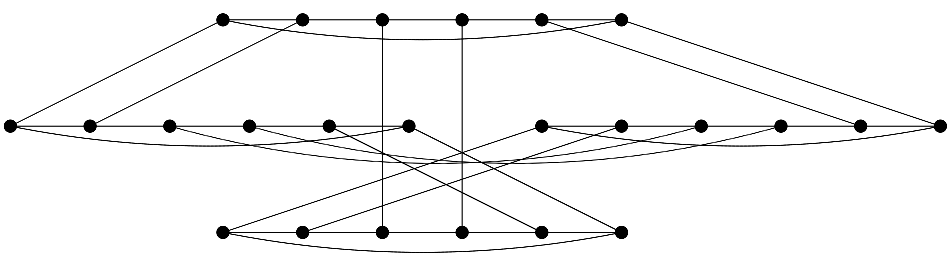

Figure 5: The subgraph Y_1

If we use the double negator $(2\ \overline{2})(3\ \overline{3})$, we obtain a graph that is isomorphic to Y_1. The same conclusions then follow. If instead we use the double negator $(1\ \overline{1})(3\ \overline{3})$, we obtain the graph Y_2 shown in Figure 6. The graph Y_2 also is not bipartite and it can be verified directly that Y_2 is Hamilton-connected.

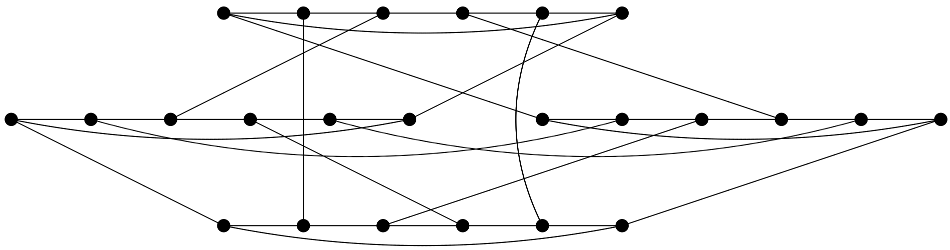

Figure 6: The subgraph Y_2

If we partition the vertices of X so that part A contains the permutations f for which $\varphi(f)$ is an even permutation in S_3, and part B contains the permutations f such that $\varphi(f)$ is an odd permutation, then the edges generated by any double transposition have one end in A and one end in B. Hence, if S contains no double negator, then X is bipartite.

Moreover, if S contains no double negator, then there must be a double transposition in S not contained in the group generated by $(1\ 2)(\overline{1}\ \overline{2})$ and $(2\ 3)(\overline{2}\ \overline{3})$. If we use the negative double transposition $(1\ \overline{2})(\overline{1}\ 2)$ to connect the components of the spanning graph shown in Figure 4, we obtain the graph Y_3 shown in Figure 7. This graph is bipartite and it is easy to verify that it is Hamilton-laceable.

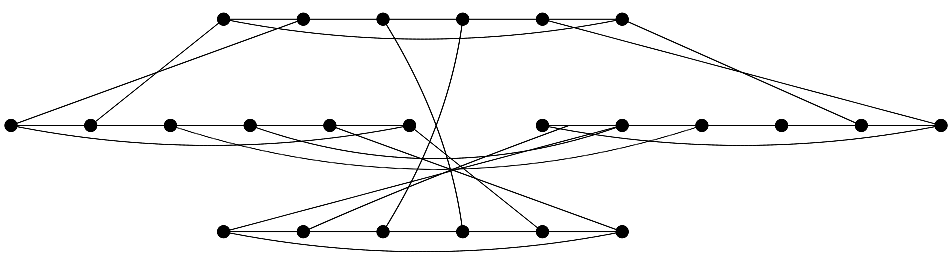

Figure 7: The subgraph Y_3

If we use the negative double transposition $(2\ \overline{3})(\overline{2}\ 3)$, we obtain a graph isomorphic to Y_3. This leaves the negative double transposition $(1\ \overline{3})(\overline{1}\ 3)$. In this case we obtain the graph Y_4 shown in Figure 8. It is bipartite and easily shown to be Hamilton-laceable.

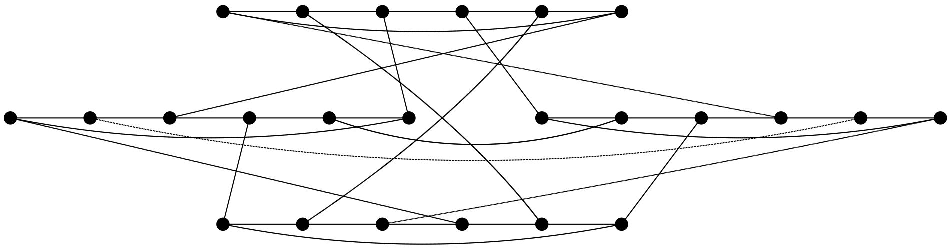

Figure 8: The subgraph Y_4

It is obvious that two Cayley graphs on the same group G, with respective connection sets S' and $gS'g^{-1}$ for $g \in G$, are isomorphic via conjugation by g. Therefore, if S contains exactly one positive double transposition, or S contains no positive double transpositions, then each Cayley graph we obtain is isomorphic via conjugation to one of those we have considered above.

This completes the analysis of the case that $n = 3$. What we have seen is that a spanning tree of $\mathrm{aux}(S)$ always produces a 2-factor composed of four 6-cycles. If any double negator lies in the connection set S, then the resulting spanning trivalent subgraph is connected, not bipartite and Hamilton-connected. This implies, of course, that X is Hamilton-connected.

Continuing in this vein, if S contains no double negators, then X is bipartite and Hamilton-laceable when it is connected. The subgraph generated by any two double transpositions τ_1, τ_2, corresponding to a spanning tree, is a 2-factor composed of four 6-cycles. Any double transposition $\tau_3 \in \langle \tau_1, \tau_2 \rangle$ results in a disconnected spanning trivalent subgraph. So for X to be connected, we need a double transposition that is not an element of the group $\langle \tau_1, \tau_2 \rangle$.

Lemma 5.1. *If $X = \mathrm{Cay}(D_n; S)$ is a Cayley graph on D_n such that S contains only double transpositions and double negators, then X is connected if and only if $\mathrm{aux}(S)$ is connected and one of the following two conditions holds:*

(a) *S contains a double negator, or*

(b) *S contains no double negators, but if $\tau_1, \tau_2, \ldots, \tau_{n-1}$ are elements of S corresponding to a spanning tree of $\mathrm{aux}(S)$, then there is a $\tau \in S$ such that τ does not belong to the group $\langle \tau_1, \tau_2, \ldots, \tau_{n-1} \rangle$ generated by $\tau_1, \tau_2, \ldots, \tau_{n-1}$.*

Moreover, X is not bipartite when $n > 2$ and (a) *holds, whereas, X is bipartite when* (b) *holds.*

Proof. Observe that if $\mathrm{aux}(S)$ is not connected, then it is obvious that X is not connected. Thus, if X is connected, then $\mathrm{aux}(S)$ also is connected.

If X is connected, let $\tau_1, \tau_2, \ldots, \tau_{n-1}$ be the double transpositions for some spanning tree of $\mathrm{aux}(S)$. The group $H = \langle \tau_1, \tau_2, \ldots, \tau_{n-1} \rangle$ is a proper subgroup of D_n by Theorem 4.1. So if S does not contain a double negator, then in order for X to be connected, there must be a double transposition τ such that $\tau \notin H$. Hence, if X is connected, then $\mathrm{aux}(S)$ is connected and at least one of the two conditions holds.

Now let $\operatorname{aux}(S)$ be connected and let condition (a) or (b) hold. Let $\tau_1, \tau_2, \ldots, \tau_{n-1}$ be as in the preceding paragraph, let X' be the subgraph of X generated by these $n-1$ double transpositions, and let Y be the component of X' containing $123\cdots n$.

First suppose that condition (a) holds and that the double negator is $(i\ \bar{i})(j\ \bar{j})$ The proof that X is connected closely mirrors the corresponding proof of Lemma 4.2. First, recall that $\mathcal{I}(A)$, where A is a subset of $\{1, 2, \ldots, n\}$, denotes the permutation which is a product of the transpositions $(i\ \bar{i})$ as i runs through A and fixes all other elements. The components of X' are then the subgraphs containing the permutations $\mathcal{I}(A)$ as A runs over all subsets of even cardinality of $\{1, 2, \ldots, n\}$. We then show that X is connected using an argument that is the same as that used for Lemma 4.2 except that we now use two coordinates at a time instead of one. This takes care of connectivity.

We cannot claim that X is not bipartite for $n = 2$ because the graph may be a cycle of length 4. To conclude the proof for condition (a), we must show that X is not bipartite when S contains a double negator and $n > 3$ ($n = 3$ was done above). Let τ denote the double negator $(i\ \bar{i})(j\ \bar{j})$. We can always find a permutation $f = a_1a_2\cdots a_n$, where a_i and a_j are fixed, with $f \in A$ by inverting two elements in two coordinates different from i and j if necessary.

Let f_1 denote a permutation in Y such that $a_i \in \{i, \bar{i}\}$, $a_j \in \{r, \bar{r}\}$, $r \neq j$, and $f_1 \in A$. There is then an edge joining f_1 and $f_1\tau$ in the left coset $\mathcal{I}(i, r)H$.

In a similar manner, there is a permutation f_2 in Y such that $a_i \in \{j, \bar{j}\}$, $a_j \in \{r, \bar{r}\}$, and $f_2 \in A$. There is then an edge joining f_2 and $f_2\tau$ in the left coset $\mathcal{I}(j, r)H$. We then find a permutation f_3 in the left coset $\mathcal{I}(i, r)H$ belonging to A with $a_i \in \{i, \bar{i}\}$ and $a_j \in \{j, \bar{j}$. This is then adjacent to $f_3\tau$ in the left coset $\mathcal{I}(j, r)H$ and $f_3\tau \in A$. The paths joining f_1 and f_2 in Y, f_3 and $f_3\tau$ in $\mathcal{I}(i, r)H$, and $f_2\tau$ and $f_3\tau$ in $\mathcal{I}(j, r)H$ all have even lengths because the vertices all belong to A. Thus, we have an odd length cycle in X so that it is not bipartite.

When condition (b) holds, then S contains no double negators but it does contain a double transposition τ not contained in the group H. The double transposition τ satisfies $\varphi(\tau) = (i\ j)$ for some $i \neq j$. We know there is an element $\tau' \in H$ such that $\varphi(\tau') = (i\ j)$ because H is isomorphic to S_n by Lemma 4.1. Moreover, Lemma 4.1 informs us that k and $\bar{k}$ are in different orbits for all k so that τ' fixes all elements not in $\{i, \bar{i}, j, \bar{j}\}$. This implies that $\tau'\tau = (i\ \bar{i})(j\ \bar{j})$ which is a double negator. Because multiplying on the right by τ' keeps one in the same left coset of H, we see that τ joins the same left cosets as the preceding double negator. Therefore, X is connected by the argument for condition (a).

Note that if we let A be all the permutations $f \in D_n$ for which $\varphi(f)$ is an even permutation and let B be all those for which $\varphi(f)$ is an odd permutation, then any edge generated by a double transposition has one end vertex in A and one end vertex in B. Thus, if S contains no double negators, then X is bipartite. □

Theorem 5.2. *If $X = \operatorname{Cay}(D_n; S)$ is a connected Cayley graph of valency at least 3 on D_n, $n \geq 2$, such that S contains only double transpositions and double negators, then X is Hamilton-laceable when it is bipartite, or Hamilton-connected when it is not bipartite.*

Proof. The results of this theorem have been proved for $n = 2$ and $n = 3$ earlier. We proceed by induction on n.

First consider the case that X is bipartite. As before, let

$$H = \langle \tau_1, \tau_2, \ldots, \tau_{n-1} \rangle,$$

where $\tau_1, \tau_2, \ldots, \tau_{n-1}$ are the double transpositions corresponding to the edges of a spanning tree of $\mathrm{aux}(S)$. We know there is a double transposition τ' not contained in H from Lemma 5.1. Furthermore, we saw in the proof of Lemma 5.1 that edges between the subgraphs induced on the left cosets of H join the same left cosets as do the edges generated by a double negator $\tau = (i\ \bar{i})(j\ \bar{j})$. For simplicity we work with τ.

As before, let A denote the elements f of $\langle S\rangle$ such that $\varphi(f)$ is an even permutation and let B denote the other elements of the group. Clearly, A and B are the two parts of the bipartition of X.

We know that $\mathcal{I}(L)H$ are the left cosets of H as L runs through all even cardinality subsets of $\{1, 2, \ldots, n\}$. If E_1 and E_2 are two subsets of the same even cardinality α and they have $\alpha - 1$ elements in common, then there is an edge joining a vertex of $\mathcal{I}(E_1)H$ and a vertex of $\mathcal{I}(E_2)H$. To see this let x, y be the two elements of $E_1 \Delta E_2$ (symmetric difference). Let $f = a_1 a_2 \cdots a_n$ be an element of H so that $a_i \in \{x, \overline{x}\}$ and $a_j \in \{y, \overline{y}\}$. Then we have $\mathcal{I}(E_1)f\tau = \mathcal{I}(E_2)f$ which implies there is an edge between the left cosets $\mathcal{I}(E_1)H$ and $\mathcal{I}(E_2)H$.

Now form a quotient graph by contracting each left coset to a single vertex making two vertices adjacent if there is an edge joining vertices of the corresponding left cosets. From the preceding paragraph, we see that two vertices corresponding to left cosets $\mathcal{I}(E_1)H$ and $\mathcal{I}(E_1)H$, $|E_1| = |E_2|$, are adjacent if E_1 and E_2 have all but one element in common. Thus, all left cosets corresponding to subsets of $\{1, 2, \ldots, n\}$ of the same cardinality k induce a subgraph containing the Johnson graph $J(n, k)$.

It is also easy to see that if E_1 is a subset of E_2 with $|E_1| = |E_2| - 2$, then there is an edge between the corresponding left cosets. Hence, the quotient graph contains a spanning subgraph isomorphic to $QJ(n, C)$, where C is the collection of all even cardinality subsets of $\{1, 2, \ldots, n\}$. This graph is Hamilton-connected by Theorem 3.2. Thus, we may employ Lemma 3.1 to easily find Hamilton path in X from $123\cdots n$ to any vertex of B in any left coset different from H.

If the target vertex v happens to lie in H, then there is a path from $123\cdots n$ to v spanning the vertices of H by induction. We then find any edge $w_1 w_2$ in this path such that w_1 and w_2 have neighbors in different left cosets of H. We then use $QJ(n, C - \emptyset)$, which also is Hamilton-connected by Theorem 3.2, along with Lemma 3.1 to find a path Q from $w_1\tau'$ to $w_2\tau'$ spanning all the vertices of the remaining left cosets. We remove the edge $w_1 w_2$ from the initial path spanning H and patch Q in to get a Hamilton path in X from $123\cdots n$ to v. This completes the bipartite case.

Now assume that S contains the double negator $\tau = (i\ \bar{i})(j\ \bar{j})$. Note that $\varphi(f)$ and $\varphi(f\tau)$ either are both even permutations or both odd permutations. Thus, an edge generated by a double negator either has both end vertices in A or both end vertices in B.

There are two cases to consider: Either $\mathrm{aux}(S)$ has a spanning tree with a leaf k different from both i and j or there is no such spanning tree. We first consider the case that there is such a tree T.

Without loss of generality we assume that n is the leaf of T different from i and j. In other words, n is fixed by τ. Let S' be the subset of S containing all double transpositions and double negators that fix the element n.

Because $\mathrm{aux}(S')$ contains a spanning tree and the double negator τ, the components of $\mathrm{Cay}(D_n; S')$ have order $2^{n-2}(n-1)!$ and are Hamilton-connected by induction. We now have exactly the same situation as in the proof of Theorem 4.3, namely, a subgraph each of whose components is composed of all the permutations whose last coordinate is constant.

There is one component for each element of $\{1, \overline{1}, \ldots, n, \overline{n}\}$. The double transposition corresponding to the edge of T incident with n is then used to connect the components together exactly as was done in the proof of Theorem 4.3. Hence, X is Hamilton-connected.

This leaves us with the other case, namely, there is no spanning tree with a leaf different from i and j. The preceding induction proof cannot be applied. It is not hard to see that this forces $\mathrm{aux}(S)$ to be a path whose end vertices are i and j.

We maintain the same notation, that is, we let $S' = \{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$ be the double transpositions corresponding to the edges of the spanning tree, $H = \langle S' \rangle$, X' be the subgraph $\mathrm{Cay}(D_n; S')$, and Y the component of X' containing $123 \cdots n$. By Lemma 4.1, X' has 2^{n-1} components. The components are bipartite and Hamilton-laceable by Theorem 2.3. Moreover, all the components are isomorphic to Y.

For each $L \subseteq \{1, 2, \ldots, n\}$, $|L|$ even, the action of the involution $\mathcal{I}(L)$ on each permutation is to negate the entries in the coordinates corresponding to the elements of L. Hence, the 2^{n-1} components of $\mathrm{Cay}(D_n; S')$ are the subgraphs induced on the left cosets $\mathcal{I}(L)H$ as L runs through all subsets of $\{1, 2, \ldots, n\}$ of even cardinality. As we saw earlier in this proof, the quotient graph of X obtained by contracting each left coset to a single vertex, deleting all loops, and replacing multiple edges by a single edge yields a spanning subgraph isomorphic to $QJ(n, C)$, where C contains all even integers between 0 and n inclusive. We employ Theorem 3.2 frequently.

We define the bipartition A and B as before. Because X is vertex-transitive, it suffices to find a Hamilton path in X from $123 \cdots n$ to any other vertex v. We shall refer to v as the target vertex.

Let v be a target vertex in the part A of any component $\mathcal{I}(L)H$ different from Y. Theorem 3.2 provides a Hamilton path in the quotient graph from the vertex corresponding to Y to the vertex corresponding to $\mathcal{I}(L)H$. Since edges generated by τ have both end vertices in either A or B, we use a (slightly) modified version of Lemma 3.1 to obtain a Hamilton path from $123 \cdots n$ to v using the fact that there are an even number of components.

Now let the target vertex v be in part B of Y. There is a path P from $123 \cdots n$ to v spanning the vertices of Y by Theorem 2.3. There must be two successive vertices w_1, w_2 on P with neighbors y_1, y_2 in different components $\mathcal{I}(a, b)H$ and $\mathcal{I}(a, c)H$ for some a, b, c. Considering the graph $QJ(n, C')$, where $C' = \{2, 4, \ldots, 2\lfloor n/2 \rfloor\}$, Theorem 3.2 and Lemma 3.1 imply there is a path Q from y_1 to y_2 spanning all the vertices of the components corresponding to C'. We then obtain a Hamilton path from $123 \cdots n$ to v by removing the edge $w_1 w_2$, adding the edges $y_1 w_1$ and $y_2 w_2$, and adding the path Q. Thus, X has a Hamilton path from $123 \cdots n$ to any vertex in part B of Y.

To complete the proof of the theorem, we must find a way to circumvent the fact that the components are only Hamilton-laceable. Before introducing the trick we use, let's review the strategy we have used so far. We find a path Q spanning Y that starts at $123 \cdots n$ and finishes at any vertex w of part B in Y. We choose w so that $w\tau$ belongs to a left coset $\mathcal{I}(L)H$ not containing the target vertex v. We then use the fact that the graph $QJ(n, C)$, where C contains all the even integers in $\{2, 3, 4, \ldots, n\}$, is Hamilton-connected so that we may apply Lemma 3.1 to find a path P from $w\tau$ to v spanning all the left cosets of H distinct from H itself. We attach P to Q using the edge from w to $w\tau$ to obtain a Hamilton path in X from $123 \cdots n$ to v. Because the number of left cosets of H distinct from H is odd (that is, the number of vertices of $Q(n, C)$ is odd), this works only for vertices in part A.

The trick we are about to introduce is based on removing either one or three special vertices from the $QJ(n,C)$ graph mentioned above. If we remove a single vertex corresponding to any 2-subset, or three vertices corresponding to three 2-subsets of the form $\{a,b\},\{a,c\},\{b,c\}$, or three 2-subsets of the form $\{a,d\},\{b,d\},\{c,d\}$, then the resulting subgraph of $QJ(n,C)$ remains Hamilton-connected. We leave this up to the reader and refer to [3] if guidance is required. The key is to prove that removing these three vertices from $J(n,2)$, $n \geq 4$, leaves a Hamilton-connected graph.

When the target vertex v is in part B of some component $\mathcal{I}(L)H$ different from Y, choose a vertex w in part B of Y such that the neighbor $w\tau$ of w belongs to a left coset $\mathcal{I}(a,b)H$ different from $\mathcal{I}(L)H$. Then choose a path Q from $123\cdots n$ to w that spans Y.

Suppose there is an edge xy on Q so that both $x\tau$ and $y\tau$ lie in the same left coset $\mathcal{C}$, and $\mathcal{C}$ does not contain v and is different from $\mathcal{I}(a,b)H$. We then remove the edge xy from Q and attach the edges to $x\tau$ and $y\tau$. We now find a path from $x\tau$ to $y\tau$ spanning the vertices of $\mathcal{C}$. This now gives us a path Q' from $123\cdots n$ to w spanning the two left cosets H and $\mathcal{C}$. The graph $QJ(n,C)$ with a single vertex corresponding to a 2-subset removed is still Hamilton-connected. However, the number of left cosets left over is now even so that when we apply the strategy outlined above, we can reach a target vertex in part B as required.

Alternatively, suppose there is an edge xy of Q so that $x\tau$ and $y\tau$ are in different left cosets $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and a third left coset $\mathcal{C}_3$ so that $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3$ are all distinct from $\mathcal{I}(L)H$ and $\mathcal{I}(a,b)H$. If there is a path from $x\tau$ to $y\tau$ spanning the vertices of the three left cosets $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3$, then we can replace the edge xy in Q with a path that spans the three left cosets, then we have a path from $123\cdots n$ to w spanning the four left cosets. The number of left cosets remaining is even and we can find a Hamilton path to a target vertex in any part B

We need to show that one of the preceding conditions holds. If the target vertex v is in a left coset $\mathcal{I}(L)H$ such that $|L|>2$, we don't concern ourselves with designating the set L. If $|L|=2$, then we let the left coset containg v be $\mathcal{I}(a,c)H$. Choose w in part B of Y so that $w\tau \in \mathcal{I}(a,b)H$, where $b \neq c$.

Let Q be a path from $123\cdots n$ to w spanning the vertices of Y. Because $n \geq 4$, there is an element d distinct from a,b,c. If there is an edge xy of Q so that both $x\tau$ and $y\tau$ lie in any single one of the left cosets $\mathcal{I}(c,d)H,\mathcal{I}(b,c)H,\mathcal{I}(b,d)H$, then the first condition above holds and there is a Hamilton path from $123\cdots n$ to v in X.

If we cannot find an edge xy such that $x\tau$ and $y\tau$ lie in the same left coset, we need to examine $\mathrm{aux}(S)$ with more care. We know $\mathrm{aux}(S)$ is a path with end vertices i and j with the double negator $\tau=(i\,\bar{i})(j\,\bar{j})$ joining the left cosets. Note that any double transposition in S alters at most one of the entries in coordinates i and j. Hence, if x and y are adjacent vertices of Y, then $x\tau$ and $y\tau$ either lie in the same left coset or lie in different left cosets $\mathcal{I}(L_1)H$ and $\mathcal{I}(L_2)H$ such that $|L_1|=|L_2|=2$, and L_1 and L_2 have one element in common.

The number of edges from Y to a fixed coset $\mathcal{I}(L)H$, where $|L|=2$, is $2(n-2)!$ which is at least 4 because $n \geq 4$. Hence, there is an internal vertex x of Q such that $x\tau \in \mathcal{I}(c,d)H$, where d is distinct from a,b and c. Consider the predecessor y of x on Q.

If $y\tau \in \mathcal{I}(c,d)H$, we are done. If $y\tau \in \mathcal{I}(a,d)H$, then the three left cosets $\mathcal{I}(a,d)H$, $\mathcal{I}(b,d)H,\mathcal{I}(c,d)H$ do the job. If $y\tau \in \mathcal{I}(b,d)H$, then the three left cosets $\mathcal{I}(b,d)H$, $\mathcal{I}(c,d)H,\mathcal{I}(b,c)H$ do the job. If $y\tau$ lies in $\mathcal{I}(b,c)H$, we also easily find three left cosets that work.

In fact, the only left coset that is bad for $y\tau$ is $\mathcal{I}(a,c)H$. But if $y\tau$ lies in the left coset

$\mathcal{I}(a,c)H$, then the successor of x on Q, call it z, cannot have $z\tau$ in $\mathcal{I}(a,c)H$ because this forces the edges yx followed by xz to be generated by the same double transposition because $\mathrm{aux}(S)$ is a path. However, double transpositions are involutions so that we cannot use them consecutively when generating distinct vertices. Thus, $x\tau$ and $z\tau$ must lie in left cosets satisfying one of the conditions above.

This leaves us with the target vertex being in part A of Y as the only missing case. Suppose that v lies in part A of Y and $v\tau \in \mathcal{I}(a,b)H$. Let w be in part B of Y such that $w\tau$ lies in $\mathcal{I}(a,b)H$ as well. Let Q be a path from $123\cdots n$ to w spanning the vertices of Y.

The vertex v is an internal vertex of Q. Remove the edge between the predecessor u of v and v. This breaks Q into a subpath Q_1 from $123\cdots n$ to u, and a subpath Q_2 from v to w. Because u is adjacent to v and by changing the labels, if necessary, we may assume that $u\tau \in \mathcal{I}(a,e)H$ for some e.

If $e = c$, then find a vertex z internal to either Q_1 or Q_2 such that $z\tau \in \mathcal{I}(b,d)H$. Just as above, one of the two neighbors of z on Q allows us to use either one or three left cosets to augment either Q_1 or Q_2. We then cover all the remaining cosets using a path with $w\tau$ and $u\tau$ as the end vertices as before because there are an even number of cosets unused. We have the desired Hamilton path.

If $e = d$ a similar argument works. If $e \notin \{b,c,d\}$ it is even simpler because we have a new symbol to work with. However, it is possible that $e = b$ so that $w\tau$ and $u\tau$ lie in the same left coset $\mathcal{I}(a,b)H$.

We now have the situation that Q_1 has been extended to part B of the left coset $\mathcal{I}(a,b)H$ via the edge to $u\tau$. Similarly, Q_2 has been extended to part B of the same left coset via the edge to $v\tau$. Now choose a vertex u_1 in part A of $\mathcal{I}(a,b)H$ whose i,j entries are disjoint from the i,j entries of $u\tau$. Then choose a path Q' from u_1 to $w\tau$ that spans the vertices of $\mathcal{I}(a,b)H$. The successor u_2 of $u\tau$ then is adjacent to a vertex in a different left coset of H than is u_1. This allows us to obtain a Hamilton path from $123\cdots n$ to v in X and completes the proof. □

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References

[1] S. Akers, D. Harel and B. Krishnamurthy, The star graph: attractive alternative to the n-cube, in: *Interconnection Networks For High-Performance Parallel Computers*, IEEE Computer Society Press, Los Alamitos, 1994, 145–152.

[2] B. Alspach, The search for long paths and cycles in vertex-transitive graphs and digraphs, in: Kevin L. McAvaney (ed.), *Combinatorial Mathematics VIII*, Lecture Notes in Mathematics **884**, Springer-Verlag, Berlin, 1981, 14–22.

[3] B. Alspach, Johnson graphs are Hamilton-connected, *Ars. Math. Contemp.* **6** (2013), 21–23.

[4] B. Alspach, C. C. Chen, and M. Dean, Hamilton paths in Cayley graphs on generalized dihedral groups, *Ars Math. Contemp.* **3** (2010), 29–47.

[5] B. Alspach and C.-Q. Zhang, Hamilton cycles in cubic Cayley graphs on dihedral groups, *Ars Combin.* **28** (1989), 101–108.

[6] A. Altshuler, Hamiltonian circuits in some maps on the torus, *Discrete Math.* **1** (1972), 299–314.

[7] T. Araki, Hyper Hamiltonian Laceability of Cayley Graphs Generated by Transpositions, *Networks* **48** (2006), 121–124.

[8] J. H. Conway, N. J. A. Sloane and A. R. Wilks, Gray Codes For Reflection Groups, *Graphs and Combin* **5** (1989), 315–325.

[9] S. Curran and J. A. Gallian, Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey. *Discrete Math.* **156** (1996), 1–18.

[10] S. J. Curran, D. Morris and J. Morris, Cayley graphs of order $16p$ are hamiltonian, *Ars Math. Contemp.* **5** (2012), 185–211.

[11] E. Ghaderpour and D. Morris, Cayley graphs on nilpotent groups with cyclic commutator subgroup are hamiltonian, *Ars Math. Contemp.* **7** (2014), 55–72.

[12] S. Hannenhali and P. A. Pevzner, Transforming cabbage into turnip: polynomial algorithm for sorting signed permutations by reversals, *Journal of the ACM* **46** (1999), 1–27.

[13] S. Johnson, Generation of permutations by adjacent transpositions, *Math. Comput.* **17** (1963), 282–285.

[14] K. Kutnar, D. Marušič, D. Morris, J. Morris and P. Šparl, Hamiltonian cycles in Cayley graphs of small order, *Ars Math. Contemp.* **5** (2012), 27–71.

[15] L. Lovász, Problem 11, *Combinatorial Structures And Their Applications*, Gordon and Breach, New York, 1970, 243–246.

[16] H. Trotter, Algorithm 115, PERM., *Commun. ACM* **5** (1962), 434–435.

[17] D. Witte and J. A. Gallian, A survey: Hamiltonian cycles in Cayley graphs. *Discrete Math.* **51** (1984), 293–304.

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Polarity graphs revisited

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Abstract

Polarity graphs, also known as Brown graphs, and their minor modifications are the largest currently known graphs of diameter 2 and a given maximum degree d such that $d - 1$ is a prime power larger than 5. In view of the recent interest in the degree-diameter problem restricted to vertex-transitive and Cayley graphs we investigate ways of turning the (non-regular) polarity graphs to large *vertex-transitive* graphs of diameter 2 and given degree.

We review certain properties of polarity graphs, giving new and shorter proofs. Then we show that polarity graphs of maximum even degree d cannot be spanning subgraphs of vertex-transitive graphs of degree at most $d + 2$. If $d - 1$ is a power of 2, there are two large vertex-transitive induced subgraphs of the corresponding polarity graph, one of degree $d - 1$ and the other of degree $d - 2$. We show that the subgraphs of degree $d - 1$ cannot be extended to vertex-transitive graphs of diameter 2 by adding a relatively small non-edge orbital. On the positive side, we prove that the subgraphs of degree $d - 2$ can be extended to the largest currently known Cayley graphs of given degree and diameter 2 found by Šiagiová and the second author [*J. Combin. Theory Ser. B* **102** (2012), 470–473].

Keywords: Graph, polarity graph, degree, diameter, automorphism, group, vertex-transitive graph, Cayley graph.

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1 Introduction

A graph of diameter 2 and maximum degree d can have at most d^2+1 vertices. This can be seen by rooting the graph at a vertex of maximum degree and observing that the vertex set of the graph is the union of vertices at distance 0, 1 and 2 from the root, giving the estimate $1+d+d(d-1) = d^2+1$, also known as the Moore bound for diameter 2. Such a graph of order d^2+1 exists if and only if $d = 2, 3, 7$ and possibly 57, by the classical result of [9]. Examples for the first three degrees – the pentagon, the Petersen graph and the Hoffman-Singleton graph – are unique, and existence of a graph of diameter 2, degree 57 and order $57^2+1 = 3250$ is still an open problem. For all other values of $d \geq 4$ it is known [5] that the largest order of a graph of diameter 2 and maximum degree d is at most d^2-1, but by [11] examples of graphs of that order are known only for $d = 4, 5$. Investigation of large graphs of given degree and diameter in general is part of the well known degree-diameter problem, surveyed in [11].

If $d = q+1$ where q is prime power such that $q \geq 7$, the largest currently known order of a graph of diameter 2 and maximum degree d is d^2-d+1 if q is odd and d^2-d+2 if q is even. Examples of graphs of such order, missing the Moore bound only by d and $d-1$, are the polarity graphs $B(q)$ for odd q and their minor modifications for even q; both will be described in the next section. Extensions of polarity graphs by adding Hamilton cycles and maximum matchings taken from the complement were considered in [15] and give graphs of maximum degree d, diameter 2 and order at least $d^2-2d^{1.525}$ for every sufficiently large d. This shows that the Moore bound can be met at least asymptotically for *all* sufficiently large degrees.

The polarity graphs $B(q)$ were first introduced in 1962 by Erdős and Rényi [6] and later in 1966 independently by Brown [3] (and considered again by Erdős, Rényi and Sós [7]) in connection with asymptotic determination of the largest number of edges in a graph of a given order without cycles of length four. The notation $B(q)$ is derived from the fact that in the degree-diameter research community these graphs have also been known as Brown graphs after their second independent discoverer. Properties of polarity graphs have been studied in considerable detail in [12], including determination of the automorphism group of these graphs and their important subgraphs. Since polarity graphs are not regular, they cannot be vertex-transitive; nevertheless they have a relatively large automorphism group which has three orbits on vertices. The role of polarity graphs in the degree-diameter problem was realized later (cf. [11]) and their modification in the case when q is a power of two comes from [5] and [4].

In view of the recent advances in constructions of large vertex-transitive graphs with given degree and diameter [11] it is of interest to ask if one can modify polarity graphs to obtain large *vertex-transitive* graphs of a given degree and diameter 2 for an infinite set of degrees. We will consider modifications by inserting new edges into a polarity graph or into an induced subgraph of a polarity graph.

It is unclear if extending polarity graphs by Hamilton cycles and maximum matchings [15] can ever produce vertex-transitive graphs. In Section 4 we show that it is not possible to extend a polarity graph $B(q)$ for odd $q \geq 37$ to a vertex-transitive graph by increasing its maximum degree by two. Regarding subgraphs, the graph $B(q)$ for q a power of 2 contains two large vertex-transitive induced subgraphs, one of degree q and order q^2 and the other of degree $q-1$ and order $q(q-1)$. We prove that the first subgraph cannot be extended to a vertex-transitive graph of diameter 2 by adding non-edge orbitals induced by the smallest transitive group of automorphisms of the graph. In contrast with this, the second subgraph is shown to be isomorphic to the Cayley graph discovered in [14], which is extendable to a Cayley graph of diameter 2 and degree $d' + O(\sqrt{d'})$ for $d' = q - 1$ and which shows that the Moore bound for diameter 2 can be approached at least asymptotically. This equips the construction of [14] with a strong geometric flavour. Presentation of these results is preceded in Section 2 by a description of polarity graphs and their properties and in Section 3 by a study of symmetry properties of polarity graphs. In the final Section 5 we discuss possible generalizations.

2 Polarity graphs and their structure

Let q be a prime power and let $F = \mathrm{GF}(q)$ be the Galois field of order q. We let $\mathrm{PG}(2, q)$ denote the standard projective plane over F, with points represented by *projective triples*, that is, equivalence classes $[a]$ of triples $a = (a_1, a_2, a_3) \neq (0, 0, 0)$ of elements of F, where two triples are equivalent if they are a non-zero multiple of each other. If $a = (a_1, a_2, a_3)$, we will simply write $[a] = [a_1, a_2, a_3]$. The vertex set of the *polarity graph* $B(q)$ is the set of all the $q^2 + q + 1$ points of $\mathrm{PG}(2, q)$, and two distinct vertices $[a]$ and $[b]$, where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, are adjacent in $B(q)$ if the corresponding triples are orthogonal, that is, if $ab^T = a_1b_1 + a_2b_2 + a_3b_3 = 0$. In the terminology of projective geometry this means that distinct vertices $[a]$ and $[b]$ are adjacent if and only if, in the projective plane $\mathrm{PG}(2, q)$, the point $[a]$ lies on the line with homogeneous coordinates $[b]$ and the point $[b]$ lies on the line with homogeneous coordinates $[a]$.

Parsons [12] derived a number of facts on the structure of polarity graphs. We will now give alternative proofs to some of the results of [12] we will need later. Our proofs are much shorter and are based on known facts about projective planes as presented in [8].

Let $[a] = [a_1, a_2, a_3]$ be a vertex of $B(q)$. Identification of neighbours of $[a]$ amounts to determining the solutions (x_1, x_2, x_3) of the linear equation $ax^T = a_1x_1 + a_2x_2 + a_3x_3 = 0$. This equation has $q^2 - 1$ non-zero solutions that represent $(q^2 - 1)/(q - 1) = q + 1$ distinct projective points, which are different from $[a]$ if and only if $aa^T = a_1^2 + a_2^2 + a_3^2 \neq 0$. It follows that a vertex $[a]$ has q or $q + 1$ neighbours in $B(q)$ according to whether aa^T is equal to zero or not. The projective triples $[a]$ of $\mathrm{PG}(2, q)$ such that $aa^T = 0$ are precisely those lying on the quadric $xx^T = 0$, which is non-degenerate (and hence a conic) if and only if q is odd. Accordingly, the vertices $[a]$ such that $aa^T = 0$ will be called *quadric vertices*.

Lemma 2.1. *The graph $B(q)$ contains exactly $q + 1$ quadric vertices.*

Proof. If q is odd, this is Theorem 5.21(i) of [8]. If q is even, a vertex $[a_1, a_2, a_3]$ is quadric if and only if $a_1^2 + a_2^2 + a_3^2 = 0$, which is in characteristic 2 equivalent to $a_1 + a_2 + a_3 = 0$, and there are exactly $(q^2 - 1)/(q - 1) = q + 1$ such projective triples in this case. □

The vertex set of $B(q)$ is thus a disjoint union of the set V of q^2 vertices of degree $q + 1$ and the set W of $q + 1$ quadric vertices, lying on the quadric $xx^T = 0$. Let V_1 be the subset of V comprising all vertices adjacent to at least one quadric vertex and let $V_2 = V \backslash V_1$. With this notation we now present further structural information on the polarity graphs $B(q)$ which we will use later.

Proposition 2.2. *For every prime power q the graph $B(q)$ has the following properties:*

(i) *The set W of quadric vertices is independent.*

(ii) *Each pair of vertices of V (adjacent or not) are connected by a unique path of length 2, while no edge incident to a quadric vertex is contained in any triangle; in particular, $B(q)$ has diameter 2.*

(iii) *If q is odd, then every vertex of V_1 is adjacent to exactly two quadric vertices, and $|V_1| = q(q+1)/2$, $|V_2| = q(q-1)/2$.*

(iv) *If q is odd, then the subgraphs of $B(q)$ induced by V_1 and V_2 are regular of degree $(q-1)/2$ and $(q+1)/2$, respectively.*

(v) *If q is even, then $|V_1| = q^2$ and V_2 is empty; moreover, V_1 contains a vertex v adjacent to all quadric vertices and every vertex in $V_1 \setminus \{v\}$ is adjacent to exactly one quadric vertex and the subgraph of $B(q)$ induced by the set $V_1 \setminus \{v\}$ is regular of degree q.*

Proof. Let $[a] = [a_1, a_2, a_3]$ and $[b] = [b_1, b_2, b_3]$ be two distinct vertices of $B(q)$, adjacent or not. Since the vectors a and b are linearly independent over F, the solution space of the linear system $ax^T = 0$, $bx^T = 0$ has dimension one. It follows that no pair of quadric vertices can be adjacent and that every pair of distinct vertices are connected by exactly one path of length two, proving (i) and (ii). Note that (ii) also follows from the property of $\mathrm{PG}(2, q)$ that any two points lie on a unique line.

Let q be odd. Invoking Chapters 7 and 8 of [8], the set W forms a conic and hence an oval. By Corollary 8.2 of [8] applied to the oval W, every vertex of V_1 and V_2 corresponds to a line of $\mathrm{PG}(2, q)$ containing exactly two points of W (a bisecant) or no point of W (an external line), respectively, and $|V_1| = q(q+1)/2$, $|V_2| = q(q-1)/2$, which proves (iii). Table 8.1 of [8] shows that a bisecant contains $(q-1)/2$ points each lying on exactly two lines determined by projective coordinates corresponding to a vertex in W, while an external line contains $(q+1)/2$ points each of which lies on no line determined by projective coordinates corresponding to a vertex in W. This exactly translates to (iv).

If q is even, the $q+1$ vertices of W have the form $[a_1, a_2, a_3]$ with $a_1 + a_2 + a_3 = 0$. The vertex $v = [1, 1, 1]$ adjacent to every vertex of W is, in the terminology of [8], the nucleus of W. By Corollary 8.8 of [8] on vertices different from the nucleus, every vertex of V_1 is incident to exactly one vertex of W and $V_2 = \emptyset$, proving (v). □

We note that the authors of [5] and [4] observed that, for q even, one may extend the polarity graph $B(q)$ by adding a vertex and making it incident to all vertices in W; the new graphs will still have diameter 2.

3 Polarity graphs and their automorphisms

The automorphism group of $B(q)$ was determined in [12]. Here we give a different and shorter proof, including a more detailed discussion on groups. The idea is to relate the polarity graphs $B(q)$ to the point-line incidence graph of $\mathrm{PG}(2, q)$. We will represent the points and lines of $\mathrm{PG}(2, q)$ by projective triples (vectors) of F^3 as in the case of vertices of $B(q)$, except that points will be represented by *row vectors* and lines will be represented by *column vectors* (distinguished by the 'transpose' superscript). In this notation, a point $[a]$ lies on a line $[b^T]$ in $\mathrm{PG}(2, q)$ if and only if $ab^T = 0$. The involution θ on the union of the point set and the line set of $\mathrm{PG}(2,q)$ that interchanges $[x]$ with $[x^T]$ is the *standard polarity* of $\mathrm{PG}(2,q)$. The *point-line incidence graph* $I(q)$ of $\mathrm{PG}(2,q)$ is the bipartite graph whose vertex set is the union of the point and line sets of $\mathrm{PG}(2,q)$, with a vertex $[a]$ adjacent to a vertex $[b^T]$ if and only if the point $[a]$ lies on the line $[b^T]$, that is, $ab^T = 0$. Observe that the standard polarity θ is an automorphism of the bipartite graph $I(q)$, interchanging its two vertex-parts.

The fundamental theorem of projective geometry (see e.g. [8]) tells us that the subgroup of all automorphisms of the graph $I(q)$ that fix each of its two vertex parts setwise is isomorphic to the extension $\mathrm{P\Gamma L}(3,q)$ of the 3-dimensional projective linear group $\mathrm{PGL}(3,q)$ over F by the group of Galois automorphisms of F over the prime field of F. Elements of $\mathrm{P\Gamma L}(3,q)$ are pairs (A,φ), where $A \in \mathrm{PGL}(3,q)$ can be identified with an invertible 3×3 matrix over F and $\varphi \in \mathrm{Gal}(F)$. The action of such an element (A,φ) on vertices $[x]$ and $[y^T]$ of $I(q)$ is given by first applying A via the assignment $[x] \mapsto [xA]$ and $[y^T] \mapsto [A^{-1}y^T]$ and then applying φ to all elements of the resulting projective triples.

We continue with a remark regarding orthogonal groups. By the 3*-dimensional projective orthogonal group* $\mathrm{PGO}(3,q)$ we mean the factor group of the subgroup of $\mathrm{GL}(3,q)$ consisting of orthogonal matrices by the centre of this subgroup (trivial if q is even and isomorphic to Z_2 if q is odd). In characteristic 2 our definition is different from what appears to be a more usual way of introducing an orthogonal group in terms of preservation of a bilinear form and having an irreducible action on a vector space; nevertheless we hope that no confusion will arise.

The obvious extension $\mathrm{P\Gamma O}(3,q)$ of $\mathrm{PGO}(3,q)$ by $\mathrm{Gal}(F)$ acts on $B(q)$ as a group of automorphisms. Indeed, an element of $\mathrm{P\Gamma O}(3,q)$ can be identified with a pair (A,φ) as above, but this time with A being a 3×3 orthogonal matrix, that is, such that $A^T = A^{-1}$, with the obvious identification of A with $-A$ if q is odd. The action is simply given by $(A,\varphi)[x] = [\varphi(xA)]$, and it preserves edges of $B(q)$ since $xy^T = 0$ is equivalent to $(xA)(yA)^T = 0$ by orthogonality of A. We begin by showing that that there are no other automorphisms of $B(q)$.

Theorem 3.1. *For every prime power q the automorphism group of the polarity graph $B(q)$ is isomorphic to* $\mathrm{P\Gamma O}(3,q)$.

Proof. Every automorphism α of $B(q)$ induces an automorphism $\tilde{\alpha}$ of $I(q)$ given by $\tilde{\alpha}[x] = \alpha[x]$ and $\tilde{\alpha}[x^T] = (\alpha[x])^T$ for every projective triple $[x]$. By the Fundamental theorem of projective geometry and our earlier discussion, the automorphism $\tilde{\alpha}$ may be represented by an action of a pair (A,φ) representing an element of $\mathrm{P\Gamma L}(3,q)$, given by $\tilde{\alpha}[x] = [\varphi(xA)]$ and $\tilde{\alpha}[x^T] = [\varphi(A^{-1}x^T)]$ for all projective triples $[x]$. But since $\tilde{\alpha}[x] = \alpha[x]$ and $\tilde{\alpha}[x^T] = (\alpha[x])^T$, we have $(\tilde{\alpha}[x])^T = \tilde{\alpha}[x^T]$ and hence $[\varphi(xA)]^T = [\varphi(A^{-1}x^T)]$. As this is valid for all pairs (A,φ), we have $[xA]^T = [A^{-1}x^T]$ for all projective triples $[x]$. Letting x be successively equal to $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$ and $(1,0,1)$ one concludes that $A^{-1} = \delta A^T$ for some non-zero $\delta \in F$. Taking determinants yields $1 = |\delta A^T A| = \delta^3 |A|^2$ and this is an equation between elements of the (cyclic) multiplicative group F^* of F.

We will show that there is a $\gamma \in F^*$ such that $\gamma^2 = \delta$. This is obvious if q is even (since in such a case every element of F^* has a unique square root), or if $\delta = 1$. If q is odd and $\delta \neq 1$, then $1 = \delta^3|A|^2$ implies that a third power of $\delta \neq 1$ is equal to 1 or to a second power of $|A|^{-1}$ if $|A| \neq 1$. This is, in the cyclic group F^* of *even* order $q-1$, possible only if $q-1$ is divisible by 3 and there is a $\gamma \in F^*$ such that $\delta = \gamma^2$ (and, in the second case, if $|A| = \gamma^{-3}$). In all circumstances we therefore have a $\gamma \in F^*$ such that $\delta = \gamma^2$.

The matrix $A_\gamma = \gamma A$ is orthogonal, i.e., $A_\gamma^{-1} = A_\gamma^T$. Since all our calculations are done with projective triples, the action of the original automorphism $\tilde{\alpha}$ may equivalently be described by $\tilde{\alpha}[x] = [\varphi(xA_\gamma)]$ and $\tilde{\alpha}[x^T] = [\varphi(A_\gamma^{-1}x^T)]$ for all projective triples $[x]$, where the pair (A_γ, φ) represents an element of $\mathrm{P\Gamma O}(3,q)$ as a subgroup of $\mathrm{P\Gamma L}(3,q)$. This shows that every automorphism of $B(q)$ is induced by an element of $\mathrm{P\Gamma O}(3,q)$. Since this group acts on $B(q)$ as we saw earlier, the result follows. □

It is known (cf. [8]) that $\mathrm{PGO}(3,q) \cong \mathrm{PGL}(2,q)$ and $\mathrm{P\Gamma O}(3,q) \cong \mathrm{P\Gamma L}(2,q)$ for every prime power q. Theorem 3.1 thus implies that if $q = p^n$ where p is a prime, then the graph $B(q)$ has exactly $nq(q^2-1)$ automorphisms.

The groups $\mathrm{P\Gamma O}(3,q)$ and $\mathrm{PGO}(3,q)$ obviously preserve the sets W, V_1 and V_2 and the sets W, $\{v\}$ and $V_1\backslash\{v\}$, depending on whether q is odd or even; it is easy to show that these sets are, in fact, orbits of $\mathrm{PGO}(3,q)$ on the vertices of $B(q)$. Corollary 7.15 of [8] tells us that the group $\mathrm{PGO}(3,q)$ is triply transitive on W. For odd q the analysis of [12] shows that $\mathrm{PGO}(3,q)$ acts arc-transitively on the subgraphs induced by the vertex set V_1 and V_2. We can say much more if q is even, extending the last result of Section 6 of [12]. Let $B_0(q)$ be the subgraph of $B(q)$ induced by the set $V_0 = V_1\backslash\{v\}$.

Theorem 3.2. *If q is a power of 2, the automorphism group of the graph $B_0(q)$ is isomorphic to $\mathrm{P\Gamma O}(3,q)$. Moreover, if $q \geq 4$, the smallest subgroup of $\mathrm{P\Gamma O}(3,q)$ acting transitively on vertices of $B_0(q)$ is the group $\mathrm{PGO}(3,q)$, which also acts regularly on arcs of $B_0(q)$. In particular, $B_0(q)$ is a vertex-transitive non-Cayley graph if $q > 2$.*

Proof. For every $w \in W$ let N_w be the set of neighbours of w in $B(q)$ distinct from v. Part (v) of Proposition 2.2 implies that the sets N_w ($w \in W$) form a partition of the vertex set of $B_0(q)$. In the subgraph $B_0(q)$ the distance of any two vertices is greater than 2 if and only if the two vertices are in the same set N_w for some $w \in W$. It follows that any automorphism of $B_0(q)$ preserves the partition $\{N_w;\ w \in W\}$ and hence extends to an automorphism of the entire polarity graph $B(q)$. Consequently, by Theorem 3.1, the automorphism group of $B_0(q)$ is isomorphic to $\mathrm{P\Gamma O}(3,q)$; this was also noted in Section 6 of [12].

The rest of the proof will address the smallest group transitive on vertices of $B_0(q)$. It is easy to check that $B_0(2)$ is isomorphic to a complete graph of order 3, admitting a regular action of a subgroup of $\mathrm{P\Gamma O}(3,q) \cong S_3$ isomorphic to Z_3. From now on we therefore assume that $q \geq 4$.

By the remark after Theorem 3.1 we know that $\mathrm{P\Gamma O}(3,q) \cong \mathrm{P\Gamma L}(2,q)$, and for $q = 2^n$ with $n \geq 2$ we have $\mathrm{P\Gamma L}(2,q) \cong \mathrm{SL}(2,q) \rtimes Z_n$, the split extension being defined by the natural action of $Z_n \cong \mathrm{Aut}(\mathrm{GF}(q))$ on $\mathrm{SL}(2,q)$. In what follows we will, for simplicity of the arguments, identify the group $\mathrm{P\Gamma O}(3,q)$ with the group $G = \mathrm{SL}(2,q) \rtimes Z_n$. We also let $K = \mathrm{SL}(2,q)$ and note that K is normal in G.

Suppose that H is a subgroup of G acting transitively on the vertex set of $B_0(q)$. Letting $H_0 = H \cap K$ and observing that H_0 is normal in H, from the second group isomorphism theorem we have $H/H_0 \cong HK/K$. It follows that the order of H/H_0 is a divisor of n. On the other hand, the transitivity assumption on H implies that the order of H is a multiple of q^2-1, the number of vertices of $B_0(q)$. These two facts imply that $|H_0| = t(q^2-1)/n$ for some positive integer t. Since H_0 has now been identified with a subgroup of $K =$

$\mathrm{SL}(2,q)$, the task reduces to identification of subgroups of the group $\mathrm{SL}(2,q) \cong \mathrm{PSL}(2,q)$, $q = 2^n$, of order $t(q^2-1)/n$.

We will show that $\mathrm{PSL}(2,q)$ admits no such *proper* subgroup H_0 if $n \geq 2$, using Dickson's classification of subgroups of $\mathrm{PSL}(2,q)$ for $q = 2^n$ as displayed in [13]. By this classification, subgroups of $\mathrm{PSL}(2,q)$ split into four classes: (a) cyclic and dihedral subgroups, (b) affine subgroups, (c) subgroups isomorphic to A_4 or A_5 for n even, and (d) subgroups of the form $\mathrm{PSL}(2,2^m)$ where m is a divisor of n. We analyse the cases separately, recalling that throughout we assume $q = 2^n$ and $n \geq 2$.

(a) *Cyclic and dihedral subgroups.* The largest order of a subgroup of $\mathrm{PSL}(2,q)$ that is cyclic or dihedral is $2(q+1)$. It is easy to see that $2(q+1) < t(q^2-1)/n$ for $n \geq 3$ and all $t \geq 1$; if $n = 2$ then t has to be even and then the same inequality holds. It follows that H_0 cannot be cyclic or dihedral.

(b) *Affine subgroups.* The largest order of an affine subgroup of $\mathrm{PSL}(2,q)$ is $q(q-1)$ and the second largest order of such a subgroup is $\sqrt{q}(\sqrt{q}-1)$ if n is even. If $q(q-1) = t(q^2-1)/n$, then $nq = t(q+1)$ and $q+1$ would have to divide n, a contradiction. For the second largest order, observe that $\sqrt{q}(\sqrt{q}-1) < q+1 \leq (q^2-1)/n$, the last inequality being a consequence of $n \geq q-1$. We conclude that H_0 cannot be affine.

(c) *The groups A_4, A_5.* We may eliminate A_4 since its order is 12 and the order of $B_0(2^2)$ is 15. Since the order of $B_0(2^4)$ is 2^8-1, the only feasible value of n for $H_0 \cong A_5$ is $n = 2$. In this case, however, $A_5 \cong \mathrm{PSL}(2,2^2)$ and so H_0 would not be a proper subgroup of $\mathrm{PSL}(2,2^2)$.

(d) *Subgroups* $\mathrm{PSL}(2,2^m)$ *where* $m \mid n$. Let $H_0 \cong \mathrm{PSL}(2,2^m)$ for m a proper divisor of n. If $n \in \{2,3\}$, then $m = 1$ and the order of $\mathrm{PSL}(2,2)$ is too small for H_0 to be transitive on vertices of $B_0(2^n)$. If $n \geq 4$, then the largest order of such a subgroup H_0 is at most $\sqrt{q}(q-1)$. It is easy to check, however, that $\sqrt{q}(q-1) < (q^2-1)/n$ for $n \geq 4$, giving a contradiction again.

The above arguments show that, for $q = 2^n$ and $n \geq 2$, the smallest subgroup of $\mathrm{P\Gamma O}(3,q) \cong \mathrm{P\Gamma L}(2,q)$ transitive on vertices of $B_0(q)$ is the group $\mathrm{PGO}(3,q) \cong \mathrm{PSL}(2,q)$. It is a matter of routine to check that this group is, in fact, regular on the arc set of $B_0(q)$. Combining the two facts we conclude that $B_0(q)$ is an arc-transitive (and, of course, vertex-transitive) non-Cayley graph for all $n \geq 2$. □

4 Vertex-transitive graphs from polarity graphs?

Polarity graphs are, of course, not vertex-transitive. Being the largest currently known examples of graphs of maximum degree $q+1$ and diameter 2, however, it is interesting to ask if one cannot add "a few" edges to a polarity graph to obtain a vertex-transitive graph. In [15] it was shown that it is impossible to construct a vertex-transitive graph of degree $q+1$ which would contain $B(q)$ as a spanning subgraph. We extend this result to the degree $q+3$ for odd $q \geq 37$.

Theorem 4.1. *For any odd prime power $q \geq 37$ there is no vertex-transitive graph of degree $q+3$ which contains the polarity graph $B(q)$ as a spanning subgraph.*

Proof. Let $B = B(q)$ and let B' be a graph containing B as a spanning subgraph. Let E and E' be the edge set of B and B', respectively; edges of the set E and $E' \backslash E$ will be called *old* and *new*, respectively. Suppose now that B' is a vertex-transitive graph of degree

$q+3$. Let u, v be vertices of B such that $e = uv$ is a new edge and let $N(u)$ and $N(v)$ be the set of neighbours of u and v in B. From the fact that any two distinct non-adjacent vertices of B are joined by a unique path of length 2 it follows that there is a unique vertex, say, w, in $N(u) \cap N(v)$. If $|N(u)| = |N(v)|$, the set of all old edges joining the set $N(u)\backslash\{w\}$ with the set $N(v)\backslash\{w\}$ forms a perfect matching between the two sets. If not, then $|N(u)|$ and $|N(v)|$ differ by one. Without loss of generality, if $|N(u)| = |N(v)| + 1$, then exactly one neighbour of u, say, t, is joined to w. In this case there is a perfect matching between the sets $N(u)\backslash\{w, t\}$ and $N(v)\backslash\{w\}$. It follows that any new edge $e = uv$ is contained in a set S_e of quadrangles such that (1) $|S_e| \geq q - 1$, and (2) any two quadrangles in S_e share just the edge e and its end-vertices u, v.

An edge $e \in E'$ will be called *thick* if there exists a set S_e of quadrangles with the properties (1) and (2) above. Note that this definition does not require e to be new, but the fact we have derived above implies that every new edge is thick. By the assumed vertex-transitivity, every vertex of B' must be incident to the same number, say, t, of thick edges. Since the degree of B' is supposed to be $q + 3$, every vertex in W is incident to at least three thick edges, so that $t \geq 3$. If $t = 3$, the three thick edges are all new and every vertex in V would be incident to exactly two new thick edges and one old thick edge. This would mean that there is a perfect matching on V formed by old thick edges, which is impossible since $|V| = q^2$ and q is odd. It follows that $t \geq 4$, every vertex in W is incident to at least one old thick edge and every vertex in V is incident to at least two old thick edges.

Properties of B imply that for any vertex $v \in V_2$ the subgraph of B induced by the set $N(v)$ is a perfect matching of $(q + 1)/2$ edges. Vertex-transitivity of B' then implies that for every vertex $w \in W$ the subgraph of B' induced by the set $N'(w)$ of all $q + 3$ neighbours of w in B' contains a subset E'_w of $(q + 1)/2$ mutually independent edges. Note that since there were no edges in $N(w)$ in the graph B, all the edges in E'_w must be new. At most three edges of E'_w join a vertex from $N(w)$ with a vertex in $N'(w)\backslash N(w)$, and so the subgraph of B' induced by the set $N(w)$ contains a subset $E_w \subset E'_w$ of least $(q + 1)/2 - 3 = (q - 5)/2$ new edges. Since any two neighbourhoods of vertices of W in the graph B intersect in exactly one vertex, the sets of new edges E_w, $w \in W$, are mutually disjoint. For counting, imagine that every new edge consists of two new half-edges incident to the corresponding end-vertices. The $q(q + 1)/2$ vertices of V_1, each incident with two new half-edges, are incident to a total of $q(q + 1)$ new half-edges. At least $(q + 1)(q - 5)$ of these are 'absorbed' by vertices of V_1 since the sets E_w, $w \in W$, are pairwise disjoint and $N(w) \subset V_1$ for every $w \in W$. It follows that there are at most $(q + 1)q - (q + 1)(q - 5) = 5(q + 1)$ new edges that join vertices of V_1 with vertices outside V_1. In particular, there are at most $5(q + 1)$ new edges between V_1 and V_2.

We know that every vertex $w \in W$ is incident to an old thick edge; let $e = wv$ be such an edge, $v \in V_1$. Let S_e be a set of quadrangles with the properties (1) and (2) introduced earlier. At most 3 quadrangles of S_e can contain a new edge incident with w, at most 2 such quadrangles can contain a new edge incident with v, and since v has at most $(q - 1)/2 + 2$ neighbours from V_1 in the graph B', at most $(q - 1)/2 + 2$ quadrangles in S_e contain a new edge having both end-vertices in V_1. Note that in each quadrangle containing three old edges the fourth edge must be new. Consequently, there are at least $(q - 1) - ((q - 1)/2 + 7) = (q - 15)/2$ quadrangles in S_e containing at least one new edge joining a vertex in V_1 with a vertex in V_2. Since our considerations are valid for every vertex $w \in W$, the neighbourhoods $N(w)$ in the graph B intersect just in one vertex, and every vertex in V_1 is incident to two new edges, we conclude that there are at least

$((q+1)(q-15)/2)/2$ new edges joining V_1 with V_2. But we have seen earlier that the number of such edges is at most $5(q+1)$. Thus, $(q+1)(q-15)/4 \leq 5(q+1)$, that is, $q \leq 35$, contrary to our assumption that $q \geq 37$. □

Another obvious method to create large vertex-transitive graphs from polarity graphs is to take a large vertex-transitive subgraph of $B(q)$ and try to extend it to a vertex-transitive graph of diameter 2 by adding edges. The hottest candidate for this is the subgraph $B_0(q)$ if q is a power of 2, which we have encountered in the previous section.

Theorem 3.2, however, is bad news for adding edges to $B_0(q)$ to produce a vertex-transitive graph of diameter two. Namely, the most natural approach would be to take the smallest group H acting transitively on the set of vertices of $B_0(q)$ and identify the smallest possible number of H-orbits of non-adjacent pairs of vertices so that making these pairs adjacent would yield a graph of diameter 2 with $B_0(q)$ as a spanning subgraph. But by Theorem 3.2, for $q \geq 4$ the smallest such subgroup H is isomorphic to $\mathrm{PGO}(3,q)$, acting transitively and with vertex stabilisers of order q. It follows that an orbit furnishing new edges would increase the degree of the resulting graph by at least q. This would make the resulting graph uninteresting from the point of view of the degree-diameter problem.

Before continuing let us comment on the isomorphism of the groups $\mathrm{PGO}(3,q)$ and $\mathrm{PGL}(2,q)$, mentioned after the proof of Theorem 3.1. In [8] an isomorphism of the two groups is given, induced by the quadratic form $x_2^2 + x_1x_3$. While for odd q such a form is equivalent to $x_1^2 + x_2^2 + x_3^2$ which we have been using, this is not true for even q since for q a power of 2 the form $x_1^2 + x_2^2 + x_3^2$ is degenerate. It is easy to check that, with respect to this form, all elements of $\mathrm{PGO}(3,q)$ for q even have the form

$$\begin{pmatrix} 1+a & 1+c & 1+a+c \\ 1+b & 1+d & 1+b+d \\ 1+a+b & 1+c+d & 1+a+b+c+d \end{pmatrix}$$

where $a,b,c,d \in F = \mathrm{GF}(q)$ and $ad+bc = 1$, which implies that $(a,b) \neq (0,0)$. Then, for even q, one may check that the mapping $\phi: \ \mathrm{PGO}(3,q) \to \mathrm{PGL}(2,q)$ given by

$$\begin{pmatrix} 1+a & 1+c & 1+a+c \\ 1+b & 1+d & 1+b+d \\ 1+a+b & 1+c+d & 1+a+b+c+d \end{pmatrix} \mapsto \begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

is a group isomorphism. We will use its inverse ϕ^{-1} in the proof of our last two result.

In contrast with Theorem 3.2, there exists a subgroup of $\mathrm{PGO}(3,q)$ that is transitive on the set $V^* = V_0 \setminus \{[t,t,1],\ t \in \mathrm{GF}(q)\}$; let $B^*(q)$ be the subgraph of $B_0(q)$ induced by the set V^*. Our last result shows that we can construct large Cayley graphs of diameter 2 and degree $d = q + O(\sqrt{q})$ by adding edges to $B^*(q)$. Since the order of $B^*(q)$ is $q(q-1) = d^2 - O(d^{3/2})$, the resulting graphs will be asymptotically approaching the Moore bound.

Theorem 4.2. *For every even prime power q there exists a Cayley graph of diameter* 2 *and degree* $q + O(\sqrt{q})$ *with* $B^*(q)$ *as a spanning subgraph.*

Proof. Let H be the subgroup of $\mathrm{PGO}(3,q)$ formed by all the matrices as in (4) for which $a+b+c+d = 0$. It is straightforward to check that $|H| = q(q-1)$ and that H acts regularly on the vertex set of the graph $B^*(q)$. It follows that $B^*(q)$ is a Cayley graph

$Cay(H, X)$ for the group H and some inverse-closed generating set $X \subset H$ such that $|X| = q - 1$, which is the degree of $B^*(q)$.

In order to create a Cayley graph of diameter 2 from $B^*(q)$ by adding $O(\sqrt{q})$ edges it is sufficient to show that one can extend the generating set X to a set $X' \supset X$ by adding $O(\sqrt{q})$ elements of H. Since we are dealing with a Cayley graph, it is sufficient to check distances from one particular vertex u, say, $u = [1, 0, 0]$. Compared with the graph $B(q)$, in our new graph $B^*(q)$ the vertex u lost two neighbours, namely, $v_1 = [0, 1, 1]$ and $v_2 = [0, 0, 1]$. We consider the effect caused by losing the two neighbours separately.

Since uv_1 is not an edge of $B^*(q)$, we lost the paths of length 2 joining u with vertices in the subset U_1 of $B^*(q)$ of the form $[1, t, t]$, $t \in F$, $t \neq 0, 1$. Consider the subgroup $H_1 < H$ formed by the matrices $\phi^{-1}(J_g)$, $g \in F^*$, where

$$J_g = \begin{pmatrix} g & 0 \\ g + g^{-1} & g^{-1} \end{pmatrix} .$$

It may be verified that $H_1 \cong F^*$ and H_1 acts regularly on the set $U_1 \cup \{u\}$. Geometrically, H_1 can be identified with the group of homologies (that is, central-axial collineations with a non-incident center-axis pair) with centre v_1 and axis vv_2. Since F^* is Abelian (in fact, cyclic), there exists an inverse-closed set X_1 of at most $\lceil 2\sqrt{q} \rceil$ elements such that the Cayley graph $Cay(H_1, X_1)$ with vertex set $U_1 \cup \{u\}$ has diameter 2.

The effect of the missing edge uv_2 is that we lost the paths of length 2 joining u with vertices in the subset U_2 of $B^*(q)$ of the form $[a + 1, a, 0]$, $a \in F^*$. Let now $H_2 < H$ be the subgroup of H consisting of the matrices $\phi^{-1}(L_a)$, $a \in F^+$, where

$$L_a = \begin{pmatrix} a + 1 & a \\ a & a + 1 \end{pmatrix} .$$

Obviously, $H_2 \cong F^+$ and H_2 is easily seen to be a regular permutation group on the set $U_2 \cup \{u\}$. From the point of view of geometry, H_2 can be identified with the group of elations (central-axial collineations with an incident center-axis pair) with centre $[1, 1, 0]$ and axis vv_2. Again, it is well known that there exists an inverse-closed set X_2 of at most $\lceil 2\sqrt{q} \rceil$ elements, making $Cay(H_2, X_2)$ a graph of diameter 2 with vertex set $U_2 \cup \{u\}$.

It is now easy to check that the graph $Cay(H, X')$ with $X' = X \cup X_1 \cup X_2$ has the required properties. □

Cayley graphs of diameter 2, order $q(q - 1)$ and degree $q + O(\sqrt{q})$ for even q have been recently constructed in [14] as follows. For q a power of 2 and $F = \mathrm{GF}(q)$ consider the one-dimensional affine group $G_q = \mathrm{AGL}(1, q) \simeq F^+ \rtimes F^*$, with elements written as pairs (g, a), $g \in F^*$, and $a \in F^+$ and with multiplication in the form $(g, a)(h, b) = (gh, ah + b)$ for $g, h \in F^*$ and $a, b \in F^+$. The set $Y_q = \{(y^2, y);\ y \in F^*\}$ is an inverse-closed generating set of G_q. The graphs of [14] are formed by taking the Cayley graph $Cay(G_q, Y_q)$ and adding a suitable set of further $O(\sqrt{q})$ generators to Y_q.

Our last result shows that, quite surprisingly, the Cayley graphs $Cay(G_q, Y_q)$ from [14] are isomorphic to the graphs $B^*(q)$. This gives the construction of [14] a strong geometric flavour.

Theorem 4.3. *If q is a power of 2, the graph $Cay(G_q, Y_q)$ is isomorphic to $B^*(q)$.*

Proof. We will use the notation introduced in the proof of Theorem 4.2, by which the graph $B^*(q)$ is isomorphic to the Cayley graph $Cay(H, X)$. It may be checked that every element of H can be written as a product $\phi^{-1}(J_g L_a)$ with $g \in F^*$ and $a \in F^+$. Let us identify this product with the ordered pair $[g, a]$. One easily verifies that in this identification the multiplication $\circ$ of elements of H is represented in the form $[g, a] \circ [h, b] = [gh, ah^{-2} + b]$. A further composition of the mapping $\phi^{-1}(J_g L_a) \mapsto [g, a]$ with $\psi : \ [g, a] \mapsto (g^{-2}, a)$ establishes an isomorphism $H \cong G_q$.

It remains to analyse the generating set X, which is uniquely determined by the elements of H that take a fixed vertex of $B^*(q)$ to all its neighbours; in what follows our fixed vertex will be $u = [1, 0, 0]$. Now, multiplication of the row vector $(1, 0, 0)$ by the matrix $\phi^{-1}(J_g L_a)$ yields the vector $(1 + ag, 1 + g + ag, 1 + g)$. This means that the element of H encoded $[g, a]$ takes the vertex u onto the vertex $[1 + ag, 1 + g + ag, 1 + g]$ of $B^*(q)$. Since the neighbours of u in $B^*(q)$ have the form $[0, 1, s]$ for $s \in F$, $s \neq 1$, the elements of H taking u to the neighbours of u are encoded by pairs $[t^{-1}, t]$ for $t \in F^*$. The generating set X of H thus consists, in our representation, of all the pairs $[t^{-1}, t]$, where $t \in F^*$. Composition with the mapping ψ introduced earlier finally establishes the isomorphism from the Cayley graph $Cay(H, X)$ onto the Cayley graph $Cay(G_q, Y_q)$ of [14]. □

5 Remarks

Adjacency in polarity graphs $B(q)$ has been defined by means of the standard dot product. For odd q the standard dot product is just a special case of a symmetric non-singular bilinear form. What happens if one uses a more general form instead? Following [2], for an odd prime power q let Q be a non-singular quadratic form over F^3 where $F = FG(q)$, and let $\beta(x, y) = Q(x + y) - Q(x) - Q(y)$ be the corresponding non-singular symmetric bilinear form. We now may let $B_\beta(q)$ be the graph on the same vertex set as $B(q)$, but with two distinct vertices $[a]$ and $[b]$ adjacent if $\beta(a, b) = 0$. It is known, however (see e.g. [2, 8]), that in dimension 3 and for odd q, all non-singular quadratic forms are equivalent and their equivalence is induced by linear transformations (i.e., by change of bases). It follows that for odd q all such graphs $B_\beta(q)$ are isomorphic to $B(q)$, with isomorphisms being provided by the corresponding linear transformations. Of course, such a correspondence between quadratic forms and bilinear forms fails in characteristic 2.

Another way of generalizing polarity graphs is to define them on more general finite projective planes. Recall that a *finite projective plane* $\mathcal{P}$ is a collection of a finite number of points and lines such that every two points are incident with a unique line, every two lines are incident with a unique point, and there are four points no three of which are incident with a line. It is well known that for any such $\mathcal{P}$ there is an integer n such that any line is incident with precisely $n + 1$ points and, dually, any point is incident with exactly $n + 1$ lines. One then speaks about a projective plane of *order* n, and it is then easy to show that the number of points and the number of lines are both equal to $n^2 + n + 1$. (An outstanding conjecture in finite geometry is that the order of a finite projective plane must be a prime power.)

Suppose now that $\mathcal{P}$ has a *polarity*, that is, a bijection π from the point set onto the line set of $\mathcal{P}$ with the property that for every two points A and B, A is incident with $\pi(B)$ if and only if B is incident with $\pi(A)$. We may then define the *generalized polarity graph* $B_{\mathcal{P},\pi}$ with vertex set equal to the point set of $\mathcal{P}$ and with two distinct points A and B adjacent if A is incident with $\pi(B)$. This graph obviously has diameter 2 by the properties of the projective plane. Observe that if $\mathcal{P} = \mathrm{PG}(2, q)$ is the standard projective plane as introduced in Section 2 and if one considers the standard polarity π interchanging projective vectors with their transposes, then the graph $B_{\mathcal{P},\pi}$ coincides with the polarity graph $B(q)$. It is of interest to point out that if $\mathcal{P}$ is a (general) finite projective plane of order n with a polarity π, then, by [1], the number $m(\pi)$ of self-conjugate points (those incident with their π-images) satisfies $m(\pi) \geq n+1$, and if $m(\pi) > n+1$ then n is a square and every prime divisor of n divides $m(\pi) - 1$. Since the corresponding generalized polarity graph $B_{\mathcal{P},\pi}$ has exactly $m(\pi)$ vertices of degree n and all the remaining vertices have degree $n+1$ it follows that for $n = q$, where q is an even power of a prime, the graph $B_{\mathcal{P},\pi}$ need not be isomorphic to $B(q)$. Investigation of such a generalization of polarity graphs may lead to interesting results.

We conclude by commenting on vertex-transitive extensions of polarity graphs. In general, by a *vertex-transitive closure* of a graph G we will understand any vertex-transitive supergraph of G on the same vertex set. We may then define $d_{vt}(G)$ to be the smallest degree of a vertex-transitive closure of G. In this terminology, our Theorem 4.1 says that $d_{vt}(B_q) \geq q + 5$ for any odd prime power $q \geq 37$. Determining or at least estimating $d_{vt}(G)$ for arbitrary graphs G appears to be an interesting problem on its own.

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References

[1] R. Baer, Polarities in finite projective planes, *Bull. Amer. Math. Soc.* **52** (1946), 77–93.

[2] S. Ball and Z. Weiner, An Introduction to Finite Geometry, Preprint (2011).

[3] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–285.

[4] C. Delorme, Examples of products giving large graphs with given degree and diameter, *Discrete Applied Math.* **37–38** (1992), 157–167.

[5] P. Erdős, S. Fajtlowicz and A. J. Hoffman, Maximum degree in graphs of diameter 2, *Networks* **10** (1980), 87–90.

[6] P. Erdős and A. Rényi, On a problem in the theory of graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 623–641.

[7] P. Erdős, A. Rényi and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

[8] J. Hirschfeld, *Projective geometries over finite fields*, Oxford University Press, 1998.

[9] A. J. Hoffman and R. R. Singleton, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497–504.

[10] B. Huppert, *Endliche Gruppen 1*, Springer, 1967.

[11] M. Miller and J. Širáň, Moore graphs and beyond: A survey of the degree-diameter problem, *Electronic J. Combin.* (2013), Dynamic survey DS14.

[12] T. D. Parsons, Graphs from projective planes, *Aequat. Math.* **14** (1976), 167–189.

[13] J. De Saedeleer and D. Leemans, On the rank two geometries of the groups $PSL(2,q)$: part I, *Ars Math. Contemp.* **3** (2010), 177–192.

[14] J. Šiagiová and J. Širáň, Approaching the Moore bound for diameter two by Cayley graphs, *J. Combin. Theory Ser. B* **102** (2012), 470–473.

[15] J. Širáň, J. Šiagiová and M. Ždímalová, Large graphs of diameter two and given degree, in: *Proc. IWONT 2010*, Univ. Politecnica de Catalunya, 2011, 347–359.

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Tight orientably-regular polytopes

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Abstract

It is known that every equivelar abstract polytope of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags. Polytopes that attain this lower bound are called *tight*. Here we investigate the conditions under which there is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$. We show that it is necessary and sufficient that whenever p_i is odd, both p_{i-1} and p_{i+1} (when defined) are even divisors of $2p_i$.

Keywords: Abstract regular polytope, equivelar polytope, flat polytope, tight polytope.

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1 Introduction

Abstract polytopes are ranked partially-ordered sets that resemble the face-lattice of a convex polytope in several key ways. Many discrete geometric objects can be viewed as an abstract polytope by considering their face-lattices, but there are also many new kinds of structures that have no immediate geometric analogue.

A *flag* of an abstract polytope is a chain in the poset that contains one element of each rank. In many ways, it is more natural to work with the flags of a polytope rather than the faces themselves. For example, every automorphism (order-preserving bijection) of a polytope is completely determined by its effect on any single flag.

Regular polytopes are those for which the automorphism group acts transitively on the set of flags. The automorphism group of a regular polytope is a quotient of some string Coxeter group $[p_1, \ldots, p_{n-1}]$, and conversely, every sufficiently nice quotient of a string

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Coxeter group appears as the automorphism group of a regular polytope. Indeed, it is actually possible to reconstruct a regular polytope from its automorphism group, so that much of the study of regular polytopes is purely group-theoretic.

We say a regular polytope $\mathcal{P}$ is of *type* (or has *Schläfli symbol*) $\{p_1, \ldots, p_{n-1}\}$ if $[p_1, \ldots, p_{n-1}]$ is the minimal string Coxeter group that covers the automorphism group of $\mathcal{P}$, in a way that $p_1, \ldots, p_{n-1}$ are the orders of the relevant generators. There is an equivalent formulation of this property that is entirely combinatorial, and hence it is possible to define a Schläfli symbol for many non-regular polytopes, including chiral polytopes (see [10]) and other two-orbit polytopes (see [7]). Any polytope with a well-defined Schläfli symbol is said to be *equivelar*.

In [3], the first author determined the smallest regular polytope (by number of flags) in each rank. To begin with, he showed that every regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags. Polytopes that meet this lower bound are called *tight*. He then exhibited a family of tight polytopes, one in each rank, of type $\{4, \ldots, 4\}$. Using properties of the automorphism groups of regular polytopes, he showed that each polytope was the smallest regular polytope in rank $n \geq 9$, and that in smaller ranks, the minimum was also attained by a tight polytope (with type or dual type $\{3\}$, $\{3, 4\}$, $\{4, 3, 4\}$, $\{3, 6, 3, 4\}$, $\{4, 3, 6, 3, 4\}$, $\{3, 6, 3, 6, 3, 4\}$ or $\{4, 3, 6, 3, 6, 3, 4\}$, respectively).

The second author showed in [5] that the bound on the number of flags extended to any equivelar polytope, regardless of regularity. Accordingly, it makes sense to extend the definition of tight polytopes to include any polytope of type $\{p_1, \ldots, p_{n-1}\}$ with $2p_1 \cdots p_{n-1}$ flags. An alternate formulation was proved as well, showing that an equivelar polytope is tight if and only if every face is incident with all faces (if any) that are 2 ranks higher.

Tightness is a restrictive property, and not every Schläfli symbol is possible for a tight polytope. In order for there to be a tight polytope of type $\{p_1, \ldots, p_{n-1}\}$, it is necessary that no two adjacent values p_i and p_{i+1} are odd. Theorem 5.1 in [5] shows that this condition is sufficient in rank 3. In higher ranks, the question of sufficiency is still open.

Constructing non-regular polytopes in high ranks is difficult. In order to determine which Schläfli symbols are possible for a tight polytope, it is helpful to begin by considering only regular polytopes. If every p_i is even, then Theorem 5.3 in [3] and Theorem 6.3 in [5] show that there is a tight regular polytope of type $\{p_1, \ldots, p_{n-1}\}$. Also the computational data from [2, 6] led the second author to conjecture that if p is odd and $q > 2p$, there is no tight regular polyhedron of type $\{p, q\}$. Although we are currently unable to prove this conjecture, we can show that for tight *orientably*-regular polyhedra, if p is odd then q must divide $2p$. Moreover, we are able to prove the following generalisation in higher ranks:

Theorem 1.1. *There is a tight orientably-regular polytope of type* $\{p_1, \ldots, p_{n-1}\}$ *if and only if each of the integers* p_{i-1} *and* p_{i+1} *(when defined) is an even divisor of* $2p_i$ *whenever* p_i *is odd, for* $1 \leq i < n$.

2 Background

Our background information is mostly taken from [8, Chs. 2, 3, 4], with a few small additions.

2.1 Definition of a polytope

Let $\mathcal{P}$ be a ranked partially-ordered set, the elements of which are called *faces*, and suppose that the faces of $\mathcal{P}$ range in rank from -1 to n. We call each face of rank j a j-*face*, and we say that two faces are *incident* if they are comparable. We also call the 0-faces, 1-faces and $(n-1)$-faces the *vertices*, *edges* and *facets* of $\mathcal{P}$, respectively. A *flag* is a maximal chain in $\mathcal{P}$. We say that two flags are *adjacent* if they differ in exactly one face, and that they are j-*adjacent* if they differ only in their j-faces.

If F and G are faces of $\mathcal{P}$ such that $F \leq G$, then the *section* G/F consists of those faces H such that $F \leq H \leq G$. If F is a j-face and G is a k-face, then we say that the *rank* of the section G/F is $k-j-1$. If removing G and F from the Hasse diagram of G/F leaves us with a connected graph, then we say that G/F is *connected.* That is, for any two faces H and H' in G/F (other than F and G themselves), there is a sequence of faces

$$H = H_0, H_1, \ldots, H_k = H'$$

such that $F < H_i < G$ for $0 \leq i \leq k$ and the faces H_{i-1} and H_i are incident for $1 \leq i \leq k$. By convention, we also define all sections of rank at most 1 to be connected.

We say that $\mathcal{P}$ is an (*abstract*) *polytope of rank* n, or briefly, an n-*polytope,* if it satisfies the following four properties:

(a) There is a unique greatest face F_n of rank n, and a unique least face F_{-1} of rank -1.

(b) Each flag has $n+2$ faces.

(c) Every section is connected.

(d) Every section of rank 1 is a diamond — that is, whenever F is a $(j-1)$-face and G is a $(j+1)$-face for some j, with $F < G$, there are exactly two j-faces H with $F < H < G$.

Condition (d) is known as the *diamond condition.* Note that this condition ensures that for $0 \leq j < n$, every flag Φ has a unique j-adjacent flag, which we denote by Φ^j.

In ranks -1, 0, and 1, there is a unique polytope up to isomorphism. Abstract polytopes of rank 2 are also called *abstract polygons*, and for each $2 \leq p \leq \infty$, there is a unique abstract polygon with p vertices and p edges, denoted by $\{p\}$.

If F is a j-face and G is a k-face of a polytope with $F \leq G$, then the section G/F itself is a $(k-j-1)$-polytope. We may identify a face F with the section F/F_{-1}, and call the section F_n/F the *co-face at* F. The co-face at a vertex F_0 is also called a *vertex-figure* at F_0.

If $\mathcal{P}$ is an n-polytope, F is an $(i-2)$-face of $\mathcal{P}$, and G is an $(i+1)$-face of $\mathcal{P}$ with $F < G$, then the section G/F is an abstract polygon. If it happens that for $1 \leq i < n$, each such section is (isomorphic to) the same polygon $\{p_i\}$, no matter which $(i-2)$-face F and incident $(i+1)$-face G we choose, then we say that $\mathcal{P}$ has *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$, or that $\mathcal{P}$ is *of type* $\{p_1, \ldots, p_{n-1}\}$. Also when this happens, we say that $\mathcal{P}$ is *equivelar*.

All sections of an equivelar polytope are themselves equivelar polytopes. In particular, if $\mathcal{P}$ is an equivelar polytope of type $\{p_1, \ldots, p_{n-1}\}$, then all its facets are equivelar polytopes of type $\{p_1, \ldots, p_{n-2}\}$, and all its vertex-figures are equivelar polytopes of type $\{p_2, \ldots, p_{n-1}\}$.

Next, let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes of the same rank. A surjective function $\gamma : \mathcal{P} \to \mathcal{Q}$ is called a *covering* if it preserves incidence of faces, ranks of faces, and adjacency of flags. If there exists such a covering $\gamma : \mathcal{P} \to \mathcal{Q}$, then we say that $\mathcal{P}$ *covers* $\mathcal{Q}$.

The *dual* of a polytope $\mathcal{P}$ is the polytope obtained by reversing the partial order. If $\mathcal{P}$ is an equivelar polytope of type $\{p_1, \ldots, p_{n-1}\}$, then the dual of $\mathcal{P}$ is an equivelar polytope of type $\{p_{n-1}, \ldots, p_1\}$.

2.2 Regularity

For polytopes $\mathcal{P}$ and $\mathcal{Q}$, an *isomorphism* from $\mathcal{P}$ to $\mathcal{Q}$ is an incidence- and rank-preserving bijection. By connectedness and the diamond condition, every polytope isomorphism is uniquely determined by its effect on a given flag. An isomorphism from $\mathcal{P}$ to itself is an *automorphism* of $\mathcal{P}$, and the group of all automorphisms of $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$. We will denote the identity automorphism by ε.

We say that $\mathcal{P}$ is *regular* if the natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$ is transitive (and hence regular, in the sense of being sharply-transitive). For convex polytopes, this definition is equivalent to any of the usual definitions of regularity.

Now let $\mathcal{P}$ be any regular polytope, and choose a flag Φ, which we call a *base flag*. Then the automorphism group $\Gamma(\mathcal{P})$ is generated by the *abstract reflections* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i maps Φ to the unique flag Φ^i that is i-adjacent to Φ. These generators satisfy $\rho_i^2 = \varepsilon$ for all i, and $(\rho_i\rho_j)^2 = \varepsilon$ for all i and j such that $|i-j| \geq 2$. Every regular polytope is equivelar, and if its Schläfli symbol is $\{p_1, \ldots, p_{n-1}\}$, then the order of each $\rho_{i-1}\rho_i$ is p_i. Note that if $\mathcal{P}$ is a regular polytope of type $\{p_1, \ldots, p_{n-1}\}$, then $\Gamma(\mathcal{P})$ is a quotient of the string Coxeter group $[p_1, \ldots, p_{n-1}]$, which is the abstract group generated by n elements $x_0, \ldots, x_{n-1}$ subject to the defining relations $x_i^2 = 1$, $(x_{i-1}x_i)^{p_i} = 1$, and $(x_ix_j)^2 = 1$ whenever $|i-j| \geq 2$.

Next, if Γ is any group generated by elements $\rho_0, \ldots, \rho_{n-1}$, we define $\Gamma_I = \langle \rho_i \mid i \in I\rangle$ for each subset I of the index set $\{0, 1, \ldots, n-1\}$. If Γ is the automorphism group $\Gamma(\mathcal{P})$ of a regular polytope $\mathcal{P}$, then these subgroups satisfy the following condition, known as the *intersection condition*:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I\cap J} \quad \text{for all } I, J \subseteq \{0, 1, \ldots, n-1\}. \tag{2.1}$$

More generally, if Γ is any group generated by elements $\rho_0, \ldots, \rho_{n-1}$ of order 2 such that $(\rho_i\rho_j)^2 = 1$ whenever $|i-j| \geq 2$, then we say that Γ is a *string group generated by involutions*, and abbreviate this to say that Γ is an *sggi*. If the sggi Γ also satisfies the intersection condition (2.1) given above, then we call Γ a *string C-group*.

There is a natural way of building a regular polytope $\mathcal{P}(\Gamma)$ from a string C-group Γ such that $\Gamma(\mathcal{P}(\Gamma)) \cong \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \cong \mathcal{P}$. In particular, the i-faces of $\mathcal{P}(\Gamma)$ are taken to be the cosets of the subgroup $\Gamma_i = \langle \rho_j \mid j \neq i\rangle$, with incidence of faces $\Gamma_i\varphi$ and $\Gamma_j\psi$ given by

$$\Gamma_i\varphi \leq \Gamma_j\psi \quad \text{if and only if} \quad i \leq j \;\text{ and }\; \Gamma_i\varphi \cap \Gamma_j\psi \neq \emptyset.$$

This construction is also easily applied when Γ is any sggi (not necessarily a string C-group), but in that case, the resulting poset is not always a polytope.

The following theory from [8] helps us determine when a given sggi is a string C-group:

Proposition 2.1. *Let* $\Gamma = \langle \rho_0, \ldots, \rho_{n-1}\rangle$ *be an sggi, and* $\Lambda = \langle \lambda_0, \ldots, \lambda_{n-1}\rangle$ *a string C-group. If there is a homomorphism* $\pi : \Gamma \to \Lambda$ *sending each* σ_i *to* λ_i*, and if* π *is one-to-one on the subgroup* $\langle \rho_0, \ldots, \rho_{n-2}\rangle$ *or the subgroup* $\langle \rho_1, \ldots, \rho_{n-1}\rangle$*, then* Γ *is a string C-group.*

Proposition 2.2. *Let* $\Gamma = \langle \rho_0, \ldots, \rho_{n-1}\rangle$ *be an sggi. If both* $\langle \rho_0, \ldots, \rho_{n-2}\rangle$ *and* $\langle \rho_1, \ldots, \rho_{n-1}\rangle$ *are string C-groups, and* $\langle \rho_0, \ldots, \rho_{n-2}\rangle \cap \langle \rho_1, \ldots, \rho_{n-1}\rangle \subseteq \langle \rho_1, \ldots, \rho_{n-2}\rangle$*, then* Γ *is a string C-group.*

Given a regular n-polytope $\mathcal{P}$ with automorphism group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, we define the *abstract rotations* $\sigma_1, \ldots, \sigma_{n-1}$ by setting $\sigma_i = \rho_{i-1}\rho_i$ for $1 \le i < n$. Then the subgroup $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ of $\Gamma(\mathcal{P})$ is denoted by $\Gamma^+(\mathcal{P})$, and called the *rotation subgroup of* $\mathcal{P}$. The index of $\Gamma^+(\mathcal{P})$ in $\Gamma(\mathcal{P})$ is at most 2, and when the index is exactly 2, then we say that $\mathcal{P}$ is *orientably-regular*. Otherwise, if $\Gamma^+(\mathcal{P}) = \Gamma(\mathcal{P})$, then we say that $\mathcal{P}$ is *non-orientably-regular*. (This notation comes from the study of regular maps.) A regular polytope $\mathcal{P}$ is orientably-regular if and only if $\Gamma(\mathcal{P})$ has a presentation in terms of the generators $\rho_0, \ldots, \rho_{n-1}$ such that all of the relators have even length. Note that every section of an orientably-regular polytope is itself orientably-regular.

2.3 Flat and tight polytopes

The theory of abstract polytopes accommodates certain degeneracies not present in the study of convex polytopes. For example, the face-poset of a convex polytope is a lattice (which means that any two elements have a unique supremum and infimum), but this need not be the case with abstract polytopes. The simplest abstract polytope that is not a lattice is the digon $\{2\}$, in which both edges are incident with both vertices. This type of degeneracy can be generalised as follows. If $\mathcal{P}$ is an n-polytope, and $0 \le k < m < n$, then we say that $\mathcal{P}$ is (k,m)*-flat* if every one of its k-faces is incident with every one of its m-faces. If $\mathcal{P}$ has rank n and is $(0, n-1)$-flat, then we also say simply that $\mathcal{P}$ is a *flat polytope*. Note that if $\mathcal{P}$ is (k,m)-flat, then $\mathcal{P}$ must also be (i,j)-flat whenever $0 \le i \le k < m \le j < n$. In particular, if $\mathcal{P}$ is (k,m)-flat, then it is also flat.

We will also need the following, taken from [8, Lemma 4E3]:

Proposition 2.3. *Let $\mathcal{P}$ be an n-polytope, and let $0 \le k < m < i < n$. If each i-face of $\mathcal{P}$ is (k,m)-flat, then $\mathcal{P}$ is also (k,m)-flat. Similarly, if $0 \le i < k < m < n$ and each co-i-face of $\mathcal{P}$ is $(k-i-1, m-i-1)$-flat, then $\mathcal{P}$ is (k,m)-flat.*

It is easy to see that the converse is also true. In other words, if $\mathcal{P}$ is (k,m)-flat, then for $i > m$ each i-face of $\mathcal{P}$ is (k,m)-flat, and for $j < k$ each co-j-face of $\mathcal{P}$ is $(k-j-1, m-j-1)$-flat.

Next, we consider tightness. An equivelar polytope $\mathcal{P}$ of type $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags, by [5, Proposition 3.3]. Whenever $\mathcal{P}$ has exactly this number of flags, we say that $\mathcal{P}$ is *tight*. It is clear that $\mathcal{P}$ is tight if and only if its dual is tight, and that in a tight polytope, every section of rank 3 or more is tight. Also we will need the following, taken from [5, Theorem 4.4]:

Theorem 2.4. *Let $n \ge 3$ and let $\mathcal{P}$ be an equivelar n-polytope. Then $\mathcal{P}$ is tight if and only if it is $(i, i+2)$-flat for $0 \le i \le n-3$.*

Later in this paper we will build polytopes inductively, and for that, the following approach is useful. We say that the regular n-polytope $\mathcal{P}$ has the *flat amalgamation property* (or FAP) with respect to its k-faces, if adding the relations $\rho_i = \varepsilon$ for $i \ge k$ to $\Gamma(\mathcal{P})$ yields a presentation for $\langle \rho_0, \ldots, \rho_{k-1} \rangle$. Similarly, we say that $\mathcal{P}$ has the FAP with respect to its co-k-faces if adding the relations $\rho_i = \varepsilon$ for $i \le k$ yields a presentation for $\langle \rho_{k+1}, \ldots, \rho_{n-1} \rangle$.

We will also use the following, taken from [8, Theorem 4F9]:

Theorem 2.5. *Suppose $m, n \ge 2$, and $0 \le k \le m-2$ where $k \ge m-n$. Let $\mathcal{P}_1$ be a regular m-polytope, and let $\mathcal{P}_2$ be a regular n-polytope such that the co-k-faces of $\mathcal{P}_1$ are* *isomorphic to the $(m-k-1)$-faces of $\mathcal{P}_2$. Also suppose that $\mathcal{P}_1$ has the FAP with respect to its co-k-faces, and that $\mathcal{P}_2$ has the FAP with respect to its $(m-k-1)$-faces. Then there exists a regular $(k+n+1)$-polytope $\mathcal{P}$ such that $\mathcal{P}$ is (k,m)-flat, and the m-faces of $\mathcal{P}$ are isomorphic to $\mathcal{P}_1$, while the co-k-faces of $\mathcal{P}$ are isomorphic to $\mathcal{P}_2$. Furthermore, $\mathcal{P}$ has the FAP with respect to its m-faces and its co-k-faces.*

3 Tight orientably-regular polyhedra

We now consider the values of p and q for which there is a tight orientably-regular polyhedron of type $\{p,q\}$. By [5, Proposition 3.5], there are no tight polyhedra of type $\{p,q\}$ when p and q are both odd. Also by [3, Theorem 5.3] and [5, Theorem 6.3], if p and q are both even then there exists a tight orientably-regular polyhedron whose automorphism group is the quotient of the Coxeter group $[p,q]$ obtained by adding the extra relation $(x_0x_1x_2x_1)^2 = 1$. (Note that this is the group of the polyhedron $\{p,q \mid 2\}$ described in [8, p. 196]; see [9] for more information on this and related polyhedra.) Indeed, for even p and q, there can often be multiple (non-isomorphic) tight polyhedra; for example, there are two of type $\{4,8\}$ that are non-isomorphic (see [2]).

When p is odd and q is even (or vice-versa), the situation is more complicated. Evidence from [2] and [6] led the second author to conjecture that there are no tight regular polyhedra of type $\{p,q\}$ if p is odd and $q > 2p$ (see [5]). We will show that this is true in the orientably-regular case. In fact, we will prove something stronger, namely that q must divide $2p$.

We start by showing that if p is odd and q is an even divisor of $2p$, then there is a tight orientably-regular polyhedron of type $\{p,q\}$. To do this, we define $\Gamma(p,q)$ as the group $\langle\, \rho_0, \rho_1, \rho_2 \mid \rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^p, (\rho_1\rho_2)^q, (\rho_0\rho_2)^2, (\rho_0\rho_1\rho_2\rho_1\rho_2)^2 \,\rangle$, which is obtainable by adding one extra relator to the Coxeter group $[p,q]$. (Note that this is the group of the polyhedron $\{p,q\}_{\cdot,2}$ described in [8, p. 196].)

Theorem 3.1. *Let $p \geq 3$ be odd, and let q be an even divisor of $2p$. Then there is a tight orientably-regular polyhedron $\mathcal{P}$ of type $\{p,q\}$ such that $\Gamma(\mathcal{P}) \cong \Gamma(p,q)$.*

Proof. Let $\Gamma(p,q) = \langle \rho_0, \rho_1, \rho_2 \rangle$. In light of the construction in Section 2.2, all we need to do is show that $\Gamma(p,q)$ is a string C-group of order $2pq$, in which the order of $\rho_0\rho_1$ is p and the order of $\rho_1\rho_2$ is q.

First, note that the element $\omega = (\rho_1\rho_2)^2$ generates a cyclic normal subgroup N of $\Gamma(p,q)$, since each of ρ_1 and ρ_2 conjugates ω to its inverse, and the extra relation $(\rho_0\rho_1\rho_2\rho_1\rho_2)^2 = 1$ implies that ρ_0 does the same. Factoring out N gives quotient $\Gamma(p,2)$, in which the extra relation $(\rho_0\rho_1\rho_2\rho_1\rho_2)^2 = 1$ is redundant. In fact $\Gamma(p,2)$ is isomorphic to the string Coxeter group $[p,2]$, which is an extension of the dihedral group of order $2p$, and has order $4p$.

In particular, $\Gamma(p,q)$ covers $\Gamma(p,2) \cong [p,2]$, and it follows that $\rho_0\rho_1$ has order p (rather than some proper divisor of p). Also the cover from $\Gamma(p,q)$ to $\Gamma(p,2)$ is one-to-one on $\langle \rho_0, \rho_1 \rangle$, and so by Proposition 2.1, we find that $\Gamma(p,q)$ is a string C-group.

Next, we observe that the dihedral group $D_q = \langle\, y_1, y_2 \mid y_1^2, y_2^2, (y_1y_2)^q \,\rangle$ is a quotient of $\Gamma(p,q)$, via an epimorphism taking $\rho_1 \mapsto y_1$, $\rho_2 \mapsto y_2$ and $\rho_0 \mapsto (y_1y_2)^{p-1}y_1$. (Note that the defining relations for $\Gamma(p,q)$ are satisfied by their images in D_q, since $(y_1y_2)^{p-1}y_1$ has order 2, the order of $(y_1y_2)^{p-1}$ divides p (as $p-1$ is even and q divides $2p$), the order of $(y_1y_2)^{p-1}y_1y_2 = (y_1y_2)^p$ divides 2 (as q divides $2p$), and $(y_1y_2)^{p-1}y_2y_1y_2$ has order 2.) In particular, the image of $\rho_1\rho_2$ is y_1y_2, which has order q, and hence $\rho_1\rho_2$ has order q.

Finally, $|\Gamma(p,q)| = |\Gamma(p,q)/N||N| = |\Gamma(p,2)||N| = 4p(q/2) = 2pq$, since $N = \langle(\rho_1\rho_2)^2\rangle$ has order $q/2$. □

We will show that in fact, the only tight orientably-regular polyhedra of type $\{p,q\}$ with p odd are those given in Theorem 3.1. We proceed with the help of a simple lemma.

Lemma 3.2. *Let $\mathcal{P}$ be an orientably-regular polyhedron of type $\{p,q\}$, with p odd, and with automorphism group $\Gamma(\mathcal{P})$ generated by the reflections ρ_0, ρ_1, ρ_2. If $\omega = (\rho_1\rho_2)^2 = \sigma_2^2$ generates a normal subgroup of $\Gamma^+(\mathcal{P})$, then ω is central, and q divides $2p$.*

Proof. For simplicity, let $x = \sigma_1 = \rho_0\rho_1$ and $y = \sigma_2 = \rho_1\rho_2$, so that $xy = \rho_0\rho_2$ and hence $x^p = y^q = (xy)^2 = 1$, and also $\omega = y^2$. By hypothesis, $\langle y^2\rangle$ is normal, and so $xy^2x^{-1} = y^{2k}$ for some k. It follows that $y^2 = x^p y^2 x^{-p} = y^{2k^p}$ and that $y^2 = (xy)^2 y^2 (xy)^{-2} = y^{2k^2}$, and therefore $2 \equiv 2k^2 \equiv 2k^p \bmod q$. Then also $2 \equiv 2k^2k^{p-2} \equiv 2k^{p-2} \bmod q$, and by induction $2 \equiv 2k^p \equiv 2k^{p-2} \equiv \cdots \equiv 2k \bmod q$, since p is odd. Thus $xy^2x^{-1} = y^{2k} = y^2$, and so $\omega = y^2$ is central. Moreover, since y^2 commutes with x (which has order p) and $xy = y^{-1}x^{-1}$, we find that $y^{2p} = x^p y^{2p} = (xy^2)^p = (y^{-1}x^{-1}y)^p = y^{-1}x^{-p}y = 1$, and so $2p$ is a multiple of q. □

We also utilise a connection between tight polyhedra and regular Cayley maps, as is explained in [4]. Specifically, suppose that the finite group G is generated by two non-involutory elements x and y such that xy has order 2, and that G can be written as AY where $Y = \langle y\rangle$ is core-free in G (that is, Y contains no non-trivial normal subgroup of G), and A is a subgroup of G such that $A \cap Y = \{1\}$. Then G is the group $\Gamma^+(M)$ of orientation-preserving automorphisms group of a regular Cayley map M for the group A. Furthermore, this map M is reflexible if and only if G admits an automorphism taking $x \mapsto xy^2$ $(= y^{-1}x^{-1}y)$ and $y \mapsto y^{-1}$.

Theorem 3.3. *Let $p \geq 3$ be odd. If $\mathcal{P}$ is a tight orientably-regular polyhedron of type $\{p,q\}$, then q is an even divisor of $2p$, and $\Gamma(\mathcal{P})$ is isomorphic to $\Gamma(p,q)$.*

Proof. Let $G = \Gamma^+(\mathcal{P})$, and let $\sigma_1 = \rho_0\rho_1$ and $\sigma_2 = \rho_1\rho_2$ be its standard generators. Also take $F = \langle\sigma_1\rangle$ and $V = \langle\sigma_2\rangle$, which are the stabilisers in $\Gamma^+(\mathcal{P})$ of a 2-face and incident vertex of $\mathcal{P}$. Then $F \cap V = \langle\varepsilon\rangle$ since $\mathcal{P}$ is a polytope, and $G = FV$ since $\mathcal{P}$ is tight.

Now, let N be the core of V in G (which is the largest normal subgroup of G contained in V), and let $\overline{G} = G/N$, $\overline{V} = V/N$ and $\overline{F} = FN/N$. Then $\overline{G} = \overline{V}\,\overline{F}$, and $\overline{V} \cap \overline{F}$ is trivial, and also $\overline{V}$ is core-free. Thus $\overline{G}$ is the orientation-preserving automorphism group of a regular Cayley map M for the cyclic group $\overline{F}$. Furthermore, since $\mathcal{P}$ is an orientably-regular polyhedron, the group $G = \Gamma^+(\mathcal{P})$ has an automorphism taking $\sigma_1 \mapsto \sigma_1\sigma_2^2$ and $\sigma_2 \mapsto \sigma_2^{-1}$, and $\overline{G}$ has the analogous property. Hence M is reflexible.

On the other hand, by [4, Theorem 3.7] we know that the only reflexible regular Cayley map for a cyclic group of odd order p is the equatorial map on the sphere, with p vertices of valence 2. Thus $|\overline{V}| = 2$, and so $q = |V|$ is even, and $N = \langle\sigma_2^2\rangle$.

In particular, $\langle\sigma_2^2\rangle$ is a normal subgroup of $\Gamma^+(\mathcal{P})$, and hence by Lemma 3.2 we also find that q divides $2p$, and that σ_2^2 is central. But $\sigma_2^2 = (\rho_1\rho_2)^2$ is inverted under conjugation by ρ_1, and now centralised by $\sigma_1 = \rho_0\rho_1$, and therefore also inverted under conjugation by ρ_0. Hence the relation $(\rho_0\rho_1\rho_2\rho_1\rho_2)^2 = \varepsilon$ holds in $\Gamma(\mathcal{P})$, so $\Gamma(\mathcal{P})$ is a quotient of $\Gamma(p,q)$. Then finally, since $\mathcal{P}$ is tight we have $|\Gamma(\mathcal{P})| = 2pq = |\Gamma(p,q)|$, and it follows that $\Gamma(\mathcal{P}) \cong \Gamma(p,q)$. □

Combining Theorem 3.1 with Theorem 3.3 and [5, Theorem 6.3], we can now draw the following conclusion:

Theorem 3.4. *There is a tight orientably-regular polyhedron of type* $\{p, q\}$ *if and only if one of the following is true*:

(a) p *and* q *are both even, or*

(b) p *is odd and* q *is an even divisor of* $2p$*, or*

(c) q *is odd and* p *is an even divisor of* $2q$*.*

4 Tight orientably-regular polytopes in higher ranks

Now that we know which Schläfli symbols appear among tight orientably-regular polyhedra, we can proceed to classify the tight orientably-regular polytopes of arbitrary rank. We will say that the $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$ is *admissible* if each of p_{i-1} and p_{i+1} (when defined) is an even divisor of $2p_i$ whenever p_i is odd.

Theorem 4.1. *If* $\mathcal{P}$ *is a tight orientably-regular polytope of type* $\{p_1, \ldots, p_{n-1}\}$*, then the* $(n-1)$*-tuple* $(p_1, \ldots, p_{n-1})$ *is admissible.*

Proof. If $\mathcal{P}$ is tight and orientably-regular, then by [5, Proposition 3.8] we know that all of its sections of rank 3 are tight and orientably-regular. Hence in particular, if p_i is odd then the sections of $\mathcal{P}$ of type $\{p_{i-1}, p_i\}$ and $\{p_i, p_{i+1}\}$ are tight and orientably-regular. The rest now follows from Theorem 3.3. □

We will prove that this necessary condition is also sufficient, which will then complete the proof of Theorem 1.1. We do this by constructing the automorphism group of a tight orientably-regular polytope of the given type.

Let $(p_1, \ldots, p_{n-1})$ be an admissible $(n-1)$-tuple. Then we define the group $\Gamma(p_1, \ldots, p_{n-1})$ to be the quotient of the string Coxeter group $[p_1, \ldots, p_{n-1}]$ obtained by adding n–2 extra relations $r_1 = \cdots = r_{n-2} = 1$, where

$$r_i = \begin{cases} (x_{i-1}x_ix_{i+1}x_i)^2 & \text{if } p_i \text{ and } p_{i+1} \text{ are both even, or} \\ (x_{i-1}x_ix_{i+1}x_ix_{i+1})^2 & \text{if } p_i \text{ is odd and } p_{i+1} \text{ is even, or} \\ (x_{i+1}x_ix_{i-1}x_ix_{i-1})^2 & \text{if } p_i \text{ is even and } p_{i+1} \text{ is odd.} \end{cases}$$

Note that if $n = 3$, this definition of $\Gamma(p_1, p_2)$ coincides with the one in the previous section. For $n \geq 4$, the group $\Gamma(p_1, \ldots, p_{n-1})$ is the amalgamation of $\Gamma(p_1, \ldots, p_{n-2})$ with $\Gamma(p_2, \ldots, p_{n-1})$ in the obvious way, subject to the extra relation $(x_0x_{n-1})^2 = 1$.

Also let $\mathcal{P}(p_1, \ldots, p_{n-1})$ be the poset obtained from $\Gamma(p_1, \ldots, p_{n-1})$, using the construction in Section 2.2. We will show that $\Gamma(p_1, \ldots, p_{n-1})$ is a string C-group of order $2p_1p_2 \ldots p_{n-1}$, and then since every relator of $\Gamma(p_1, \ldots, p_{n-1})$ has even length, it follows that $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$.

We start by considering the order of $\Gamma(p_1, \ldots, p_{n-1})$.

Proposition 4.2. *Let* p_i *be even. Then every element of* $\Gamma(p_1, \ldots, p_{n-1})$ *either commutes with* $(x_{i-1}x_i)^2$ *or inverts it by conjugation. In particular, the square of every element of* $\Gamma(p_1, \ldots, p_{n-1})$ *commutes with* $(x_{i-1}x_i)^2$*.*

Proof. Let $\omega = (x_{i-1}x_i)^2$. If $j \le i-3$ or $j \ge i+2$, then x_j commutes with both x_{i-1} and x_i, and so commutes with ω. Also it is clear that x_{i-1} and x_i both conjugate ω to ω^{-1}, so it remains to consider only x_{i-2} and x_{i+1}. Now since p_i is even, the relator r_{i-1} is either $(x_{i-2}x_{i-1}x_ix_{i-1})^2$ or $(x_{i-2}x_{i-1}x_ix_{i-1}x_i)^2$. In the first case, x_{i-2} commutes with $x_{i-1}x_ix_{i-1}$ and hence with $(x_{i-1}x_ix_{i-1})x_i = \omega$, while in the second case, we have $(x_{i-2}\omega)^2 = 1$ and so x_{i-2} conjugates ω to ω^{-1}. Similarly, the relator r_i is $(x_{i-1}x_ix_{i+1}x_i)^2$ or $(x_{i+1}x_ix_{i-1}x_ix_{i-1})^2$, and in these two cases we find that $x_{i+1}\omega x_{i+1} = \omega$ or ω^{-1}, respectively. Thus every generator x_j of $\Gamma(p_1, \dots, p_{n-1})$ either commutes with $(x_{i-1}x_i)^2$ or inverts it by conjugation, and it follows that the same is true for every element of $\Gamma(p_1, \dots, p_{n-1})$. The rest follows easily. □

Proposition 4.3. *Let $(p_1, \dots, p_{n-1})$ be an admissible $(n-1)$-tuple with the property that for every i strictly between 1 and n–1, either $p_i = 2$ or $p_{i-1} = p_{i+1} = 2$. Then $y_i = x_{i-1}x_i$ has order p_i for all i, and $|\Gamma(p_1, \dots, p_{n-1})| = 2p_1 \cdots p_{n-1}$.*

Proof. We use induction on n, together with the observation that if $p_j = 2$, then $1 = (x_{j-1}x_j)^2$, so that x_{j-1} commutes with x_j, and therefore $\langle x_0, \dots, x_{j-1}\rangle$ centralises $\langle x_j, \dots, x_{n-1}\rangle$. First, if $p_1 = 2$, then $\Gamma(p_1, p_2, \dots, p_{n-1}) = \Gamma(2, p_2, \dots, p_{n-1}) \cong \langle x_0\rangle \times \Gamma(p_2, \dots, p_{n-1})$, and so $|\Gamma(p_1, \dots, p_{n-1})| = 2|\Gamma(p_2, \dots, p_{n-1})| = 4p_2 \cdots p_{n-1} = 2p_1p_2 \cdots p_{n-1}$. Otherwise $p_2 = 2$ and $\Gamma(p_1, p_2, \dots, p_{n-1}) = \Gamma(p_1, 2, p_3, \dots, p_{n-1}) \cong \Gamma(p_1) \times \Gamma(p_3, \dots, p_{n-1})$, and therefore $|\Gamma(p_1, \dots, p_{n-1})| = |\Gamma(p_1)|\,|\Gamma(p_3, \dots, p_{n-1})| = 2p_1\,2p_3 \cdots p_{n-1} = 2p_1p_2 \cdots p_{n-1}$. The claim about the orders of the elements $y_i = x_{i-1}x_i$ follows easily by induction as well. □

Lemma 4.4. *Let q_i be the order of $x_{i-1}x_i$ in $\Gamma(p_1, \dots, p_{n-1})$, for $1 \le i < n$. Then $q_i = p_i$ whenever p_i is odd, and also $|\Gamma(p_1, \dots, p_{n-1})| = 2q_1 \cdots q_{n-1}$, which divides $2p_1 \cdots p_{n-1}$.*

Proof. Let $k_i = p_i$ when p_i is odd, or 2 when p_i is even. Then since k_i divides p_i for all i, there exists an epimorphism $\pi : \Gamma(p_1, \dots, p_{n-1}) \to \Gamma(k_1, \dots, k_{n-1})$. Also the $(n-1)$-tuple $(k_1, \dots, k_{n-1})$ is admissible, and indeed k_{i-1} and k_{i+1} are both 2 whenever k_i is odd (since p_{i-1} and p_{i+1} are both even whenever p_i is odd). Thus $(k_1, \dots, k_{n-1})$ satisfies the hypotheses of Proposition 4.3, and so $|\Gamma(k_1, \dots, k_{n-1})| = 2k_1 \cdots k_{n-1}$.

Moreover, Proposition 4.3 tells us that when p_i is odd, the order of the image of x_i in $\Gamma(k_1, \dots, k_{n-1})$ is $k_i = p_i$, and so $q_i = p_i$; on the other hand, if p_i is even, then the order of the image of x_i in $\Gamma(k_1, \dots, k_{n-1})$ is $k_i = 2$, and so q_i is even in that case.

Now the kernel of the epimorphism π is the smallest normal subgroup of $\Gamma(p_1, \dots, p_{n-1})$ containing the elements $(x_{i-1}x_i)^2$ for those i such that p_i is even. By Proposition 4.2, however, the subgroup N generated by these elements is normal in $\Gamma(p_1, \dots, p_{n-1})$, and abelian. Hence in particular, $N = \ker \pi$, and also by the intersection condition, $|N|$ is the product of the numbers $q_i/2$ over all i for which p_i is even. Thus $|\Gamma(p_1, \dots, p_{n-1})| = 2q_1 \cdots q_{n-1}$. □

In order to use Theorem 2.5 to build our tight regular polytopes recursively, we need two more observations. The first concerns the flat amalgamation property (FAP):

Proposition 4.5. *If p_2 is even, then $\mathcal{P}(p_1, \dots, p_{n-1})$ has the FAP with respect to its 2-faces, and if p_{n-2} is even, then $\mathcal{P}(p_1, \dots, p_{n-1})$ has the FAP with respect to its co-$(n-3)$-faces.*

Proof. Let p_2 be even, and consider the effect of killing the generators x_i of $\Gamma(p_1, \ldots, p_{n-1})$, for $i \geq 2$ (that is, by adding the relations $x_i = 1$ to the presentation for $\Gamma(p_1, \ldots, p_{n-1})$). Each of the relators $r_3, \ldots, r_{n-2}$ contains only generators x_i with $i \geq 2$, so becomes redundant, and may be removed. The relator r_2 reduces to x_1^2 or x_1^4, while r_1 reduces to $(x_0 x_1^2)^2$, which is equivalent to x_0^2, and hence all of these become redundant too. Thus adding the relations $x_i = 1$ to $\Gamma(p_1, \ldots, p_{n-1})$ has the same effect as adding the relations $x_i = 1$ to the string Coxeter group $[p_1, \ldots, p_{n-1}]$. It is easy to see that this gives the quotient group with presentation $\langle\, x_0, x_1 \mid x_0^2, x_1^2, (x_0 x_1)^{p_1} \,\rangle$, which is the automorphism group of the 2-faces of $\mathcal{P}(p_1, \ldots, p_{n-1})$. Thus $\mathcal{P}(p_1, \ldots, p_{n-1})$ has the FAP with respect to its 2-faces. The second claim can be proved by a dual argument. □

Proposition 4.6. *Let $\mathcal{P}$ be an equivelar n-polytope with tight m-faces and tight co-k-faces, where $m \geq k + 3$. Then $\mathcal{P}$ is tight.*

Proof. Since the m-faces are tight, they are $(i, i{+}2)$-flat for $0 \leq i \leq m{-}3$, by Theorem 2.4, and then by Proposition 2.3, the polytope $\mathcal{P}$ is $(i, i{+}2)$-flat for $0 \leq i \leq m{-}3$. Similarly, the co-k-faces are $(i, i{+}2)$-flat for $0 \leq i \leq n{-}k{-}4$, and $\mathcal{P}$ is $(i, i+2)$-flat for $k{+}1 \leq i \leq n{-}3$. Finally, since $m \geq k + 3$, we see that $\mathcal{P}$ is $(i, i + 2)$-flat for $0 \leq i \leq n - 3$, and again Theorem 2.4 applies, to show that $\mathcal{P}$ is tight. □

We can now prove the following.

Theorem 4.7. *Let $(p_1, \ldots, p_{n-1})$ be an admissible $(n{-}1)$-tuple, with $n \geq 4$. Also suppose that p_{i-1} and p_{i+1} are both even, for some i (with $2 \leq i \leq n{-}2$). If $\mathcal{P}(p_1, \ldots, p_i)$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_i\}$, and $\mathcal{P}(p_i, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_i, \ldots, p_{n-1}\}$, then $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$.*

Proof. Let $\mathcal{P}_1 = \mathcal{P}(p_1, \ldots, p_i)$ and $\mathcal{P}_2 = \mathcal{P}(p_i, \ldots, p_{n-1})$, which by hypothesis are tight orientably-regular polytopes of the appropriate types. Since p_{i-1} and p_{i+1} are even, Proposition 4.5 tells us that $\mathcal{P}_1$ has the FAP with respect to its co-$(i-2)$-faces, and $\mathcal{P}_2$ has the FAP with respect to its 2-faces. Then by Theorem 2.5, there exists a regular polytope $\mathcal{P}$ with $(i{+}1)$-faces isomorphic to $\mathcal{P}_1$ and co-$(i{-}2)$-faces isomorphic to $\mathcal{P}_2$. Moreover, since $\mathcal{P}_1$ and $\mathcal{P}_2$ are both tight, Proposition 4.6 implies that $\mathcal{P}$ is also tight, and then since $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$, we find that $|\Gamma(\mathcal{P})| = 2p_1 \cdots p_{n-1}$. But also the $(i+1)$-faces of $\mathcal{P}$ are isomorphic to $\mathcal{P}_1$, and the co-$(i-2)$-faces are isomorphic to $\mathcal{P}_2$, and so the standard generators of $\Gamma(\mathcal{P})$ must satisfy all the relations of $\Gamma(p_1, \ldots, p_{n-1})$. In particular, $\Gamma(\mathcal{P})$ is a quotient of $\Gamma(p_1, \ldots, p_{n-1})$, and so $|\Gamma(p_1, \ldots, p_{n-1})| \geq 2p_1 \cdots p_{n-1}$. On the other hand, $|\Gamma(p_1, \ldots, p_{n-1})| \leq 2p_1 \cdots p_{n-1}$ by Lemma 4.4. Thus $|\Gamma(p_1, \ldots, p_{n-1})| = 2p_1 \cdots p_{n-1}$, and hence also $\Gamma(\mathcal{P}) \cong \Gamma(p_1, \ldots, p_{n-1})$, and $\mathcal{P} \cong \mathcal{P}(p_1, \ldots, p_{n-1})$. Thus $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight polytope of type $\{p_1, \ldots, p_{n-1}\}$, and finally, since all the defining relations of $\Gamma(p_1, \ldots, p_{n-1})$ have even length, we find that $\mathcal{P}(p_1, \ldots, p_{n-1})$ is orientably-regular. □

Note that the above theorem helps us deal with a large number of possibilities, once we have enough 'building blocks' in place. We are assuming that the $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$ is admissible, so that p_{i-1} and p_{i+1} are even divisors of $2p_i$ whenever p_i is odd. Now suppose that $n \geq 6$. If p_2 and p_4 are both even, then Theorem 4.7 will apply, and if not, then one of them is odd, say p_2, in which case p_1 and p_3 must both be even, and again Theorem 4.7 will apply. Hence this leaves us with just a few cases to verify, namely admissible $(n-1)$-tuples $(p_1, \ldots, p_{n-1})$ with $n = 4$ or 5 for which there is no i such that p_{i-1} and p_{i+1} are both even.

The only such cases are as follows:

- $n = 4$, with p_1 odd, p_2 even and p_3 even, or dually, p_1 even, p_2 even and p_3 odd,
- $n = 4$, with p_1 odd, p_2 even and p_3 odd,
- $n = 5$, with p_1 odd, p_2 even, p_3 even and p_4 odd.

We start with the cases where $n = 4$:

Proposition 4.8. *If p_1 is odd, p_2 is an even divisor of $2p_1$, and $p_3 \geq 2$, then $\Gamma(p_1, p_2, p_3)$ is a string C-group, and $\mathcal{P}(p_1, p_2, p_3)$ is a tight orientably-regular polytope of type $\{p_1, q, p_3\}$, for some even q dividing p_2.*

Proof. Let $\Gamma = \Gamma(p_1, p_2, p_3) = \langle x_0, x_1, x_2, x_3 \rangle$, and let $\overline{\Gamma} = \Gamma(p_1, 2, p_3) = \langle y_0, y_1, y_2, y_3 \rangle$, where $y_i = \overline{x_i}$ is the image of x_i under the natural epimorphism $\pi : \Gamma(p_1, p_2, p_3) \to \Gamma(p_1, 2, p_3)$, for $0 \leq i \leq 3$. By Proposition 4.3, we know that $\Gamma(p_1, 2, p_3) \cong [p_1, 2, p_3]$. Also the subgroup $\langle x_0, x_1, x_2 \rangle$ of Γ covers $[p_1, 2]$, and this cover is one-to-one on $\langle x_0, x_1 \rangle$, so $\langle x_0, x_1, x_2 \rangle$ is a string C-group, by Proposition 2.1. A similar argument shows that $\langle x_1, x_2, x_3 \rangle$ is a string C-group. Now the intersection of these two string C-groups is $\langle x_1, x_2 \rangle$, since the intersection of their images in $\overline{\Gamma}$ is $\langle y_0, y_1, y_2 \rangle \cap \langle y_1, y_2, y_3 \rangle = \langle y_1, y_2 \rangle$, and the kernel of π is $\langle (x_1 x_2)^2 \rangle$. Hence by Proposition 2.2, $\Gamma(p_1, p_2, p_3)$ is a string C-group, and the rest follows easily from Lemma 4.4. □

Lemma 4.9. *If $p_i = 2$ for some i, then in $\Gamma(p_1, \ldots, p_{n-1}) = \langle x_0, \ldots, x_{n-1} \rangle$, we have $\langle x_0, \ldots, x_i \rangle \cong \Gamma(p_1, \ldots, p_i)$ and $\langle x_{i-1}, \ldots, x_{n-1} \rangle \cong \Gamma(p_i, \ldots, p_{n-1})$.*

Proof. First consider $\Lambda = \langle x_0, \ldots, x_i \rangle$. This is obtainable as a quotient of $\Gamma(p_1, \ldots, p_i)$ by adding extra relations. But also it can be obtained from the group $\Gamma(p_1, \ldots, p_{n-1})$ by killing the unwanted generators $x_{i+1}, \ldots, x_{n-1}$. (Note that for $i+1 \leq j \leq n-1$, the relation $r_j = 1$ and all relations of the form $(\rho_j \rho_k)^m = 1$ become redundant and may be removed, and the same holds for the relations $r_i = 1$ and $r_{i-1} = 1$ since the assumption that $p_i = 2$ implies that $[x_{i-1}, x_i] = (x_{i-1}, x_i)^2 = 1$ and hence that $x_0, \ldots, x_{i-1}$ commute with $x_i, \ldots, x_{n-1}$.) It follows that $\Lambda \cong \Gamma(p_1, \ldots, p_i)$. Also $\langle x_{i-1}, \ldots, x_{n-1} \rangle \cong \Gamma(p_i, \ldots, p_{n-1})$, by the dual argument. □

Theorem 4.10. *If p_1 is odd, p_2 is an even divisor of $2p_1$, and p_3 is even, then $\mathcal{P}(p_1, p_2, p_3)$ is a tight orientably-regular polytope of type $\{p_1, p_2, p_3\}$.*

Proof. First, the group $\Gamma(p_1, p_2, p_3) = \langle x_0, x_1, x_2, x_3 \rangle$ covers $\Gamma(p_1, p_2, 2) = \langle y_0, y_1, y_2, y_3 \rangle$, say, since p_3 is even. Also by Lemma 4.9 we know that $\langle y_0, y_1, y_2 \rangle$ is isomorphic to $\Gamma(p_1, p_2)$, and hence in particular, $y_1 y_2$ has order p_2. It follows that the order of $x_1 x_2$ is also p_2, and then the conclusion follows from Proposition 4.8. □

Theorem 4.11. *If p_1 and p_3 are odd, and p_2 is an even divisor of both $2p_1$ and $2p_3$, then $\mathcal{P}(p_1, p_2, p_3)$ is a tight orientably-regular polytope of type $\{p_1, p_2, p_3\}$.*

Proof. Under the given assumptions, the group $\Gamma(p_1, p_2, p_3)$ is obtained from the string Coxeter group $[p_1, p_2, p_3] = \langle x_0, x_1, x_2, x_3 \rangle$ by adding the two extra relations $(x_0 x_1 x_2 x_1 x_2)^2 = 1$ and $(x_3 x_2 x_1 x_2 x_1)^2 = 1$. These imply that the element $\omega = (x_1 x_2)^2$

is inverted under conjugation by each of x_0 and x_3, and hence by all the x_i. Now let $y_i = x_{i-1}x_i$ for $1 \le i \le 3$. These elements generate the orientation-preserving subgroup $\Gamma^+(p_1, p_2, p_3)$, and they all centralise $\omega = y_2^2$. It follows that $\Gamma^+(p_1, p_2, p_3)$ has presentation

$$\langle\, y_1, y_2, y_3 \mid y_1^{p_1}, y_2^{p_2}, y_3^{p_3}, (y_1y_2)^2, (y_2y_3)^2, (y_1y_2y_3)^2, [y_1, y_2^2], [y_3, y_2^2] \,\rangle.$$

We now exhibit a permutation representation of this group on the Cartesian product $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, by letting each y_i induce the permutation π_i, where

$$(j, k)^{\pi_2} = (j, k+1) \text{ for all } (j, k),$$

$$(j,k)^{\pi_1} = \begin{cases} (j+1, k) & \text{for } k \text{ even} \\ (j-1, k-2) & \text{for } k \text{ odd,} \end{cases} \qquad (j,k)^{\pi_3} = \begin{cases} (j, k-2j) & \text{for } k \text{ even} \\ (j, k+2(j-1)) & \text{for } k \text{ odd.} \end{cases}$$

It is easy to see that π_1 and π_2 have orders p_1 and p_2 respectively (since p_2 divides $2p_1$), and that the order of π_3 divides $p_2/2$ and hence divides p_3. It is also easy to verify that they satisfy the other defining relations for $\Gamma^+(p_1, p_2, p_3)$, and thus we do have a permutation representation.

In particular, since π_2 has order p_2, so does $y_2 = x_1x_2$, and again the conclusion follows from Proposition 4.8. □

We now handle the remaining case.

Theorem 4.12. *If p_1 and p_4 are odd, p_2 is an even divisor of $2p_1$, and p_3 is an even divisor of $2p_4$, then $\mathcal{P}(p_1, p_2, p_3, p_4)$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_4\}$.*

Proof. Take $\Gamma = \Gamma(p_1, \ldots, p_4) = \langle x_0, \ldots, x_4\rangle$, and $\Lambda = \Gamma(p_1, p_2, 2, p_4) = \langle y_0, \ldots, y_4\rangle$. Then Γ covers Λ, and this induces a cover from $\langle x_0, \ldots, x_3\rangle$ to $\langle y_0, \ldots, y_3\rangle$, which is isomorphic to $\Gamma(p_1, p_2, 2)$ by Lemma 4.9. Similarly we have a cover from $\langle x_0, x_1, x_2\rangle$ to $\langle y_0, y_1, y_2\rangle$, which is isomorphic to $\Gamma(p_1, p_2)$. But on the other hand, $\langle x_0, x_1, x_2\rangle$ is a quotient of $\Gamma(p_1, p_2)$, and hence these two groups are isomorphic. In particular, the cover from $\langle x_0, \ldots, x_3\rangle$ to $\Gamma(p_1, p_2, 2)$ is one-to-one on the facets, so $\langle x_0, \ldots, x_3\rangle$ is a string C-group. By a dual argument, $\langle x_1, \ldots, x_4\rangle$ is also a string C-group.

Next, let $\Delta = \Gamma(p_1, 2, 2, p_4) = \langle z_0, \ldots, z_4\rangle$, and let π be the covering homomorphism from Γ to Δ. The kernel of π is the subgroup generated by $(x_1x_2)^2$ and $(x_2x_3)^2$, since the defining relations for $\Gamma = \Gamma(p_1, \ldots, p_4)$ imply that these two elements are centralised or inverted under conjugation by each generator x_i. In particular, $\ker \pi \subseteq \langle x_1, x_2, x_3\rangle$. As also the intersection of $\langle z_0, \ldots, z_3\rangle$ and $\langle z_1, \ldots, z_4\rangle$ in Δ is $\langle z_1, z_2, z_3\rangle$, it follows that intersection of $\langle x_0, \ldots, x_3\rangle$ and $\langle x_1, \ldots, x_4\rangle$ is $\langle x_1, x_2, x_3\rangle$. Hence Γ is a string C-group.

Now $\langle x_0, x_1, x_2\rangle \cong \Gamma(p_1, p_2)$, and by Theorem 3.1 we know the polytope $\mathcal{P}(p_1, p_2)$ has type $\{p_1, p_2\}$. Similarly $\langle x_2, x_3, x_4\rangle \cong \Gamma(p_3, p_4)$, and $\mathcal{P}(p_3, p_4)$ has type $\{p_3, p_4\}$. It follows that $\mathcal{P}(p_1, p_2, p_3, p_4)$ is an orientably-regular polytope of type $\{p_1, \ldots, p_4\}$. In particular, the order of $x_{i-1}x_i$ is p_i (for $1 \le i \le 4$), and so by Lemma 4.4, $\mathcal{P}(p_1, p_2, p_3, p_4)$ is tight. □

This gives us all the building blocks we need. With the help of Theorem 4.7, we now know that $\mathcal{P}(p_1, \ldots, p_{n-1})$ is a tight orientably-regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ whenever $(p_1, \ldots, p_{n-1})$ is admissible, and the proof of Theorem 1.1 is complete.

5 Tight non-orientably-regular polytopes

We have not yet been able to completely characterise the Schläfli symbols of tight, non-orientably-regular polytopes, but we have made some partial progress. For example, we can easily find an infinite family of tight, non-orientably-regular polyhedra.

Theorem 5.1. *For every odd positive integer k, there exists a non-orientably-regular tight polyhedron of type $\{3k, 4\}$, with automorphism group $\Lambda(k)$ having presentation*

$$\langle\, \rho_0, \rho_1, \rho_2 \mid \rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^{3k}, (\rho_0\rho_2)^2, (\rho_1\rho_2)^4, \rho_0\rho_1\rho_2\rho_1\rho_0\rho_1\rho_2\rho_1\rho_2 \,\rangle.$$

Proof. We note that $\Lambda(1)$ is the automorphism group of the hemi-octahedron (of type $\{3, 4\}$), and that $\Lambda(k)$ covers $\Lambda(1)$, for every k. Hence in each $\Lambda(k)$, the order of $\rho_1\rho_2$ is 4 (and not 1 or 2). Next, because the covering is one-to-one on the vertex-figures, it follows from Proposition 2.1 that $\Lambda(k)$ is a string C-group. Also $\Lambda(k)$ has a relation of odd length, and so it must be the automorphism group of a non-orientably-regular polyhedron.

Now let N be the subgroup generated by the involutions ρ_2 and $\rho_1\rho_2\rho_1$. Since their product has order 2, this is a Klein 4-group. Moreover, N is normalised by ρ_1 and ρ_2, and also by ρ_0 since ρ_0 centralises ρ_2 and the final relation in the definition of $\Lambda(k)$ gives $(\rho_1\rho_2\rho_1)^{\rho_0} = \rho_1\rho_2\rho_1\rho_2$. It is now easy to see that N is the normal closure of $\langle\rho_2\rangle$. The quotient $\Lambda(k)/N$ is isomorphic to $\langle\, \rho_0, \rho_1 \mid \rho_0^2, \rho_1^2, (\rho_0\rho_1)^{3k} \,\rangle$, with the final relator for $\Lambda(k)$ becoming trivial, and so $\Lambda(k)/N$ is dihedral of order $6k$. In particular, this shows that $\rho_0\rho_1$ has order $3k$ (and that $|\Lambda(k)| = |\Lambda(k)/N||N| = 24k$). □

The computational data that we have on polytopes with up to 2000 flags (obtained with the help of MAGMA [1]) suggests that these polyhedra are the only tight non-orientably-regular polyhedra of type $\{p, q\}$ with p odd.

Using Theorem 5.1, it is possible to build tight, non-orientably-regular polytopes in much the same way as we did in Theorem 4.7. In particular, the regular polytope with automorphism group $\Lambda(k)$ has the FAP with respect to its 2-faces, and its dual has the FAP with respect to its vertex-figures (co-0-faces). Then by Theorem 2.5, we know there are tight, non-orientably-regular polytopes of type $\{4, 3k, 4\}$ for each odd k, and of type $\{4, 3k, r\}$ for each odd k and each even r dividing $3k$. It is possible to continue in this fashion, building tight non-orientably-regular polytopes of every rank.

Finally, just as with orientably-regular polytopes, there are some kinds of Schläfli symbol for which no examples can be constructed using Theorem 2.5. In fact (and in contrast with the situation for orientably-regular polytopes), there seem to be no tight non-orientably-regular polytopes of some of these types at all. For example, there are no tight regular polytopes of type $\{3, 4, r\}$ with $r \geq 3$ and with 2000 flags or fewer, but the reason for this is not clear.

References

[1] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I, The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[2] M. Conder, Lists of all regular polytopes with up to 2000 flags, `http://www.math.auckland.ac.nz/~conder`.

[3] M. Conder, The smallest regular polytopes of any given rank, *Adv. Math.* **236** (2013), 92–110.

[4] M. D. E. Conder and T. W. Tucker, Regular Cayley maps for cyclic groups, *Trans. Amer. Math. Soc.* **366** (2014), 3585–3609.

[5] G. Cunningham, Minimal equivelar polytopes, *Ars Math. Contemp.* **7** (2014), 299–315.

[6] M. I. Hartley, An atlas of small regular abstract polytopes, *Period. Math. Hung.* **53** (2006), 149–156.

[7] I. Hubard, Two-orbit polyhedra from groups, *Eur. J. Comb.* **31** (2010), 943–960.

[8] P. McMullen and E. Schulte, *Abstract regular polytopes*, Encyclopedia of Mathematics and its Applications, vol. 92, Cambridge University Press, Cambridge, 2002.

[9] E. Schulte, Amalgamation of regular incidence-polytopes, *Proc. London Math. Soc.* **3** (1988), 303–328.

[10] E. Schulte and A. Ivić Weiss, Chiral polytopes, *Applied geometry and discrete mathematics*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 493–516.

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Reachability relations, transitive digraphs and groups

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Abstract

In [6] it was shown that properties of digraphs such as growth, property $\mathbf{Z}$, and number of ends are reflected by the properties of certain reachability relations defined on the vertices of the corresponding digraphs.

In this paper we study these relations in connection with certain properties of automorphism groups of transitive digraphs. In particular, one of the main results shows that if a transitive digraph admits a nilpotent subgroup of automorphisms with finitely many orbits, then its nilpotency class and the number of orbits are closely related to particular properties of reachability relations defined on the digraphs in question.

The obtained results have interesting implications for Cayley digraphs of certain types of groups such as torsion-free groups of polynomial growth.

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1 Introduction and preliminaries

In [2], highly arc-transitive digraphs were considered from several different viewpoints, leading to – besides many nice results – a number of interesting problems. One of these problems, which remained open for a very long time and was finally settled in [4], concerned a certain reachability relation defined on the edges of digraphs. A subset of the authors of this paper also worked on this 'reachability problem' [5] and several other questions concerning highly-arc-transitive digraphs. In [6], as an offspring of our considerations, we became interested in reachability relations defined on vertices rather than edges, which we review in the sequel.

A *digraph* is an ordered pair $D = (V(D), E(D))$, where $V(D)$ is the vertex-set and $E(D) \subseteq V(D) \times V(D)$ is the edge-set. Note that a digraph can have loops (v, v) as well as pairs of 'oppositely directed' edges of the form (u, v) and (v, u). We also emphasize that with this definition our digraphs are always simple in the sense that between two vertices there can be at most one edge in each direction. Digraphs considered in this paper are connected in the sense that their underlying undirected graphs are connected.

By $\mathrm{Aut}(D)$ we denote the automorphism group of a digraph D. We say that D is *transitive* if some $H \subseteq \mathrm{Aut}(D)$ acts transitively on the vertices of D. Also, if $g \in \mathrm{Aut}(D)$, then ${}^{g}v$ denotes the image of $v \in V(D)$ under g and ${}^{H}v$ denotes the orbit of v under some subset $H \subseteq \mathrm{Aut}(D)$.

To make sure that no ambiguity arises, we explicitely define Cayley digraphs as they are understood in this paper. The *Cayley digraph* $Cay(G, S)$ of a group G with respect to a generating set S has the group G as its vertex set and the edges are given by right multiplication by the generators: $E(Cay(G, S)) = \{(g, gs) | s \in S\}$. If $Cay(G, S)$ is defined in this way, then G acts regularly on $Cay(G, S)$ by left multiplication.

A *walk* $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ from v_0 to v_n of length $n \geq 0$ (denoted by $|W|$) is a sequence of $n + 1$ (not necessarily pairwise distinct) vertices $v_0, v_1, \ldots, v_n \in V(D)$, and n indicators $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ such that for all $j \in \{1, 2, \ldots, n\}$ we have

$$\varepsilon_j = 1 \quad \Rightarrow \quad (v_{j-1}, v_j) \in E(W),$$
$$\varepsilon_j = -1 \quad \Rightarrow \quad (v_j, v_{j-1}) \in E(W).$$

W is called a *closed walk* if $v_0 = v_n$. Intuitively, a walk is a traversal in the digraph from vertex to vertex along edges, where indicators 1 and -1 tell whether the traversal respects the direction of edges or not. The formal definition of a walk as above has been chosen in order to make proofs unambiguous. If the vertices of a walk W are pairwise different then W is called a *path*. A walk (or a path) is *directed* if its indicators are all equal to 1 or to -1, and is *alternating* if the values of the indicators alternate.

Let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ be a walk. We let the *inverse walk* of W be $W^{-1} = (v_n, -\varepsilon_n, v_{n-1}, \ldots, -\varepsilon_1, v_0)$. Moreover, for $0 \leq i \leq j \leq n$, the subsequence

$${}_iW_j = (v_i, \varepsilon_{i+1}, \ldots, \varepsilon_j, v_j)$$

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of W is called a *subwalk*. Furthermore, let $W' = (u_0, \delta_1, u_1, \ldots, \delta_m, u_m)$ be a walk such that $u_0 = v_n$. Then the *concatenation* of W and W' is the walk

$$W \cdot W' = (v_0, \varepsilon_1, v_1, \ldots, v_{n-1}, \varepsilon_n, u_0, \delta_1, u_1, \ldots, \delta_m, u_m)$$

of length $n + m$.

We now introduce two families of reachability relations defined on vertices of a digraph. Let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ be a walk. The *weight* of the walk W is defined as

$$\zeta(W) = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n.$$

Let k be a nonnegative integer. We say that a vertex $u \in V(D)$ is R_k^+*-related* to a vertex $v \in V(D)$, in symbols

$$uR_k^+ v,$$

if there exists a walk W from u to v such that $\zeta(W) = 0$, and that for every $0 \leq j \leq |W|$ we have $\zeta({}_0W_j) \in [0, k]$. For a given pair of vertices u, v, the set of all such walks is denoted by $R_k^+[u, v]$. Analogously we say that u is R_k^-*-related* to v, in symbols $uR_k^- v$, if there exists a walk W such that $\zeta(W) = 0$, and that for every $0 \leq j \leq |W|$ we have $\zeta({}_0W_j) \in [-k, 0]$. For a given pair of vertices u, v, the set of all such walks is denoted by $R_k^-[u, v]$. Note that R_k^+ and R_k^- are equivalence relations. Their equivalence classes are denoted by $R_k^+(v)$ and $R_k^-(v)$, $v \in V(D)$, respectively. If D is transitive, then the equivalence classes of R_k^+ (and similarly of R_k^-) form an imprimitivity system for $\mathrm{Aut}(D)$. Observe that the sequences $(R_k^+)_{k \in \mathbb{Z}^+}$ and $(R_k^-)_{k \in \mathbb{Z}^+}$ are ascending: for all k we have $R_k^+ \subseteq R_{k+1}^+$ and $R_k^- \subseteq R_{k+1}^-$. Their respective unions

$$R^+ = \bigcup_{k \in \mathbb{Z}^+} R_k^+ \quad \text{and} \quad R^- = \bigcup_{k \in \mathbb{Z}^+} R_k^-$$

are thus also equivalence relations, and their equivalence classes form imprimitivity systems for $\mathrm{Aut}(D)$ whenever D is transitive. As was shown in [6], $R^+ = R_k^+$ holds whenever $R_k^+ = R_{k+1}^+$. In this case, the smallest nonnegative integer k such that $R_k^+ = R^+$ holds is called the *exponent* $\exp^+(D)$ of D. If $R_k^+ \neq R^+$ for all k, then we set $\exp^+(D) = \infty$. The exponent $\exp^-(D)$ is defined analogously. We say that the relation R_k^+ (R^+, R_k^-, R^-) is *universal* if $uR_k^+ v$ $(uR^+ v, uR_k^- v, uR^- v)$ holds for any pair $u, v \in V(D)$. We mention (see [6]) that all of the above relations are universal, provided that the digraph in question is connected and has a loop at every vertex.

In [6] it was also shown that properties of the two sequences of equivalence relations $(R_k^+)_{k \in \mathbb{Z}^+}$ and $(R_k^-)_{k \in \mathbb{Z}^+}$ are strongly related to other properties of digraphs such as having property **Z**, the number of ends, growth conditions and vertex degree.

Furthermore, in [8] the relations $R_{a,b}$ were studied, where a is a non-positive integer or $a = -\infty$ and b is a non-negative integer or $b = \infty$. We say that a vertex u is $R_{a,b}$-related to a vertex v if there exists a 0-weighted walk from u to v such that every subwalk starting at u has weight in the interval $[a, b]$.

The *distance* $\mathrm{dist}_D(u, v)$ between vertices u and v in a connected digraph D is the length of a shortest path from u to v in the underlying undirected graph. The *growth function* $f_D(v, n), n \geq 0$, with respect to some $v \in V(D)$ is given by

$$f_D(v, n) = |\{u \in V(D) \mid \mathrm{dist}_D(v, u) \leq n\}|.$$

If D is transitive, then this function does not depend on a particular vertex $v \in V(D)$. In this case we denote it by $f_D(n)$.

We say that a transitive digraph D has *polynomial growth* if there are positive constants c and d such that

$$f_D(n) \leq cn^d$$

holds for all $n \geq 0$. The digraph D has *exponential growth* if there is a constant $c > 1$ such that

$$f_D(n) > c^n$$

holds for all $n \geq 0$. If the growth function of a digraph D grows faster than any polynomial but D does not have exponential growth, then we say that D has *intermediate growth*. In the case of polynomial growth it can be shown that there always exist constants c_1, c_2 and an integer $d \geq 1$ such that

$$c_1 n^d \leq f_D(n) \leq c_2 n^d$$

holds for all $n \geq 0$. We call this integer d the *growth degree* of D. We remark that the definitions concerning growth coincide with the usual definitions in the context of undirected graphs.

Let D be a digraph and let τ be a partition of the vertex set of D. The *quotient digraph* D_τ of D with respect to τ is the digraph with vertex set τ and $(u_\tau, v_\tau) \in E(D)$ if and only if there exist vertices $u \in u_\tau$ and $v \in v_\tau$ such that $(u, v) \in E(D)$. If $W = (v_0, \varepsilon_1, v_1, \varepsilon_2, \ldots, \varepsilon_n, v_n)$ is a walk in D, then the *quotient walk* W_τ of W is defined to be the walk $W = ((v_0)_\tau, \varepsilon_1, (v_1)_\tau, \varepsilon_2, \ldots, \varepsilon_n, (v_n)_\tau)$. Note that for every j, $0 \leq j \leq |W| = |W_\tau|$, we have $\zeta({}_0W_j) = \zeta({}_0(W_\tau)_j)$. We emphasize that we consider these quotient digraphs as simple digraphs in the sense that if there are several edges in the same direction between two sets in τ, then the quotient digraph contains exactly one directed edge between the respective vertices. But of course these quotient graphs might contain loops if there is an edge $(u, v) \in E(D)$ for some $u \in v_\tau$.

Let G be a group acting transitively on D and let H be a normal subgroup of G. Then the orbits of H on $V(D)$ give rise to an imprimitivity system τ of G on $V(D)$. The respective quotient digraph D_τ is usually denoted by D_H.

As mentioned above, if D is transitive, then R^+ and R^- give rise to imprimitivity systems of $\mathrm{Aut}(D)$ on D. The respective quotient digraphs are denoted by D/R^+ and D/R^- and can be described easily (see e. g. [8]). The digraph D/R^+ either is (1) a finite directed cycle or (2) a two-way infinite directed line or (3) an infinite regular directed tree with indegree 1 and outdegree > 1. Considering R^- the first two possibilities are the same, but if D/R^- is neither of these digraphs, then it is a regular tree with outdegree 1 and indegree > 1.

2 Motivation and main result

The aim of this paper is to investigate the interplay between properties of groups and properties of reachability relations in their Cayley digraphs.

For example, as a consequence of the last paragraph of the previous section, we immediately see that the quotient digraphs with respect to R^+ of Cayley digraphs of finitely generated groups with polynomial or intermediate growth are either finite directed cycles or directed lines. Further, from [6, Theorem 4.12] we know that a finitely generated group

G has exponential growth if for at least one Cayley digraph D of G, at least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite.

Additionally, by Gromov's important result [3], a finitely generated group has polynomial growth if and only if it contains a normal nilpotent subgroup of finite index. Hence the following question arises naturally: What can be said about properties of our reachability relations in Cayley digraphs of finitely generated groups with polynomial growth?

In fact we carry out our considerations by assuming that nilpotent groups act with finitely many orbits on digraphs. The results for Cayley digraphs of groups with polynomial growth are then obtained as corollaries. The main result of this paper is the following theorem.

To avoid ambiguity, we recall the definition of nilpotent groups: For a group $G = G^0$, let $G^{i+1} = [G^0, G^i]$ for $i \geq 0$. If $G = G^0 \triangleright G^1 \triangleright \ldots \triangleright G^r \triangleright G^{r+1} = 1$ then we say that G is *nilpotent of class* r.

Theorem 2.1. *Let a group G act transitively on a connected digraph D, and let $N \trianglelefteq G$ be a normal nilpotent subgroup of class r acting with m orbits on D, where $1 \leq m < \infty$. Then* $\exp^+(D) = \exp^-(D) \leq m(r+2) - 1$.

Although we are mainly interested in properties of our relations in Cayley digraphs of finitely generated groups, we emphasize that - with the exception of those results explicitely formulated for finitely generated groups - we never assume that the graphs in consideration are locally finite.

3 Auxiliary results

In this section we prove several results which will be useful for our main investigations, carried out in Section 4.

Lemma 3.1. *Let D be a digraph with minimal in- and outdegree at least 1 and let $k \geq 1$ be an integer. Then for any two vertices $u, v \in V(D)$ we have that uR_k^+v if and only if there exists a walk $W \in R_k^+[u,v]$ that is a concatenation of walks of the form $(u_0, 1, u_1, 1, \ldots, 1, u_k, -1, u_{k+1}, -1, \ldots, -1, u_{2k})$. An analogous result holds for the relation R_k^-.*

Proof. We prove the assertion for the relation R_k^+ and leave the analogous proof for R_k^- to the reader.

To this end suppose uR_k^+v and let $W' = (u_0, \varepsilon_1, u_1, \varepsilon_2, u_2, \varepsilon_3, \ldots, \varepsilon_n, u_n) \in R_k^+[u,v]$. Observe that, since the minimal in- and outdegrees of D are at least 1, there is a directed walk of any prescribed positive or negative weight starting at any vertex of D.

A walk $W \in R_k^+[u,v]$, as described in the statement of the lemma, can now be obtained from W' inductively by inserting a concatenation of such a directed walk of appropriate length with its inverse at each vertex u_i for which $\varepsilon_i \neq \varepsilon_{i+1}$ and $\zeta({}_0W_i')$ is not 0 or k.

□

For a group G, a positive integer k, and subsets $S, T \subseteq G$ let $ST = \{st | s \in S, t \in T\}$, $S^k = \underbrace{S \cdots S}_{k}$ and $S^{-k} = \underbrace{S^{-1} \cdots S^{-1}}_{k}$.

Corollary 3.2. *Let $D = \mathrm{Cay}(G, S)$ be a Cayley digraph of a group G with respect to the generating set S. Then for any integer $k \geq 1$ and any $g \in G$ we have that $R_k^+(g) = gR_k^+(1) = g\langle S^kS^{-k}\rangle$ and $R_k^-(g) = gR_k^-(1) = g\langle S^{-k}S^k\rangle$.*

Proof. We prove the assertions for R_k^+ and leave the analogous proof for R_k^- to the reader. The fact that $R_k^+(g) = gR_k^+(1)$ is obvious since G has a natural left regular action on D while $R_k^+(1) = \langle S^k S^{-k} \rangle$ follows from Lemma 3.1. □

Lemma 3.3. *Let D be a digraph and let τ be a partition of the vertex set of D. Suppose that for each $u, v \in V(D)$ with $(u_\tau, v_\tau) \in E(D_\tau)$ there exist $u' \in u_\tau$ and $v' \in v_\tau$ such that (u, v') and (u', v) are arcs of D. Then for each $k \geq 1$ and each $u, v \in V(D)$ we have that $u_\tau R_k^+ v_\tau$ if and only if there exists some $w \in v_\tau$ such that $uR_k^+ w$. An analogous result holds for the relation R_k^-.*

Proof. We prove the result for R_k^+ and leave the analogous proof for R_k^- to the reader. Let $k \geq 1$ and let $u, v \in V(D)$.

Suppose first that for some $w \in v_\tau$ we have that $uR_k^+ w$ and let $W \in R_k^+[u, w]$. Since $v_\tau = w_\tau$, the walk W_τ is contained in $R_k^+[u_\tau, v_\tau]$. This proves one implication.

Suppose now that $u_\tau R_k^+ v_\tau$ and let $\bar{W} = (u_\tau, \varepsilon_1, \bar{x}_1, \varepsilon_2, \ldots, \bar{x}_n, \varepsilon_{n+1}, v_\tau) \in R_k^+[u_\tau, v_\tau]$. Then by assumption one can successively find representatives $x_i \in \bar{x}_i$ and $w \in v_\tau$ such that $W = (u, \varepsilon_1, x_1, \varepsilon_2, x_2, \varepsilon_3, \ldots, x_n, \varepsilon_{n+1}, w) \in R_k^+[u, w]$. □

Remark 3.4. Observe that the condition of the above lemma is satisfied if τ consists of the orbits of some group acting on D.

Lemma 3.5. *Let a group G act transitively on a digraph D and let H be a normal subgroup of G such that each of its subgroups is normal in G. Then* $\exp^+(D) \leq \exp^+(D_H) + 1$ *and* $\exp^-(D) \leq \exp^-(D_H) + 1$.

Proof. We prove the result for $\exp^+(D)$. The proof for $\exp^-(D)$ is analogous and is left to the reader. If $\exp^+(D_H) = \infty$, there is nothing to prove. We may thus assume that $\exp^+(D_H) = k$ for some integer $k \geq 0$.

To show that $\exp^+(D) \leq k + 1$ let $u \in V(D)$ and $v \in R^+(u)$ be arbitrary. Consider the equivalence class $B = R_{k+1}^+(u)$ and the H-orbit ${}^H u$. Note that both of these sets are blocks of imprimitivity for the action of G on $V(D)$. Let K be the setwise stabilizer in H of the set B. Note that the K-orbit of u is ${}^K u = {}^H u \cap B$ and is thus a block of imprimitivity for G. Moreover, by assumption on H the subgroup K is normal in G, and so the block system generated by the block ${}^K u$ coincides with the block system given by the orbits of K. Consequently, any two vertices within the same H-orbit are R_{k+1}^+ related if and only if they belong to the same K-orbit.

We first show that $\exp^+(D_K) \leq k$. If this is not the case, then there exists ${}^K w \in V(D_K)$ such that ${}^K w \in R_{k+1}^+({}^K u) \setminus R_k^+({}^K u)$. By Lemma 3.3 there exists $w' \in {}^K w$ such that $uR_{k+1}^+ w'$. Moreover, since $\exp^+(D_H) = k$ there exists $z \in {}^H w' = {}^H w$ such that $uR_k^+ z$. But then $zR_{k+1}^+ w'$, and so ${}^K z = {}^K w' = {}^K w$, implying that ${}^K w \in R_k^+({}^K u)$, a contradiction.

Hence $\exp^+(D_K) \leq k$. But then ${}^K v \in R_k^+({}^K u) = R_{k+1}^+({}^K u)$ in D_K, and by Lemma 3.3 there exists some $x \in {}^K v$ such that $uR_{k+1}^+ x$. Since $x \in {}^K v$ we have that $xR_{k+1}^+ v$, and so $uR_{k+1}^+ v$ holds.

Since u and v were arbitrary subject to the condition that $uR^+ v$, this shows that $\exp^+(D) \leq k + 1$. □

Lemma 3.6. *Let a group G act transitively on a digraph D with finite exponents* $\exp^+(D)$ *and* $\exp^-(D)$. *Furthermore, let τ denote the imprimitivity system of G on $V(D)$ which is* *induced by the equivalence classes with respect to R^+ or R^-. Then every $g \in G$ which leaves invariant at least one block of τ leaves invariant all blocks of τ.*

Proof. Since the exponents $\exp^+(D)$ and $\exp^-(D)$ are both finite, [6, Corollary 3.5] implies that $R^+ = R^-$, and so the discussion from the last paragraph of the first section implies that D_τ is a finite cycle or the two-way infinite directed line. Hence, the only automorphism of D_τ which fixes a vertex is the identity. On the other hand, every automorphism $g \in G$ which leaves invariant a block of τ induces an automorphism of D_τ fixing a vertex of D_τ, and the result follows. □

4 R^+ and R^- in transitive digraphs

We start with a simple observation concerning Cayley digraphs of abelian groups.

Proposition 4.1. *Let G be an abelian group acting transitively on a digraph D. Then* $\exp^+(D) = \exp^-(D) = 1$.

Proof. Since G is abelian, D is a Cayley graph of G. Then Corollary 3.2 implies that $R_1^+ = R_1^-$ and [6, Corollary 3.4] implies that $R^+ = R_1^+ = R_1^- = R^-$, as claimed. □

We now generalise this result to nilpotent groups.

Theorem 4.2. *Let G be a nilpotent group of class r acting transitively on a digraph D. Then* $\exp^+(D) = \exp^-(D) \leq r + 1$.

Proof. We first show that $\exp^+(D) \leq r + 1$. The proof is carried out by induction on r. If $r = 0$, then G is an abelian group and Proposition 4.1 applies.

Suppose now that $r \geq 1$. As $G^{(r+1)} = 1$, we have that $H = G^{(r)}$ is contained in the center of G, and so each of its subgroups is normal in G. Hence Lemma 3.5 implies that $\exp^+(D) \leq \exp(D_H) + 1$. Now, the quotient group G/H is a nilpotent group of class $r - 1$ and acts transitively on the quotient digraph D_H. By induction hypothesis we thus have that $\exp^+(D_H) \leq r$. Consequently, $\exp^+(D) \leq r + 1$, as claimed.

The fact that $\exp^-(D) \leq r + 1$ follows by analogous arguments. Then [6, Corollary 3.5], implies that $\exp^+(D) = \exp^-(D)$. □

The next example shows that the bound from the above theorem is tight, that is, for every positive integer r there exists a nilpotent group G of class r and a digraph D on which G acts transitively such that $\exp^+(D) = r + 1 = \exp^-(D)$ holds.

Example 4.3. Already for the smallest nonabelian finitely generated nilpotent group, the dihedral group D_8 of order 8 (of nilpotency class 1), this is the case. Let us write $D_8 = \langle f, a_1, a_2 | f^2 = a_1^2 = a_2^2 = 1, f a_1 f^{-1} = a_1 a_2, f a_2 = a_2 f, a_1 a_2 = a_2 a_1 \rangle$. Then for the Cayley digraph $D = Cay(D_8, \{f, f a_1\})$ we clearly have that $\exp^+(D) = \exp^-(D) = 2$.

In fact, this example happens to be the smallest member of the following infinite family. Let $n \geq 1$ be an integer and let G_n be the semidirect product of the elementary abelian group $\mathbb{Z}_2^n$ by the cyclic group $\mathbb{Z}_{2^{n-1}}$ generated by $G_n = \langle f, a_1, a_2, \ldots, a_n \rangle$, where f is of order 2^{n-1}, the a_i are involutions commuting with each other and $f a_i f^{-1} = a_i a_{i+1}$ holds for all i, $1 \leq i < n$, while f and a_n commute. One can verify that for $S = \{f, f a_1 a_2 \cdots a_n\}$ we have that $\langle S^i S^{-i} \rangle = \langle a_1, a_2, \ldots, a_i \rangle$ holds for all i, $1 \leq i \leq n$, and so Corollary 3.2 implies that $\exp^+(Cay(G_n, S)) = n$. Moreover, it can be verified that G_n is nilpotent of class $n - 1$. Indeed, we have that $G^{(i)} = \langle a_{i+1}, a_{i+2}, \ldots, a_n \rangle$ holds for each i, $1 \leq i \leq n-1$, and of course then $G^{(n)} = 1$. The Cayley graph $Cay(G_n, S)$ thus attains the bound from the above theorem.

We shall now see, that the above theorem cannot be generalized to solvable groups.

Example 4.4. The lamplighter group L is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. The standard representation for L is

$$\langle a, t | a^2, [t^m a t^{-m}, t^n a t^{-n}], m, n \in \mathbb{Z} \rangle.$$

If we consider the Cayley digraph of L with respect to the generating set $S = \{t, at\}$, then this Cayley digraph is the horocyclic product of two directed trees with indegree 1 and outdegree 2. In this digraph $R_k^+ \neq R^+$ clearly holds for all $k \in \mathbb{Z}^+$. This shows that for solvable groups we cannot expect a result like Theorem 4.2.

As was shown in [6], a connected, locally finite, transitive digraph D has exponential growth if at least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite. Hence these exponents must be finite if a connected, locally finite, transitive digraph D does not have exponential growth. So the question arises if we can find a bound on $\exp^+(D)$ and $\exp^-(D)$ which depends on the growth rate of D or on certain properties of groups acting transitively on D. In the sequel we show that this is indeed possible.

We first consider the case where a digraph D allows a transitive action of a group G containing a normal abelian subgroup, acting with finitely many orbits on D, thereby obtaining a tight bound for $\exp^+(D)$ and $\exp^-(D)$. We then explore a more general situation where a transitive group G contains a normal nilpotent subgroup acting with finitely many orbits on D. We start by proving two auxiliary results.

Lemma 4.5. *Let D be a connected digraph, and let G be a transitive subgroup of $\mathrm{Aut}(D)$ having a normal subgroup $H \triangleleft G$ with m, $1 \leq m < \infty$, orbits on D. If for some (and hence every) $u \in V(D)$ the set $R_1^+(u)$ is contained in ${}^H u$, then the following hold:*

- *(i) For every $v \in V(D)$ the set $R^+(v)$ is contained in ${}^H v$.*
- *(ii) The quotient digraph D_H is a directed cycle.*

Proof. Observe that if $m = 1$ there is nothing to prove, so we may assume $m \geq 2$.

To prove (i) we show that $R_k^+(v) \subseteq {}^H v$ for all $v \in V(D)$ and all k. We do that by induction on k. The base of induction ($k = 1$) holds by assumption. Let now $k \geq 1$ and suppose that $R_j^+(v) \subseteq {}^H v$ holds for all $j \leq k$. Pick an arbitrary vertex $v \in V(D)$ and let $w \in R_{k+1}^+(v)$. Let $v = v_0$, $w = v_n$ and choose a walk $W = (v_0, 1, v_1, \ldots, v_{n-1}, -1, v_n) \in R_{k+1}^+[v, w]$. Suppose first that for all i, $0 < i < n$, we have that $\zeta({}_0W_i) > 0$. In this case $v_1 R_k^+ v_{n-1}$, and so induction hypothesis implies that $v_{n-1} \in {}^H v_1$, that is, $v_{n-1} = {}^h v_1$ for some $h \in H$. Then $({}^h v_0, v_{n-1}) \in E(D)$, and so ${}^h v_0 R_1^+ v_n$. Then, by assumption, we have that ${}^h v_0 \in {}^H v_n$, and so $v \in {}^H w$ (recall that $v = v_0$ and $w = v_n$). Suppose now that $0 < i_1 < i_2 < \cdots < i_t = n$ are such that $\zeta({}_0W_{i_j}) = 0$. By the above argument $v_{i_1} \in {}^H v$, $v_{i_2} \in {}^H v_{i_1}, \ldots, w \in {}^H v_{i_{t-1}}$. Hence $v \in {}^H w$, which proves (i).

We now prove (ii). Let ${}^H v$ be an H-orbit. Since D is connected and H has at least two orbits which are blocks of imprimitivity for G, there exists an H-orbit ${}^H w \neq {}^H v$ such that $({}^H w, {}^H v) \in E(D_H)$. It follows that there exists a vertex $w' \in {}^H w$ with $(w', v) \in E(D)$. Consequently, the quotient digraph D_H must have indegree one (for otherwise we obtain a vertex $x \notin {}^H w$ which is R_1^+-related to w'). Since D_H is finite, it is a simple directed cycle. □

Lemma 4.6. *Let D be a digraph, and let G be a transitive subgroup of $\mathrm{Aut}(D)$ having an abelian normal subgroup $H \lhd G$ with m, $1 \leq m < \infty$, orbits on D. If for some (and hence any) $u \in V(D)$ the set $R_1^+(u)$ is contained in ${}^H u$, then $exp^+(D) = exp^-(D) \leq m$.*

Proof. We prove that $exp^+(D) \leq m$ and leave the analogous proof that $exp^-(D) \leq m$ to the reader.

If $m = 1$, then D is a Cayley digraph of an abelian group, so Proposition 4.1 applies. We can thus assume that $m \geq 2$. Let $\Delta = {}^H u$ for some $u \in V(D)$.

We first construct an auxiliary digraph D^* with vertex set Δ and an edge (w, v) whenever there exists a directed path of length m in D from w to v. The restriction of H on Δ acts regularly on Δ. The digraph D^* thus is a Cayley digraph of an abelian group (possibly disconnected). Therefore $\exp^+(D^*) \leq 1$ by Proposition 4.1.

Now, let vR^+w for some $v, w \in V(D)$ and let us show that in this case vR_m^+w holds. By definition of R^+ we have that vR_k^+w holds for some integer k. Then Lemma 3.1 implies that there exists a walk in $R_k^+[v, w]$ which is a concatenation of walks of the form $W = (v_0, 1, v_1, 1, \ldots, 1, v_k, -1, v_{k-1}, -1, \ldots, -1, v_{2k})$. By transitivity it suffices to prove that $v_0 R_m^+ v_{2k}$. Let t, r with $0 \leq r < m$ be the integers such that $k = tm + r$. By Lemma 4.5 the vertices $v_0, v_m, v_{2m}, \ldots, v_{tm}$ and $v_{2k}, v_{2k-m}, v_{2k-2m}, \ldots, v_{2k-tm}$ all belong to the H-orbit ${}^H v_0$. Hence $v_{2k} = {}^h v_0$, $v_{tm} = {}^{h_1} v_0$ and $v_{2k-tm} = {}^{h_2} v_0$ for some $h, h_1, h_2 \in H$. Now, ${}_0W_{tm} \cdot ({}^{h_1 h_2^{-1}}({}_{(2k-tm)}W_{2k}))$ is a walk from v_0 to $x = {}^{h_1 h_2^{-1}} v_{2k} = {}^{h_1 h_2^{-1} h} v_0$. As H is abelian, $x = {}^{h h_2^{-1} h_1} v_0$. Therefore, ${}^{h h_2^{-1}}({}_{tm}W_{2k-tm}) \in R_r^+[x, v_{2k}]$, and so $r < m$ implies that $v_0 R_m^+ v_{2k}$ if and only if $v_0 R_m^+ x$, that is, we can assume $r = 0$. Since $v_0 \in {}^H v$, we have $v_0 = {}^{h_0} v$ for some $h_0 \in H$. It follows that the walk W corresponds to a walk $W^* \in R_t^+[h_0, h_1 h_2^{-1} h h_0]$ in D^*. Since $\exp^+(D^*) \leq 1$, the walk W^* can be replaced by a walk in $R_1^+[h_0, h_1 h_2^{-1} h h_0]$, implying that W can be replaced by a walk in $R_m^+[v_0, v_{2k}]$.

Therefore, $R^+ \subseteq R_m^+$, implying that $R^+ = R_m^+$. Analogously, it can be shown that $R^- = R_m^-$. Then [6, Corollary 3.5] completes the proof. □

To prove the next theorem we need the following result from [6].

Proposition 4.7. *([6], Proposition 3.11) Let D be a digraph, let τ be the set of equivalence classes of R_1^+, and let $u \in V(D)$. Then, for any $v \in V(D)$ and any $k \geq 2$ we have that uR_k^+v if and only if $u_\tau R_{k-1}^+ v_\tau$. An analogous assertion holds for R_k^- when taking the quotient with respect to R_1^-.*

Theorem 4.8. *Let D be a digraph and let $G \leq \mathrm{Aut}(D)$ be a transitive subgroup having an abelian normal subgroup H acting with m, $1 \leq m < \infty$, orbits on $V(D)$. Then $\exp^+(D) = \exp^-(D) \leq m$.*

Proof. We prove that $\exp^+(D) \leq m$ and leave the analogous proof for $\exp^-(D)$ to the reader. We proceed by induction on m. If $m = 1$, then the result follows from Proposition 4.1. Suppose the assertion holds for all $n < m$, $m \geq 2$, and suppose that H has m orbits on D. If for some $u \in V(D)$ the set $R_1^+(u)$ is contained in ${}^H u$, then Lemma 4.6 applies.

Assume now that the equivalence classes with respect to R_1^+ are not contained in the orbits of H and consider the quotient digraph D/R_1^+. Let K be the kernel of the action of G on D/R_1^+ and let $N = HK/K \cong H/(H \cap K)$ be the induced faithful action of H on D/R_1^+. Observe that, since the R_1-equivalence classes are not fully contained in the H-orbits, N acts with at most $\frac{m}{2}$ orbits on D/R_1^+. By induction hypothesis (note that N is an abelian normal subgroup of G/K) we have that $\exp^+(D/R_1^+) \leq \frac{m}{2}$. By Proposition 4.7 it follows that $\exp^+(D) = \exp^+(D/R_1^+) + 1 \leq \frac{m+2}{2} \leq m$.

Analogously it can be shown that $\exp^-(D) \leq m$. Then again [6, Corollary 3.5] completes the proof. □

Proof of Theorem 2.1

Let a group G act transitively on a connected digraph D, and let $N \trianglelefteq G$ be a normal nilpotent subgroup of class r acting with m orbits on D, where $1 \leq m < \infty$.

We first prove that $\exp^+(D) \leq m(r+2) - 1$. The proof is done by induction on m.

If $m = 1$, then the result holds by Theorem 4.2. If $m \geq 2$ we distinguish two cases, depending on the structure of D_N.

Case 1. D_N is not isomorphic to a directed cycle on $m \geq 2$ vertices.

In this case Lemma 4.5 implies that, for any $v \in V(D)$, the set $R_1^+(v)$ is not completely contained in one orbit of N. Let τ denote the imprimitivity system of G on D consisting of the equivalence classes with respect to R_1^+. Then the permutation group G_τ, induced by the action of G on τ, acts transitively on D_τ. Furthermore, N_τ acts with at most $\frac{m}{2}$ orbits. In addition N_τ is nilpotent of class at most r. Then, by induction hypothesis, $\exp^+(D_\tau) \leq \frac{m}{2}(r+2) - 1$ holds and the result follows by Proposition 4.7.

Case 2. D_N is isomorphic to a directed cycle $C = (c_1, \ldots, c_m)$ on $m \geq 2$ vertices.

Let $O_1, \ldots, O_m$ denote the orbits of N on $V(D)$ which correspond to the vertices $c_1, \ldots, c_m \in D_N$. Then of course there is no edge in D which connects two vertices which are both contained in the same orbit. Furthermore, all edges of D are directed from O_i to O_{i+1}, $1 \leq i \leq m$, where indices are taken modulo m. Then for every $v \in O_i$, $1 \leq i \leq m$, $R^+(v) \subseteq O_i$ holds. Of course $\exp^+(D) \leq m - 1$ holds if $R_{m-1}^+(v) = O_i$ for some $v \in O_i$ and some i, $1 \leq i \leq m$.

Hence we only have to consider the case when $R_{m-1}^+(v)$ is properly contained in O_i for every i, $1 \leq i \leq m$, and every vertex $v \in O_i$. By B_ι, $\iota \in \mathcal{I}$, we denote the equivalence classes of R_{m-1}^+ on O_1. For $v \in O_1$, let $\mathcal{P}(v)$ denote the set of all directed paths starting at v and containing exactly one vertex from each orbit O_i, $1 \leq i \leq m$. Since D_N is isomorphic to a directed cycle with m vertices and N acts transitively on each of its orbits, $\mathcal{P}(v) \neq \emptyset$ for all $v \in O_1$. Furthermore, for $\iota \in \mathcal{I}$ let S_ι be the subdigraph of D induced by the vertices of the union $\bigcup_{v \in B_\iota} \mathcal{P}(v)$. Note that since the sets B_ι are different equivalence classes with respect to R_{m-1}^+, the digraphs S_ι, $\iota \in \mathcal{I}$, are pairwise disjoint.

We first define $\mathcal{P}^m$ as the set of all directed paths $P = (v_1, \ldots, v_{m+1})$ in D where $v_j \in O_j$ for $1 \leq j \leq m$ and $v_{m+1} \in O_1$. Analogously we define $\mathcal{P}^{-m}$ as the set of all inverses of the paths in $\mathcal{P}^m$. Furthermore, let $\mathcal{R}_{m-1}^+$ denote the set of all walks which are contained in $R_{m-1}^+[u, v]$ for some vertices $u, v \in O_1$.

Let $v_1, v_2 \in O_1$ now satisfy $v_1 R^+ v_2$. If v_1 and v_2 are both contained in one and the same set B_ι, $\iota \in \mathcal{I}$, then of course $v_1 R_{m-1}^+ v_2$ holds.

Now let $v_1 \in B_{\iota_1}$ and $v_2 \in B_{\iota_2}$, $\iota_1 \neq \iota_2$. Then there is a walk $W \in R^+[v_1, v_2]$ which is the concatenation of finitely many paths and walks which are contained in $\mathcal{P}^m$, $\mathcal{P}^{-m}$ or $\mathcal{R}_{m-1}^+$. Let D' now be the digraph with vertex set $\mathcal{I}$ with $(\iota_1, \iota_2) \in E(D')$ whenever there exists a path $P \in \mathcal{P}^m$ with origin in B_{ι_1} and terminal vertex in B_{ι_2}. Observe that, in general, the digraph D' might not be locally finite. Nevertheless, the restriction of N to O_1 induces a transitive group acting on D' which is nilpotent of class at most r. Thus Theorem 4.2 implies that $\exp^+(D') \leq r + 1$.

Observe that, by Lemma 3.1, we can assume that the walk W is the concatenation of t paths from $\mathcal{P}^m$, followed by a walk in $\mathcal{R}^+_{m-1}$ and then t paths from $\mathcal{P}^{-m}$, for some non-negative integer t. Let $u_0, u_1, \ldots, u_{2t+1}$ be the vertices of W, contained in O_1, given in the order they are met when traversing W. Thus $u_0, u_1, \ldots, u_{t-1}$ are the origins of the paths from $\mathcal{P}^m$ while the vertices $u_{t+1}, u_{t+2}, \ldots, u_{2t}$ are the origins of the paths from $\mathcal{P}^{-m}$. The walk W thus naturally gives rise to the walk $W' = (\iota_0, \iota_1, \ldots, \iota_{t-1}, \iota_{t+1}, \iota_{t+2}, \ldots, \iota_{2t+1})$ in D', where for each i we have that $u_i \in B_{\iota_i}$ (observe that $u_t \in B_{\iota_{t+1}} = B_{\iota_t}$). Of course $W' \in R^+[\iota_0, \iota_{2t+1}]$, and so $\exp^+(D') \leq r + 1$ implies that there is a walk $\bar{W}' \in R^+_{r+1}[\iota_0, \iota_{2t+1}]$. Since the sets B_{ι_i} are equivalence classes of the relation R^+_{m-1} on D it is now clear that this walk gives rise to some walk in $R^+_{m(r+2)-1}[v_1, v_2]$.

Since $\exp^-(D) \leq m(r+2) - 1$ holds by similar arguments, [6, Corollary 3.5] implies that $\exp^+(D) = \exp^-(D)$. □

Corollary 4.9. *Let G be a finitely generated group, let N be a normal nilpotent subgroup of finite index m in G and let D denote a Cayley digraph of G with respect to some finite generating set S. Then $\exp^+(D) = \exp^-(D) \leq m(r+2) - 1$ holds, where r is the nilpotency class of N.*

It is natural to ask if this bound is tight. All examples we know in fact satisfy the inequality $\exp^+(D) \leq m(r+1)$. We thus pose the following problem.

Problem 4.10. Is it true that $\exp^+(D) = \exp^-(D) \leq m(r+1)$ holds for the Cayley digraphs of groups described in Corollary 4.9?

For Cayley digraphs D of finitely generated torsion-free groups G with polynomial growth we even obtain bounds for $\exp^+(D)$ and $\exp^-(D)$ which only depend on the growth degree. To formulate the result we first have to consider $\mathrm{GL}(n, \mathbb{Z})$.

Theorem 4.11. *(see e.g. [7]) The orders of the finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ are bounded by some function g(n) of n alone.*

Theorem 4.12. *(see e.g. [7]) Let G be a finitely generated torsion-free group with polynomial growth of degree d. Then G contains a normal nilpotent subgroup of class less than $\sqrt{2d}$ and index at most $g(d)$, where $g(d)$ is the function of Theorem 4.11.*

Corollary 4.13. *Let G be a finitely generated torsion-free group with polynomial growth of degree d. Then for any Cayley digraph D of G, $\exp^+(D)$ and $\exp^-(D)$ are bounded by $g(d)(\sqrt{2d} + 2) - 1$, where $g(d)$ is the function of Theorem 4.11*

We conclude the paper with the following observations. Let $G \leq \mathrm{Aut}(D)$ act transitively on a digraph D with finite exponents $\exp^+(D)$ and $\exp^-(D)$. Then Lemma 3.6 implies that the equivalence classes of the relation $R^+ = R^-$ are orbits of a normal subgroup of G. Thus, if this relation is not universal and if the digraph has indegree or outdegree at least 2, then this normal subgroup of G is proper and not trivial. As a consequence, if G is simple, the relation $R^+ = R^-$ is universal on D. As was already mentioned above it was shown in [6, Theorem 4.12] that a connected infinite locally finite transitive digraph D has exponential growth if at least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite. At this point we recall the following problem from combinatorial group theory (see e.g. [1]), which was originally posed by R. I. Grigorchuk.

Problem 4.14. Does every finitely generated infinite simple group have exponential growth?

The following proposition then allows to formulate a conjecture which closely relates this problem to reachability relations.

Proposition 4.15. *If a finitely generated infinite simple group G does not have exponential growth, then for every finite generating set S of G there is a finite integer $k_S \geq 1$, such that $R^+_{k_S} = R^-_{k_S}$ is universal in $C(G, S)$.*

Proof. Follows immediately from [6, Theorem 4.12] and Lemma 3.6. □

Conjecture 4.16. *Let G be a finitely generated infinite group. Then there is a finite generating set S of G such that for the Cayley digraph D of G with respect to S one of the following holds:*

- *At least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite and hence D has exponential growth.*
- *Both, $\exp^+(D)$ and $\exp^-(D)$ are finite and the reachability relations R^+ and R^- are not universal on D.*

Observe that by Proposition 4.15 the validity of this conjecture would provide a positive answer to Grigorchuk's problem.

References

[1] G. Baumslag, A. G. Myasnikov and V. Shpilrain, Open problems in combinatorial group theory. Second edition, Combinatorial and geometric group theory, S. Cleary, R. Gilman, A. G. Myasnikov, V. Shpilrain, eds., *Contemporary Mathematics* **296**, AMS 2002.

[2] P. J. Cameron, C. E. Praeger and N. C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs, *Combinatorica* **13** (1993), 377–396.

[3] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Etudes Sci. Publ. Math.* **53** (1981), 53–78.

[4] M. DeVos, B. Mohar and R. Samal, Reachability in highly arc-transitive digraphs, arXiv:1110. 2945.

[5] A. Malnič, D. Marušič, R. Möller, N. Seifter, V. Trofimov and B. Zgrablič, Highly arc transitive digraphs: reachability, topological groups, *Europ. J. Combinatorics* **26** (2005), 19–28.

[6] A. Malnič, D. Marušič, N. Seifter, P. Šparl and B. Zgrablič, Reachability relations in digraphs, *Europ. J. Combin.* **29** (2008), 1566–1581.

[7] A. Mann, *How groups grow*, LMS Lecture Notes Ser. 395, Cambridge Uni. Press, 2012.

[8] N. Seifter and V. I. Trofimov, Reachability relations and the structure of transitive digraphs, *Electron. J. Combin.* **16** (2009), R26.

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The clone cover*

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Abstract

Each finite graph on n vertices determines a special $(n-1)$-fold covering graph that we call the *clone cover*. Several equivalent definitions and basic properties about this remarkable construction are presented. In particular, we show that for $k \geq 2$, the clone cover of a k-connected graph is k-connected, the clone cover of a planar graph is planar and the clone cover of a hamiltonian graph is hamiltonian. As for symmetry properties, in most cases we also understand the structure of the automorphism groups of these covers. A particularly nice property is that every automorphism of the base graph lifts to an automorphism of its clone cover. We also show that the covering projection from the clone cover onto its corresponding 2-connected base graph is never a regular covering, except when the base graph is a cycle.

Keywords: Covering projection, canonical cover, regular cover, automorphisms.

Math. Subj. Class.: 57M10, 20B25, 05E18, 05C75, 05C76

*Dedicated to Dragan Marušič at the occasion of his 60th birthday.

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1 Introduction

Some coverings are "natural" or *canonical* in the sense that they are determined by the graph itself. A typical example is the universal cover, which is a tree, usually, an infinite tree [6]. Another such example is the so-called canonical double cover, or the Kronecker cover. It can be described as the tensor product of the graph in question by K_2 [5]. And there is also the trivial cover with the identity mapping as the covering projection.

In this paper we describe another canonical cover, an $(n-1)$-fold covering of a graph on n vertices, called the *clone cover*.[1] The clone cover of a graph X will be denoted by $\mathrm{Clone}(X)$. We present four equivalent definitions and several basic properties of this canonical covering graph. An application in mathematical chemistry can be found in [3]. As usual in the theory of covering graphs we will always assume that X is connected.

First, we study graph-theoretical properties such as connectedness, genus, and hamiltonicity. It turns out that the clone cover of a planar graph is also planar, the clone cover of a k-connected graph for $k \geq 2$ is also k-connected, and the clone cover of a hamiltonian graph is also hamiltonian. All these properties are far from being guaranteed for general covering graphs. In the second part of the paper we study automorphisms of such covers. Each automorphism of a graph X lifts to an automorphism of $\mathrm{Clone}(X)$, and moreover, the automorphism group of X embeds isomorphically in the automorphism group of $\mathrm{Clone}(X)$. This also is not true for general coverings. In most cases the covering projection $\mathrm{Clone}(X) \to X$ is irregular, with trivial group of covering transformations. Finally, there is a natural quotient projection $\mathrm{Clone}(X) \to X$, called contraction, that is different from the covering projection. This enables us to determine the full automorphism group of $\mathrm{Clone}(X)$ for certain classes of 2-connected graphs X.

2 Preliminaries

In this section we review some basic definitions and elementary properties of covering graphs. The most frequent descriptions and constructions of coverings use *voltage graphs*. These were first introduced by Gross and Tucker and popularized in their classic text [4]. In this paper a slightly different but equivalent approach is taken, following [9]. There are two differences in the approaches. While [4] requires a choice of directions of edges in the base graph, the approach in [9] mantains the base graph as completely undirected. The other advantage of [9] is that the base graph may also be a pregraph, that is, a graph with pending semi-edges. Pregraphs, however, will not be used in this paper.

A *graph* X is a quadruple $X = (V, S, i, r)$ where V is a finite set of *vertices*, S is a finite set of *arcs*, i is a mapping $S \to V$, specifying the *initial vertex* of each arc, while the reversal involution $r : S \to S$ is an involution without fixed points. The *terminal vertex* of an arc is then specified by the mapping $t : S \to V$, $t(s) = i(r(s))$. An arc s and its reverse $r(s)$ form an *edge* with *endvertices* $i(s)$ and $i(r(s))$. Two vertices are *adjacent* if they are the endvertices of a common edge. If every edge of the graph has two distinct endvertices and no two edges have the same endvertices, the graph is *simple*. We consider only simple graphs, with at least one edge, to avoid trivialities.

We will use the following notation. The set of vertices of a graph X will be denoted by $V(X)$, the set of its arcs by $S(X)$, and the set of its edges by $E(X)$. In a simple graph every edge is uniquely determined by its endvertices. Therefore we will denote an edge with endvertices u and v by $\{u, v\}$. An arc with initial vertex u and terminal vertex v will

[1] Note that this construction was previously called TheCover.

be denoted by $[u, v]$, or more briefly by uv. The set of vertices, adjacent to a vertex v of X, will be denoted by $N(v)$.

Let X and Y be graphs. A mapping $p : Y \to X$ that takes vertices to vertices and arcs to arcs is called a *homomorphism* if $p(i(s)) = i(p(s))$ and $p(r(s)) = r(p(s))$ for every $s \in S$. A surjective homomorphism $p : Y \to X$ is called a *covering projection* if the set $N(v)$ is mapped bijectively onto the set $N(p(v))$, for each vertex $v \in V(Y)$. The graph X is usually referred to as the *base graph* and Y as the *covering graph*. We call $p^{-1}(u)$ the *fiber* over $u \in V(X)$, and $p^{-1}(s)$ the *fiber* over $s \in S(X)$. We will assume that X is connected. This implies that all the fibers are of the same size.

By [4, Theorem 2.4.5], every covering graph can be obtained as follows. Let L be a finite (labeling) set and X a finite connected simple graph. Let $\tau\colon X \to \mathrm{Sym}(L)$ be a *permutation voltage assignment* on X, defined by $\tau(s) \in \mathrm{Sym}(L)$ for each arc s in X, and satisfying the condition $\tau(r(s)) = \tau^{-1}(s)$. The graph X together with the assignment τ is called a *permutation voltage graph* (X, τ). To a permutation voltage graph (X, τ) we associate a *derived graph* $Y = \mathrm{Cov}_\tau(X)$, with vertex set $V(Y) = V(X) \times L$, arc set $S(Y) = S(X) \times L$, and mappings i, r satisfying

$$\begin{aligned} i(s,j) &= (i(s),j) \quad \text{for any } (s,j) \in S(Y) \text{ and} \\ r(s,j) &= (r(s),j^{\tau(s)}) \quad \text{for any } (s,j) \in S(Y). \end{aligned}$$

In other words, with each arc $uv \in S(X)$ and each $j \in L$ we associate the arc $(uv, j) = [(u, j), (v, j^{\tau(u,v)})]$ from $(u, j) \in V(Y)$ to $(v, j^{\tau(uv)}) \in V(Y)$. Note that its reverse arc is $(vu, j^{\tau(uv)}) = [(v, j^{\tau(uv)}), (u, j)]$, from $(v, j^{\tau(uv)})$ to (u, j). Hence these opposite arcs form an edge "over" the edge $\{u, v\}$. Therefore the graph Y is a covering graph over the base graph X, with the *natural* covering projection $p : Y \to X$ taking a vertex $(u, j) \in V(Y)$ to $u \in V(X)$ and an arc (uv, j) in Y to the arc uv in X.

3 Constructions

Let X be a connected graph on $n \geq 2$ vertices. We begin by constructing a canonical n-fold covering graph of X, where we use $V(X)$ as the labeling set L. Let $\tau\colon X \to \mathrm{Sym}(V(X))$ be a permutation voltage assignment on X defined by the transposition

$$\tau(uv) = (u, v) \in \mathrm{Sym}(V(X))$$

for each arc uv in X. The associated covering graph is denoted by Cov(X). The vertex set of Cov(X) is $V(X) \times V(X)$ while $E(X) \times V(X)$ is the edge set. The edge set can be naturally partitioned into three subsets, namely, the subset of

- diagonal edges $\{(u, u), (v, v)\}$,
- connecting edges $\{(u, v), (v, u)\}$, $u \neq v$, and
- inner edges $\{(u, w), (v, w)\}$, $w \neq u, v$.

We call this partition the *fundamental edge partition*. The three different types of edges in the fiber over one edge are shown in Figure 1.

Example 3.1. Let us consider the graph $K_{2,3}$. The voltage assignment in Figure 2 determines the 5-fold covering graph Cov$(K_{2,3})$ in Figure 3.

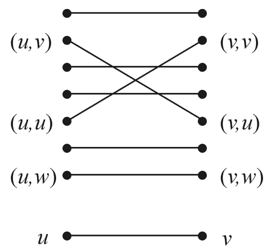


Figure 1: The lift of an edge.

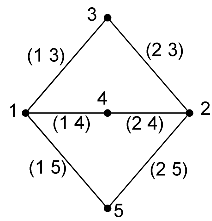


Figure 2: A voltage graph $K_{2,3}$ for Cov$(K_{2,3})$.

3.1 The construction of $\mathrm{Clone}(X)$ via $\mathrm{Cov}(X)$

Let X be a connected graph on $n \geq 2$ vertices. The following proposition carves $\mathrm{Clone}(X)$ out of the auxiliary graph Cov(X).

Proposition 3.2. Cov(X) *is a disjoint union of two covering graphs of* X*. One is isomorphic to* Cov$_{\mathrm{id}}(X) = X$.

Proof. The subgraph of Cov(X) induced by the diagonal vertices $\{(v,v),\ v \in X\}$ (and diagonal edges) is isomorphic to X, and the restriction of the covering projection to it gives Cov$_{\mathrm{id}}(X)$. □

The subgraph of Cov(X) that does not contain the diagonal vertices is called the *clone cover* and denoted by $\mathrm{Clone}(X)$. Clearly, $\mathrm{Clone}(X)$ is an $(n-1)$-fold covering graph over X. We call the subgraph of $\mathrm{Clone}(X)$, spanned by the vertices $\{(v,i); v \in V(X)\backslash i\}$, the i-th *layer* of $\mathrm{Clone}(X)$.

Example 3.3. Let X be the cycle on n vertices for $n \geq 3$. Then $\mathrm{Clone}(X)$ is the cycle on $n(n-1)$ vertices.

(a) (b)

Figure 3: Cov$(K_{2,3})$ has two components: (a) Cov$_{\mathrm{id}}(K_{2,3})$, (b) $\mathrm{Clone}(K_{2,3})$.

3.2 A direct permutation voltage graph construction

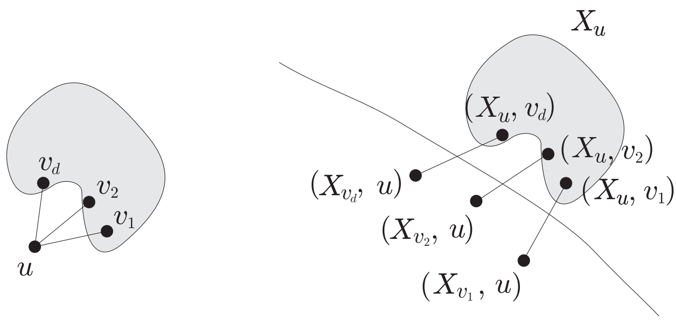
This construction depends on the choice of the base vertex b of X. The permutation voltages are taken from $\mathrm{Sym}(V(X) - \{b\})$. They are defined as follows. The permutation voltages on arcs incident with the vertex b are equal to the identity while the voltages of arcs uv not involving b are, as before, equal to the transposition (u, v). The corresponding covering graph is denoted, for the time being, by $\mathrm{Clone}_d(X, b)$.

3.3 Combinatorial Construction

Let X be a connected graph on n vertices. We define the graph $\mathrm{Clone}_c(X)$ as follows. The vertex set W of $\mathrm{Clone}_c(X)$ consists of all $n(n-1)$ pairs of vertices $(u, v) \in V(X) \times V(X)$ with $u \neq v$. There are two sets of edges. Each edge $\{u, v\}$ from X gives rise to the edge $\{(u, v), (v, u)\}$ in $\mathrm{Clone}_c(X)$ (these will correspond to the connecting edges). For each $w \in V(X)$, different from u and v, we get in total $(n-2)$ (inner) edges $\{(u, w), (v, w)\}$. It is not hard to show that the projection from W to $V(X)$ defined by $(u, v) \mapsto u$ is an $(n-1)$-fold covering projection.

3.4 Graphical Construction

Let X_v denote the graph X with vertex v removed. The graph $\mathrm{Clone}_g(X)$ is obtained from the collection of n *vertex-deleted subgraphs* $X_v = X - v$ by adding, for each edge $\{u, v\}$ of X, an edge joining the vertex u in X_v to the vertex v in X_u; see Figure 4.



(a) (b)

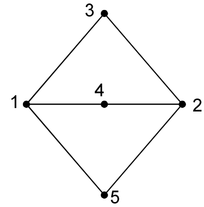
Figure 4: (a) The graph X and one of its vertices u. $\mathrm{Clone}_g(X)$ is obtained in such a way that each vertex u is replaced by a vertex deleted subgraph X_u. In (b), this is shown for the vertex u.

The edges of $\mathrm{Clone}_g(X)$ can be naturally partitioned (or colored) into two classes: the edges belonging to each vertex-deleted subgraph X_v, and the connecting edges. Each edge of X lifts to one connecting edge and $(n-2)$ original edges.

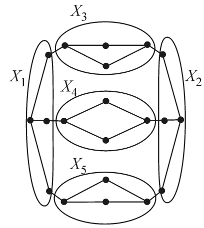
Example 3.4. Figure 5 shows the graphical construction of $\mathrm{Clone}(K_{2,3})$.

Proposition 3.5. *Let X be a 2-connected graph. Then X is a minor of* $\mathrm{Clone}_g(X)$.

Proof. If X is 2-connected, then for every vertex $u \in V(X)$ the vertex-deleted subgraph X_u is connected. If for each u we contract the edges of the copy of X_u from $\mathrm{Clone}_g(X)$,



(a) $K_{2,3}$



(b) $\mathrm{Clone}(K_{2,3})$.

Figure 5: The graphical construction of $\mathrm{Clone}(K_{2,3})$.

then this copy of X_u is contracted to a single vertex and the resulting graph is isomorphic to X. Hence X is a minor of $\mathrm{Clone}_g(X)$. □

3.5 Equivalence of the four constructions

Here we prove that the above four definitions are equivalent.

Theorem 3.6. *The covers* $\mathrm{Clone}(X), \mathrm{Clone}_d(X, b), \mathrm{Clone}_c(X)$*, and* $\mathrm{Clone}_g(X)$ *are isomorphic.*

Proof. It is easy to see that $\mathrm{Clone}(X)$ and $\mathrm{Clone}_c(X)$ are isomorphic since they have the same vertex set and the same edge set. Also the mapping that sends the vertex (X_u, v) of $\mathrm{Clone}_g(X)$ to the vertex (v, u) in $\mathrm{Clone}(X)$ is an isomorphism.

To finish the proof we show that $\mathrm{Clone}_c(X)$ and $\mathrm{Clone}_d(X, b)$ are isomorphic. Define the mapping $\varphi : V(\mathrm{Clone}_c(X)) \to V(\mathrm{Clone}_d(X, b))$ by

$$\varphi(u, v) = \begin{cases} (u, v) & \text{if } v \neq b, \\ (u, u) & \text{if } v = b. \end{cases}$$

This is obviously bijective. The edges of the form $\{(u, b), (v, b)\}$ are mapped to the edges of the form $\{(u, u), (v, v)\}$, while all other edges of $\mathrm{Clone}_c(X)$ are mapped to the edges with the same labels in $\mathrm{Clone}_d(X, b)$. This shows that φ is an isomorphism. In particular, the choice of the vertex b in $\mathrm{Clone}_d(X) = \mathrm{Clone}_d(X, b)$ is irrelevant. □

3.6 Lifts of cycles

Recall from general theory [4] that any voltage assignment can be naturally extended from arcs to walks by successively multiplying voltages of arcs encountered along the walk. The voltage of a walk actually tells how this walk lifts to the corresponding covering graph. We are particularly interested in how a given cycle of the base graph lifts. Clearly, a cycle lifts to a collection of cycles.

Theorem 3.7. [4, Theorem 2.4.3] *Consider a covering projection* $p\colon \tilde{X} \to X$ *arising from a permutation voltage assignment in* S_n *on* X*. If* C *is a cycle of length* k *in* X *whose voltage has cycle structure* $(c_1, \ldots, c_n)$*, then the preimage of* C *in the derived graph has* $c_1 + \cdots + c_n$ *components, consisting of exactly* c_j *cycles of length* kj*, for* $j = 1, \ldots, n$*.*

Let X be a graph on n vertices, with the voltage assignment $\tau(uv) = (u, v) \in S(V(X))$ for each arc uv in X. Recall that this assignment gives rise to the covering graph Cov(X) which consists of an isomorphic copy of X, and Clone(X). In this particular setting it is easy to see that the voltage of a directed cycle $C = v_0v_1\ldots v_mv_0$ in X, rooted at v_0, is then

$$(v_m, v_{m-1}, \ldots, v_1)(v_0) \in \mathrm{Sym}(V(X)).$$

The following proposition is therefore a direct consequence of Theorem 3.7 .

Proposition 3.8. *Let X be a connected graph. A k-cycle in X based at u lifts in* Clone(X) *to one "long" cycle of length $k(k-1)$ based at (u, v), where $v \neq u$ is any vertex in the cycle, and $n-k$ "short" cycles of length k based at (u, v) where v is any vertex not in the cycle.*

Corollary 3.9. *Let X be a connected graph on n vertices. If X contains a cycle of length $k < n$ then also* Clone(X) *contains a cycle of length k.*

4 Graph-theoretical properties

A natural problem to consider is the impact of a given graph invariant of a graph such as girth, connectivity or diameter, on its clone cover. Some invariants are easy to determine. For instance, girth is a well-known graph invariant measuring the lengh of the shortest cycle in a graph. Any connected graph that is not a cycle has the same girth as its clone cover by Corollary 3.9, and the girth of the clone cover of a cycle on n vertices is $n(n-1)$ by Proposition 3.8. In this section some other graph invarants are studied.

4.1 Connectivity

The graph Clone(X) can be connected or disconnected, with an easy test for connectivity. Recall that a *block* of a graph X is a maximal connected subgraph of X without a cut-vertex. If X contains no cut-vertex, then X itself is called a block.

Theorem 4.1. *Let X be a connected graph. Then* Clone(X) *is connected if and only if X is a block. Moreover, if X is k-connected, where $k \geq 2$, then* Clone(X) *is also k-connected.*

Proof. In this proof we will use the graphical construction of Clone. Suppose X has a cut-vertex v, and let the vertices v_1 and v_2 be in different blocks of X. Then the vertices (X_{v_1}, v_2) and (X_{v_2}, v_1) are in different components of Clone(X), since every path between them would pass through X_v, and in X_v there is no edge between the vertices of the blocks of v_1 and v_2. Therefore Clone(X) is not connected.

If X is a block that is not 2-conneced, then it is isomorphic to the complete graph on two vertices. So Clone(X) is isomorphic to X and hence connected.

Suppose now that X is k-connected, where $k \geq 2$. We will prove that Clone(X) is k-connected (and therefore also connected). By the global version of Menger's theorem, it is enough to prove that for any two distinct vertices in Clone(X) there exist k internally disjoint paths between them. Note that each of the subgraphs X_u of Clone(X) is connected since X is 2-connected.

We use the following notation. Let $P = u_1u_2\ldots u_t$ be a path in X. Then $P(u_i, \ldots, u_j)$ denotes the part of P between the vertices u_i and u_j for $1 \leq i \leq j \leq t$. Let $u \in V(X)$ be distinct from $u_1, u_2, \ldots, u_t$. By $\tilde{P}_u(u_1, u_2, \ldots, u_t)$, or more briefly, by $\tilde{P}_u$, we denote the path $(X_u, u_1)(X_u, u_2)\ldots(X_u, u_t)$ in Clone(X) that is contained in X_u. We denote a path in $\mathrm{Clone}(X)$ between the vertices (X_{u_1}, u_2) and (X_{u_t}, u_{t-1}) of the form

$$(X_{u_1}, u_2)(X_{u_2}, u_1) \quad \ldots \quad (X_{u_2}, u_3)(X_{u_3}, u_2)\ldots \\ (X_{u_{t-2}}, u_{t-1})(X_{u_{t-1}}, u_{t-2})\ldots(X_{u_{t-1}}, u_t)(X_{u_t}, u_{t-1})$$

by $\tilde{P}(u_1, u_2, \ldots, u_t)$, or more briefly, by $\tilde{P}$. The walks in the same copy of any vertex deleted subgraph can be arbitrary paths.

Let (X_u, v) and (X_w, z) be two distinct vertices of $\mathrm{Clone}(X)$. We now construct k internally disjoint paths between them. We distinguish two cases.

Case 1. Suppose $u = w$. Let $P_1, \ldots, P_k$ be k internally disjoint paths between v and z in X. If none of them contains u, we have k disjoint paths between (X_u, v) and (X_u, z) in $\mathrm{Clone}(X)$ which are all contained in X_u. If one of the paths, say P_1, contains u, we have only $k-1$ internally disjoint paths contained in X_u. We now construct another path that will also use other vertex-deleted subgraphs. Let $P = P_1 = vu_1 \ldots u_t u u_{t+1} \ldots u_s z$. Let Q be a path between u and u_{t+1} in X that does not contain u_t, and let R be a path between u_t and u in X that does not contain u_{t+1}. Since X is 2-connected, such paths exist. Then

$$\tilde{P}_u(v, u_1, \ldots, u_t)(X_{u_t}, u)\tilde{Q}_{u_t}(X_{u_{t+1}}, u_t)\tilde{R}_{u_{t+1}}(X_u, u_{t+1})\tilde{P}_u(u_{t+1} \ldots, u_t, z)$$

is a path between (X_u, v) and (X_u, z) that is internally disjoint from each of $P_2, \ldots, P_k$.

Case 2. Suppose $u \neq w$. Let $P^i = uu_1^i \ldots u_{t_i}^i, w$ be k internally disjoint paths between u and w in X. By Dirac's Fan Lemma (see, for example, [10, Theorem 4.2.23]), there exist internally disjoint paths from v to $u_1^1, \ldots, u_1^k$ in X, say $Q^1, \ldots, Q^k$. Similarly, there exist internally disjoint paths from $u_{t_1}^1, \ldots, u_{t_k}^k$ to z in X, say $R^1, \ldots, R^k$. Suppose first that u is not contained in any of the paths Q^i or R^i. Then for $i = 1, \ldots, k$ the paths

$$S^i = \tilde{Q}_u^i \tilde{P}^i(u, u_1^i, \ldots, u_{t_i}^i, w)\tilde{R}_w^i$$

are k internally disjoint paths between (X_u, v) and (X_w, z); see the top path in Figure 6.

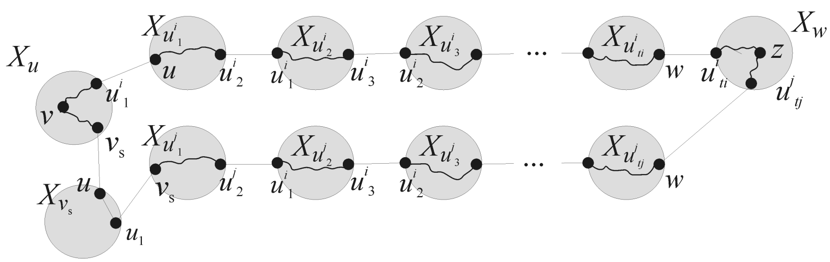


Figure 6: Two of internally disjoint paths in $\mathrm{Clone}(X)$.

Suppose now that u belongs to Q^j for some j. Let $Q = Q^j = vv_1 \ldots v_s u u_1^j$. If v_s belongs to P^j, we just interchange the roles of u_1^j and v_s and construct S^j as before. Otherwise, we may assume that v_s does not belong to any of the paths P^i for $i \neq j$. We can do this since if for some i the vertex v_s belongs to P^i, so P^i is of the form $uu_1^i \ldots v_s \ldots u_{t_i}^i w$, then we may replace P^i by the path $uv_s \ldots u_{t_i}^i w$ that is also internally disjoint from the other $k-1$ paths between u and w in X. We define

$$S^j = \tilde{Q}_u^j(v, v_1, \ldots, v_s)(X_{v_s}, u)(X_{v_s}, u_1^j)(X_{u_1^j}, v_s)\ldots(X_{u_1^j}, u_2^j)\tilde{P}^i(u_2^j, \ldots, u_{t_j}^j, w)\tilde{R}_w^j.$$

Note that the subgraph X_{v_s} is not used by any of the other paths S^i for $i \neq j$, so S^j is internally disjoint with them; see the bottom path in Figure 6. If u belongs to some R^ℓ, we modify S^ℓ in a similar way as above. Again, we have k internally disjoint paths between (X_u, v) and (X_w, z). □

If X is connected, every block of X is either a maximal 2-connected subgraph or a bridge. Different blocks can have at most one vertex in common, which is then a cut-vertex of X. Therefore every edge lies in a unique block, and X is the union of its blocks. We denote by $X_1 \oplus_v X_2$, or just $X_1 \oplus X_2$, the union of two graphs with a common vertex v. We denote by $X_1 \sqcup X_2$ the disjoint union of two graphs.

Lemma 4.2. *Let $X = B \oplus_v C$ be composed of two blocks B and C with a common vertex v. Let $\{u_1, u_2, \ldots, u_p, v\}$ be the vertex set of B and let $\{w_1, w_2, \ldots, w_q, v\}$ be the vertex set of C. Then* $\mathrm{Clone}(X)$ *is isomorphic to the following graph:*

$$(\mathrm{Clone}(B) \oplus_{(v,u_1)} C \oplus_{(v,u_2)} C \oplus \cdots \oplus_{(v,u_p)} C) \sqcup \\ (\mathrm{Clone}(C) \oplus_{(v,w_1)} B \oplus_{(v,w_2)} B \oplus \cdots \oplus_{(v,w_q)} B).$$

Proof. As in the proof of Theorem 4.1 we see that $\mathrm{Clone}(B)$ and $\mathrm{Clone}(C)$ are connected and lie in different components of $\mathrm{Clone}(X)$. The claim follows from the fact that every vertex deleted subgraph X_{u_i} contains a copy of C, and every vertex deleted subgraph X_{w_i} contains a copy of B. □

Corollary 4.3. *Let X be a connected graph and let $B_1, B_2, \ldots, B_k$ be the blocks of X. In other words, $X = B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Let C_i consist of the blocks of X different from i, for $i = 1, \ldots, k$. Then* $\mathrm{Clone}(X)$ *is isomorphic to the following graph (with blocks attached at appropriate vertices):*

$$(\mathrm{Clone}(B_1) \oplus_{i=1}^{k-1} (C_1)) \sqcup (\mathrm{Clone}(B_2) \oplus_{i=1}^{k-1} (C_2)) \sqcup \cdots \sqcup (\mathrm{Clone}(B_k) \oplus_{i=1}^{k-1} (C_k)).$$

In particular, the number of components of $\mathrm{Clone}(X)$ *is equal to the number of blocks of X.*

4.2 Bipartiteness

Any covering graph of a bipartite graph is obviously bipartite. The graph $\mathrm{Clone}(C_n)$ is the cycle $C_{n(n-1)}$, hence it is bipartite. It turns out that odd cycles are the only non-bipartite graphs for which the clone cover is bipartite.

Proposition 4.4. *Let X be a graph that is not a cycle. Then* $\mathrm{Clone}(X)$ *is bipartite if and only if X is bipartite.*

Proof. Suppose X is not bipartite, and let C be an odd cycle in X as short as possible. Since X is not a cycle, there exists a vertex v of X not in C. In $\mathrm{Clone}(X)$ there is a copy of C in the layer corresponding to v, thus also $\mathrm{Clone}(X)$ is not bipartite.

Conversely, suppose that $\mathrm{Clone}(X)$ is not bipartite. Then it contains an odd cycle. By Proposition 3.8, this must come from a cycle of the same odd length in X (since $k(k-1)$ is even for all k), and therefore X is not bipartite. □

4.3 Hamiltonicity

Although the cycle structure of the clone covers is fairly well understood, no complete characterization of hamiltonian clone covers is known. However, there is a simple sufficient condition for the base graph to have a hamiltonian clone cover. By Proposition 3.8, a Hamilton cycle (of length n) in the base graph X lifts to one cycle of length $n(n-1)$ in $\mathrm{Clone}(X)$. The cycle of length $n(n-1)$ is a Hamilton cycle in $\mathrm{Clone}(X)$. We record this formally.

Theorem 4.5. *Let X be a hamiltonian graph. Then* $\mathrm{Clone}(X)$ *is hamiltonian.*

We only have a partial converse of this result. However, no examples are known of a 2-connected non-hamiltonian graph for which the clone cover is hamiltonian.

Proposition 4.6. *Let X be a non-hamiltonian graph of minimal degree at most three. Then* $\mathrm{Clone}(X)$ *is also non-hamiltonian.*

Proof. Let v be a vertex of degree at most three in X, and suppose that $\mathrm{Clone}(X)$ has a Hamilton cycle H. Denote by H_v the subgraph of H restricted to X_v. Since X_v is connected to the rest of the graph by at most three edges, H_v forms a Hamilton path in X_v. Denote the vertices of degree 1 of this path by (X_v, u) and (X_v, w). Then u and w are neighbors of v in X. By adding edges $\{v, u\}$ and $\{v, w\}$ to the projection of H_v on X, we obtain a Hamilton cycle in X. A contradiction. □

Proposition 4.7. *Let X be a graph and let $v \in V(X)$ be a vertex of degree k such that $X \backslash N(v)$ has more than k components (X is not hamiltonian). Then* $\mathrm{Clone}(X)$ *is also not hamiltonian.*

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and let $U = \{(v, v_1), (v, v_2), \ldots, (v, v_k)\}$. Then $\mathrm{Clone}(X) \backslash U$ has more than k components since $X_v \backslash U$ has at least k components and is not connected to the rest of the graph. Therefore $\mathrm{Clone}(X)$ is not hamiltonian. □

4.4 Planarity

Recall the graphical construction of $\mathrm{Clone}(X)$: the graph $\mathrm{Clone}(X)$ can be obtained from the graph X by "replacing" each vertex v by $X_v = X - v$. Using this fact, we make the following observations.

Theorem 4.8. *Let X be a graph. Then* $\mathrm{Clone}(X)$ *is planar if and only if X is planar.*

Proof. Let X be a planar graph and let Y be a planar embedding of X. We choose an orientation of the plane. Let u be a vertex of X and let Y^u be an embedding of X such that u lies on the outer face with the cyclic order of the neighbors of each vertex reversed with regard to Y. Then the order of the neighbors of u along the outer face of $Y^u - u$ is the same as the order of the neighbors of u in Y. Therefore it is possible to replace u in Y by the graph $Y^u - u$, and connect each of the neighbors of u in Y by the corresponding neighbor of u in Y^u such that this replacement yields a plane graph again. Doing this for each vertex of Y we obtain a planar embedding of $\mathrm{Clone}(X)$.

Conversely, if X is not planar, then it contains a copy of $K_{3,3}$ or K_5 as a minor. If X is 2-connected, then it is a minor of $\mathrm{Clone}(X)$ by Proposition 3.5, and therefore also $\mathrm{Clone}(X)$ contains a copy of $K_{3,3}$ or K_5 as a minor. If X is not 2-connected, then $\mathrm{Clone}(X)$ contains each block of X as a subgraph by Corollary 4.3. In this case, at least one block of X is not planar and therefore also $\mathrm{Clone}(X)$ is not planar. □

Similarly we can give an upper bound on the genus of $\mathrm{Clone}(X)$ in terms of genus of X.

Theorem 4.9. *Let X be a graph on n vertices. The following bound holds for the genus γ:*

$$\gamma(\mathrm{Clone}(X)) \leq (n+1)\gamma(X).$$

The same inequality holds for the non-orientable genus.

Proof. Let m denote the number of edges of X and let $g = \gamma(X)$. Let f denote the number of faces in the genus embedding of X, and let f_i be the number od faces of length i, for $i \geq 3$. Suppose first that all the faces of X are cycles. Recall that the voltage of a cycle of length k is a cycle of length $k-1$ in $\mathrm{Sym}(V(X)) \cong S_n$ fixing $n-k+1$ symbols. By Proposition 3.8, such a cycle lifts to one cycle of length $k(k-1)$ and $n-k$ cycles of length k in $\mathrm{Clone}(X)$.

Take an embedding of $\mathrm{Clone}(X))$ in which the cyclic order of the edges around each vertex is the same as in the genus embedding of X. Then a face of X lifts to a face of $\mathrm{Clone}(X))$. Denote by n', m', f', g' the number of vertices, number of edges, number of faces, and the genus of this embedding of $\mathrm{Clone}(X))$, respectively. Then $n' = (n-1)n$, $m' = (n-1)m$, and

$$f' = \sum_{i\geq 3}(n-i+1)f_i = \sum_{i\geq 3}(n+1)f_i - \sum_{i\geq 3} if_i = (n+1)f - 2m.$$

Now we can compute g':

$$\begin{aligned} g' &= (2+m'-n'-f')/2 = (2+m(n-1)-n(n-1)-f(n+1)+2m)/2 \\ &= (2n+2+m(n+1)-n(n+1)-f(n+1))/2 = (n+1)g. \end{aligned}$$

If a face of X of length k is not a cycle, it lifts to more than $n-k+1$ faces, which makes the genus of such an embedding of $\mathrm{Clone}(X)$ even smaller than $(n+1)g$. In any case we have $\gamma(\mathrm{Clone}(X)) \leq (n+1)g$.

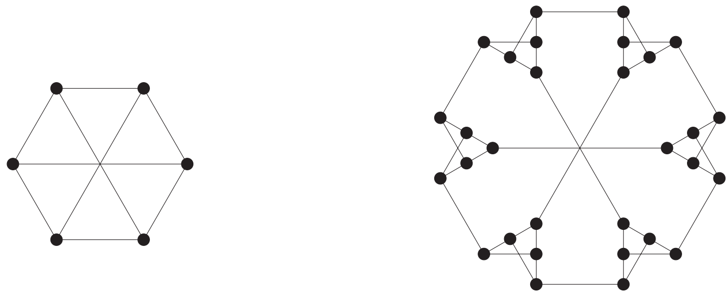
The same reasoning holds also for the nonorientable embeddings. □

Example 4.10. The genus of $K_{3,3}$, which is a graph on 6 vertices, is 1. By Theorem 4.9, the genus of $\mathrm{Clone}(K_{3,3})$ is at most 7; see Figure 7.

5 Algebraic properties

A reasonable assumption when studying algebraic properties of covering graphs, or indeed graphs in general, is to restrict our considerations to connected covering graphs – which in our case translates to requiring that the base graphs are at least 2-connected, in view of Theorem 4.1 and the assumption that the base graph is not K_2.

There are two different kinds of automorphisms of a covering graph: the ones that are lifts of some automorphism of the base graph, and the ones that are not. Along these lines we consider certain structural properties of the automorphism group of $\mathrm{Clone}(X)$, edge- and vertex-transitivity of $\mathrm{Clone}(X)$, and regularity of the covering projection $\mathrm{Clone}(X) \to X$. Automorphisms that respect the fundamental edge partition, see Subsection 5.2 below, will play a significant role in this context.



(a) (b)

Figure 7: (a) The graph $K_{3,3}$, (b) $\mathrm{Clone}(K_{3,3})$.

5.1 Lifts of automorphisms along the covering projection

Certain automorphisms of a covering graph can be studied in terms of automorphisms of the base graph. Such automorphisms arise as lifts of automorphisms, a concept we shall now define. Let $p : \tilde{X} \to X$ be a covering projection of graphs, and let f be an automorphism of X. We say that f *lifts* if there exists an automorphism $\tilde{f}$ of $\tilde{X}$, a *lift* of f, such that the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

is commutative; in other words, $f \circ p = p \circ \tilde{f}$. Observe that a lift of an automorphism maps bijectively fibers over vertices (resp. edges) to fibers over vertices (resp. edges), in particular, any lift of f maps the fiber over a vertex $u \in V(X)$ to the fiber over the vertex $f(u)$.

Suppose that all the elements of a subgroup $G \leq \mathrm{Aut}(X)$ have a lift. Then the lifts of all automorphisms from G form a subgroup of $\mathrm{Aut}(\tilde{X})$ which we denote by $\tilde{G}$. In particular, the lift of the trivial group is known as the group of *covering transformations* and is denoted by $\mathrm{CT}(p)$. Further, if G lifts, then there exists an epimorphism $p_G : \tilde{G} \to G$ with $\mathrm{CT}(p)$ as its kernel. Hence $\tilde{G}$ is an extension of $\mathrm{CT}(p)$ by G, and the set of all lifts of a given $f \in \mathrm{Aut}(X)$ is a coset of $\mathrm{CT}(p)$ in $\tilde{G}$. As a final opening remark in this section, recall from general theory that $\mathrm{CT}(p)$ acts semiregularly on the covering graph $\tilde{X}$ whenever $\tilde{X}$ is connected; that is, $\mathrm{CT}(p)$ acts without fixed points both on vertices and on arcs of $\tilde{X}$. Moreover, each lift is uniquely determined by the mapping of a single vertex. For a background on lifting automorphisms in terms of voltages we refer the reader to [8].

In general, not every automorphism of the base graph X lifts. This is not the case with the natural covering projection $p\colon \mathrm{Clone}(X) \to X$, $p = \mathrm{pr}_1\colon (u, v) \mapsto u$. To this end let us introduce, for each automorphism f of the graph X, a mapping $\bar{f}\colon \mathrm{Clone}(X) \to \mathrm{Clone}(X)$ which we call the *diagonal* mapping, defined by

$$\bar{f}\colon (u, i) \mapsto (f(u), f(i)).$$

Theorem 5.1. *Let f be an automorphism of X. Then the map $\bar{f}$ is an automorphism of*

$\mathrm{Clone}(X)$ *and is a lift of* f.

Proof. Obviously, $\bar{f}$ is well defined since $i \neq u$ implies $f(i) \neq f(u)$, and moreover, it is bijective on the vertex set of $\mathrm{Clone}(X)$.

We will show that $\bar{f}$ maps arcs to arcs, so it is an automorphism. Let uv be an arc in X, and let a be an arc in $\mathrm{Clone}(X)$ from (u,i) to $(v,i^{(u,v)})$. The vertex (u,i) is mapped by $\bar{f}$ to $(f(u), f(i))$. Since $f(i^{(u,v)}) = f(i)^{(f(u),f(v))}$ and the vertex $(v, i^{(u,v)})$ is mapped by $\bar{f}$ to the vertex $(f(v), f(i^{(u,v)}) = (f(v), f(i)^{(f(u),f(v))})$, it follows that $f(a)$ is an arc in $\mathrm{Clone}(X)$, as required.

Let (u,i) be a vertex from $\mathrm{Clone}(X)$. Then $f \circ p(u,i) = f(u) = p(f(u), f(i)) = p \circ \bar{f}(u,i)$. Let a be an arc in $\mathrm{Clone}(X)$ from (u,i) to $(v, i^{(u,v)})$. Then $f \circ p(a) = f([u,v]) = [f(u), f(v)] = p([(f(u), f(i)), (f(v)f(i)^{(u,v)})]) = p \circ \bar{f}(a)$. Thus, $\bar{f}$ is an automorphism and clearly a lift of f.

□

In view of the above theorem, $\mathrm{Clone}(X)$ inherits all the symmetries of X in the sense that there is a natural injection of $\mathrm{Aut}(X)$ into $\mathrm{Aut}(\mathrm{Clone}(X))$ taking $f \mapsto \bar{f}$. This injection is actually a group homomorphism, as we shall see shortly. For convenience we denote by $A = \mathrm{Aut}(X)$ the full automorphism group of X and $\tilde{A}$ its lift.

Proposition 5.2. *The set* $\bar{A}$ *of all diagonal mappings is a subgroup of* $\mathrm{Aut}(\mathrm{Clone}(X))$, *isomorphic to* A, *and* $\tilde{A} = \mathrm{CT}(p) \rtimes \bar{A}$.

Proof. The set $\bar{A}$ is clearly a complete system of coset representatives of $\mathrm{CT}(p)$ within $\tilde{A}$ since for each $f \in A$ there is only one diagonal mapping. Moreover, from the definition of the diagonal mapping it easily follows that $\overline{fg} = \bar{f}\bar{g}$. Hence $\bar{A}$ is a complement to $\mathrm{CT}(p)$ within $\tilde{A}$, and so $\tilde{A} = \mathrm{CT}(p) \rtimes \bar{A}$, as required. □

From now we shall be explicitly assuming that the base graph X is 2-connected, as already anticipated at the beginning of this section. The simplest base graphs of this kind are the cycles.

Proposition 5.3. *Let* $p\colon \mathrm{Clone}(C_n) \to C_n$ *be the covering projection for the* n*-cycle* C_n*, for* $n \geq 3$*. Then* $\mathrm{CT}(p)$ *is isomorphic to* $\mathbb{Z}_{n-1}$*, and* $\mathrm{Aut}(\mathrm{Clone}(C_n)) \cong D_{n(n-1)} \cong \mathbb{Z}_{n-1} \rtimes D_n$.

Proof. Recall that $\mathrm{Clone}(C_n) = C_{n(n-1)}$ which has the automorphism group isomorphic to the dihedral group $D_{n(n-1)}$ of order $2n(n-1)$. The subgroup of $\mathrm{Aut}(\mathrm{Clone}(C_n))$ generated by the n-step rotation of $C_{n(n-1)}$ is isomorphic to $\mathbb{Z}_{n-1}$. It is easy to see that each of these automorphisms is a covering transformation. Since $A = \mathrm{Aut}(C_n)$ is isomorphic to the dihedral group of order $2n$, it follows that $\tilde{A} = \mathrm{CT}(p) \rtimes \bar{A}$ has order $2n(n-1)$. Hence $\mathrm{Aut}(\mathrm{Clone}(C_n)) = \tilde{A}$. □

$\mathrm{Clone}(C_n)$ appears to be rather special, since every vertex deleted subgraph of a cycle is acyclic. In all other cases where the clone cover is connected, the group of covering transformations is trivial, and this fact has strong impact on symmetry properties of Clone.

Theorem 5.4. *Let* X *be a* 2*-connected graph that is not a cycle. Then* $\mathrm{CT}(p)$ *is trivial, and hence the lifted group is equal to the group of diagonal mappings – that is,* $\tilde{A} = \bar{A}$.

Proof. From the assumptions it easily follows that in X there exist two distinct vertices, connected with three internally disjoint paths P, Q, R. Let C be the cycle formed by the paths P and Q, and let C' be the cycle formed by P and R. Let k and k' be the lengths of C and C', respectively. Then C lifts to one $k(k-1)$ cycle and to $n-k$ cycles of length k, while C' lifts to one $k'(k'-1)$ cycle and $n-k'$ cycles of length k', where $n = |V(X)|$. Denote the $k(k-1)$ cycle over C by $\tilde{C}$, and the $k'(k'-1)$ cycle over C' by $\tilde{C}'$.

Let now $f \in \mathrm{CT}(p)$ be a covering transformation. Since f permutes the cycles over C we have $f(\tilde{C}) = \tilde{C}$. Similarly, $f(\tilde{C}') = \tilde{C}'$. Denote the union of C and C' by $Y = C \cup C'$, and let $\tilde{Y}$ be the connected component of the preimage $p^{-1}(Y)$ containing $\tilde{C} \cup \tilde{C}'$. Note that $\tilde{Y} = \mathrm{Clone}(Y)$. Moreover, $f(\tilde{Y}) = \tilde{Y}$. Further, the restriction of f to $\tilde{Y}$ is a covering transformation of the projection $\mathrm{Clone}(Y) \to Y$. Since $\mathrm{CT}(p)$ acts without fixed points, it is enough to show that the group of covering transformations of the projection $\mathrm{Clone}(Y) \to Y$ is trivial.

To formally prove the above assertion we shall actually prove that the group of covering transformations of the auxiliary covering $\bar{p}\colon \mathrm{Cov}(Y) \to Y$ (which is isomorphic to that of $\mathrm{Clone}(Y) \to Y$) must be trivial.

Let $V(Y) = \{u_0, u_1 \ldots, u_s, x_{s+1}, \ldots, x_{k-1}, y_{s+1}, \ldots, y_{k'-1}\}$ be the vertex set of $Y = C \cup C'$, and let $C = u_0 u_1 \ldots u_s x_{s+1} \ldots x_{k-1} u_0$ and $C' = u_0 u_1 \ldots u_s y_{s+1} \ldots y_{k'-1} u_0$ be the corresponding directed cycles, rooted at u_0. The first cycle has voltage $\alpha = (x_{k-1}, \ldots, x_{s+1}, u_s, \ldots, u_1)$ while $\beta = (y_{k'-1}, \ldots, y_{s+1}, u_s, \ldots, u_1)$ is the voltage of the second one. From general theory [8, Corollary 7.3] it easily follows that the group of covering transformations of a connected cover given by permutation voltages is isomorphic to the centralizer in the symmetric group of the subgroup generated by the voltages of all closed walks at a chosen vertex. Hence in our case $\mathrm{CT}(\bar{p})$ is isomorphic to the centralizer of α and β in $\mathrm{Sym}(V(Y))$. Let τ commute with both α and β. If we represent these permutations graphically as a colored digraph on the vertex set $V(Y)$, then τ corresponds to a color- and direction-preserving automorphism of this digraph. From the structure of the above colored 'permutation digraph', it is now immediate that τ must be trivial, as required. □

Recall that a cover is *regular* if the fiber-preserving automorphisms act transitively on each fiber. The three canonical covers mentioned in the introduction, namely the universal cover, the Kronecker cover, and the trivial cover, are all regular covers for all base graphs. However, the clone cover is in most cases an irregular covering.

Theorem 5.5. *Let X be a 2-connected graph. Then $\mathrm{Clone}(X) \to X$ is not a regular covering projection unless $X = C_n$.*

Proof. By Theorem 5.4 we know that the group of covering transformations is trivial, except when $X = C_n$. This completes the proof. □

In particular, the graph $\mathrm{Clone}(K_n)$, $n \geq 4$, is an irregular cover of K_n.

5.2 Automorphisms that respect the fundamental edge partition

We will say that an automorphism of $\mathrm{Clone}(X)$ *respects the fundamental edge partition* if it takes inner edges to inner edges, and connecting edges to connecting edges. From the graphical construction of $\mathrm{Clone}(X)$ it follows that inner edges can be naturally partitioned into layers – which are nothing but the vertex deleted subgraphs of X. Consequently, the property of preserving the edge partition is equivalent to requiring that layers are mapped to layers, at least when $\mathrm{Clone}(X)$ is connected.

Proposition 5.6. *Let X be a 2-connected graph. An automorphism of $\mathrm{Clone}(X)$ respects the fundamental edge partition if and only if it maps layers to layers.*

Proof. Let f be an automorphism of $\mathrm{Clone}(X)$. Suppose f maps layers to layers. Then it maps inner edges to inner edges. Hence it must also map connecting edges to connecting edges, and must therefore respect the fundamental edge partition.

Conversely, suppose that f respects the fundamental edge partition. Then it maps inner edges to inner edges. Since X is 2-connected, every layer of $\mathrm{Clone}(X)$, which is just a vertex-deleted subgraph of X, is connected. Take two vertices (u, v) and (w, v) from the same layer of $\mathrm{Clone}(X)$. Then there exists a path between them, consisting only of inner edges. But then also $f(u, v)$ and $f(w, v)$ are connected by a path consisting only of inner edges. Therefore $f(u, v)$ and $f(w, v)$ are in the same layer of $\mathrm{Clone}(X)$. This shows that f takes layers to layers. □

Note that all automorphisms that respect the fundamental edge partition form a subgroup in $\mathrm{Aut}(\mathrm{Clone}(X))$ which we denote $\mathcal{E}$. We are now going to explicitly describe the structure of this group whenever $\mathrm{Clone}(X)$ is connected. To start with, note that there is a natural mapping $\mathrm{contr}\colon \mathrm{Clone}(X) \to X$, called *contraction*, defined by collapsing each vertex-deleted subgraph to its corresponding vertex v. To put it differently, contraction is in fact the projection $\mathrm{contr} = \mathrm{pr}_2\colon (u, v) \mapsto v$ onto the second coordinate – in contrast with the covering projection which is the projection onto the first coordinate. Let now f and $\hat{f}$ be automorphisms of X and $\mathrm{Clone}(X)$, respectively, such that the following diagram

$$\begin{array}{ccc} \mathrm{Clone}(X) & \xrightarrow{\ \hat{f}\ } & \mathrm{Clone}(X) \\ \mathrm{contr}\big\downarrow & & \big\downarrow\mathrm{contr} \\ X & \xrightarrow{\ f\ } & X \end{array}$$

is commutative. We then say that f *lifts* and that $\hat{f}$ *projects along the contraction*. In a similar fashion we speak about lifting and projecting groups. In view of Proposition 5.6 we have the following obvious characterization of the fundamental edge partition preserving subgroup.

Proposition 5.7. *Let X be a 2-connected graph. Then $\mathcal{E}$ is precisely the subgroup of automorphisms of $\mathrm{Clone}(X)$ that projects along the contraction. Moreover, the contraction induces a group homomorphism $\mathcal{E} \to \mathrm{Aut}(X)$.*

We have already remarked that $\mathrm{Clone}(C_n)$ is rather special for several reasons. Apart from the fact that its group of covering transformations is not trivial, it is also true that it does not respect the fundamental edge partition, in view of the next result and Theorem 5.2.

Theorem 5.8. *Let X be a 2-connected graph and let $A = \mathrm{Aut}(X)$. Then the maximal subgroup in the lifted group $\tilde{A}$ that respects the fundamental edge partition is the group $\bar{A}$ of diagonal mappings. In particular, the induced homomorphism $\mathcal{E} \to A$ is surjective.*

Proof. Choose an arc uv in X and its corresponding unique connecting arc $[(u,v),(v,u)]$ in $\mathrm{Clone}(X)$. If $\tilde{f} \in \tilde{A}$ preserves the fundamental edge partition, then it must map $[(u,v),(v,u)]$ to the unique connecting arc over $[f(u), f(v)]$, that is, to $[(f(u), f(v)), (f(v), f(u))]$. It follows that $\tilde{f}(u,v) = (f(u), f(v)) = \bar{f}(u,v)$, where $\bar{f}$ is the diagonal mapping. Since the covering graph is connected, a lift of $f \in \mathrm{Aut}(X)$ is uniquely determined by the mapping of a single vertex. Hence $\tilde{f} = \bar{f}$.

The final statement obviously holds since each $f \in A$ lifts to $\bar{f} \in \mathcal{E}$. This completes the proof. □

In order to identify the group $\mathcal{E}$, we need another definition. For a vertex $v \in V(X)$, let $A_{N(v)} \leq A_v$ denote the subgroup in the stabilizer of v fixing all vertices in the neighborhood $N(v)$ point-wise. For each $f \in A_{N(v)}$ let

$$f^\sharp(x,i) = \begin{cases} (x,i) & \text{if } i \neq v, \\ (f(x),v) & \text{if } i = v. \end{cases}$$

Clearly, $f^\sharp$ is an automorphism of $\mathrm{Clone}(X)$ that preserves the fundamental edge partition; its projection along the contraction is the identity automorphism of X, but $f^\sharp$ does not project along the covering projection; see below. Let $A^\sharp_{N(v)} = \{f^\sharp \mid f \in A_{N(v)}\}$. Note that $A^\sharp_{N(v)} \cong A_{N(v)}$. We are now ready to identify the fundamental edge partition preserving subgroup $\mathcal{E}$.

Theorem 5.9. *Let X be a 2-connected graph. Denote its automorphism group $Aut(X)$ by A, and let $\bar{A}$ be the group of diagonal mappings of* $\mathrm{Clone}(X)$*. Then $\mathcal{E}$ is the internal semi-direct product*

$$\Pi_{v\in V(X)} A^\sharp_{N(v)} \rtimes \bar{A}.$$

Proof. For each vertex $v \in V = V(X)$, a typical element of the Cartesian product of groups $\Pi_{v\in V} A^\sharp_{N(v)}$ has the form $\Pi_{v\in V} f^\sharp_v$, where $f_v \in A_{N(v)}$. Note further that $\Pi_{v\in V} A^\sharp_{N(v)}$ indeed exists as a group of automorphisms of $\mathrm{Clone}(X)$. Since each automorphism $f^\sharp_v$ projects to the identity along the contraction, we know that $\Pi_{v\in V} A^\sharp_{N(v)}$ is contained in the kernel $\mathcal{K}$ of the homomorphism $\mathcal{E} \to A$ induced by the contraction.

Conversely, let $\hat{f} \in \mathcal{E}$ be in the kernel $\mathcal{K}$. Then $\hat{f}$ fixes each layer set-wise, and its restriction to the v-th layer induces an automorphism f_v of the vertex deleted subgraph X_v. It follows that $\hat{f}$ can be written as the product $\hat{f} = \Pi_{v\in V} f^\sharp_v$, with commuting factors. Hence $\hat{f}$ is an element of the Cartesian product of subgroups $A^\sharp_{N(v)}$, $v \in V$, and so the kernel is precisely this group.

Since the homomorphism $\mathcal{E} \to A$ is surjective, $\mathcal{E}$ is an extension of its kernel $\mathcal{K}$ by A. Observe that the diagonal mapping $\bar{f} \in \bar{A}$ projects to f both along the covering projection and along the contraction. This means that the group $\bar{A}$ is a system of coset representatives of $\mathcal{K}$, and so $\mathcal{E}$ is a semi-direct product of $\Pi_{v\in V} A^\sharp_{N(v)}$ by $\bar{A}$. □

5.3 The full automorphism group of Clone

Let X be a 2-connected graph. We have seen that $A = \mathrm{Aut}(X)$ embeds in $\mathrm{Aut}(\mathrm{Clone}(X))$ as the group of diagonal mappings $\bar{A}$. Now $\mathrm{Clone}(X)$ may have other automorphisms, for two reasons.

Firstly, the group of covering transformations is not always trivial; this happens only when X is a cycle, and in that case the lifted group is in fact the full automorphism group of $\mathrm{Clone}(X)$. Secondly, the fundamental edge partition preserving subgroup $\mathcal{E}$ may be larger than $\bar{A}$. In view of Theorem 5.9, the latter happens if and only if there exists a nontrivial automorphism of X fixing a vertex and its neighboring vertices point-wise. Example 5.10 provides an instance of such a case.

Example 5.10. The automorphism group of $K_{2,3}$ is $S_2 \times S_3$ and has 12 elements. On the other hand, $\mathrm{Aut}(\mathrm{Clone}(K_{2,3}))$ has additional automorphisms of order two, and has 96 elements; see Figure 3. In fact, the full automorphism group is equal to $\mathcal{E}$, which is by Theorem 5.9 isomorphic to $\mathbb{Z}_2^3 \rtimes (S_3 \times \mathbb{Z}_2)$.

The case when when the full automorphism group of $\mathrm{Clone}(X)$ is equal to $\mathcal{E}$ is particularly interesting, since then the group $\mathrm{Aut}(\mathrm{Clone}(X))$ can be determined using Theorem 5.9. Moreover, the fact that every automorphism respects the fundamental edge-partition has strong impact on the transitivity properties of $\mathrm{Clone}(X)$.

Theorem 5.11. *Let X be a 2-connected graph such that every automorphism of $\mathrm{Clone}(X)$ respects the fundamental edge partition. Then the following hold.*

(a) $\mathrm{Clone}(X)$ *is not edge-transitive.*

(b) $\mathrm{Clone}(X)$ *is not vertex-transitive unless $X = K_n$, for some $n \geq 4$.*

Proof. If every automorphism of $\mathrm{Clone}(X)$ respects the fundamental edge partition, then no inner edge can be mapped by an automorphism to a connecting edge, and vice versa. Therefore $\mathrm{Clone}(X)$ is not edge-transitive. Also in this case, $\mathrm{Clone}(X)$ can be vertex-transitive only if all the vertex-deleted subgraphs of X are vertex-transitive, and isomorphic to each other. This only happens when X is a complete graph on more than 3 vertices. □

In certain cases it is easy to show that any automorphism of $\mathrm{Clone}(X)$ preserves the fundamental edge partition.

Proposition 5.12. *Let X be a 2-connected graph of girth g such that each edge of each vertex deleted subgraph is contained in a cycle of length at most $2g - 1$. Then each automorphism of $\mathrm{Clone}(X)$ preserves the fundamental edge partition.*

Proof. Since a layer naturally corresponds to a vertex deleted subgraph of X, each inner edge of $\mathrm{Clone}(X)$ belongs to a cycle of length at most $2g - 1$. On the other hand, no connecting edge belongs to such a cycle. The conclusion follows. □

There are several families of graphs which fulfill the conditions of Proposition 5.12. We state the following without proof.

Proposition 5.13.

- *The complete graph K_n for $n \geq 4$ has girth 3. Each vertex deleted subgraph of K_n is isomorphic to K_{n-1}; any edge of a complete graph on at least 3 vertices is contained in a 3-cycle.*

- *A hypercube Q_n for $n \geq 3$ has girth 4 and every edge in any vertex-deleted subgraph of Q_n lies in a 4-cycle.*
- *A cartesian product of 2-connected graphs has girth at most 4 and every edge in any vertex-deleted subgraph of such a graph lies in a 4-cycle.*
- *A generalized Petersen graph $G(n, 2)$ for $n > 8$ has girth 5 and every edge in any vertex-deleted subgraph lies in a 5-cycle or an 8-cycle.*

Moreover, the above families of graphs are all 2-connected.

Since several other generalized Petersen graphs also fulfill the conditions of Proposition 5.12, it would be interesting to characterize them. Such a characterization has to take into account the girth of every single member of the family of generalized Petersen graphs, which, in turn, was calculated in [1]. On the other hand, many interesting families of 2-connected graphs do not satisfy the condition of Proposition 5.12, yet their clone covers still have the required property, for instance, the wheel graphs W_n, $n \geq 6$. In contrast, the clone cover of a cycle is different: the fundamental edge partition preserving subgroup has index $n - 1$ in the automorphism group of $\mathrm{Clone}(C_n)$. We believe that this is the only exception.

Conjecture 5.14. *Let X be a 2-connected graph that is not a cycle. Then any automorphism of $\mathrm{Clone}(X)$ preserves the fundamental edge partition.*

6 Concluding remarks

Note that $\mathrm{Clone}(K_n)$ has yet another nice description, namely, as the line graph of the first subdivision of K_n. Along these lines, $\mathrm{Clone}(K_n)$ was considered in [7], where it was shown that with few exceptions, $\mathrm{Clone}(K_n)$ is the only m-sheeted covering graph of K_n, for $m \leq n - 1$, such that the full automorphism group of K_n has a lift.

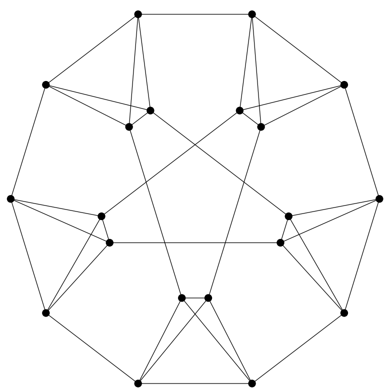


Figure 8: $\mathrm{Clone}(K_5)$ is not a regular cover. It is vertex- but not edge-transitive.

On the other hand, $\mathrm{Clone}(K_5)$ was studied in connection with a graph-theoretical interpretation of the Jahn-Teller effect [2]. In order to clarify the role of degenerate eigenvalues (that is, eigenvalues having higher multiplicities) in the Jahn-Teller distortion, graphs with symmetry group S_5 were sought [3]. It turned out that both K_5 and $\mathrm{Clone}(K_5)$ have their automorphism group isomorphic to S_5. Our paper gives, among other results, a theoretical background for this result. As noted in Proposition 5.13, the conditions of Proposition 5.12 are satisfied by K_n for $n \geq 4$. Hence according to Theorem 5.9, $\mathrm{Aut}(K_n)$ is isomorphic to $\mathrm{Aut}(\mathrm{Clone}(K_n))$. Figure 8 depicts the graph $\mathrm{Clone}(K_5)$, which was used in [3].

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References

[1] M. Boben, T. Pisanski and A. Žitnik, I-graphs and the corresponding configurations, *J. Combin. Designs* **13** (2005), 406–424.

[2] A. Ceulemans, E. Lijnen, P. W. Fowler, R. B. Mallion and T. Pisanski, Graph theory and the Jahn-Teller theorem, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **468** (2012), 971–989.

[3] A. Ceulemans, E. Lijnen, P. W. Fowler, R. B. Mallion and T. Pisanski, S_5 graphs as model systems for icosahedral Jahn-Teller problems, *Theor. Chem. Acc.* **131** (2012), article 1246 (10 pages).

[4] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience, New York, 1987.

[5] W. Imrich and T. Pisanski, Multiple Kronecker coverings of graphs, *Europ. J. Combin.* **29** (2008), 1116–1122.

[6] F. T. Leighton, Finite common coverings of graphs, *J. Combin. Theory ser. B* **33** (1982), 231–238.

[7] A. Malnič and D. Marušič, Imprimitive groups and graph coverings, in: D. Jungnickel, S. A. Vanstone (eds), *Coding Theory, Design Theory, Group Theory, Proc. Marshall Hall Conf.*, J. Wiley and Sons, New York, 1993, pp. 227–235.

[8] A. Malnič, R. Nedela and M. Škoviera, Lifting graph automorphisms by voltage assignments, *Europ. J. Combin.* **21** (2000), 927–947.

[9] T. Pisanski, A classification of cubic bicirculants, *Discrete Math.* **307** (2007), 567–578.

[10] D. B. West, *Introduction to Graph Theory*, Second Edition, Pearson Education, 2002.

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Unstable graphs: A fresh outlook via TF-automorphisms

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Abstract

In this paper, we first establish the very close link between stability of graphs, a concept first introduced by Marušič, Scapellato and Zagaglia Salvi and studied most notably by Surowski and Wilson, and two-fold automorphisms. The concept of two-fold isomorphisms, as far as we know, first appeared in Zelinka's work on isotopies of digraphs and later studied formally by the authors with a greater emphasis on undirected graphs. We then turn our attention to the stability of graphs which have every edge on a triangle, but with the fresh outlook provided by TF-automorphisms. Amongst such graphs are strongly regular graphs with certain parameters. The advantages of this fresh outlook are highlighted when we ultimately present a method of constructing and generating unstable graphs with large diameter having every edge lying on a triangle. This was a rather surprising outcome.

Keywords: Graphs, canonical double covers, two-fold isomorphisms.

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1 General Introduction and Notation

Let G and H be simple graphs, that is, undirected and without loops or multiple edges. Consider the edge $\{u, v\}$ to be the set of arcs $\{(u, v),\ (v, u)\}$. A *two-fold isomorphism* or TF-*isomorphism* from G to H is a pair of bijections $\alpha, \beta\colon V(G) \to V(H)$ such that (u, v) is an arc of G if and only if $((\alpha(u), \beta(v))$ is an arc of H. When such a pair of bijections exist, we say that G and H are TF-*isomorphic* and the TF-isomorphism is denoted by (α, β). The inverse of (α, β), that is, $(\alpha^{-1}, \beta^{-1})$ is a TF-isomorphism from H to G. Furthermore, if (α_1, β_1) and (α_2, β_2) are both TF-isomorphisms from G to H then so is $(\alpha_1\alpha_2, \beta_1\beta_2)$. When $\alpha = \beta$, the TF-isomorphism can be identified with the isomorphism α.

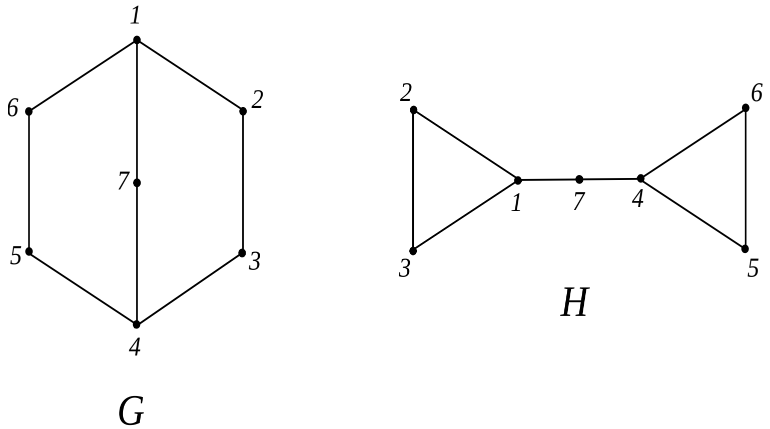

Figure 1: G and H are two non-isomorphic TF-isomorphic graphs.

The two graphs G, H in Figure 1, which have the same vertex set $V(G) = V(H)$, are non-isomorphic and yet they are TF-isomorphic. In fact (α, β) where $\alpha = (2\ 5)$ and $\beta = (1\ 4)(3\ 6)$ is a TF-isomorphism from G to H.

The concept was first studied by Zelinka [15, 16] in the the context of isotopy of digraphs. We shall extend it further to the case of mixed graphs (see next section).

Some graph properties are preserved by a TF-isomorphism. Such is the case with the degree sequence, as illustrated by Figure 1. We also know that two graphs are TF-isomorphic if and only if that they have isomorphic canonical double covers [4]. Alternating paths or **A**-trails, which we shall define in full below, are invariant under TF-isomorphism. For instance, the alternating path $5 \longrightarrow 6 \longleftarrow 1 \longrightarrow 2$ in G is mapped by (α, β) to the similarly alternating path $2 \longrightarrow 3 \longleftarrow 1 \longrightarrow 2$ which we shall later be calling "semi-closed".

2 Notation

A *mixed graph* is a pair $G = (V(G), A(G))$ where $V(G)$ is a set and $A(G)$ is a set of ordered pairs of elements of $V(G)$. The elements of $V(G)$ are called *vertices* and the elements of $A(G)$ are called *arcs*. When referring to an arc (u, v), we say that u is *adjacent to* v and v is *adjacent from* u. Sometimes we use $u \longrightarrow v$ to represent an arc $(u, v) \in$

$V(G)$. The vertex u is the *start-vertex* and v is the *end-vertex* of a given arc (u, v). An arc of the form (u, u) is called a *loop*. A mixed graph cannot contain multiple arcs, that is, it cannot contain the arc (u, v) more than once. A mixed graph G is called *bipartite* if there is a partition of $V(G)$ into two sets X and Y, which we call *colour classes*, such that for each arc (u, v) of G the set $\{u, v\}$ intersects both X and Y. A set S of arcs is *self-paired* if, whenever $(u, v) \in S$, (v, u) is also in S. If $S = \{(u, v), (v, u)\}$, then we consider S to be the unordered pair $\{u, v\}$; this unordered pair is called an *edge*.

It is useful to consider two special cases of mixed graphs. A *graph* is a mixed graph without loops whose arc-set is self-paired. The edge set of a graph is denoted by $E(G)$. A *digraph* is a mixed graph with no loops in which no set of arcs is self-paired. The *inverse* G' of a mixed graph G is obtained from G by reversing all its arcs, that is $V(G') = V(G)$ and (v, u) is an arc of G' if and only if (u, v) is an arc of G. A digraph G may therefore be characterised as a mixed graph for which $A(G)$ and $A(G')$ are disjoint. Given a mixed graph G and a vertex $v \in V(G)$, we define the *in-neighbourhood* $N_{in}(v)$ by $N_{in}(v) = \{x \in V(G) | (x, v) \in A(G)\}$. Similarly we define the *out-neighbourhood* $N_{out}(v)$ by $N_{out}(v) = \{x \in V(G) | (v, x) \in A(G)\}$. The *in-degree* $\rho_{in}(v)$ of a vertex v is defined by $\rho_{in}(v) = |N_{in}(v)|$ and the *out-degree* $\rho_{out}(v)$ of a vertex v is defined by $\rho_{out}(v) = |N_{out}(v)|$. When G is a graph, these notions reduce to the usual neighbourhood $N(v) = N_{in}(v) = N_{out}(v)$ and degree $\rho(v) = \rho_{in}(v) = \rho_{out}(v)$.

Let G be a graph and let $v \in V(G)$. Let $N(v)$ be the neighbourhood of v. We say that G is *vertex-determining* if $N(x) \neq N(y)$ for any two distinct vertices x and y of G [8].

A set P of arcs of G is called a *trail* of *length* k if its elements can be ordered in a sequence $a_1, a_2, \ldots, a_k$ such that each a_i has a common vertex with a_{i+1} for all $i = 1, \ldots, k-1$. If u is the vertex of a_1, that is not in a_2 and v is the vertex of a_k which is not in a_{k-1}, then we say that P *joins* u and v; u is called the *first vertex* of P and v is called the *last vertex* with respect to the sequence $a_1, a_2, \ldots, a_k$. If, whenever $a_i = (x, y)$, either $a_{i+1} = (x, z)$ or $a_{i+1} = (z, y)$ for some new vertex z, P is called an *alternating trail* or **A**-*trail*.

If the first vertex u and the start-vertex v of an **A**-trail P are different, then P is said to be *open*. If they are equal then we have to distinguish between two cases. When the number of arcs is even then P is called *closed* while when the number of arcs is odd then P is called *semi-closed*. Note that if P is semi-closed then either (i) $a_1 = (u, x)$ for some vertex x and $a_k = (y, u)$ for some vertex y or (ii) $a_1 = (x, u)$ and $a_k = (u, y)$. If P is closed then either $a_1 = (u, x)$ and $a_k = (u, y)$ or $a_1 = (x, u)$ and $a_k = (y, u)$. Observe also that the choice of the first (equal to the last) vertex for a closed **A**-trail is not unique but depends on the ordering of the arcs. However, this choice is unique for semi-closed **A**-trails as this simple argument shows:
Suppose P is semi-closed and the arcs of P are ordered such that u is the unique (in that ordering) first and last vertex, that is, it is the unique vertex such as the first and the last arcs in the ordering in P do not alternate in direction at the meeting point u. Therefore, it is easy to see that both $\rho_{in}(u)$ and $\rho_{out}(u)$ (degrees taken in P as a subgraph induced by its arcs) are odd whereas any other vertex v in the trail has both $\rho_{in}(v)$ and $\rho_{out}(v)$ even. This is because, in the given ordering, arcs have to alternate in direction at v and therefore in-arcs

of the form (x, v) are paired with out-arcs of the form (v, y). Therefore, in no ordering of the arcs of P can u be anything but the only vertex at which the first and last arcs do not alternate. The same argument holds for open **A**-trails.

Any other graph theoretical terms which we use are standard and can be found in any graph theory textbook such as [1]. For information on automorphism groups, the reader is referred to [7].

Let G and H be two mixed graphs and suppose that α, β are bijections from $V(G)$ to $V(H)$. The concept of TF-isomorphisms may be extended from graphs to mixed graphs. In fact, we can say that the pair (α, β) is a *two-fold isomorphism* (or TF-*isomorphism*) from G to H if the following holds: (u, v) is an arc of G if and only if $(\alpha(u), \beta(v))$ is an arc of H. We then say that G and H are TF-*isomorphic* and write $G \cong^{\text{TF}} H$. Note that when $\alpha = \beta$ the pair (α, β) is a TF-isomorphism if and only if α itself is an isomorphism. If $\alpha \neq \beta$, then the given TF-isomorphism (α, β) is essentially different from a usual isomorphism and hence we call (α, β) a *non-trivial* TF-*isomorphism*. If (α, β) is a non-trivial TF-isomorphism from a mixed graph G to a mixed graph H, the bijections α and β need not necessarily be isomorphisms from G to H. This is illustrated by the graphs in Figure 1, examples found in [5], and also others presented below.

When $G = H$, (α, β) is said to be a TF-*automorphism* and it is again called non-trivial if $\alpha \neq \beta$. The set of all TF-automorphisms of G with multiplication defined by $(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \beta\delta)$ is a subgroup of $S_{V(G)} \times S_{V(G)}$ and it is called the *two-fold automorphism group* of G and is denoted by $\text{Aut}^{\text{TF}}(G)$. Note that if we identify an automorphism α with the TF-automorphism (α, α), then $\text{Aut}(G) \subseteq \text{Aut}^{\text{TF}}(G)$. When a graph has no non-trivial TF-automorphisms, $\text{Aut}(G) = \text{Aut}^{\text{TF}}(G)$. It is possible for an asymmetric graph G, that is a graph with $|\text{Aut}(G)| = 1$, to have non-trivial TF-automorphisms. This was one of our main results in [5].

The main theme of this paper is *stability of graphs*, an idea introduced by Marušič et al. [8] and studied extensively by others, most notably by Wilson [14] and Surowski [12], [13]. Let G be a graph and let $\mathbf{CDC}(G)$ be its *canonical double cover* or *duplex*. This means that $V(\mathbf{CDC}(G)) = V(G) \times \mathbb{Z}_2$ and $\{(u, 0), (v, 1)\}$ and $\{(u, 1), (v, 0)\}$ are edges of $\mathbf{CDC}(G)$ if and only if $\{u, v\}$ is an edge of G. One may think of the second entry in the notation used for vertices of $\mathbf{CDC}(G)$, that is 0 or 1 as colours. For the reader familiar with products of graphs we wish to add that $\mathbf{CDC}(G)$ is the direct product $G \times K_2$ of G by K_2. Recall that the graph $\mathbf{CDC}(G)$ is bipartite and we may denote its colour classes by $V_0 = V \times \{0\}$ and $V_1 = V \times \{1\}$ containing vertices of the type $(u, 0)$ and $(u, 1)$ respectively. A graph is said to be *unstable* if $\text{Aut}(G) \times \mathbb{Z}_2$ is a proper subgroup of the set $\text{Aut}(\mathbf{CDC}(G))$. The elements of $\text{Aut}(\mathbf{CDC}(G)) \setminus \text{Aut}(G) \times \mathbb{Z}_2$ will be called *unexpected automorphisms* of $\mathbf{CDC}(G)$. In other words, a graph G is unstable if at least one element of $\text{Aut}(\mathbf{CDC}(G))$ is not a lifting of some element of $\text{Aut}(G)$. In this paper, we shall investigate the relationship between the stability of the graph G and its two-fold automorphism group $\text{Aut}^{\text{TF}}(G)$.

3 Unstable Graphs and TF-automorphisms

Consider Aut($\mathbf{CDC}(G)$). Now, let Σ be the set-wise stabiliser of V_0 in Aut($\mathbf{CDC}(G)$), which of course coincides with the set-wise stabiliser of V_1. Note that every $\sigma \in \Sigma$ also fixes V_1 set-wise. We will show that it is the structure of Σ which essentially determines whether $\mathbf{CDC}(G)$ has unexpected automorphisms which cannot be lifted from automorphisms of G. The following result, which is based on [9], Lemma 2.1, implies that these unexpected automorphisms of $\mathbf{CDC}(G)$ arise if the action of σ on V_0 is not mirrored by its action of V_1.

Lemma 3.1. *Let $f : \Sigma \to Sym(V) \times Sym(V)$ be defined by $f : \sigma \mapsto (\alpha, \beta)$ where $(\alpha(v), 0) = \sigma(v, 0)$ and $(\beta(v), 1) = \sigma(v, 1)$, that is α, β extract from σ its action on V_0 and V_1 respectively. Then:*

1. *f is a group homomorphism;*
2. *f is injective and therefore $f : \Sigma \to f(\Sigma)$ is a group automorphism;*
3. *$f(\Sigma) = \{(\alpha, \beta) \in Sym(V) \times Sym(V) : x$ is adjacent to y in G if and only if $\alpha(x)$ is adjacent to $\beta(y)$ in $G\}$ that is, $f(\Sigma) = \mathrm{Aut}^{TF}(G)$ that is, (α, β) (the ordered pair of separate actions of σ on the two classes) is a TF-automorphism of G.*

Proof. The fact that f is a group homomorphism, that is, that f is a structure preserving map from Σ to $Sym(V) \times Sym(V)$ follows immediately from the definition since for any σ_1, $\sigma_2 \in \Sigma$ where $f(\sigma_1) = (\alpha_1, \beta_1)$ and $f(\sigma_2) = (\alpha_2, \beta_2)$, $f(\sigma_1)f(\sigma_2) = (\alpha_1\beta_1)(\alpha_2\beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2) = f(\sigma_1\sigma_2)$. This map is clearly injective and therefore $f : \Sigma \to f(\Sigma)$ is a group automorphism.

Consider an arc $((u, 0), (v, 1))$. Since $\sigma \in \Sigma \subseteq \mathrm{Aut}(\mathbf{CDC}(G))$, $(\sigma(u), 0), (\sigma(v), 1))$ is also an arc of $\mathbf{CDC}(G)$. By definition, this arc may be denoted by $((\alpha(u), 0), (\beta(v), 1))$ and, following the definition of $\mathbf{CDC}(G)$, it exists if and only if $(\alpha(u), \beta(v))$ is an arc of G. Hence f maps elements of Σ to (α, β) which clearly take arcs of G to arcs of G. This implies that (α, β) is a TF-automorphism of G and hence $f(\Sigma) \subseteq \mathrm{Aut}^{TF}(G)$.

Conversely, let $(\alpha, \beta) \in \mathrm{Aut}^{TF}(G)$. Define σ by $\sigma(v, 0) = (\alpha(v), 0)$ and $\sigma(v, 1) = (\beta(v), 1))$, then $f(\sigma) = (\alpha, \beta)$. Hence, $\mathrm{Aut}^{TF}(G) \subseteq f(\Sigma)$. Therefore $f(\Sigma)$= $\mathrm{Aut}^{TF}(G)$. □

Theorem 3.2. *Let G be a graph. Then*

$$\mathrm{Aut}(\mathbf{CDC}(G)) = \mathrm{Aut}^{TF}(G) \rtimes \mathbb{Z}_2.$$

Furthermore, G is unstable if and only if it has a non-trivial TF-automorphism.

Proof. From Lemma 3.1, $f(\Sigma) = \mathrm{Aut}^{TF}(G)$ which must have index 2 in Aut $(\mathbf{CDC}(G))$. The permutation $\delta(v, \varepsilon) \mapsto (v, \varepsilon + 1)$ is an automorphism of $\mathbf{CDC}(G)$ and $\delta \notin f(\Sigma)$. Then Aut($\mathbf{CDC}(G)$) is generated by $f(\Sigma)$ and δ. Furthermore, $f(\Sigma) \cap \langle\delta\rangle = \mathrm{id}$ and $f(\Sigma) \lhd \mathrm{Aut}\ (\mathbf{CDC}(G))$ being of index 2.

Since $\text{Aut}(\mathbf{CDC}(G)) = \text{Aut}^{\text{TF}}(G) \rtimes \mathbb{Z}_2$, G is stable if and only if $\text{Aut}^{\text{TF}}(G) = \text{Aut}(G)$. $\square$

As shown in [5], Proposition 3.1, if $(\alpha, \beta) \in \text{Aut}^{\text{TF}}(G)$ then $(\gamma, \gamma^{-1}) \in \text{Aut}^{\text{TF}}(G)$ (where $\gamma = \alpha\beta^{-1}$). This means that for any edge $\{x, y\}$ of G, $\{\gamma(x), \gamma^{-1}(y)\}$ is also an edge. A permutation γ of $V(G)$ with this property is called an *anti-automorphism*. Such maps possess intriguing applications to the study of cancellation of graphs in direct products with arbitrary bipartite graphs, that is, the characterisation of those graphs G for which $G \times C \simeq H \times C$ implies $G \simeq H$, whenever C is a bipartite graph (see [2], Chapter 9). The second part of Theorem 3.2 could be rephrased as follows: "G is unstable if and only if it has an anti-automorphism of order different from 2". Note that the existence of an anti-automorphisms of order 2 does not imply instability since such a map corresponds to a trivial TF-automorphisms.

Notice that $\text{Aut}^{\text{TF}}(G)$ is the edge set of the graph factorial $G!$ introduced in [2] in the wake of investigations of anti-automorphisms. $G!$ shares many properties with factorials of natural numbers and is interesting per se. As $\mathbf{CDC}(G) = G \times K_2$ one could thus rephrase Theorem 3.2 in the language of [2] as $\text{Aut}(G \times K_2) = E(G!) \rtimes Z_2$.

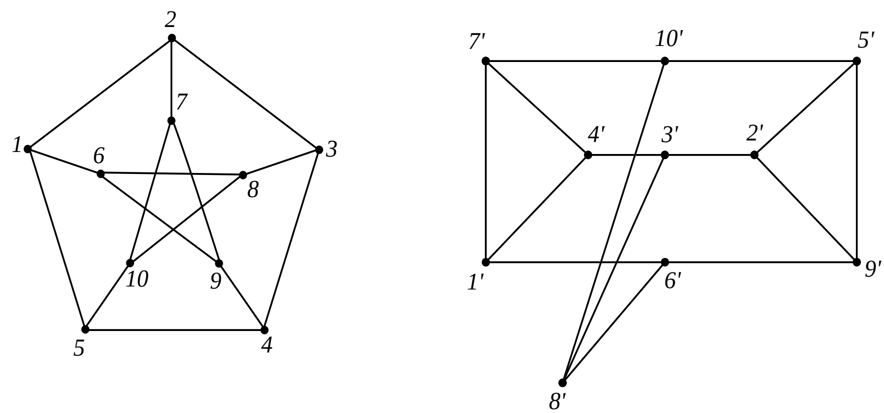

Figure 2: A stable and unstable graph which are TF-isomorphic.

It is natural to ask whether it can happen that a stable graph is TF-isomorphic to an unstable one. The answer is yes and an example is shown in Figure 2 with the Petersen graph being stable and the other graph which is TF-isomorphic to it being unstable. Both graphs have the same bipartite canonical double cover since they are TF-isomorphic. If loops are allowed, then the smallest pair of TF-isomorphic graphs, only one of which is stable is given by $G = K_3$ and H the path of three vertices with a loop at each of the end-vertices.

Porcu, motivated by the quest of finding pairs of co-spectral graphs, was the first to study non-isomorphic graphs having the same canonical double cover [11], but his work was overlooked by several mathematicians interested in these questions. Porcu was the first author to study the example given in Figure 1. The same example was re-discovered much later in [3]. Pacco and Scapellato [10], answering a question raised by Porcu, proved a result giving the number of graphs having a given bipartite graph B as canonical double covering terms of involutions in Aut(B). A later extension of their result appeared in [2], Theorem 9.15.

We should point out here (as noted by Surowski in [12]) that if a graph G is stable, that is, Aut$(\mathbf{CDC}(G))$ = Aut$(G) \rtimes \mathbb{Z}_2$, then the semi-direct product must be a direct product because Aut(G) is normal in Aut$(\mathbf{CDC}(G))$ since it has index 2 and also $\mathbb{Z}_2$ is normal since its generator commutes with every element of Aut$(\mathbf{CDC}(G))$, by stability. In fact, as Surowski comments, the stability of G is equivalent to the centrality of $\mathbb{Z}_2$ in Aut$(\mathbf{CDC}(G))$, which is the lift of the identity in Aut(G).

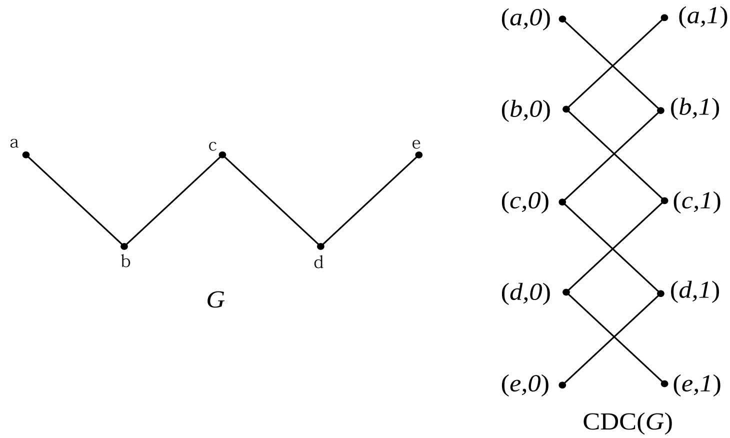

Figure 3: An example used to show how a TF-automorphism (α, β) of G can be obtained from an automorphism σ of $\mathbf{CDC}(G)$.

It is worth noting that the ideas explored in the proof of Lemma 3.1 may be used to extract TF-automorphisms of a graph G from automorphisms of $\mathbf{CDC}(G)$ which fix the colour classes. In fact, let σ be such an automorphism. Define the permutations α and β of $V(G)$ as follows: $\alpha(x) = y$ if and only if $\sigma(x, 0) = (y, 0)$ and $\beta(x) = y$ if and only if $\sigma(x, 1) = (y, 1)$. Then (α, β) is a TF-automorphism of G. We remark that α and β are not necessarily automorphisms of G as we shall show in the example shown in Figure 3. The automorphism σ is chosen so that it fixes one component of $\mathbf{CDC}(G)$ whilst being an automorphism of the other component. In order to have a more concise representation, we denote vertices of $\mathbf{CDC}(G)$ of the form $(u, 0)$, that is, elements of the colour class V_0 by u_0 and similarly denote vertices of the form $(u, 1)$ in V_1 by u_1. Using this notation, $\sigma = (a_0)(b_1)(c_0)(d_1)(e)(a_1\ e_1)(b_0\ d_0)(c_1)$. The permutations α and β of G are extracted from σ as described in the proof of Lemma 3.1. For instance, to obtain α, we restrict the action of σ to the elements of V_0, that is, those vertices of the form $(v, 0)$ or v_0 when using the new notation and then drop the subscript. Similarly, the permutation β is obtained from the action of σ restricted to V_1. Therefore, $\alpha = (a)(c)(e)(b\ d)$ and $\beta = (b)(d)(a\ e)(c)$. Note that neither α nor β is an automorphism of G, but (α, β) is a TF-automorphism of G which in turn can be lifted to the unexpected automorphism σ of $\mathbf{CDC}(G)$. This example illustrates Lemma 3.1 since the graph G is unstable and has a non-trivial TF-automorphism.

The result of Lemma 3.1 and the subsequent example lead us to other questions regarding the nature of the permutations α and β which, as discussed in the preceding example given in Figure 3, may not be automorphisms of G.

If (α, id) is a non-trivial TF-automorphism of a graph G, then G is not vertex-determining. In fact, since $\alpha \neq \mathrm{id}$ then $\alpha(u) = v$ for some $u \neq v$ and the TF-automorphism (α, id) fixes the neighbours of u and takes u to v. Hence u and v must have the same neighbourhood set, which implies that G is not vertex-determining.

We shall use this idea to prove some results below. An alternative way of looking at this is to consider Lemma 3.1 and to note that a graph G is stable if and only if given $\sigma(v, 0) = (\alpha(v), 0)$, there exists no $\beta \neq \alpha$ such that $(\beta(v), 1)) = \sigma(v, 0)$. Hence, as implied by Theorem 3.2 a graph G is stable if and only if $f(\Sigma) \subseteq \Delta_V$ where Δ_V is the diagonal group of (α, β), α, β automorphisms of G, with $\alpha = \beta$.

Proposition 3.3. *If (α, β) is a non-trivial TF-automorphism of a graph G but α and β are automorphisms of G, then G is not vertex-determining.*

Proof. Since α is an automorphism of G and (α, β) is a TF-automorphism, the group $\mathrm{Aut}^{\mathrm{TF}}(G)$ must also contain $(\alpha, \beta)(\alpha^{-1}, \alpha^{-1}) = (\mathrm{id}, \beta\alpha^{-1})$. Since $\alpha\beta^{-1} \neq \mathrm{id}$, let u be a vertex such that $v = \alpha\beta^{-1}(u)$ is different from u. Then for each neighbour w of u the arc (w, u) is taken to the arc (w, v), so that $N(u)$ is contained in $N(v)$ and vice versa. Therefore u and v have the same neighbourhood, so G is not vertex-determining. □

Proposition 3.4. *If (α, β) is a non-trivial TF-automorphism of a graph G and α, β have a different order, then G is not vertex-determining.*

Proof. Let (α, β) be an element of $\mathrm{Aut}^{\mathrm{TF}}(G)$ with the orders of α and β being p and q respectively and assume without loss of generality that $p < q$. Since $\mathrm{Aut}^{\mathrm{TF}}(G)$ is a group, $(\alpha, \beta)^p = (\alpha^p, \beta^p) = (\mathrm{id}, \beta^p)$ must also be in $\mathrm{Aut}^{\mathrm{TF}}(G)$. The same argument used in the proof of Proposition 3.3 holds since $\beta^p \neq \mathrm{id}$. Hence G is not vertex-determining. □

Proposition 3.3 and Proposition 3.4 are equivalent to their counterparts in [8] in which they are stated in terms of adjacency matrices. In [8], it is shown that if a graph G is unstable but vertex-determining and (α, β) is a non-trivial TF-automorphism of G, then α and β must not be automorphisms of G and must have the same order. This then gives us information about automorphisms σ of $\mathbf{CDC}(G)$ which are liftings of TF-automorphisms of G.

4 Triangles

In this section we shall study the behaviour of a non-trivial TF-automorphism of a graph G acting on a subgraph of G isomorphic to K_3 with the intent of obtaining information regarding the stability of graphs which have triangles as a basic characteristic of their structure. Strongly regular graphs in which every pair of adjacent vertices has a common neighbour are an example. The stability of such graphs has been studied by Surowski [12]. We believe that this section is interesting because it is a source of simple examples of unstable graphs and also because a detailed analysis of what happens to triangles can throw more light on TF-automorphisms of graphs.

Proposition 4.1. *Let (α, β) be a TF-isomorphism from a graph G to a graph G'. The action of (α, β) on some subgraph $H \cong K_3$ yields either (a) a closed* **A***-trail of length 6 with no repeated vertices, (b) a pair of oriented* **A***-connected triangles with exactly one common vertex, (c) a pair of oriented triangles with exactly two common vertices or (d) an undirected triangle as illustrated in Figure 4(a),(b),(c) and (d) respectively.*

Proof. Let $V(H) = \{1, 2, 3\}$. The semi-closed **A**-trials $1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 1$ and $1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 1$ of H are taken by (α, β) to $\alpha(1) \longrightarrow \beta(2) \longleftarrow \alpha(3) \longrightarrow \beta(1)$ and $\beta(1) \longleftarrow \alpha(2) \longrightarrow \beta(3) \longleftarrow \alpha(1)$, respectively; together, these two **A**-trails form a closed **A**-trail consisting of six arcs.

Consider the possible equalities $\alpha(v) = \beta(v)$ for $v \in V(H)$. Up to a reordering of the vertices, we have the following four cases, corresponding to the four cases stated above:

(a) $\alpha(v) \neq \beta(v)$ for all v.

(b) $\alpha(v) \neq \beta(v)$ for $v = 1, 3$ but $\alpha(2) = \beta(2)$.

(c) $\alpha(v) \neq \beta(v)$ for $v = 1$ but $\alpha(v) = \beta(v)$ for $v = 2,\ 3$.

(d) $\alpha(v) = \beta(v)$ for all v.

For each case, the corresponding graph is shown in Figure 4. □

In general, when a TF-automorphism (α, β) acts on the arcs of G it maps any triangle H into another triangle if $\alpha(x) = \beta(x)$ for every vertex x of the triangle or it fits one of the other configurations described by Proposition 4.1. If a graph G in which every edge lies in a triangle is unstable, then it must have a non-trivial TF-automorphism which follows one of the configurations illustrated in Figure 4(a), (b) and (c).

Proposition 4.2. *Let (α, β) be a TF-isomorphism from G to G'. When the TF-isomorphism acting on K_3, a subgraph of G, yields a closed* **A***-trail of length 6 with no repeated vertices as shown in Figure 4(a), either two triangles with no common vertex or a triangle and a closed* **A***-trail with* 6 *arcs are mapped to a subgraph isomorphic to C_6. In the cases when the TF-isomorphism acting on a K_3 yields the images illustrated in Figure 4 (b), (c), the pair of triangles which are either mapped to two triangles with exactly one common vertex or to two triangles with exactly one common edge must have a common vertex.*

Proof. Refer to Figure 4. In the case illustrated in Figure 4(a) the arcs of one closed **A**-trail P of length 6 can be the co-domain the arcs of a triangle H. The **A**-trail P' obtained by reversing the arcs of P can be the co-domain of another triangle K. We claim that H and K are vertex disjoint. In fact, suppose not and assume that the two triangles have a common vertex u. The pair of vertices $\alpha(u)$ and $\beta(u)$ where $\alpha(u) \neq \beta(u)$ are in both P and P' and this is contradiction as the in-degree of $\alpha(u)$ and similarly the out-degree of $\beta(u)$ must be zero and this makes it impossible to identify arcs of P with arcs of P' to form the edges of a C_6. Figure 5 shows an example where setting $\alpha(3) = \beta(5)$, $\beta(2) = \alpha(6)$, $\alpha(1) = \beta(4)$, $\beta(3) = \alpha(5)$, $\alpha(2) = \beta(6)$, $\beta(1) = \alpha(4)$ would be one way of associating one directed

C_6 with the other so that the alternating connected circuits form an undirected C_6. The other possibility is illustrated in Figure 6. In this example $\beta(1) = \alpha(4)$, $\alpha(2) = \beta(5)$, $\beta(3) = \alpha(6)$, $\alpha(1) = \beta(7)$, $\beta(2) = \alpha(8)$ and $\alpha(3) = \beta(9)$ so that a closed **A**-trail of length 6 covering a K_3 is mapped to a closed **A**-trail of length 6 covering half of the arcs of an undirected C_6 whilst the rest of the arcs come from an **A**-trail of length 6 covering half of the arcs of another subgraph isomorphic to C_6. The K_3 and the C_6 in the domain of the TF-isomorphism cannot have a common vertex and the proof is analogous to the one concerning the former case.

The proof for the remaining cases may be carried over along the same lines as the above. Refer to Figure 7. We observe that the **A**-trail P_1 described by $\beta(3) \longleftarrow \alpha(2) \longrightarrow \beta(1)$, the **A**-trail P_2 described by $\alpha(1) \longrightarrow \beta(2) \longleftarrow \alpha(3)$ and the **A**-trails P_1' and P_2' obtained by reversing the arcs of P_1 and P_2 respectively would imply by the preservation of **A**-trails, that in the pre-image of the subgraph, there are four **A**-trails passing through the vertex 2. This is only possible if the triangles in the pre-image have a common vertex. □

Figure 8 shows an example to illustrate the result of Proposition 4.2 where $\alpha(1) = \beta(4)$, $\beta(3) = \alpha(5)$, $\alpha(3) = \beta(5)$ and $\beta(1) = \alpha(4)$. Figure 8 shows the smallest unstable graphs which have a TF-automorphism which takes a triangle to a pair of directed triangles with a common vertex as illustrated in Figure 4(b). Figure 9 shows the smallest graph which has a nontrivial TF-automorphism which maps a triangle to the mixed graph illustrated in Figure 4(c).

5 Unstable graphs of arbitrarily large diameter

In this section, we present a method of constructing unstable graphs of an arbitrarily high diameter.

If H, K are graphs, let $[H, K]$ be the graph whose vertex set is the union $V(H) \cup V(K)$ and whose edge set is the union of $E(H)$, $E(K)$ plus the edges of the complete bipartite graph with classes $V(H)$ and $V(K)$. More generally, if H_i are graphs, where $i \in \mathbb{Z}_m$, let $G = [H_0,\ H_1,\ \ldots,\ H_{m-1}]$ be the graph whose vertex set is the union of all $V(H_i)$ and whose edge set is the union of all $E([H_i, H_{(i+1)}])$. In other words, G contains all vertices and edges of the graphs H_i, plus all edges of the complete bipartite graph connecting two consecutive H_i's.

Now, assume that none of the H_i has isolated vertices. Let $(\alpha_i, \beta_i) : H_i \to H_{i+1}$ be TF-isomorphisms as i runs over $\mathbb{Z}_m$. Assume that the product

$$(\alpha_0, \beta_0)(\alpha_1, \beta_1)\ldots(\alpha_{m-1}, \beta_{m-1}) = (\mathrm{id}, \mathrm{id}).$$

Note that the latter assumption is not a restriction, because one can always take (α_0, β_0) as the inverse of the product of the remaining TF-isomorphisms.

Theorem 5.1. *With the above assumptions, let $G = [H_0, H_1, ..., H_{m-1}]$. Define two permutations α, β of $V(G)$ as follows. For $v \in V(G)$, let i be such that $v \in V(H_i)$; then set $\alpha(v) = \alpha_i(v)$ and $\beta(v) = \beta_i(v)$.*

Then the following hold:

1. *(α, β) is a TF-automorphism of G;*

2. *if $m \geq 4$, diam $G = (m - e)/2$, where $e = 0$ if m is even and $e = 1$ if m is odd;*

3. *each edge of G belongs to a triangle.*

Proof. First note that if (u, v) is an arc of G and both u, v belong to the same H_i, then the image of (u, v) is an arc of H_{i+1}, hence of G, because (α, β) acts like (α_i, β_i) in H_i. If u, v do not belong to the same H_i, then they belong to consecutive graphs, say H_i, H_{i+1}, so $\alpha(u)$ and $\beta(u)$ belong to the consecutive graphs H_{i+1}, H_{i+2}, and are adjacent because all the arcs between these two graphs belong to G. This proves (1).

Concerning distance, a path from u in H_0 to v in H_s, where $s \leq (m - e)/2$, must pass through all graphs $H_1, H_2, ..., H_{s-1}$ or else $H_{m-1}, H_{m-2}, ..., H_{s+1}$. Since such a path can be found, $\mathrm{d}(u, v) = s$. For two vertices in, say, H_i and H_j with $i \neq j$, the same argument shows that $d(u, v)$ cannot exceed $(m - e)/2$. Finally, if $u,\ v$ lie in the same H_i, they have a common neighbour in H_{i+1}, then $\mathrm{d}(u, v)$ is less or equal than 2 (regardless to their distance within H_i). This proves (2).

What about triangles? If $\{u, v\}$ is an edge of some H_i then letting $w \in H_{i+1}$ the vertex w is adjacent to both u and v, hence $\{u, v, w\}$ is a triangle. If $\{u, v\}$ is an edge of some $[H_i, H_{i+1}]$, say with $u \in H_i$ and $v \in H_{i+1}$, take any neighbour w of u in H_i (recall the assumption about no isolated vertices) and get the triangle $\{u, v, w\}$. This proves (3). □

Until now, we did not mention that the concerned TF-isomorphisms are non-trivial, so all the above would work fine for the case of isomorphisms too. But adding the hypothesis that at least one of them is non-trivial, the obtained graph G has a non-trivial TF-isomorphism, namely (α, β) as described above. Theorem 5.1 shows that there are unstable graphs of arbitrarily high diameter, where each edge belongs to a triangle.

Surowski [12, 13] proved various results concerning graph stability. In [12], Proposition 2.1, he claims that if G is a connected graph of diameter $d \geq 4$ in which every edge lies in a triangle, then G is stable. However, by taking $m \geq 7$ in Theorem 5.1 we get infinitely many counterexamples to this claim by taking all the H_i isomorphic to the same vertex-determining bipartite graph, because such a vertex-transitive graph is unstable, therefore one can find a non-trivial TF-isomorphism from H_i to H_{i+1}, and since these H_i are isomorphic, the resulting graph G is vertex-determining, has diameter $k \geq 4$, and is unstable. One of these counterexamples is illustrated in Figure 10.

We detected one possible flaw in Surowski's proof. It is claimed in [12] that whenever an automorphism of **CDC**(G) fixes $(v, 1)$ it also fixes $(v, -1)$. We have not seen a proof of this result. Besides, in our last example, G has a non-trivial fixed-point-free TF-automorphism, which implies that $\mathbf{CDC}(G)$ has a fixed-point-free automorphism that fixes the colour classes. This claim is also used in [12] Proposition 2.2 which states that if G is a strongly regular graph with $k > \mu \neq \lambda \geq 1$, then G is stable. Hence, we believe that at this point, the stability of strongly regular graphs with these parameters requires further investigation.

6 Concluding Remarks

The use of TF-isomorphisms in the study of stability of graphs provides a fresh outlook which allows us to view facts within a more concrete framework and also provides tools to obtain new results. For instance, we can investigate the structure of the given graph without actually requiring to lift the graph to its canonical double cover, but only having to reason within the original graph. Furthermore, the insights that we already have about TF-isomorphisms of graphs may be considered to be new tools added to a limited toolkit. In particular, let us mention the idea of graph invariants under the action of TF-isomorphisms, such as **A**-trails, a topic which we have started to study in [6]. To be able to find out how the subgraphs of a graph are related to other subgraphs within the graph itself in the case of unstable graphs fills a gap in our understanding of graph stability and using TF-isomorphisms appears to be a promising approach in this sense. We believe that this paper substantiates these claims. Furthermore, it motivates us to carry out further investigations. Some pending questions such as those concerning the stability of certain strongly regular graphs have already been indicated. The study of how TF-isomorphisms act on common subgraphs such as triangles is another useful lead. Nevertheless, the more ambitious aim would be the classification of unstable graphs in terms of the types of TF-automorphisms which they admit.

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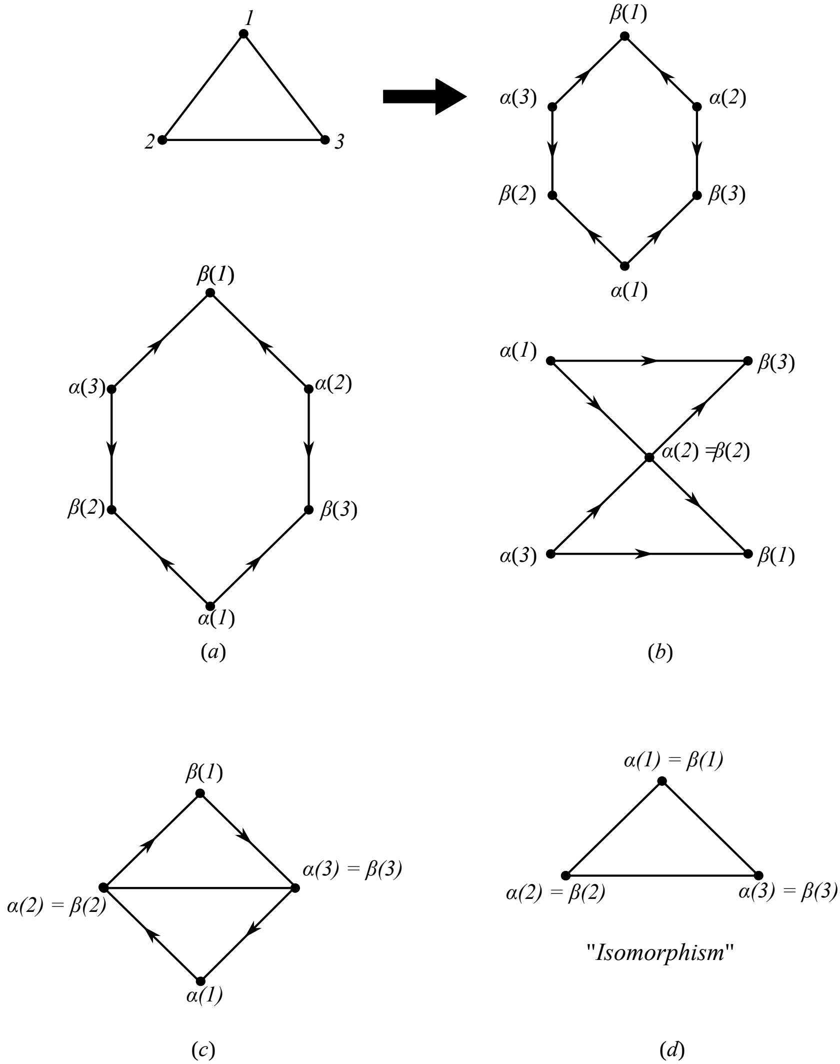

Figure 4: The configurations of possible images of a triangle under the action of a TF-automorphism as described in Proposition 4.1.

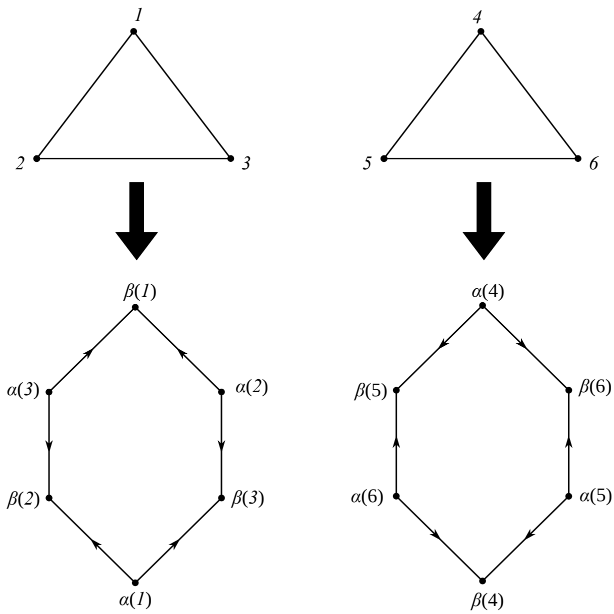

Figure 5: Two triangles mapped to a C_6.

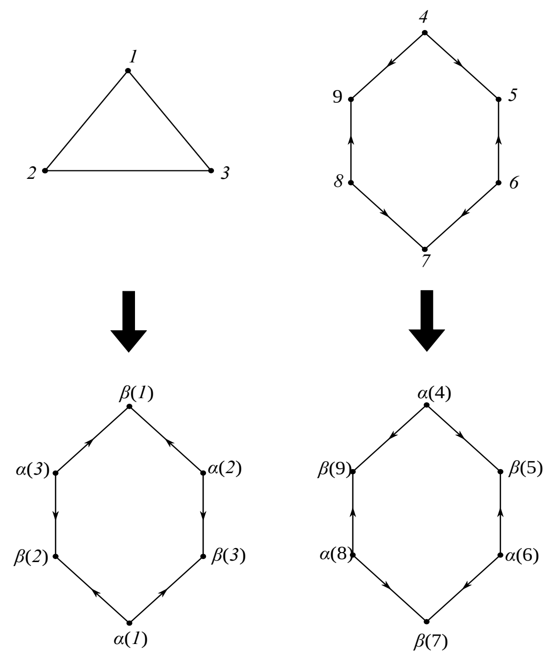

Figure 6: A triangle and one closed **A**-trail of length 6 covering the edges of a C_6 mapped to a C_6.

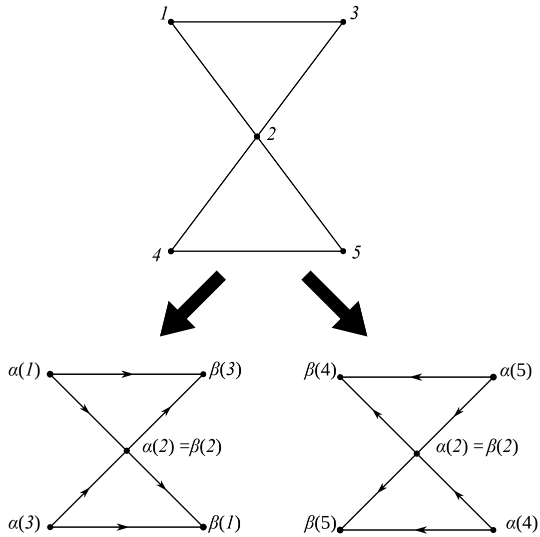

Figure 7: An example to illustrate the result of Proposition 4.2.

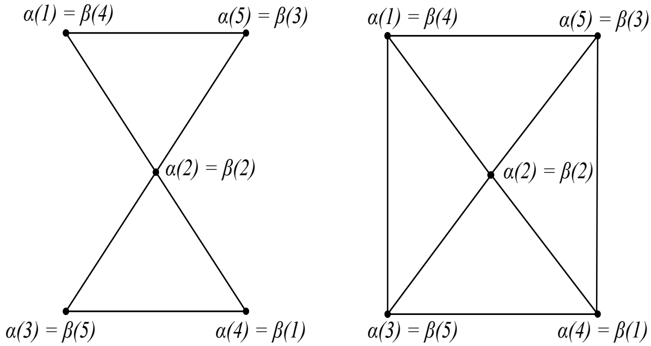

Figure 8: The smallest unstable graphs where a triangle is taken to a two directed triangles sharing a vertex.

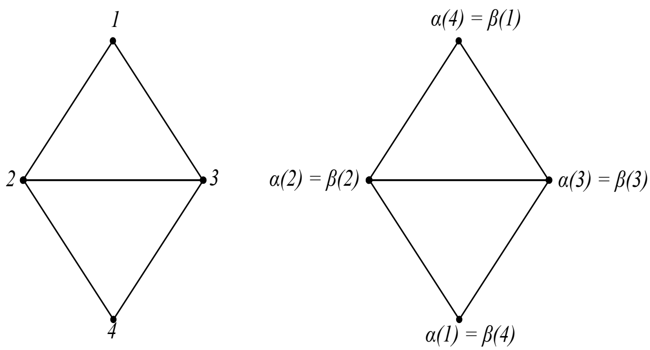

Figure 9: The smallest unstable graph which has a TF-automorphism taking a triangle to the mixed graph as illustrated in Figure 4(c).

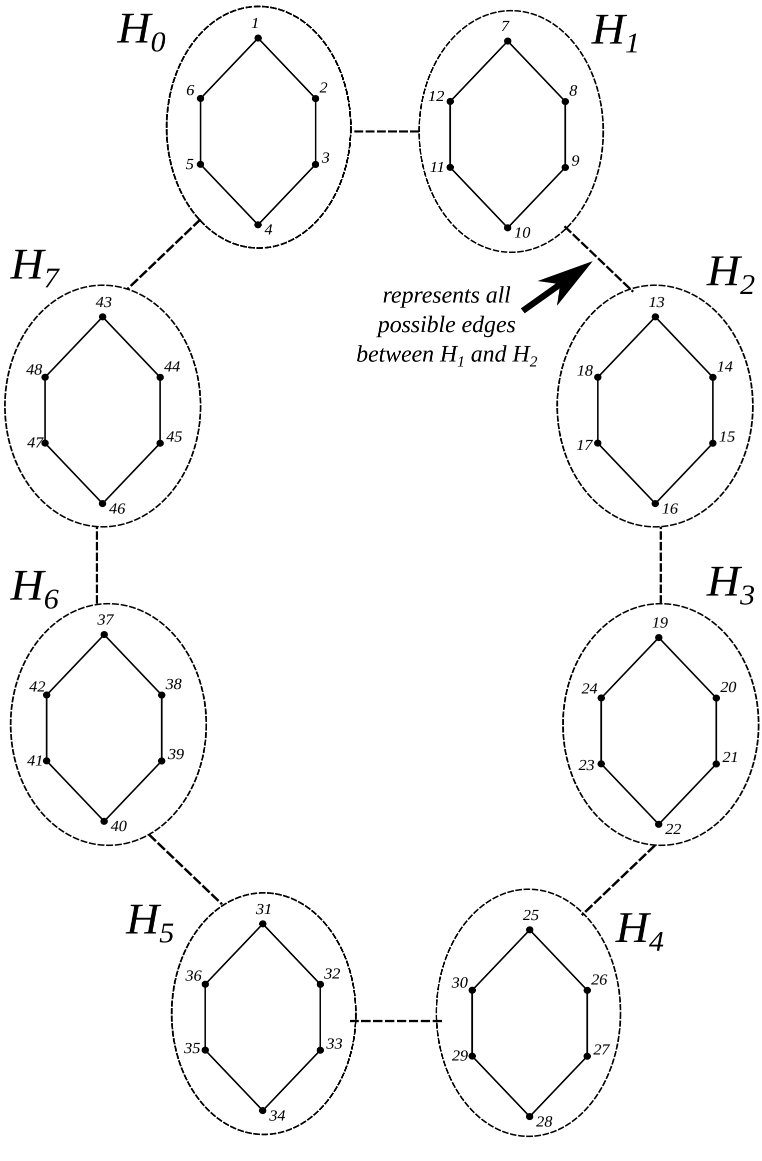

Figure 10: A graph constructed using Theorem 5.1.

References

[1] A. Bondy and U. Murty, *Graph Theory (Graduate Texts in Mathematics)*, Springer, 2008.

[2] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs, Second Edition*, Discrete Mathematics and Its Applications, Taylor & Francis, 2011.

[3] W. Imrich and T. Pisanski, Multiple Kronecker covering graphs, *Eur. J. Comb.* **29** (2008), 1116–1122, doi:10.1016/j.ejc.2007.07.001.

[4] J. Lauri, R. Mizzi and R. Scapellato, Two-fold orbital digraphs and other constructions., *International J. of Pure and Applied Math.* **1** (2004), 63–93.

[5] J. Lauri, R. Mizzi and R. Scapellato, Two-fold automorphisms of graphs., *Australasian J. Combinatorics.* **49** (2011), 165–176.

[6] J. Lauri, R. Mizzi and R. Scapellato, A generalisation of isomorphisms with applications., *Preprint* (2014).

[7] J. Lauri and R. Scapellato, *Topics in Graph Automorphisms and Reconstruction*, Cambridge University Press, 2003.

[8] D. Marušič, R. Scapellato and N. Zagaglia Salvi, A characterization of particular symmetric $(0, 1)$-matrices, *Linear Algebra Appl.* **119** (1989), 153–162, doi:10.1016/0024-3795(89)90075-X.

[9] D. Marušič, R. Scapellato and N. Zagaglia Salvi, Generalized Cayley graphs, *Discrete Mathematics* **102** (1992), 279 – 285, doi:http://dx.doi.org/10.1016/0012-365X(92)90121-U.

[10] W. Pacco and R. Scapellato, Digraphs having the same canonical double covering, *Discrete Math.* (1997), 291–296.

[11] L. Porcu, Sul raddoppio di un grafo, *Att. Ist. Lombardo (Rend. Sci.)* **A** (1976), 353–360.

[12] D. B. Surowski, Stability of arc-transitive graphs, *J. Graph Theory* **38** (2001), 95–110, doi:10.1002/jgt.1026.

[13] D. B. Surowski, Automorphism groups of certain unstable graphs, *Math. Slovaca* **53** (2003), 215–232.

[14] S. Wilson, Unexpected symmetries in unstable graphs, *J. Combin. Theory Ser. B* **98** (2008), 359–383, doi:10.1016/j.jctb.2007.08.001.

[15] B. Zelinka, The group of autotopies of a digraph, *Czechoslovak Mathematical Journal* **21** (1971), 619–624.

[16] B. Zelinka, Isotopy of digraphs, *Czechoslovak Mathematical Journal* **22** (1972), 353–360.

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A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two

Dedicated to Dragan Marušič on the occasion of his 60th birthday

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Abstract

A complete list of all connected arc-transitive asymmetric digraphs of in-valence and out-valence 2 on up to 1000 vertices is presented. As a byproduct, a complete list of all connected 4-valent graphs admitting a $\frac{1}{2}$-arc-transitive group of automorphisms on up to 1000 vertices is obtained. Several graph-theoretical properties of the elements of our census are calculated and discussed.

Keywords: Graph, digraph, edge-transitive, vertex-transitive, arc-transitive, half-arc-transitive.

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1 Introduction

Recall that a graph Γ is called $\frac{1}{2}$*-arc-transitive* provided that its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on its edge-set $\mathrm{E}(\Gamma)$ and on its vertex-set $\mathrm{V}(\Gamma)$ but intransitively on its arc-set $\mathrm{A}(\Gamma)$. More generally, if G is a subgroup of $\mathrm{Aut}(\Gamma)$ such that G acts transitively on $\mathrm{E}(\Gamma)$ and $\mathrm{V}(\Gamma)$ but intransitively on $\mathrm{A}(\Gamma)$, then G is said to *act* $\frac{1}{2}$*-arc-transitively* on Γ and we say that Γ is $(G, \frac{1}{2})$*-arc-transitive*. To shorten notation, we shall say that a $\frac{1}{2}$-arc-transitive graph is a *HAT* and that a graph admitting a $\frac{1}{2}$-arc-transitive group of automorphisms is a *GHAT*. Clearly, any HAT is also a GHAT. Conversely, a GHAT is either a HAT or arc-transitive.

The history of GHATs goes back to Tutte who, in his 1966 paper [38, 7.35, p.59], proved that every GHAT is of even valence and asked whether HATs exist at all. The first examples of HATs were discovered a few years later by Bouwer [6]. After a short break, interest in GHATs picked up again in the 90s, largely due to a series of influential papers of Marušič concerning the GHATs of valence 4 (see [1, 17, 21, 24], to list a few). For a nice survey of the topic, we refer the reader to [16], and for an overview of some more recent results, see [13, 22].

To shorten notation further, we shall say that a connected GHAT (HAT, respectively) of valence 4 is a 4-GHAT (4-HAT, respectively). The main result of this paper is a compilation of a complete list of all 4-GHATs with at most 1000 vertices. This improves an unpublished work [23], where all 4-HATs of order up to 869 vertices (with the exception of order 512) with the vertex-stabiliser of order 2 were determined.

Our result was obtained indirectly using an intimate relation between 4-GHATs and connected arc-transitive asymmetric digraphs of in- and out-valence 2 (we shall call such digraphs 2-ATDs for short) – see Section 2.2 for details on this relationship. These results can be succinctly summarised as follows:

Theorem 1.1. *There are precisely 26457 pairwise non-isomorphic* 2*-ATDs on at most* 1000 *vertices, and precisely 11941* 4*-GHATs on at most* 1000 *vertices, of which 8695 are arc-transitive and 3246 are* $\frac{1}{2}$*-arc-transitive.*

The actual lists of (di)graphs, together with a spreadsheet (in a “comma separated values” format) with some graph theoretical invariants, is available at [28].

The rest of this section is devoted to some interesting facts gleaned from these lists. All the relevant definitions that are omitted here can be found in Section 2. In Section 3, we explain how the lists were computed and present the theoretical background which assures that the computations were exhaustive. In Section 4, information about the format of the files available on [28] is given.

We now proceed with a few comments on the census of 4-HATs. By a *vertex-stabiliser* of a vertex-transitive graph or digraph Γ, we mean the stabiliser of a vertex in $\mathrm{Aut}(\Gamma)$. Even though it is known that a vertex-stabiliser of a 4-HAT can be arbitrarily large (see [18]), not many examples of 4-HATs with vertex-stabilisers of order larger than 2 were known, and all known examples had a very large number of vertices. Recently, Conder and Šparl (see also [8]) discovered a 4-HAT on 256 vertices with vertex-stabiliser of order 4 and proved that this is the smallest such example. This fact is confirmed by our census; in fact, the following theorem can be deduced from the census.

Theorem 1.2. *Amongst the 3246* 4*-HATs on at most* 1000 *vertices, there are seventeen with vertex-stabiliser of order* 4*, three with vertex-stabiliser of order* 8*, and none with* *larger vertex-stabilisers. The smallest* 4*-HAT with vertex-stabiliser of order* 4 *has order* 256 *and the smallest two with vertex-stabilisers of order* 8 *have* 768 *vertices; the third* 4*-HAT with vertex-stabiliser of order* 8 *has* 896 *vertices.*

Another curiosity about 4-HATs is that those with a non-abelian vertex-stabiliser tend to be very rare (at least amongst the "small" graphs). The first known 4-HAT with a non-abelian vertex-stabiliser was discovered by Conder and Marušič (see [7]) and has 10752 vertices. Further examples of 4-HATs with non-abelian vertex-stabilisers were discovered recently (see [8]), including one with a vertex-stabiliser of order 16. However, the one on 10752 vertices remains the smallest known example. Using our list, the following fact is easily checked.

Theorem 1.3. *Every* 4*-HAT with a non-abelian vertex-stabiliser has more than* 1000 *vertices.*

In fact, there are strong indications that the graph on 10752 vertices discovered by Conder and Marušič is the smallest 4-HAT with a non-abelian vertex-stabiliser.

We will call a 4-HAT with a non-solvable automorphism group a *non-solvable* 4*-HAT*. The first known non-solvable 4-HAT was constructed by Marušič and Xu [24]; and its order is $7!/2$. An infinite family of non-solvable 4-HATs were constructed later by Malnič and Marušič [15]. The smallest member of this family has an even larger order, namely $11!/2$. To the best of our knowledge, no smaller non-solvable 4-HATs appeared in literature. Perhaps surprisingly, small examples of non-solvable 4-HATs seem not to be too rare, as can be checked from our census (as well as from the unpublished work [23]).

Theorem 1.4. *There are thirty-two non-solvable* 4*-HATs with at most* 1000 *vertices. The smallest one, named HAT[480,44], has order* 480*, girth* 5*, radius* 5*, attachment number* 2*, alter-exponent* 2*, and alter-perimeter* 1*. It is non-Cayley and non-bipartite.*

(The terms *radius*, *attachment number*, *alter-exponent*, and *alter-perimeter* appearing in the statement of Theorem 1.4 are defined in Sections 4.2 and 4.3.) Let us now continue with a few comments on the census of 2-ATDs. All the undefined notions mentioned in the theorems below are explained in Sections 2, 4.2 and 4.3. It is not surprising that, apart from the generalised wreath digraphs (see Section 2.3 for the definition), very few of the 2-ATDs on at most 1000 vertices are 2-arc-transitive. In fact, the following can be deduced from the census.

Theorem 1.5. *Out of the 26457* 2*-ATDs on at most* 1000 *vertices, 961 are generalised wreath digraphs. Of the remaining 25496, only 1199 are* 2*-arc-transitive (the smallest having order* 18*), only 255 are* 3*-arc-transitive (the smallest having order* 42*), only 61 are* 4*-arc-transitive (the smallest having order* 90*), and only 6 are* 5*-arc-transitive (the smallest two having order* 640*); none of them is* 6*-arc-transitive.*

Note that the non-existence of a 6-arc-transitive non-generalised-wreath 2-ATD on at most 1000 vertices follows from a more general result (see Corollary 3.2).

Recall that there is no 4-HAT on at most 1000 vertices with a non-abelian vertex-stabiliser (Theorem 1.3). Consequently (see Section 2.2), every 2-ATD on at most 1000 vertices with a non-abelian vertex-stabiliser has an arc-transitive underlying graph; and there are indeed such examples. In fact, the following holds (see Section 2.1 for the definition of *self-opposite*).

Theorem 1.6. *There are precisely forty-five* 2*-ATDs on at most* 1000 *vertices with a non-abelian vertex-stabiliser. They are all self-opposite, at least* 3*-arc-transitive, have non-solvable automorphism groups, and radius* 3*. The smallest of these digraphs has order* 42*, and the smallest that is* 4*-arc-transitive has order* 90*. There are no* 5*-arc-transitive* 2*-ATDs with a non-abelian vertex-stabiliser and order at most* 1000*.*

If a 2-ATD is self-opposite, then the isomorphism between the digraph and its opposite digraph is an automorphism of the underlying graph, making the underlying graph arc-transitive. Hence, self-opposite 2-ATDs always yield arc-transitive 4-GHATs. However, the converse is not always true: there are 2-ATDs that are not self-opposite, but have an arc-transitive underlying graph. In this case, the index of the automorphism group of the 2-ATD in the automorphism group of its underlying graph must be larger than 2 (for otherwise the former would be normal in the latter and thus any automorphism of the underlying graph would either preserve the arc-set of the digraph, or map it to the arc-set of the opposite digraph). It is perhaps surprising that there are not many small examples of such behaviour.

Theorem 1.7. *There are precisely fifty-two* 2*-ATDs on at most* 1000 *vertices that are not self-opposite but have an arc-transitive underlying graph. The smallest two have order* 21*. None of these digraphs is* 2*-arc-transitive. The index of the automorphism group of these digraphs in the automorphism group of the underlying graphs is always* 8*.*

We finish this section by mentioning an earlier attempt of Stephen Wilson and the first author of this paper to compile a census of all small edge-transitive graphs of valence 4, and thus, in particular, of all 4-GHATs; see [31]. The results of this paper confirm that the list given in [31] contains all 4-GHATs of order at most 100.

2 Notation and definitions

2.1 Digraphs and graphs

A *digraph* is an ordered pair (V, A) where V is a finite non-empty set and $A \subseteq V \times V$ is a binary relation on V. We say that (V, A) is *asymmetric* if A is asymmetric, and we say that (V, A) is a *graph* if A is irreflexive and symmetric. If $\Gamma = (V, A)$ is a digraph, then we shall refer to the set V and the relation A as the *vertex-set* and the *arc-set* of Γ, and denote them by $\mathrm{V}(\Gamma)$ and $\mathrm{A}(\Gamma)$, respectively. Members of V and A are called *vertices* and *arcs*, respectively. If (u, v) is an arc of a digraph Γ, then u is called the *tail*, and v the *head* of (u, v). If Γ is a graph, then the unordered pair $\{u, v\}$ is called an *edge* of Γ and the set of all edges of Γ is denoted $\mathrm{E}(\Gamma)$.

If Γ is a digraph, then the *opposite digraph* Γ^{opp} has vertex-set $\mathrm{V}(\Gamma)$ and arc-set $\{(v, u) : (u, v) \in \mathrm{A}(\Gamma)\}$. The *underlying graph* of Γ is the graph with vertex-set $\mathrm{V}(\Gamma)$ and with arc-set $\mathrm{A}(\Gamma) \cup \mathrm{A}(\Gamma^{\mathrm{opp}})$. A digraph is called *connected* provided that its underlying graph is connected.

Let v be a vertex of a digraph Γ. Then the *out-neighbourhood* of v in Γ, denoted by $\Gamma^+(v)$, is the set of all vertices u of Γ such that $(v, u) \in \mathrm{A}(\Gamma)$, and similarly, the *in-neighbourhood* $\Gamma^-(v)$ is defined as the set of all vertices u of Γ such that $(u, v) \in \mathrm{A}(\Gamma)$. Further, we let $\mathrm{val}^+(v) = |\Gamma^+(v)|$ and $\mathrm{val}^-(v) = |\Gamma^-(v)|$ be the *out-valence* and *in-valence* of Γ, respectively. If there exists an integer r such that $\mathrm{val}^+(v) = \mathrm{val}^-(v) = r$ for every $v \in \mathrm{V}(\Gamma)$, then we say that Γ is *regular* of *valence* r, or simply that Γ is an *r-valent* digraph.

An s-arc of a digraph Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of Γ, such that (v_{i-1}, v_i) is an arc of Γ for every $i \in \{1, \ldots, s\}$ and $v_{i-1} \neq v_{i+1}$ for every $i \in \{1, \ldots, s-1\}$. If $x = (v_0, v_1, \ldots, v_s)$ is an s-arc of Γ, then every s-arc of the form $(v_1, v_2, \ldots, v_s, w)$ is called a *successor* of x.

An *automorphism* of a digraph Γ is a permutation of $\mathrm{V}(\Gamma)$ which preserves the arc-set $\mathrm{A}(\Gamma)$. Let G be a subgroup of the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. We say that Γ is *G-vertex-transitive* or *G-arc-transitive* provided that G acts transitively on $\mathrm{V}(\Gamma)$ or $\mathrm{A}(\Gamma)$, respectively. Similarly, we say that Γ is *(G, s)-arc-transitive* if G acts transitively on the set of s-arcs of Γ. If Γ is a graph, we say that it is *G-edge-transitive* provided that G acts transitively on $\mathrm{E}(\Gamma)$. When $G = \mathrm{Aut}(\Gamma)$, the prefix G in the above notations is usually omitted.

If Γ is a digraph and $v \in \mathrm{V}(\Gamma)$, then a *v-shunt* is an automorphism of Γ which maps v to an out-neighbour of v.

2.2 From 4-GHATs to 2-ATDs and back

If Γ is a connected 4-valent $(G, \frac{1}{2})$-arc-transitive graph, then G has two paired orbits on the arc-set of Γ, each orbit having the property that each vertex of Γ is the head of precisely two arcs, and also the tail of precisely two arcs of the orbit. By taking any of these two orbits as an arc-set of a digraph on the same vertex-set, one thus obtains a 2-ATD whose underlying graph is Γ, and admitting G as an arc-transitive group of automorphisms.

Conversely, the underlying graph of a G-arc-transitive 2-ATD is a $(G, \frac{1}{2})$-arc-transitive 4-GHAT. In this sense the study of 4-GHATs is equivalent to the study of 2-ATDs.

In Section 3, we explain how a complete list of all 2-ATDs on at most 1000 vertices was obtained. The above discussion shows how this yields a complete list of all 4-GHATs on at most 1000 vertices.

2.3 Generalised wreath digraphs

Let n be an integer with $n \geq 3$, let $V = \mathbb{Z}_n \times \mathbb{Z}_2$, and let $A = \{((i, a), (i+1, b)) : i \in \mathbb{Z}_n, a, b \in \mathbb{Z}_2\}$. The asymmetric digraph (V, A) is called a *wreath digraph* and denoted by $\vec{\mathrm{W}}_n$.

If Γ is a digraph and r is a positive integer, then the *r-th partial line digraph* of Γ, denoted $\mathrm{Pl}^r(\Gamma)$, is the digraph with vertex-set equal to the set of r-arcs of Γ and with (x, y) being an arc of $\mathrm{Pl}^r(\Gamma)$ whenever y is a successor of x. If $r = 0$, then we let $\mathrm{Pl}^r(\Gamma) = \Gamma$.

Let r be a positive integer. The $(r-1)$-th partial line digraph $\mathrm{Pl}^{r-1}(\vec{\mathrm{W}}_n)$ of the wreath digraph $\vec{\mathrm{W}}_n$ is denoted by $\vec{\mathrm{W}}(n, r)$ and called a *generalised wreath digraph*. Generalised wreath digraphs were first introduced in [32], where $\vec{\mathrm{W}}(n, r)$ was denoted $C_n(2, r)$. It was proved there that $\mathrm{Aut}(\vec{\mathrm{W}}(n, r)) \cong C_2 \wr C_n \cong C_2^n \rtimes C_n$ and that $\mathrm{Aut}(\vec{\mathrm{W}}(n, r))$ acts transitively on the $(n-r)$-arcs but not on the $(n-r+1)$-arcs of $\vec{\mathrm{W}}(n, r)$ [32, Theorem 2.8]. In particular, $\vec{\mathrm{W}}(n, r)$ is arc-transitive if and only if $n \geq r+1$. Note that $|\mathrm{V}(\vec{\mathrm{W}}(n, r))| = n2^r$, and thus $|\mathrm{Aut}(\vec{\mathrm{W}}(n, r))_v| = n2^n / n2^r = 2^{n-r}$.

The underlying graph of a generalised wreath digraph will be called a *generalised wreath graph*.

2.4 Coset digraphs

Let G be a group generated by a core-free subgroup H and an element g with $g^{-1} \notin HgH$. One can construct the *coset digraph*, denoted $\mathrm{Cos}(G, H, g)$, whose vertex-set is the set G/H of right cosets of H in G, and where (Hx, Hy) is an arc if and only if $yx^{-1} \in HgH$. Note that the condition $g^{-1} \notin HgH$ guarantees that the arc-set is an asymmetric relation. Moreover, since $G = \langle H, g\rangle$, the digraph $\mathrm{Cos}(G, H, g)$ is connected.

The digraph $\mathrm{Cos}(G, H, g)$ is G-arc-transitive (with G acting upon G/H by right multiplication), and hence $\mathrm{Cos}(G, H, g)$ is a G-arc-transitive and G-vertex-transitive digraph with g being a v-shunt (where $v = H \cdot 1 \in G/H$). On the other hand, it is folklore that every such graph arises as a coset digraph.

Lemma 2.1. *If Γ is a connected G-arc-transitive and G-vertex-transitive digraph, v is a vertex of Γ, and g is a v-shunt contained in G, then $\Gamma \cong \mathrm{Cos}(G, G_v, g)$.*

3 Constructing the census

If Γ is a G-vertex-transitive digraph with n vertices, then $|G| = n|G_v|$. If one wants to use the coset digraph construction to obtain all 2-ATDs on n vertices, one thus needs to consider all groups G of order $n|G_v|$ that can act as arc-transitive groups of 2-ATDs. In order for this approach to be practical, two issues must be resolved:

First, one must get some control over $|G|$ and thus over $|G_v|$. (Recall that in $\vec{\mathrm{W}}(n, r)$, $|G_v|$ can grow exponentially with $|\mathrm{V}(\vec{\mathrm{W}}(n, r))|$, as $n \to \infty$ and r is fixed). Second, one must obtain enough structural information about G to be able to construct all possibilities.

Fortunately, both of these issues were resolved successfully. The problem of bounding $|G_v|$ was resolved in a recent paper [37] and details can be found in Section 3.1. The second problem was dealt with in [19], and later, in greater generality in [30] (both of these papers rely heavily on a group-theoretical result of Glauberman [12]); the summary of relevant results is given in Section 3.2.

3.1 Bounding the order of the vertex-stabiliser

The crucial result that made our compilation of a complete census of all small 2-ATDs possible is Theorem 3.1, stated below, which shows that the generalised wreath digraphs (defined in Section 2.3) are very special in the sense of having large vertex-stabilisers. In fact, together with the correspondence described in Section 2.2, [37, Theorem 9.2] has the following corollary:

Theorem 3.1. *Let Γ be a G-arc-transitive 2-ATD on at most m vertices and let t be the largest integer such that $m > t2^{t+2}$. Then one of the following occurs:*

1. *$\Gamma \cong \vec{\mathrm{W}}(n, r)$ for some $n \geq 3$ and $1 \leq r \leq n - 1$,*
2. *$|G_v| \leq \max\{16, 2^t\}$,*
3. *(Γ, G) appears in the last line of [37, Table 1]. In particular, $|\mathrm{V}(\Gamma)| = 8100$.*

The following is an easy corollary:

Corollary 3.2. *Let Γ be a G-arc-transitive 2-ATD on at most 1000 vertices. Then either $|G_v| \leq 32$ or $\Gamma \cong \vec{\mathrm{W}}(n, r)$ for some $n \geq 3$ and $1 \leq r \leq n - 1$.*

3.2 Structure of the vertex-stabiliser

Definition 3.3. Let s and α be positive integers satisfying $\frac{2}{3}s \le \alpha \le s$, and let c be a function assigning a value $c_{i,j} \in \{0,1\}$ to each pair of integers i, j with $\alpha \le j \le s-1$ and $1 \le i \le 2\alpha - 2s + j + 1$. Let $A^c_{s,\alpha}$ be the group generated by $\{x_0, x_1, \ldots, x_{s-1}, g\}$ and subject to the defining relations:

- $x_0^2 = x_1^2 = \cdots = x_{s-1}^2 = 1$;
- $x_i^g = x_{i+1}$ for $i \in \{0, 1, \ldots, s-2\}$;
- if $j < \alpha$, then $[x_0, x_j] = 1$;
- if $j \ge \alpha$, then $[x_0, x_j] = x_{s-\alpha}^{c_{1,j}} x_{s-\alpha+1}^{c_{2,j}} \cdots x_{j-s+\alpha}^{c_{2\alpha-2s+j+1,j}}$.

Furthermore, let $\mathcal{A}_{s,\alpha}$ be the family of all groups $A^c_{s,\alpha}$ for some c. It was proved in [19] (see also [30]) that every group G acting arc-transitively on a 2-ATD is isomorphic to a quotient of some $A^c_{s,\alpha}$. More precisely, the following can be deduced from [19] or [30].

Theorem 3.4. *Let* Γ *be a* G*-arc-transitive* 2*-ATD, let* $v \in \mathrm{V}(\Gamma)$ *and let* s *be the largest integer such that* G *acts transitively on the set of* s*-arcs of* Γ*. Then there exists an integer* α *satisfying* $\frac{2}{3}s \le \alpha \le s$*, a function* c *as in Definition 3.3, and an epimorphism* $\wp\colon A^c_{s,\alpha} \to G$*, which maps the group* $\langle x_0, \ldots, x_{s-1}\rangle$ *isomorphically onto* G_v *and the generator* g *to some* v*-shunt in* G*. In particular,* $|G_v| = 2^s$*.*

In this case, we will say that (Γ, G) is of *type* $A^c_{s,\alpha}$, and call the group $A^c_{s,\alpha}$ the *universal group* of the pair (Γ, G).

For s, α, and a function c satisfying the conditions of Definition 3.3, let c' be the function defined by $c'_{i,j} = c_{2\alpha-2s+j+2-i,j}$. The relationship between c and c' can be visualised as follows: if one fixes the index j and views the function $i \mapsto c_{i,j}$ as the sequence $[c_{1,j}, c_{2,j}, \ldots, c_{2\alpha-2s+j+1,j}]$, then the sequence for c' is obtained by reversing the one for c. If $\tilde{G} = A^c_{s,\alpha}$ then we denote the *opposite type* $A^{c'}_{s,\alpha}$ by $\tilde{G}^{\mathrm{opp}}$.

Observe that if (Γ, G) is of type $\tilde{G}$, then $(\Gamma^{\mathrm{opp}}, G)$ is of type $\tilde{G}^{\mathrm{opp}}$. A class of groups, obtained from $\mathcal{A}_{s,\alpha}$ by taking only one group in each pair $\{\tilde{G}, \tilde{G}^{\mathrm{opp}}\}$, $\tilde{G} \in \mathcal{A}_{s,\alpha}$, will be denoted $\mathcal{A}^{\mathrm{red}}_{s,\alpha}$. (Note that some groups $\tilde{G}$ might have the property that $\tilde{G} = \tilde{G}^{\mathrm{opp}}$.)

In view of Corollary 3.2, we shall be mainly interested in the universal groups $A^c_{s,\alpha}$ with $s \le 5$ (as, excluding generalised wreath digraphs, these are the only types of 2-ATDs of order at most 1000). We list the relevant classes $\mathcal{A}^{\mathrm{red}}_{s,\alpha}$ for $s \le 5$ explicitly in Table 1. Groups in $\mathcal{A}^{\mathrm{red}}_{s,\alpha}$, for a fixed s will be named by A^i_s, where i will be a positive integer, where groups with larger α will be indexed with lower i. Also, the generators x_0, x_1, x_2, x_3, and x_4 will be denoted a, b, c, d, and e, respectively.

3.3 The algorithm and its implementation

We now have all the tools required to present a practical algorithm that takes an integer m as input and returns a complete list of all 2-ATDs on at most m vertices (see Algorithm 1). It is based on the fact that every such digraph can be obtained as a coset digraph of some group G (see Lemma 2.1), and that G is in fact an epimorphic image of some group $A^c_{s,\alpha}$ (see Theorem 3.4) with G_v and the shunt being the corresponding images of $\langle x_0, \ldots, x_{s-1}\rangle$ and g in $A^c_{s,\alpha}$.

Moreover, if Γ is not a generalised wreath digraph or the exceptional digraph on 8100 vertices mentioned in part 3 of Theorem 3.1, then the parameter s satisfies $s2^{s+2} < m$, and

name	$\tilde{G}$
A_1^1	$\langle a, g \mid a^2 \rangle$
A_2^1	$\langle a, b, g \mid a^2, b^2, a^g b, [a, b] \rangle$
A_3^1	$\langle a, b, c, g \mid a^2, b^2, c^2, a^g b, b^g c, [a, b], [a, c] \rangle$
A_3^2	$\langle a, b, c, g \mid a^2, b^2, c^2, a^g b, b^g c, [a, b], [a, c]b \rangle$
A_4^1	$\langle a, b, c, d, g \mid a^2, b^2, c^2, d^2, a^g b, b^g c, c^g d, [a, b], [a, c], [a, d] \rangle$
A_4^2	$\langle a, b, c, d, g \mid a^2, b^2, c^2, d^2, a^g b, b^g c, c^g d, [a, b], [a, c], [a, d]b \rangle$
A_4^3	$\langle a, b, c, d, g \mid a^2, b^2, c^2, d^2, a^g b, b^g c, c^g d, [a, b], [a, c], [a, d]bc \rangle$
A_5^1	$\langle a, b, c, d, e, g \mid a^2, b^2, c^2, d^2, e^2, d^2, a^g b, b^g c, c^g d, d^g e, [a, b], [a, c], [a, d], [a, e] \rangle$
A_5^2	$\langle a, b, c, d, e, g \mid a^2, b^2, c^2, d^2, e^2, d^2, a^g b, b^g c, c^g d, d^g e, [a, b], [a, c], [a, d], [a, e]b \rangle$
A_5^3	$\langle a, b, c, d, e, g \mid a^2, b^2, c^2, d^2, e^2, d^2, a^g b, b^g c, c^g d, d^g e, [a, b], [a, c], [a, d], [a, e]c \rangle$
A_5^4	$\langle a, b, c, d, e, g \mid a^2, b^2, c^2, d^2, e^2, d^2, a^g b, b^g c, c^g d, d^g e, [a, b], [a, c], [a, d], [a, e]bc \rangle$
A_5^5	$\langle a, b, c, d, e, g \mid a^2, b^2, c^2, d^2, e^2, d^2, a^g b, b^g c, c^g d, d^g e, [a, b], [a, c], [a, d], [a, e]bd \rangle$
A_5^6	$\langle a, b, c, d, e, g \mid a^2, b^2, c^2, d^2, e^2, d^2, a^g b, b^g c, c^g d, d^g e, [a, b], [a, c], [a, d], [a, e]bcd \rangle$

Table 1: Universal groups of 2-ATDs with $|\tilde{G}_v| \leq 32$

the order of the epimorphic image G is bounded by $2^s m$ (see Theorem 3.1). The algorithm thus basically boils down to the task of finding normal subgroups of bounded index in the finitely presented groups $A_{s,\alpha}^c$.

Practical implementations of this algorithm have several limitations. First, the best known algorithm for finding normal subgroups of low index in a finitely presented group is an algorithm due to Firth and Holt [11]. The only publicly available implementation is the `LowIndexNormalSubgroups` routine in MAGMA [5] and the most recent version allows one to compute only the normal subgroups of index at most $5 \cdot 10^5$; hence only automorphisms groups of order $5 \cdot 10^5$ can possibly be obtained in this way.

More importantly, even when only normal subgroups of relatively small index need to be computed, some finitely presented groups are computationally difficult. For example, finding all normal subgroups of index at most 2048 of the group $A_1^1 \cong C_2 * C_\infty$ seems to represent a considerable challenge for the `LowIndexNormalSubgroups` routine in MAGMA. In order to overcome this problem, we have used a recently computed catalogue of all $(2,*)$-groups of order at most 6000 [29], where by a $(2,*)$-*group* we mean any group generated by an involution x and one other element g. Since A_1^1 is a $(2,*)$-group and every non-cyclic quotient of a $(2,*)$-group is also a $(2,*)$-group, this catalogue can be used to obtain all the quotients of A_1^1 of order up to 6000. Consequently, all 2-ATDs admitting an arc-regular group of automorphisms of order at most 3000 can be obtained. Similarly, since A_2^1 is also a $(2,*)$-group, we can use this catalogue to obtain all the 2-ATDs of order at most 1500 admitting an arc-transitive group G with $|G_v| = 4$.

It should be mentioned that the concept of a $(2,*)$-group is equivalent to that of a *rotary map*, that can be described as groups generated by two elements the product of which is

Algorithm 1 2-ATDs on at most m vertices.

Require: positive integer m
Ensure: $\mathcal{D} = \{\Gamma : \Gamma \text{ is 2-ATD}, |V(\Gamma)| \leq m\}$

```
Let t be the largest integer such that m > t2^{t+2};
Let D be the list of all arc-transitive generalised wreath digraphs on at most m vertices;
If m ≥ 8100, add to D the exceptional digraph Γ on 8100 vertices, mentioned in part 3
of Theorem 3.1;
for s ∈ {1, ..., max{4, t}} do
  for α ∈ {⌈(2/3)s⌉, ⌈(2/3)s⌉ + 1, ..., s} do
    for G̃ ∈ A^red_{s,α} do
      Let 𝒩 be the set of all normal subgroups of G̃ of index at most 2^s m;
      for N ∈ 𝒩 do
        Let G := G̃/N and let ℘: G̃ → G be the quotient projection;
        Let H := ℘(⟨x_0, ..., x_{s−1}⟩);
        if H is core-free in G and |H| = 2^s and ℘(g)^{−1} ∉ H℘(g)H then
          Let C := cos(G, H, ℘);
          for Γ ∈ {C, C^opp} do
            if Γ is not isomorphism to any of the digraphs in D then
              add Γ to the list D;
            end if
          end for
        end if
      end for
    end for
  end for
end for
```

an involution. A catalogue of all $(2,*)$-groups of order at most 2000 could thus be derived from Conder's catalogue of rotary maps with at most 1000 edges [9]. Conversely, the catalogue of $(2,*)$-group of order up to 6000 [29] extends the list in [9] up to 3000 edges in the orientable case and to 1500 in the non-orientable case.

Like A_1^1 and A_2^1, the groups with $\langle x_0, \ldots, x_{s-1}\rangle$ abelian (namely those with $\alpha = s$ and $c_{i,j} = 0$ for all i, j) are also computationally very difficult. One can make the task easier by dividing it into cases, where the order of g is fixed in each case. Since g represents a shunt, it can be proved that its order cannot exceed the order of the digraph (see, for example, [27, Lemma 13]). Cases can then be run in parallel on a multi-core computer.

4 The census and accompanying data

Using Algorithm 1, we found that there are exactly 26457 2-ATDs of order up to 1000. Following the recipe explained in Section 2.2, we have also computed all the 4-GHATs, which we split in two lists: 4-HATs and arc-transitive 4-GHATs.

The data about these graphs, together with MAGMA code that generates them, is available on-line at [28]. The package contains ten files. The file "Census-ATD-1k-README.txt" is a text file containing information similar to the information in this section. The remaining nine files come in groups of three, one group for each of the three lists (2-ATDs, arc-transitive 4-GHATs, 4-HATs). In each group, there is a $*$.mgm file, a $*$.txt file and a $*$.csv file.

The $*$.mgm file contains MAGMA code that generates the corresponding digraphs. After loading the file in MAGMA, a double sequence is generated (named either ATD, GHAT, or HAT, depending on the file). The length of each double sequence is 1000 and the n-th component of the sequence is the sequence of all the corresponding digraphs of order n, with the exception of the generalised wreath digraphs. Thus, ATD[32,2] will return the second of the four non-generalised-wreath 2-ATDs on 32 vertices (the ordering of the digraphs in the sequence ATD[32] is arbitrary). In order to include the generalised wreath digraphs into the corresponding sequence, one can call the procedure `AddGWD(~ATD,GWD)` in the case of the 2-ATDs, or `AddGWG(~GHAT,GWG)` in the case of the 4-GHATs (note that a generalised wreath graph is never $\frac{1}{2}$-arc-transitive).

The $*$.txt file contains the list of neighbours of each digraph. This file is needed when the $*$.mgm file is loaded into MAGMA, but, being an ASCII file, it can be used also by other computer systems to reconstruct the digraphs. For the details of the format, see the "README" file.

Finally, the $*$.csv file is a "comma separated values" file representing a spreadsheet containing some precomputed graph invariants. We shall first introduce some of these invariants and then discuss each $*$.csv separately.

4.1 Walks and cycles

Let Γ be a digraph. A *walk* of length n in Γ is an $(n+1)$-tuple $(v_0, v_1, \ldots, v_n)$ of vertices of Γ such that, for any $i \in \{1, \ldots n\}$, either (v_{i-1}, v_i) or (v_i, v_{i-1}) is an arc of Γ. The walk is *closed* if $v_0 = v_n$ and *simple* if the vertices v_i are pairwise distinct (with the possible exception of the first and the last vertex when the walk is closed).

A closed simple walk in Γ is called a *cyclet*. The *inverse* of a cyclet $(v_0, \ldots, v_{n-1}, v_0)$ is the cyclet $(v_0, v_{n-1}, \ldots, v_1, v_0)$, and a cyclet $(v_0, \ldots, v_{n-1}, v_0)$ is said to be a *shift* of a cyclet $(u_0, \ldots, u_{n-1}, u_0)$ provided that there exists $k \in \mathbb{Z}_n$ such that $u_i = v_{i+k}$ for all

$i \in \mathbb{Z}_n$. Two cyclets W and U are said to be *congruent* provided that W is a shift of either U or the inverse of U. The relations of "being a shift of" and "being congruent to" are clearly equivalence relations, and their equivalence classes are called *oriented cycles* and *cycles*, respectively. With a slight abuse of terminology, we shall sometimes identify a (oriented) cycle with any of its representatives.

4.2 Alter-equivalence, alter-exponent, alter-perimeter, and alter-sequence

Let Γ be an asymmetric digraph. The *signature* of a walk $W = (v_0, v_1, \ldots, v_n)$ is an n-tuple $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, where $\epsilon_i = 1$ if (v_{i-1}, v_i) is an arc of Γ, and $\epsilon_i = -1$ otherwise. The signature of a walk W will be denoted by $\sigma(W)$. The sum of all the integers in $\sigma(W)$ is called the *sum* of the walk W and denoted by $s(W)$; similarly, the k^{th} *partial sum* $s_k(W)$ is the sum of the initial walk $(v_0, v_1, \ldots, v_k)$ of length k. By convention, we let $s_0(W) = 0$.

The *tolerance* of a walk W of length n, denoted $T(W)$, is the set $\{s_k(W) : k \in \{0, 1, \ldots, n\}\}$. Observe that the tolerance of a walk is always an interval of integers containing 0. Let t be a positive integer or ∞. We say that two vertices u and v of Γ are *alter-equivalent with tolerance* t if there is a walk from u to v with sum 0 and tolerance contained in $[0, t]$; we shall then write $u\mathcal{A}_t v$. The equivalence class of $\mathcal{A}_t$ containing a vertex v will be denoted by $\mathcal{A}_t(v)$.

Since we assume that Γ is a finite digraph, there exists an integer $e \geq 0$ such that $\mathcal{A}_e = \mathcal{A}_{e+1}$ (and then $\mathcal{A}_e = \mathcal{A}_\infty$). The smallest such integer is called the *alter-exponent* of Γ and denoted by $\exp(\Gamma)$.

The number of equivalence classes of $\mathcal{A}_\infty$ is called the *alter-perimeter* of Γ. The name originates from the fact that the quotient digraph of Γ with respect to $\mathcal{A}_\infty$ is either a directed cycle or the complete graph K_2 or the graph K_1 with one vertex.

If e is the alter-exponent of a (vertex-transitive) digraph Γ, then the finite sequence $[|\mathcal{A}_1(v)|, |\mathcal{A}_2(v)|, \ldots, |\mathcal{A}_e(v)|]$ is called the *alter-sequence* of Γ.

Several interesting properties of the alter-exponent can be proved (see [20] for example). For example, if Γ is connected and G-vertex-transitive, then $\exp(\Gamma)$ is the smallest positive integer e such that the setwise stabiliser $G_{\mathcal{A}_e(v)}$ is normal in G. The group $G_{\mathcal{A}_e(v)}$ is the group generated by all vertex-stabilisers in G and $G/G_{\mathcal{A}_e(v)}$ is a cyclic group.

All notions defined in this section for digraphs generalise to half-arc-transitive graphs, where instead of the graph one of the two natural arc-transitive digraphs are considered. As was shown in [20], all the parameters defined here remain the same if instead of a digraph, its opposite digraph is considered. The notions defined in this section were later generalised in the context of infinite digraphs [14].

4.3 Alternating cycles – radius and attachment number

A walk W in an asymmetric digraph is called *alternating* if its tolerance is either $[0, 1]$ or $[-1, 0]$ (that is, if the signs in its signature alternate). Similarly, a cycle is called *alternating* provided that any (and thus every) of its representatives is an alternating walk.

This notion was introduced in [17] and used to classify the so-called *tightly attached* 4-GHATs and 4-HATs of odd radius. The concept of alternating cycles was explored further in a number of papers on 4-HATs (see for example [21, 34]).

Let Γ be a 2-ATD, let $\mathcal{C}$ be the set of all alternating cycles of Γ, and let $G = \mathrm{Aut}(\Gamma)$. The set $\mathcal{C}$ is clearly preserved by the action of G upon the cycles of Γ. Moreover, since Γ is arc-transitive, G acts transitively on $\mathcal{C}$. In particular, all the alternating cycles of Γ are of equal length. Half of the length of an alternating cycle is called the *radius* of Γ.

Since Γ is 2-valent, every vertex of Γ belongs to precisely two alternating cycles. It thus follows from vertex-transitivity of Γ that any (unordered) pair of intersecting cycles can be mapped to any other such pair, implying that there exists a constant a such that any two cycles meet either in 0 or in a vertices. The parameter a is then called the *attachment number* of Γ. In general, the attachment number divides the length of the alternating cycle (twice the radius), and there are digraphs where a equals this length; they were classified in [17, Proposition 2.4], where it was shown that their underlying graphs are always arc-transitive. A 2-valent asymmetric digraph with attachment number a is called *tightly attached* if a equals the radius, is called *antipodally attached* if $a = 2$, and is called *loosely attached* if $a = 1$. Note that tightly attached 2-ATDs are precisely those with alter-exponent 1.

4.4 Consistent cycles

Let Γ be a graph and let $G \leq \mathrm{Aut}(\Gamma)$. A (oriented) cycle C in a graph Γ is called G-consistent provided that there exists $g \in G$ that preserves C and acts upon it as a 1-step rotation. A G-orbit of G-consistent oriented cycles is said to be *symmetric* if it contains the inverse of any (and thus each) of its members, and is *chiral* otherwise.

Consistent oriented cycles were first introduced by Conway in a public lecture [10] (see also [3, 25, 26]). Conway's original result states that in an arc-transitive graph of valence d, the automorphism group of the graph has exactly $d - 1$ orbits on the set of oriented cycles. In particular, if Γ is 4-valent and G-arc-transitive, then there are precisely three G-orbits of G-consistent oriented cycles. Since chiral orbits of G-consistent cycles come in pairs of mutually inverse oriented cycles, this implies that there must be at least one symmetric orbit, while the other two are either both chiral or both symmetric.

Conway's result was generalised in [4] to the case of $\frac{1}{2}$-arc-transitive graphs by showing that if Γ is a 4-valent $(G, \frac{1}{2})$-arc-transitive graph, then there are precisely four G-orbits of G-consistent oriented cycles, all of them chiral. These four orbits of oriented cycles thus constitute precisely two G-orbits of G-consistent (non-oriented) cycles.

4.5 Metacirculants

A *weak metacirculant* is a graph whose automorphism group contains a vertex-transitive metacyclic group G, generated by ρ and σ, such that the cyclic group $\langle \rho \rangle$ is semiregular on the vertex-set of the graph, and is normal in G. This notion was introduced by Marušič and Šparl [22] and generalises that of a *metacirculant* introduced by Alspach and Parsons [2]. Metacirculants admitting $\frac{1}{2}$-arc-transitive groups of automorphisms were first investigated in [33]. Recently, the interesting problem of classifying all 4-HATs that are weak metacirculants was considered in [22, 35, 36]. Such 4-HATs fall into four (not necessarily disjoint) classes (called Class I, Class II, Class III, and Class IV), depending on the structure of the quotient by the orbits of the semiregular element ρ. For a precise definition of the *class* of a 4-HAT weak metacirculant see, for example, [35, Section 2]. Since a given 4-HAT may admit several vertex-transitive metacyclic groups, a fixed graph can fall into several of these four classes. Several interesting facts about 4-HAT (weak) metacirculants are known. For example, tightly attached 4-HATs are precisely the 4-HATs that are weak metacirculants of Class I.

4.6 The data on 2-ATDs

The "Census-ATD-1k-data.csv" file concerns 2-ATDs. Each line of the file represents one of the digraphs in the census, and has 19 fields described below. Since this file is in "csv" format, every occurrence of a comma in a field is substituted with a semicolon.

- `Name`: the name of the digraph (for example, ATD[32,2]);
- |`V`|: the order of the digraph;
- `SelfOpp`: contains "yes" if the digraph is isomorphic to its opposite digraph and "no" otherwise;
- `Opp`: the name of the opposite digraph (the same as "Name" if the digraph is self-opposite);
- `IsUndAT`: "yes" if the underlying graph is arc-transitive, "no" otherwise;
- `UndGrph`: the name of the underlying graph, as given in the files "Census-HAT-1k-data.csv" and "Census-GHAT-1k-data.csv" – if the underlying graph is generalized wreath, then this is indicated by, say, "GWD(m,k)" where m and k are the defining parameters.
- `s`: the largest integer s, such that the digraph is s-arc-transitive;
- `GvAb`: "Ab" if the vertex-stabiliser in the automorphism group of the digraph is abelian, otherwise "n-Ab";
- |`Tv:Gv`|: the index of the automorphism group G of the digraph in the smallest arc-transitive group T of the underlying graph that contains G – if there's no such group T, then 0;
- |`Av:Gv`|: the index of the automorphism group of the digraph in the automorphism group of the underlying graph;
- `Solv`: this field contains "solve" if the automorphism group of the digraph is solvable and "n-solv" otherwise;
- `Rad`: the *radius*, that is, half of the length of an alternating cycle;
- `AtNo`: the *attachment number*, that is, the size of the intersection of two intersecting alternating cycles;
- `AtTy`: the *attachment type*, that is: "loose" if the attachment number is 1, "antipodal" if 2, and "tight" if equal to the radius, otherwise "—";
- |`AltCyc`|: the number of alternating cycles;
- `AltExp`: the alter-exponent;
- `AltPer`: the alter-perimeter;
- `AltSeq`: the alter-sequence;
- `IsGWD`: "yes" if the digraph is generalized wreath, and "no" otherwise.

4.7 The data on arc-transitive 4-GHATs

The "Census-GHAT-1k-data.csv" file concerns arc-transitive 4-GHATs . Each line of the file represents one of the graphs in the census, and has nine fields, described below. Note, however, that the file does not contain the generalised wreath graphs.

- `Name`: the name of the graph (for example GHAT[9,1]);
- $|\mathtt{V}|$: the order of the graph;
- `gir`: the girth (length of a shortest cycle) of the graph;
- `bip`: this field contains "b" if the graph is bipartite and "nb" otherwise;
- `CayTy`: this field contains "Circ" if the graph is a circulant (that is, a Cayley graph on a cyclic group), "AbCay" if the graph is Cayley graph on an abelian group, but not a circulant, and "Cay" if it is a Cayley but not on an abelian group – it contains "n-Cay" otherwise;
- $|A_v|$: the order of the vertex-stabiliser in the automorphism group of the graph;
- $|G_v|$: a sequence of the orders of vertex-stabilisers of the maximal half-arc-transitive subgroups of the automorphism group – up to conjugacy in the automorphism group;
- `solv`: this field contains "solve" if the automorphism group of the graph is solvable and "n-solv" otherwise;
- $[|\mathtt{ConCyc}|]$: the sequence of the lengths of A-consistent oriented cycles of the graph (one cycle per each A-orbit, where A is the automorphism group of the graph) – the symbols "c" and "s" indicate whether the corresponding cycle is chiral or symmetric – for example, $[4c; 4c; 10s]$ means there are two chiral orbits of A-consistent cycles, both containing cycles of length 4, and one orbit of symmetric consistent cycles, containing cycles of length 10.

4.8 The data on 4-HATs

The "Census-HAT-1k-data.csv" file concerns 4-HATs. Each line of the file represents one of the graphs in the census, and has 16 fields. The fileds $|\mathtt{V}|$, `gir`, `bip`, and `Solv` are as in Section 4.7, and the fields `Rad, AtNo, AtTy, AltExp, AltPer` and `AltSeq` are as in Section 4.6. The remaining fileds as follows:

- `Name`: the name of the graph (for example HAT[27,1]);
- `IsCay`: this field contains "Cay" if the graph is Cayley and "n-Cay" otherwise;
- $|G_v|$: the order of the vertex-stabiliser in the automorphism group of the graph;
- `CCa`: the length of a shortest consistent cycle;
- `CCb`: the length of a longest consistent cycle;
- `MetaCircCl`: "{}" if the graph is not a meta-circulant; otherwise a set of classes of meta-circulants that represents the graph.

References

[1] B. Alspach, D. Marušič, L. Nowitz, Constructing graphs which are $\frac{1}{2}$-transitive, *J. Austral. Math. Soc. Ser. A* **56** (1994), 391–402.

[2] B. Alspach, T. D. Parsons, A construction for vertex-transitive graphs *Canad. J. Math.* **34** (1982), 307–318.

[3] N. Biggs, *Aspects of symmetry in graphs*, Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), pp. 27–35, Colloq. Math. Soc. János Bolyai, 25, North-Holland, Amsterdam-New York, 1981.

[4] M. Boben, Š. Miklavič, P. Potočnik, Consistent cycles in $\frac{1}{2}$-arc-transitive graphs, *Electron. J. Combin.* **16** (2009), # R5, 1–10.

[5] W. Bosma, J. Cannon and C. Playoust, The MAGMA Algebra System I: The User Language, *J. Symbolic Comput.* **24** (1997), 235–265.

[6] I. Z. Bouwer, Vertex and edge-transitive but not 1-transitive graphs, *Canad. Math. Bull.* **13** (1970), 231–237.

[7] M. D. E. Conder, D. Marušič, A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer, *J. Combin. Theory Ser. B* **88** (2003) 67–76.

[8] M. D. E. Conder, P. Potočnik, P. Šparl, Some recent discoveries about half-arc-transitive graphs, *Ars. Math. Contemp.* **8** (2005)

[9] M. D. E. Conder, All rotary maps on closed surfaces with up to 1000 edges, `https://www.math.auckland.ac.nz/~conder/`https://www.math.auckland.ac.nz/∼conder/, accessed October 2013.

[10] J. H. Conway, Talk given at the Second British Combinatorial Conference at Royal Holloway College, Egham, 1971.

[11] D. Firth, *An algorithm to find normal subgroups of a finitely presented group up to a given index*, PhD Thesis, University of Warwick, 2005.

[12] G. Glauberman, Isomorphic subgroups of finite p-subgroups, *Canad. J. Math.* **23** (1971), 983–1022.

[13] K. Kutnar, D. Marušič, P. Šparl, R.-J. Wang, M.-Y. Xu, Classification of half-arc-transitive graphs of order $4p$, *European J. Combin.* **34** (2013), 1158–1176.

[14] A. Malnič, D. Marušič, N. Seifter, P. Šparl, B. Zgrablić, Reachability relations in digraphs, *Europ. J. Combinatorics* **29** (2008), 1566–1581.

[15] A. Malnič, D. Marušič, Constructing 4-valent $\frac{1}{2}$-Transitive Graphs with a Nonsolvable Automorphism Group, *J. Combin. Theory Ser. B* **75** (1999), 46–55.

[16] D. Marušič, Recent developments in half-transitive graphs, *Discrete Math.* **182** (1998), 219–231.

[17] D. Marušič, Half-transitive group actions on finite graphs of valency 4, *J. Combin. Theory Ser. B* **73** (1998), 41–76.

[18] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.* **299** (2005), 180–193.

[19] D. Marušič, R. Nedela, On the point stabilizers of transitive groups with non-self-paired suborbits of length 2, *J. Group Theory* **4** (2001), 19–43.

[20] D. Marušič, P. Potočnik, Bridging Semisymmetric and Half-Arc-Transitive Actions on Graphs, *European J. Combin.* **23** (2002), 719–732.

[21] D. Marušič, C. E. Praeger, Tetravalent graphs admitting half-transitive group actions: alternating cycles, *J. Combin. Theory Ser. B* **75** (1999), 188–205.

[22] D. Marušič, P. Šparl, On quartic half-arc-transitive metacirculants, *J. Algebraic Combin.* **28** (2008), 365–395.

[23] D. Marušič, A. Orbanić, P. Šparl, A list of $\frac{1}{2}$-arc-transitive tetravalent graphs with the vertex-stabiliser of order 2, unpublished work, available upon request from P. Šparl, `primoz.sparl@pef.uni-lj.si`.

[24] D. Marušič, M.-Y. Xu, $\frac{1}{2}$-transitive graph of valency 4 with a nonsolvable group of automorphisms, *J. Graph Theory* **25** (1997), 133–138.

[25] Š. Miklavič, A note on a conjecture on consistent cycles, *Ars Math. Contemp.* **6** (2013), 389–392.

[26] Š. Miklavič, P. Potočnik, S. Wilson, Overlap in consistent cycles, *J. Graph Theory* **55** (2007), 55–71.

[27] P. Potočnik, P. Spiga, G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *J. Symbolic Comput.* **50** (2013), 465–477.

[28] P. Potočnik, P. Spiga, G. Verret, Census of 2-valent arc-transitive digraphs, `http://www.fmf.uni-lj.si/~potocnik/work.htm`, accessed October 2013.

[29] P. Potočnik, P. Spiga, G. Verret, Census of $(2,*)$-groups, `http://www.fmf.uni-lj.si/~potocnik/work.htm`, accessed October 2013.

[30] P. Potočnik, G. Verret, On the vertex-stabiliser in arc-transitive digraphs, *J. Combin. Theory Ser. B.* **100** (2010), 497–509.

[31] P. Potočnik, S. Wilson, Census of tetravalent edge-transitive graphs, `https://jan.ucc.nau.edu/~swilson/C4Site/BigTable.html`, accessed October 2013.

[32] C. E. Praeger, Highly arc transitive digraphs, *European J. Combin.* **10** (1989), 281–292.

[33] M. Šajna, Half-transitivity of some metacirculants, *Discrete Math.* **185** (1998), 117–136.

[34] P. Šparl, A classification of tightly attached half-arc-transitive graphs of valency 4, *J. Combin. Theory Ser. B* **98** (2008), 1076–1108.

[35] P. Šparl, On the classification of quartic half-arc-transitive metacirculants, *Discrete Math.* **309** (2009), 2271–2283.

[36] P. Šparl, Almost all quartic half-arc-transitive weak metacirculants of Class II are of Class IV, *Discrete Math.* **310** (2010), 1737–1742.

[37] P. Spiga, G. Verret, On the order of vertex-stabilisers in vertex-transitive graphs with local group $\mathrm{C}_p \times \mathrm{C}_p$ or $\mathrm{C}_p \wr \mathrm{C}_2$, `http://arxiv.org/pdf/1311.4308.pdf`.

[38] W. T. Tutte, *Connectivity in Graphs*, University of Toronto Press, Toronto (1966).

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Some recent discoveries about half-arc-transitive graphs

Dedicated to Dragan Marušič on the occasion of his 60th birthday

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Abstract

We present some new discoveries about graphs that are half-arc-transitive (that is, vertex- and edge-transitive but not arc-transitive). These include the recent discovery of the smallest half-arc-transitive 4-valent graph with vertex-stabiliser of order 4, and the smallest

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with vertex-stabiliser of order 8, two new half-arc-transitive 4-valent graphs with dihedral vertex-stabiliser D_4 (of order 8), and the first known half-arc-transitive 4-valent graph with vertex-stabiliser that is neither abelian nor dihedral. We also use half-arc-transitive group actions to provide an answer to a recent question of Delorme about 2-arc-transitive digraphs that are not isomorphic to their reverse.

Keywords: Graph, edge-transitive, vertex-transitive, half-arc-transitive.

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1 Introduction

A graph X is said to be *vertex-transitive*, *edge-transitive*, or *arc-transitive*, if its automorphism group $\mathrm{Aut}\, X$ acts transitively on the set $V(X)$ of all vertices of X, the set $E(X)$ of all edges of X, or the set $A(X)$ of all arcs (ordered pairs of adjacent vertices) of X, respectively. An arc-transitive graph is also called *symmetric*. The graph X is called *half-arc-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive.

(A related class of graphs consists of those which are regular and edge-transitive but not vertex-transitive. Any such graph is called *semi-symmetric*. Semi-symmetric 3-valent graphs will be the topic of another forthcoming paper by the first two authors.)

Every half-arc-transitive graph X is regular, with even valency — indeed $\mathrm{Aut}\, X$ has two orbits on arcs, with half the arcs emanating from any vertex v lying in each orbit. The smallest possible valency of a half-arc-transitive graph is 4, and this is the valency of the smallest half-arc-transitive graph, a graph on 27 vertices constructed independently by Doyle [7] and Holt [10].

Furthermore, examples exists for every even valency greater than 2. This was proved (in answer to a question by Tutte [21]) by Bouwer [2], who constructed a family of examples of valency $2k$ for all $k \geq 2$, with the property that the stabiliser in $\mathrm{Aut}\, X$ of every vertex v induces the symmetric group S_k on the neighbourhood of v, acting with two orbits of length k. (The latter property was not explicitly mentioned by Bouwer, but may be easily deduced from his construction.)

For valency greater than 4, the vertex-stabilisers in Bouwer's examples are non-abelian. In contrast, for quite some time all known examples of 4-valent half-arc-transitive graphs had vertex-stabilisers that are abelian, or more precisely, elementary abelian 2-groups.

The first known example of a 4-valent half-arc-transitive graph with non-abelian vertex-stabiliser was found in 1999 by Conder and Marušič, who produced an example of order 10752 from a transitive permutation group of degree 32 and order 86016, with dihedral point stabiliser (of order 8) having a non-self-paired sub-orbit; see [4]. Until recently, however, this was the only known example, and no examples were known of 4-valent half-arc-transitive graphs with vertex-stabiliser that is neither abelian nor dihedral.

In this paper, we exhibit a number of new examples with particular properties.

In Section 3 we give the smallest half-arc-transitive 4-valent graph (on 256 vertices) with vertex-stabiliser of order 4, and also the two smallest half-arc-transitive 4-valent graphs (on 768 vertices) with vertex-stabiliser of order 8. The fact that the former has

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order 256 disproves the conjecture by Feng, Kwak, Xu and Zhou [8] that every 4-valent half-arc-transitive graph of prime-power order has vertex stabilisers of order 2. Then in Section 4 we describe two new half-arc-transitive 4-valent graphs with vertex-stabiliser D_4 (of order 8), and in Section 5 we produce the first known half-arc-transitive 4-valent graph with vertex-stabiliser that is neither abelian nor dihedral. (In this last example, the vertex-stabiliser is isomorphic to $D_4 \times C_2$, of order 16.)

In Section 6 we use half-arc-transitive group actions to provide an answer to a recent question of Charles Delorme [5] about 2-arc-transitive digraphs that are not isomorphic to their reverse. The *reverse* $\mathrm{Rev}(D)$ of a digraph D is obtained by reversing all the arcs of D, and then D is called *self-reverse* if $\mathrm{Rev}(D)$ is isomorphic to D. Also a 2-arc in a digraph D is a directed walk (v_0, v_1, v_2) of length 2 on three vertices, and then D is called arc-transitive or 2-arc-transitive if its automorphism group $\mathrm{Aut}\, D$ is transitive on the set of all arcs in D or the set of all 2-arcs in D, respectively. In [5, §5.2], Delorme asked this question: *Do finite digraphs exist that are vertex-transitive, arc-transitive and 2-arc-transitive, but not self-reverse?* We answer this question in the affirmative. Finally, we make some concluding remarks and pose some further questions in Section 7.

2 Further background

Before proceeding, we provide some more background information. Throughout this paper, graphs are assumed to be finite and simple, and unless otherwise specified, also undirected and connected.

For any graph (or digraph) X, we let $V(X)$, $E(X)$ and $A(X)$ be the vertex-set, edge-set, and arc-set of X, respectively, and we let $\mathrm{Aut}\, X$ be the automorphism group of X. As mentioned earlier, we say that X is *vertex-transitive*, *edge-transitive*, or *arc-transitive*, if $\mathrm{Aut}\, X$ is transitive on $V(X)$, $E(X)$ or $A(X)$, respectively, and we say that the graph X is *half-arc-transitive* if it is vertex- and edge-transitive but not arc-transitive.

More generally, if G is any group of automorphisms of X (that is, any subgroup of $\mathrm{Aut}\, X$), then G is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* on X if G acts transitively on $V(X)$, $E(X)$ or $A(X)$, respectively, and G is *half-arc-transitive* on X if G is vertex- and edge-transitive but not arc-transitive on X. In the latter case, we also say that X is $(G, \frac{1}{2})$*-arc-transitive*, or $(G, \frac{1}{2}, H)$*-arc-transitive* when it needs to be stressed that the stabiliser G_v of a given vertex v is isomorphic to a particular subgroup H of G.

Next, we repeat an explanation given in [14] of a connection between half-arc-transitive group actions and transitive permutation groups with a non-self-paired sub-orbit.

Let G be a transitive permutation group acting on a set V. An *orbital* of G is an orbit of the natural action of G on the Cartesian product $V \times V$, and a *sub-orbit* of G on V is an orbit of the stabiliser G_v of a given point $v \in V$. For any given point $v \in V$ there is a 1-to-1 correspondence between the set of all orbitals of G and the set of all sub-orbits of G on V (with regard to v), with the orbital containing the pair $(v, w) \in V \times V$ corresponding to the orbit of G_v containing w. In particular, the diagonal orbital $\{(w, w) : w \in V\}$ corresponds to the trivial sub-orbit $\{v\}$.

For any sub-orbit W of G (with regard to v), let Δ_W be the corresponding orbital of G, which contains the pair (v, w) for each $w \in W$. Then the *orbital digraph* $X(G, V; W)$ *of* (G, V) *relative to* W is the digraph (or oriented graph) with vertex-set V and arc-set Δ_W. The underlying undirected graph, with orientations of all arcs ignored, is denoted by $X^*(G, V; W)$.

The *paired orbital* of a given orbital Δ is the orbital $\Delta' = \{(v, w) : (w, v) \in \Delta\}$. The orbital Δ is said to be *self-paired* if $\Delta' = \Delta$, and *non-self-paired* otherwise; in the latter case $\Delta \cap \Delta' = \emptyset$. This notion of pairing also carries over to sub-orbits in a natural way. It is important to note that for a non-self-paired sub-orbit W of G, the orbital digraph $X(G, V; W)$ is an oriented graph, while the underlying undirected graph $X^*(G, V; W)$ admits a half-arc-transitive action of G.

In the special case where V is the set $(G : H)$ of all (right) cosets of a subgroup H of G, and W is a non-self-paired sub-orbit of the action of G on V (by right multiplication), the graph $X^*(G, V; W)$ is $(G, \frac{1}{2}, H)$-arc-transitive, with valency $2|W|$. This graph might or might not be half-arc-transitive, depending on whether it admits any additional automorphisms that reverse an arc. The example in [4] began with H being dihedral of order 8, and had G as its full automorphism group.

For some further references on half-arc-transitive graphs (which are often referred to simply as *half-transitive* graphs), see the survey paper by Marušič [15], and for the theory of permutation groups, see [6, 22].

3 The smallest half-arc-transitive 4-valent graphs with vertex-stabilisers of order 4 and 8

In this section we present the smallest 4-valent half-arc-transitive graphs for which the vertex-stabilisers have order 4 and 8, respectively. These graphs have 256 and 768 vertices, and will also appear in a recently computed census of all 4-valent half-arc-transitive graphs of order at most 1000 (see [18]).

3.1 Stabiliser of order 4

If G is the automorphism group of a half-arc-transitive 4-valent graph such that the stabiliser H in G of a vertex v has order 4, then H is isomorphic to the Klein group V_4, and so is generated by two commuting involutions p and q. At the same time, G is generated by H and an automorphism a mapping v to an out-neigbour w of v (in one of the two corresponding orbital digraphs). Generators p and q of H can be chosen in such a way that $q = a^{-1}pa$. In addition, since H is the stabiliser in a transitive permutation group G, the core of H in G is trivial.

Conversely, given a group G generated by two commuting involutions p and q, and an element a such that $q = a^{-1}pa$, where p and q generate a core-free subgroup H of G, one can construct the orbital digraph $X = X^*(G, V; W)$, where V is the coset-space $(G : H)$, with G acting upon V by right-multiplication, and W is the sub-orbit of this action containing the coset Ha. It can be seen that such a sub-orbit is necessarily non-self-paired, implying that X is 4-valent and admits G as a half-arc-transitive group of automorphisms. In particular, if X admits no additional automorphisms, then X is a half-arc-transitive graph with valency 4 and vertex-stabiliser V_4. This happens quite frequently, although not for small orders.

Candidates for G of order up to 1023 can be checked quite easily using the small groups database in MAGMA [1], but none of them has the required properties. Order 1024 is more challenging by this approach, since there are 49 487 365 422 groups of order 1024, and they are not easily available.

Instead, we can use an algorithm for finding all normal subgroups of given index in a finitely-presented group, as described in [3], or better still, [9]. The latter version is implemented in MAGMA [1] as the `LowIndexNormalSubgroups` procedure. We can apply this to the finitely-presented group $F = \langle\, p, q, a \,|\, p^2 = q^2 = (pq)^2 = a^{-1}paq = 1 \,\rangle$, and for every normal subgroup N found, consider the quotient F/N as a candidate for G.

It turns out there are 3102 normal subgroups of index up to 1024 in F, and remarkably, four of them give good candidates. The quotient via one such normal subgroup is the group G used in the following:

Theorem 3.1. *Let G be the group with presentation*
$\langle\, p, q, a \,|\, p^2 = q^2 = (pq)^2 = a^{-1}paq = a^8 = (pa^{-2}qa^2)^2 = a^3qapa^{-1}pqa^{-2}qa^{-1}q = 1 \,\rangle$, *and let W be the sub-orbit with regard to $H = \langle p, q\rangle$ containing the coset Ha. Then G has order* 1024*, and $X^*(G, H; W)$ is a* 4*-valent half-arc-transitive graph of order* 256*, girth* 8 *and diameter* 8*, with automorphism group G, and vertex-stabiliser G_v isomorphic to V_4.*

Indeed the sub-orbit W containing the coset Ha is $\{Ha, Hap\}$, of size 2, and is not self-paired, its paired sub-orbit being $\{Ha^{-1}, Ha^{-1}q\}$. The graph $X = X^*(G, H; W)$ can easily be constructed with the help of MAGMA, and the fact that $\mathrm{Aut}\, X = G$ verified using the function `AutomorphismGroup`.

As mentioned in the first section, the fact that the order of this graph is $256 = 2^8$ shows that the conjecture by Feng, Kwak, Xu and Zhou [8] (that every 4-valent half-arc-transitive graph of prime-power order has vertex stabilisers of order 2) is false.

By inspecting normal subgroups of $G = \mathrm{Aut}\, X$, one finds that G has an elementary abelian normal subgroup K of order 16, and another normal subgroup J isomorphic to $C_4 \times C_4 \times C_2$. With respect to these two normal subgroups, X is an abelian regular cover of two smaller graphs, namely the Rose-window graph $\mathrm{RW}_8(6, 5)$, as defined in [23, §5], and the "doubled" 8-cycle DC_8 (that is, the (multi)graph obtained from the directed cycle C_8 by doubling each edge). Accordingly, the graph X has at least two alternative constructions as a covering graph.

Theorem 3.2. *The graph $X = X^*(G, H; W)$ in Theorem* 3.1 *is an abelian regular cover of the Rose-window graph* $\mathrm{RW}_8(6, 5)$*, with covering group $K \cong C_2^4$, and also an abelian regular cover of the doubled cycle DC_8, with covering group $J \cong C_4 \times C_4 \times C_2$.*

In fact, the Rose-Window graph $\mathrm{RW}_8(6, 5)$ is isomorphic to the Cartesian product $C_4 \,\Box\, C_4$ of two cycles of length 4, and moreover, the quotient projection $X \to X/K$ is a covering projection, and G/K is the largest subgroup of $\mathrm{Aut}\,(X/K)$ acting half-arc-transitively on X/K. (For background information on quotients and covering projections of graphs, see [12, 20] for example.)

The construction of X as an abelian regular cover of DC_8 can be described as follows. Let

$$J = \langle a, b, c \,|\, a^4 = b^4 = c^2 = [a, b] = [a, c] = [b, c] = 1\rangle,$$

which is abelian of order 32, and isomorphic to $C_4 \times C_4 \times C_2$, and let φ be the automorphism of J of order 8 that takes a to b, and b to ac, and c to a^2c, or equivalently, is given by

$$\varphi(a^i b^j c^k) = a^{j+2k} b^i c^{j+k} \quad \text{for} \quad 0 \le i, j \le 3 \ \text{ and } \ 0 \le k \le 1.$$

Now label the vertices of DC_8 with the elements of $\mathbb{Z}_8$ in a natural way, and label the edges with the elements of $\{e_i \,:\, i \in \mathbb{Z}_8\} \cup \{f_i \,:\, i \in \mathbb{Z}_8\}$ in such a way that the two edges incident to both i and $i + 1$ are labelled e_i and f_i, for each $i \in \mathbb{Z}_8$. We can now describe a certain voltage assignment of the voltages from J to the set of the arcs of DC_8. (For the terminology on graph coverings via voltage assignments, see [13].) For each $i \in \mathbb{Z}_8$, we put the trivial voltage 1 on all of the arcs corresponding to the edge f_i, and put the voltage $\varphi^i(a)$ on the arc corresponding to e_i (directed from i to $i+1$). It can easily be verified that the covering graph obtained in this way is a 4-valent half-arc-transitive graph of order 256, with vertex stabilisers of order 4, as above.

Similarly, the other three normal subgroups giving good candidates for G all give rise to half-arc-transitive graphs that are isomorphic to X.

Let us mention also that 4-valent half-arc-transitive graphs with vertex stabilisers of order 4 have been known to exist for some time. In fact, an infinite family of such graphs was constructed in each of the papers [11] and [16]. These graphs, however, are very large. The smallest graph in the first family has order $\frac{17!}{4} > 8 \cdot 10^{13}$, and the smallest one in the second family has order 9 979 200.

3.2 Stabiliser of order 8

As was shown in [17] (see also [19]), if the vertex-stabiliser $H = G_v$ in a half-arc-transitive group of automorphisms G of a connected tetravalent graph X has order 8, then H is either elementary abelian or dihedral. Moreover, as in the case where $|H| = 4$, the group G is generated by H and any element a of G that maps the vertex v to an out-neighbour of v.

Using these observations we can prove the following:

Theorem 3.3. *Let X be a 4-valent half-arc-transitive graph with automorphism group G, and suppose the stabiliser $H = G_v$ of a vertex v has order 8.*

(a) *If H is dihedral, then G is a quotient of the finitely generated group*

$\mathcal{G}_{3,1} = \langle p, q, r, a \mid p^2 = q^2 = r^2 = [p, q] = [q, r] = 1, [p, r] = q, q = a^{-1}pa, r = a^{-1}qa\rangle$

of order $8|V(X)|$. There is no such graph X of order up to 768.

(b) *If H is abelian, then G is a quotient of the finitely generated group*

$\mathcal{G}_{3,0} = \langle p, q, r, a \mid p^2 = q^2 = r^2 = [p, q] = [p, r] = [q, r] = 1, q = a^{-1}pa, r = a^{-1}qa\rangle$

of order $8|V(X)|$. There is no such graph of order less than 768, but there are two (non-isomorphic) examples of order 768, say X_1 and X_2, which arise from orbital digraphs for quotients of $\mathcal{G}_{3,0}$ with respect to the additional relator sets

$$\mathcal{R}_1 = \{a^{12}, (ra^2pa^{-2})^2, a^{-1}pa^{-4}rqpa^6pqa^{-1}, a^{-1}rapa^{-2}pa^{-1}rpapa^2papa^{-1}\}$$

and $$\mathcal{R}_2 = \{a^{12}, (ra^2pa^{-2})^2, rapa^{-1}ra^2qpa^{-4}ra^2\},$$ *respectively.*

Proof.

(a) If H is dihedral, then the generators p, q, r of $H = G_v$ can be chosen so that they satisfy the relations $p^2 = q^2 = r^2 = [p, q] = [q, r] = 1$ and $[p, r] = (pr)^2 = q$, and the automorphism a (moving v to an out-neighbour of v) chosen so that $q = a^{-1}pa$ and $r = a^{-1}qa$. Thus G is a quotient of $\mathcal{G}_{3,1}$, of order $8|V(X)|$. Inspection of the normal subgroups of index at most $8 \cdot 768 = 6144$ in $\mathcal{G}_{3,1}$, and the corresponding orbital digraphs, shows there are no such X of order up to 768 (with $H = G_v$ dihedral of order 8).

(b) If H is abelian, then G is a quotient of $\mathcal{G}_{3,0}$, by a similar argument to that in part (a). Computation of normal subgroups of index at most 6144 in $\mathcal{G}_{3,0}$ shows there is no such X of order less than 768 (with $H = G_v$ abelian of order 8), but there are two (non-isomorphic) examples which arise from normal subgroups of index 768, namely the normal closures of the sets $\mathcal{R}_1$ and $\mathcal{R}_2$ as given. In each case, it is easy to check using MAGMA that the corresponding quotient of $\mathcal{G}_{3,0}$ is the full automorphism group of the graph. □

Theorem 3.4. *The graphs X_1 and X_2 in Theorem 3.3 are non-isomorphic regular covers of the doubled cycle DC_3, as well as elementary abelian regular covers of the so-called Hill Capping $HC(Q_3)$ of the cube Q_3 (see [24]).*

Proof. The automorphism group of X_1 has a normal subgroup K_1 of index 24 and order 256, generated by

$$x = apa^{-1}r,\ y = a^3qra^3rq = a^6[a^3, rq],\ z = a^3pqra^3rqp = a^6[a^3, rqp] \text{ and } u = a^3,$$

all of which have order 4, with x, y and z generating an abelian normal subgroup of order 64, and $x^u = xy^2z^2$, $y^u = x^2y^{-1}z^2$ and $z^u = z^{-1}$. The centre and derived subgroup of K_1 are the elementary abelian 2-subgroups $\langle x^2, y^2, z^2, u^2\rangle$ and $\langle x^2, y^2, z^2\rangle$, of orders 16 and 8 respectively. (In particular, K_1 has nilpotency class 2.)

This subgroup K_1 acts semi-regularly on $V(X_1)$, and the quotient graph X_1/K_1 is isomorphic to the doubled cycle DC_3. It follows that X_1 can be reconstructed as a regular cover of DC_3, with covering group K_1. This can be achieved as follows. First, label the vertices of DC_3 with the elements of $\mathbb{Z}_3$, and label the edges of DC_3 with elements of the set $\{e_i : i \in \mathbb{Z}_3\} \cup \{f_i : i \in \mathbb{Z}_3\}$, so that e_i and f_i are the two edges incident to both i and $i+1$ (for each $i \in \mathbb{Z}_3$). Also let $\vec{e}_i$ and $\vec{f}_i$ be the corresponding arcs from i to $i+1$, for $i \in \mathbb{Z}_3$. Then the graph X_1 is isomorphic to the cover of DC_3 obtained from the voltage assignment φ on DC_3 defined by

$$\varphi(e_0) = \varphi(e_2) = 1,\ \varphi(e_1) = u,\ \varphi(f_0) = x,\ \varphi(f_1) = uxy \ \text{ and } \ \varphi(f_2) = (yz)^{-1}.$$

Similarly, the automorphism group of X_2 has a normal subgroup K_2 of index 24 and order 256, and nilpotency class 2, acting semi-regularly on $V(X_2)$. The quotient graph X_2/K_2 is again isomorphic to the doubled cycle DC_3, and hence X_2 can be reconstructed as a regular cover of DC_3, with covering group K_2 (not isomorphic to K_1).

Finally, each of Aut X_1 and Aut X_2 contains an elementary abelian normal subgroup K of order 16, with respect to which the quotient graphs X_1/K and X_2/K are both isomorphic to $HC(Q_3)$. Hence X_1 and X_2 can be reconstructed as elementary abelian covers of $HC(Q_3)$. □

Here we note that the graph $HC(Q_3)$ happens to be the unique tetravalent arc-transitive graph of order 48, girth 4, and diameter 6.

4 Two new half-arc-transitive graphs with vertex-stabiliser D_4

The first (and until recently, the only) known example of a half-arc-transitive 4-valent graph with non-abelian vertex-stabiliser is described in [4].

This graph has 10 752 vertices, and its automorphism group G of order 86 016 is generated by two elements a and b of orders 8 and 24, which satisfy the (defining) relations

$$a^8 = (ab^{-1})^2 = a^{-2}bab^{-2}ab = (ab^3ab^2ab^2)^2$$
$$= a^{-3}ba^2b^{-3}a^2b = (a^2ba^2babab)^2 = a^3b^2a^2b^2aba^3bababab^2 = 1.$$

The graph is the underlying graph of the orbital digraph $X(G, V; W)$ where V is the coset space $(G\!:\!H)$ for the subgroup H generated by $p = a^{-1}b$ and $q = a^{-1}pa$ and $r = a^{-1}qa$, and W is the non-self-paired sub-orbit $\{Ha, Hb\}$.

In response to a comment made by Dragan Marušič about this graph being somewhat unique, in a lecture at a workshop at the Fields Institute in October 2011, the first author decided to look for more examples. Somewhat surprisingly, it turns out there is another example on 10 752 vertices (with vertex-stabiliser D_4), not isomorphic to the first. Also there exists an example on 21 870 vertices, with similar properties.

Theorem 4.1. *There are at least two non-isomorphic half-arc-transitive* 4*-valent graphs of order* 10 752 *with non-abelian vertex-stabiliser of order* 8*.*

The automorphism group of the new one of order 10 752 is a different group G of order 86 016, generated by two elements a and b of orders 16 and 12 which satisfy the (defining) relations

$$\begin{aligned} a^{16} &= b^{12} = (ab^{-1})^2 = a^{-2}bab^{-2}ab = (ab^3ab^2ab^2)^2 \\ &= a^{-3}b^2ab^{-3}ab^2 = (a^2b^2abab^2ab)^2 = a^5b^3a^5bab = 1. \end{aligned}$$

Again if V is the coset space $(G : H)$ for the subgroup H generated by $p = a^{-1}b$ and $q = a^{-1}pa$ and $r = a^{-1}qa$, and W is the non-self-paired sub-orbit $\{Ha, Hb\}$, then the 4-valent underlying graph of the orbital digraph $X(G, V; W)$ is half-arc-transitive, but it is not isomorphic to the first example.

Theorem 4.2. *There exists at least one half-arc-transitive* 4*-valent graph of order* 21 870 *with non-abelian vertex-stabiliser of order* 8*.*

The automorphism group of the one we found has order 174 960, and is generated by two elements a and b of orders 8 and 24 which satisfy the (defining) relations

$$\begin{aligned} a^8 &= (ab^{-1})^2 = a^{-2}bab^{-2}ab = a^{-3}ba^2b^{-3}a^2b \\ &= (a^3b)^5 = a^3b^{-1}a^{-3}b^2ababab^2ab^2a^2b = 1. \end{aligned}$$

Both of these new examples were found with the help of MAGMA [1], in the same way as the first one in [4]. Also MAGMA can be used to verify that the full automorphism group of the graph is as stated, in each case.

5 A half-arc-transitive 4-valent graph with a non-abelian and non-dihedral vertex-stabiliser

In his 2011 lecture at the Fields Institute (mentioned in the previous section), Dragan Marušič made the observation that no half-arc-transitive 4-valent graph was known with vertex-stabiliser that is neither abelian nor dihedral. We provide an example here.

We begin with an arc-transitive 4-valent graph on 90 vertices, which can be constructed from a group $\overline{G}$ of order 1440, generated by two elements c and d subject to the relations

$$\begin{aligned} c^8 &= d^{10} = (cd)^6 = (cd^{-1})^2 = (cd^2)^4 = c^{-2}dcd^{-2}cd \\ &= c^3d^{-1}c^{-2}dcdc^{-2}d^{-1} = d^2c^{-3}d^{-1}c^4d^{-1}c^{-3}d^2 = 1. \end{aligned}$$

Let $\overline{H}$ be the subgroup generated by $\overline{p} = c^{-1}d$ and conjugates $\overline{q} = c^{-1}\overline{p}c$, $\overline{r} = c^{-1}\overline{q}c$ and $\overline{s} = c^{-1}\overline{r}c$. Then $\overline{H}$ is isomorphic to the direct product $D_4 \times C_2$, or order 16. If $\overline{V}$

is the coset space $(\overline{G}:\overline{H})$, and $\overline{W}$ is the non-self-paired sub-orbit $\{\overline{H}c, \overline{H}d\}$, then the 4-valent underlying graph $\overline{X}$ of the orbital digraph $X(\overline{G}, \overline{V}, \overline{W})$ is arc-transitive, but admits a half-arc-transitive action of $\overline{G}$. (In fact, its full automorphism group has order 2880.)

The half-arc-transitive action of $\overline{G}$ on $\overline{X}$ lifts to the action of a larger group G on a regular cover of $\overline{X}$, which we will prove is half-arc-transitive.

The group we take is the transitive permutation group G of degree 60, generated by the elements

$$a = (1,2)(3,4,6,8,12,17,21,27,36,44,53,60,59,52,47,37,45,35,26,20,16,11,7,5)\\(9,13,18,23,30,40,51,48,57,58,50,39,29,22,28,38,34,25,33,24,19,15,10,14)\\(31,41,49,42)(32,43)(46,55,54,56)$$

and

$$b = (1,2,5,3,14,6,8,29,17,47,55,46,37,36,48,53,60,30,52,21,56,54,27,45,38,\\26,20,19,11,7)(4,9,13,42,31,50,39,12,22,51,44,57,58,41,49,18,23,59,40,28,\\35,34,25,43,32,33,24,16,15,10).$$

This group G is imprimitive, with blocks of sizes 3, 6 and 30. The action of G on the blocks of size 3 (which are $\{1, 55, 56\}$ and its images) gives an epimorphism from G to the group $\overline{G}$ above, with elementary abelian kernel K of order 3^{10}.

Let $p = a^{-1}b$, $q = a^{-1}pa$, $r = a^{-1}qa$ and $s = a^{-1}ra$. These elements satisfy the relations $p^2 = q^2 = r^2 = s^2 = b^{-1}pbq = b^{-1}qbr = b^{-1}pqrbs = (rs)^2 = (qs)^2 = 1$, as well as $pa^{-1}b = a^{-1}paq = a^{-1}qar = a^{-1}ras = 1$ and others. The subgroup H generated by p, q, r and s has order 16, and just like $\overline{H}$ above, is isomorphic to $D_4 \times C_2$.

Theorem 5.1. *With the notation above, let X be the underlying graph of the orbital digraph $X(G, V; W)$, where V is the coset space $(G:H)$ and W is the sub-orbit $\{Ha, Hb\}$. Then X is a 4-valent half-arc-transitive graph of order $90 \cdot 3^{10}$, with automorphism group G, and vertex-stabiliser $H = G_v \cong D_4 \times C_2$. In fact, X is a regular cover of $\overline{X}$, and the action of G on X projects to the action of $\overline{G}$ on $\overline{X}$.*

Proof. The sub-orbit $W = \{Ha, Hb\}$ is non-self-paired, and so X is 4-valent, and G acts half-arc-transitively on X, with vertex-stabiliser $H \cong D_4 \times C_2$. The group G has order $1440 \cdot 3^{10}$, and so the order of X is as given. This order makes X too large to construct and analyse easily using MAGMA, but nevertheless we can study the permutation representation of G on the right coset space V closely enough to prove that G is the full automorphism group of X. We will not provide all details, but we explain most of the argument below.

First, let '1' be the vertex H in X, and let x_1, x_2, x_3 and x_4 be the four neighbours of 1 (which are the cosets Ha, Hb, Ha^{-1} $(= Hb^{-1})$ and $Ha^{-1}s$). Then by vertex-transitivity, the edges of X are images of the edges $\{1, x_i\}$ under the action of elements of G.

We can use that fact to find all vertices within a given small distance from vertex 1. The numbers of vertices at distances 0 to 9 from vertex 1 are 1, 4, 12, 36, 108, 324, 972, 2 916, 8 748 and 26 050, respectively. It turns out that no two vertices at distance 8 from vertex 1 are adjacent, and that no three such vertices have a common neighbor at distance 9 from vertex 1. It follows that the girth of X is 18, and that there are $3 \cdot 8748 - 26050 = 194$ girth cycles (of length 18) in X containing the vertex 1.

Moreover, we find easily that a given 1-arc with initial vertex 1 lies in 97 girth cycles, a given 2-arc with initial vertex 1 lies in 31 or 35 girth cycles, a given 3-arc with initial vertex 1 lies in 10, 11 or 13 girth cycles, a given 4-arc with initial vertex 1 lies in 3, 4 or 7 girth cycles, and a given 5-arc with initial vertex 1 lies in 1, 2, 3 or 5 girth cycles. In fact, just one of the 2-arcs extending a given 1-arc $(1, x_i)$ lies in 35 girth cycles, with the other two lying in 31.

Now if a 2-arc of the form (u, v, w) lies in exactly 35 girth cycles, let us call the vertex w the 'twin' of vertex u at vertex v. This gives a 'twinning' (or pairing) of neighbours at each vertex in the usual way, but an important point is that this is a property of X (rather than just the action of G). Also it shows that the stabiliser in Aut X of any vertex of X is a 2-group.

Label the neighbours of vertex 1 so that $\{x_1, x_3\}$ and $\{x_2, x_4\}$ are pairs of twins, and for each x_i, let y_i be the twin of vertex 1 at x_i, and let z_i and w_i be the other two (twin) neighbours of x_i.

Also let us call a 5-arc 'special' if it lies in a unique girth cycle. If a 5-arc $(u, x_i, 1, x_j, v)$ with middle vertex 1 is special, then the analysis shows that $\{x_i, x_j\}$ is not a twin pair. Note that Aut X must permute the special 5-arcs among themselves.

Next, let B be the ball of radius 2 centred at vertex 1, and consider the pointwise stabiliser of B in Aut X. If the automorphism h fixes every vertex in B, then for every 2-arc of the form $(1, x_i, v)$, we see that h fixes the twin of x_i at vertex w, and for every special 5-arc $(u, x_i, 1, x_j, v)$ with middle vertex 1, also h fixes every vertex in the unique girth cycle that contains it. These two observations are enough to show that h fixes every vertex at distance 3 from vertex 1. Then by induction and connectedness, the automorphism h is trivial. Hence every automorphism of X is completely determined by its effect on B.

Now consider the permutations induced by each of p, q, r, s and a on B. We can choose the labels x_i, y_i, z_i and w_i of the 16 vertices of $B \setminus \{1\}$ such that

- p induces $(x_1, x_3)(y_1, y_3)(z_1, z_3)(w_1, w_3)(z_4, w_4)$, and fixes all other vertices of B,
- q induces $(z_1, w_1)(z_3, w_3)$, and fixes all other vertices of B,
- r induces $(z_2, w_2)(z_4, w_4)$, and and fixes all other vertices of B,
- s induces $(x_2, x_4)(y_2, y_4)(z_2, z_4)(w_2, w_4)(z_3, w_3)$, and fixes all other vertices of B,

and similarly,

- a induces a permutation of the form $(1, x_1, z_1, ..., z_2, x_2)(x_3, w_2, ..., y_2)(x_4, y_1, ..., w_2)$ $(y_3, ...)(y_4, ...)(z_3, ...)(z_4, ...)(w_3, ...)(w_4, ...)...$.

Next, suppose there is some automorphism g of X that fixes vertex 1 and all its neighbours x_i, and the twins y_i of vertex 1 at those neighbours, but does not lie in the subgroup generated by q and r. Note that this automorphism g would have to fix or swap each pair $\{z_i, w_i\}$. Multiplying by q or r or qr if necessary, we can suppose that g fixes also the vertices z_1, w_1, z_2 and w_2, and so g induces either (z_3, w_3) or (z_4, w_4) or $(z_3, w_3)(z_4, w_4)$ on the 2-ball B.

Then in particular, by considering the permutations induced by various elements on the 2-ball B, and the fact that the point wise stabiliser of B is trivial, we deduce that g centralises q and r, and conjugates p to either p or pq, and conjugates s to s or sr. Similarly, g conjugates a to something of the form ah, where h fixes every vertex in B with the possible exception of z_3, w_3, z_4 and w_4, and in particular, it follows that h commutes with each of q and r, and h conjugates p to p or pq, and conjugates s to s or sr, as well. But $q = p^a$, and so if g conjugates p to pq, we find that $q = q^g = (p^a)^g = (p^g)^{a^g} = (pq)^{ah} = (qr)^h = qr$, which forces r to be trivial, contradiction. Thus $p^g = p$. Since p induces $(x_1, x_3)(y_1, y_3)(z_1, z_3)(w_1, w_3)(z_4, w_4)$, it follows that g fixes z_3 and w_3, and hence g induces (z_4, w_4) on B.

Thus every automorphism of X that fixes all the vertices of the 2-ball B apart from (possibly) z_3, w_3, z_4 and w_4, must fix z_3 and w_3, and so equals the identity or g. In particular, $h = 1$ or g, and so g conjugates a to either a or ag. But also we know that g centralizes each of p, q and r, and conjugates s to sr, and hence $sr = s^g = (r^a)^g = (r^g)^{a^g} = r^{ah} = s^h$. Thus h conjugates s to sr as well, and so h cannot be trivial, so $h = g$. In particular, g conjugates a to ag. But that means $g^{-1}ag = ag$, which forces g^{-1} to be trivial, contradiction.

Hence no such g exists.

Next, for the moment, suppose that X is half-arc-transitive, but G is not the full automorphism group of X. Then there must be some automorphism g of X fixing the vertex 1 and its neighbours x_i (and their twins y_i), but not lying in G. By what we just showed above, however, this is impossible. Hence if $G \neq \mathrm{Aut}\, X$, then X must be arc-transitive, and $\mathrm{Aut}\, X$ contains G as a subgroup of index 2. In particular, G is normal in $\mathrm{Aut}\, X$.

Finally, suppose X is arc-transitive, and let t be any automorphism of X taking the arc $(1, x_1)$ to the arc $(1, x_2)$. Then by the 'twinning' property of X, we know that t must also take x_3 to x_4, and so t induces either (x_1, x_2, x_3, x_4) or $(x_1, x_2)(x_3, x_4)$ on the neighbours of 1. Multiplying by p if necessary, we may assume that t induces the double transposition $(x_1, x_2)(x_3, x_4)$ on the neighbours of 1, and hence also that t induces $(y_1, y_2)(y_3, y_4)$ on their 'twins'. Then since t swaps x_1 with x_2 and swaps y_1 with y_2, it must swap the set $\{z_1, w_1\}$ of the other two neighbours of vertex x_1 with the set $\{z_2, w_2\}$ of the other two neighbours of vertex x_2, and it follows that t induces either $(z_1, z_2)(w_1, w_2)$ or (z_1, z_2, w_1, w_2) or (z_1, w_2, w_1, z_2) on those four vertices. Similarly, t induces either $(z_3, z_4)(w_3, w_4)$ or (z_3, z_4, w_3, w_4) or (z_3, w_4, w_3, z_4) on $\{z_3, w_3, z_4, w_4\}$.

This gives 16 possibilities for the effect of t on the 2-ball B, but since we can multiply t by q or r or qr (all of which fix the vertices 1, x_i and y_i for all i), we may reduce the possibilities for t to these four:

$$t_1 = (x_1, x_2)(x_3, x_4)(y_1, y_2)(y_3, y_4)(z_1, z_2)(z_3, z_4)(w_1, w_2)(w_3, w_4),$$
$$t_2 = (x_1, x_2)(x_3, x_4)(y_1, y_2)(y_3, y_4)(z_1, z_2, w_1, w_2)(z_3, z_4)(w_3, w_4),$$
$$t_3 = (x_1, x_2)(x_3, x_4)(y_1, y_2)(y_3, y_4)(z_1, z_2)(z_3, w_4)(w_1, w_2)(w_3, z_4),$$
$$t_4 = (x_1, x_2)(x_3, x_4)(y_1, y_2)(y_3, y_4)(z_1, z_2, w_1, w_2)(z_3, w_4)(w_3, z_4).$$

In each case, it is easy to check that the candidate for t conjugates the permutation induced by q on the 2-ball to the permutation induced by r, and vice versa. Hence $t^{-1}qt = r$, and $t^{-1}rt = q$. Similarly, we find that $t^{-1}pt = s$ and $t^{-1}st = p$ when $t = t_1$, while $t^{-1}pt = s$ and $t^{-1}st = pq$ when $t = t_2$, and $t^{-1}pt = rs$ and $t^{-1}st = pq$ when $t = t_3$, and $t^{-1}pt = rs$ and $t^{-1}st = p$ when $t = t_4$.

Now if $t = t_1$, we find that $atat^{-1}$ fixes all the vertices $1, x_i$ and y_i, and so by our earlier observations, $atat^{-1}$ lies in the subgroup generated by q and r. Since also $atat^{-1}$ fixes z_2 and w_2, we deduce that $atat^{-1} = 1$ or q, so $tat^{-1} = a^{-1}$ or $a^{-1}q$. Rearranging these (and using the fact that $t^{-1}qt = r$), we find that also $t^{-1}at = a^{-1}$ or $a^{-1}q$. But a MAGMA computation shows there is no automorphism of the group G taking a to a^{-1}, and p to s (and q to r, etc.), and also there is no automorphism of G taking a to $a^{-1}q$, and p to s (and q to r, etc.). Hence $t \neq t_1$.

Similarly, if $t = t_2$, we find that $t^{-1}at = sa^{-1}$ or sra^{-1}, which upon rearrangement gives $t^{-1}at = a^{-1}r$ or $a^{-1}rq$, but another MAGMA computation shows there is no automorphism of G taking a to $a^{-1}r$ or $a^{-1}rq$, and p to s (and q to r, etc.). Hence $t \neq t_2$.

If $t = t_3$, then $t^{-1}at = a^{-1}$ or $a^{-1}q$, as in the case of t_1. But if t conjugates a to a^{-1}, and conjugates p to rs (as t_3 does), then t conjugates $q = p^a$ to $(rs)^{a^{-1}} = arsa^{-1} = qr$, rather than r, contradiction. Similarly, if t conjugates a to $a^{-1}q$, and conjugates p to rs (as t_3 does), then t conjugates q to rq, again a contradiction. Hence $t \neq t_3$.

Similarly, if $t = t_4$, then $t^{-1}at = a^{-1}r$ or $a^{-1}rq$, but both of these possibilities give contradictions when combined with the facts that t conjugates p and q to rs and r, so $t \neq t_4$.

Thus no such t exists, and so X is half-arc-transitive, as claimed. □

6 Answer to a question of Delorme

We begin this section with the following:

Proposition 6.1. *Let X be a half-arc-transitive graph of valence $2k$, and let $G = \mathrm{Aut}\, X$. Then if A is one of the two orbits of G on the arcs of X, then the digraph D with vertex-set $V(X)$ and arc-set A is regular, and admits G as a group of automorphisms, but D is not isomorphic to its reverse.*

Proof. First, D has out-valence k and in-valence k, and obviously G acts on D as a group of automorphisms. If D were isomorphic to its reverse digraph $\mathrm{Rev}(D)$, then the isomorphism from D to $\mathrm{Rev}(D)$ would be an automorphism of X not contained in G, which is a contradiction. Hence D is not self-reverse. □

Next, recall that an s-arc in a digraph D is a sequence $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices such that any two consecutive vertices v_{i-1} and v_i form an arc (v_{i-1}, v_i).) We note that if k (the in- and out-valence of D) is 2, and s is the largest positive integer such that G acts transitively on the s-arcs of D, then it is well known that $|G_v| = 2^s$ (see for example [19]).

As a consequence of these things, we have an answer to Delorme's question in [5]:

Theorem 6.2. *Let X be the unique half-arc-transitive 4-valent graph of order 256, with automorphism group G of order 1024 and vertex-stabiliser $G_v \cong V_4$. Then the corresponding digraph D is vertex-transitive, arc-transitive and 2-arc-transitive, but is not self-reverse. Moreover, this is the smallest 4-valent digraph with these properties.*

The example provided by Theorem 6.2 is the smallest such digraph that comes from a half-arc-transitive 4-valent graph with vertex-stabiliser of order 4, but in principle, a 2-arc-transitive non-self-reverse digraph could also come from an arc-transitive 4-valent graph X that admits a half-arc-transitive group action. (Indeed there is an arc-transitive 4-valent graph Y on 21 vertices with vertex-stabiliser D_4 such that $\mathrm{Aut}\, Y$ contains a half arc-transitive subgroup G with $G_v \cong \mathbb{Z}_2$, such that the corresponding orbital digraph $D = X(G, V; W)$ is not self-reverse; but this example is not 2-arc-transitive.) An inspection of the database of all such graphs of small order [18], however, shows that there is no such graph with fewer than 256 vertices, and so the above example of order 256 is the smallest.

On the other hand, there are infinitely many such examples, since there are infinitely many half-arc-transitive 4-valent graphs with vertex-stabiliser V_4; see [11, 16].

In fact, there is an infinite family of half-arc-transitive covers of the smallest example given above. With the help of the Reidemeister-Schreier process (implemented as the `Rewrite` command in MAGMA [1]), it can be shown that the kernel of the epimorphism from the group $\langle\, p, q, a \,|\, p^2 = q^2 = (pq)^2 = a^{-1}paq = a^8 = (pa^{-2}qa^2)^2 = 1\,\rangle$ to the group of order 1024 above has abelianisation $\mathbb{Z}_2^3 \oplus \mathbb{Z}^{10}$. It follows that for every odd positive integer m, there is a half-arc-transitive regular cover of order $256m^{10}$, with abelian covering group $K \cong \mathbb{Z}_m^{10}$.

7 Final remarks

In this paper, we have described the smallest 4-valent half-arc-transitive graphs with vertex-stabilisers of order 4 and 8, respectively. In each case, the vertex-stabiliser is abelian.

It is also known that for any positive integer s, there is at least one 4-valent half-arc-transitive graph with abelian vertex-stabilisers of order 2^s; see [16]. The graphs constructed in [16], however, have very large orders. Hence the following question arises naturally.

Question 7.1. What is the order of a smallest 4-valent half-arc-transitive graph with abelian vertex-stabiliser of order 2^s, for each $s \geq 4$?

Next, we have also found a new 4-valent half-arc-transitive graph with vertex-stabilisers isomorphic to the dihedral group D_4, having the same order as the smallest known such graph (on 10 752 vertices). It is not yet known, however, if these two graphs are the smallest such examples. Hence we pose the following question.

Question 7.2. Is there a 4-valent half-arc-transitive graph of order less than 10 752, with non-abelian vertex-stabilisers of order 8?

We have also constructed the first known example of a 4-valent half-arc-transitive graph with vertex-stabilisers that are non-abelian and non-dihedral. This has order 5 314 410, with vertex-stabilisers of order 16. We conclude the paper with the following two questions.

Question 7.3. Is there a 4-valent half-arc-transitive graph of order less than 5 314 410, with non-abelian, non-dihedral vertex-stabilisers?

Question 7.4. Does there exist a 4-valent half-arc-transitive graph with non-abelian vertex-stabilisers of order 2^s, for every $s \geq 3$?

References

[1] W. Bosma, J. Cannon and C. Playoust, The MAGMA Algebra System I: The User Language, *J. Symbolic Comput.* **24** (1997), 235–265.

[2] I. Z. Bouwer, Vertex and edge transitive, but not 1-transitive, graphs. *Canad. Math. Bull.* **13** (1970), 231–237.

[3] M. Conder and P. Dobcsányi, Applications and adaptations of the low index subgroups procedure, *Math. Comp.* **74** (2005), 485-497.

[4] M. D. E. Conder and D. Marušič, A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer, *J. Combin. Theory Ser. B* **88** (2003), 67–76.

[5] C. Delorme, Cayley digraphs and graphs, *European J. Combin.* **34** (2013), 1307–1315.

[6] J. D. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, New York, 1996.

[7] P. G. Doyle, On transitive graphs, Senior Thesis, Harvard College, 1976.

[8] Y. Q. Feng, J. H. Kwak, M. Y. Xu and J. X. Zhou, Tetravalent half-arc-transitive graphs of order p^4, *Eur. J. Comb.* **29** (2008), 555–567.

[9] D. Firth, *An algorithm to find normal subgroups of a finitely presented group up to a given index*, PhD Thesis, University of Warwick, 2005.

[10] D. F. Holt, A graph which is edge transitive but not arc transitive, *J. Graph Theory* **5** (1981), 201–204.

[11] A. Malnič and D. Marušič, Constructing $\frac{1}{2}$-transitive graphs of valency 4 and vertex stabilizer $\mathbb{Z}_2 \times \mathbb{Z}_2$, *Discrete Math.* **245** (2002), 203–216.

[12] A. Malnič, D. Marušič, P. Potočnik, Elementary abelian covers of graphs, *J. Alg. Combin.* **20** (2004), 71–97.

[13] A. Malnič, R. Nedela and M. Škoviera, Lifting graph automorphisms by voltage assignments, *Eur. J. Comb.*, **21** (2000), 927–947.

[14] D. Marušič, Half-transitive group actions on finite graphs of valency 4, *J. Combin. Theory Ser. B* **73** (1998), 41–76.

[15] D. Marušič, Recent developments in half-transitive graphs, *Discrete Math.* **182** (1998), 219–231.

[16] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.* **299** (2005), 180–193.

[17] D. Marušič and R. Nedela, On the point stabilizers of transitive permutation groups with non-self-paired suborbits of length 2, *J. Group Theory* **4** (2001), 19–43.

[18] P. Potočnik, P. Spiga, G. Verret, A census of arc-transitive digraphs of valence 2 and tetravalent graphs admitting a $\frac{1}{2}$-arc-transitive group action, *Ars. Math. Contemp.* **8** (2015), 133–148.

[19] P. Potočnik, G. Verret, On the vertex-stabiliser in arc-transitive digraphs, *J. Combin. Theory Ser. B.* **100** (2010), 497–509.

[20] R. Požar, Sectional split extensions arising from lifts of groups, *Ars. Math. Contemp.* **6** (2013), 393–408.

[21] W. T. Tutte, *Connectivity in graphs*, University of Toronto Press, Toronto, 1966.

[22] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.

[23] S. Wilson, Rose window graphs, *Ars Mathematica Contemporanea* **1** (2008), 7–19.

[24] S. Wilson, A census of edge-transitive tetravalent graphs, `http://jan.ucc.nau.edu/~swilson/C4Site/BigTable.html`.

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A note on m-factorizations of complete multigraphs arising from designs*

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Abstract

Some new infinite families of simple, indecomposable m-factorizations of the complete multigraph λK_v are presented. Most of the constructions come from finite geometries.

Keywords: Graph factorization, affine and projective spaces, spread.

Math. Subj. Class.: 05C70, 51E23

1 Introduction

The *complete multigraph* λK_v has v vertices and λ edges joining each pair of vertices. An m*-factor* of the complete multigraph λK_v is a set of pairwise vertex-disjoint m-regular subgraphs, which induce a partition of the vertices. An m*-factorization* of λK_v is a set of pairwise edge-disjoint m-factors such that these m-factors induce a partition of the edges. An m-factorization is called *simple* if the m-factors are pairwise distinct. Furthermore, an m-factorization of λK_v is *decomposable* if there exist positive integers μ_1 and μ_2 such

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that $\mu_1 + \mu_2 = \lambda$ and the factorization is the union of the m-factorizations $\mu_1 K_v$ and $\mu_2 K_v$, otherwise it is called *indecomposable*. There is no direct correspondence between simplicity and indecomposability.

Many papers deal with m-factorizations of graphs and multigraphs. This is an interesting problem in its own right, but it is motivated by several applications, too. In particular if $m = 1$, then a one-factorization of K_v corresponds to a schedule of a round robin tournament. For a comprehensive survey on one-factorizations we refer to [29]. A special case of 2-factorizations is the famous *Oberwolfach problem*, see e.g. [2, 8]. Several authors investigated 3-factorizations of λK_v with a certain automorphism group, see e.g. [1, 24]. In general, decompositions of λK_v is also a widely studied problem, see e.g. [12, 13, 18, 27]. As m increases, the structure of an arbitrary m-factor of λK_v can be much more complicated and the existence problem becomes much more difficult. In this paper we restrict ourselves to construct factorizations in which all factors are regular graphs of degree m whose connected components are complete graphs on $(m + 1)$ vertices. In the case $m = 1$ an indecomposable one-factorization of λK_{2n} is denoted by $\mathrm{IOF}(2n, \lambda)$. Only a few conditions on the parameters are known: if $\mathrm{IOF}(2n, \lambda)$ exists, then $\lambda < 1 \cdot 3 \cdot \ldots \cdot (2n - 3)$ [4]; each $\mathrm{IOF}(2n, \lambda)$ can be embedded in a simple $\mathrm{IOF}(2s, \lambda)$, provided that $\lambda < 2n < s$ [16]. Six infinite classes of indecomposable one-factorizations have been constructed so far, namely a simple $\mathrm{IOF}(2n, n-1)$ when $2n-1$ is a prime [16], $\mathrm{IOF}(2(\lambda+p), \lambda)$ where $\lambda > 2$ and p is the smallest prime wich does not divide λ [3] (an improvement of this result can be found in [15]), a simple $\mathrm{IOF}(2^h + 2, 2)$ where h is a positive integer [28], $\mathrm{IOF}(q^2 + 1, q - 1)$ where q is an odd prime number [26], a simple $\mathrm{IOF}(q^2 + 1, q + 1)$ for any odd prime power q [25], and a simple $\mathrm{IOF}(q^2, q)$ for any even prime power q [25]. Most of these constructions arise from finite geometry.

The aim of this paper is to construct new simple and indecomposable m-factorizations of λK_v for different values of m, λ and v. In Section 2 we recall the basic combinatorial properties of designs and the geometric properties of finite affine and projective spaces. We also describe a general construction method of m-factorizations which is based on spreads of block designs. In Sections 3 and 4 affine spaces and projective spaces, respectively, are the key objects. We present several new multigraph factorizations using subspaces, subgeometries and other configurations of these structures.

2 Preliminaries

In this section we collect some concepts and results from design theory. For a detailed introduction to block designs we refer to [14].

2.1 Designs

Let v, b, k, r and λ be positive integers with $v > 1$. Let $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a triple consisting of a set $\mathcal{P}$ of v distinct objects, called points of D, a set $\mathcal{B}$ of b distinct objects, called blocks of D, and an incidence relation I, a subset of $\mathcal{P} \times \mathcal{B}$. We say that x is incident with y (or y is incident with x) if and only if the ordered pair (x, y) is in I. D is called a $2 - (v, b, k, r, \lambda)$ *design* if it satisfies the following axioms.

(a) Each block of D is incident with exactly k distinct points of D.

(b) Each point of D is incident with exactly r distinct blocks of D.

(c) If x and y are distinct points of D, then there are exactly λ blocks of D incident with both x and y.

A $2-(v,b,k,r,\lambda)$ design is called a balanced incomplete block design and is denoted by (v,k,λ)-design, too. The parameters of a $2-(v,b,k,r,\lambda)$ design are not all independent. The two basic equations connecting them are the following:

$$vr = bk \quad \text{and} \quad r(k-1) = \lambda(v-1). \tag{2.1}$$

These necessary conditions are not sufficient, for example no $2-(43,43,7,7,1)$ design exists.

2.2 Resolvability

A *resolution class* (or, a parallel class) of a (v,k,λ)-design is a partition of the point-set of the design into blocks. In general, an *f-resolution class* of a design is a collection of blocks, which together contain every point of the design exactly f times. A *resolution* of a design is a partition of the block-set of the design into r resolution classes. A (v,k,λ)-design with a resolution is called *resolvable*.

Necessary conditions for the existence of a resolvable (v,k,λ)-design are $\lambda(v-1)\equiv 0$ (mod $(k-1)$), $v\equiv 0$ (mod k) and $b\geq v+r-1$, (see [9]).

Let $D=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a (v,k,λ)-design, where $\mathcal{P}=\{p_1,p_2,\ldots,p_v\}$ is the set of its points and $\mathcal{B}=\{B_1,B_2,\ldots,B_b\}$ is the set of its blocks. Identify the points of D with the vertices of the complete multigraph λK_v. Then in the natural way, the set of points of each block of D induces in λK_v a subgraph isomorphic to K_k. For $B_i\in\mathcal{B}$, let G_i be the subgraph of λK_v induced by B_i. Then it follows from the properties of D that a resolution class of D gives a $(k-1)$-factor of λK_v and a resolution of D gives a $(k-1)$-factorization of λK_v. Hence we get the following well-known fact.

Lemma 2.1 (Basic Construction)**.** *The existence a resolvable (v,k,λ)-design is equivalent to the existence of a $(k-1)$-factorization of the complete multigraph λK_v.*

2.3 Projective and affine spaces

Most of our factorizations come from finite geometries. In this subsection we collect the basic properties of these objects. For a more detailed introduction we refer to the book of Hirschfeld [22].

Let V_{n+1} be an $(n+1)$-dimensional vector space over the finite field of q elements, $\mathrm{GF}(q)$. The *n-dimensional projective space* $\mathrm{PG}(n,q)$ is the geometry whose k-dimensional subspaces for $k=0,1,\ldots,n$ are the $(k+1)$-dimensional subspaces of V_{n+1} with the zero deleted. A k-dimensional subspace of $\mathrm{PG}(n,q)$ is called k-space. In particular subspaces of dimension zero, one and two are respectively a *point,* a *line* and a *plane,* while a subspace of dimension $n-1$ is called a *hyperplane.*

The relation $\sim$

$$\mathbf{x}\sim\mathbf{y}\Leftrightarrow \exists\, 0\neq\alpha\in\mathrm{GF}(q)\,:\,\mathbf{x}=\alpha\mathbf{y}$$

is an equivalence relation on the elements of $V_{n+1}\setminus\mathbf{0}$ whose equivalence classes are the points of $\mathrm{PG}(n,q)$. Let $\mathbf{v}=(v_0,v_1,\ldots,v_n)$ be a vector in $V_{n+1}\setminus\mathbf{0}$. The equivalence class of $\mathbf{v}$ is denoted by $[\mathbf{v}]$. The homogeneous coordinates of the point represented by $[\mathbf{v}]$ are $(v_0:v_1:\ldots:v_n)$. Hence two $(n+1)$-tuples $(x_0:x_1:\ldots:x_n)$ and $(y_0:y_1:\ldots:y_n)$ represent the same point of $\mathrm{PG}(n,q)$ if and only if there exists $0\neq\alpha\in\mathrm{GF}(q)$ such that $x_i=\alpha y_i$ holds for $i=0,1,\ldots,n$.

A k-space contains those points whose representing vectors $\mathbf{x}$ satisfy the equation $\mathbf{x}A = \mathbf{0}$, where A is an $(n+1) \times (n-k)$ matrix of rank $n-k$ with entries in $\mathrm{GF}(q)$. In particular a hyperplane contains those points whose homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$ satisfy a linear equation

$$u_0 x_0 + u_1 x_1 + \cdots + u_n x_n = 0$$

where $u_i \in \mathrm{GF}(q)$ and $(u_0, u_1, \ldots, u_n) \neq \mathbf{0}$.

The basic combinatorial properties of $\mathrm{PG}(n,q)$ can be described by the q-nomial coefficients. $\begin{bmatrix} n \\ k \end{bmatrix}_q$ equals to the number of k-dimensional subspaces in an n-dimensional vector space over $\mathrm{GF}(q)$, hence it is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n-1)(q^n-q)\ldots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\ldots(q^k-q^{k-1})}.$$

The proof of the following proposition is straightforward.

Proposition 2.2.

- *The number of k-dimensional subspaces in* $\mathrm{PG}(n,q)$ *is* $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.
- *The number of k-dimensional subspaces of* $\mathrm{PG}(n,q)$ *through a given d-dimensional ($d \leq k$) subspace in PG(n,q) is* $\begin{bmatrix} n-d \\ k-d \end{bmatrix}_q$.
- *In particular the number of k-dimensional subspaces of* $\mathrm{PG}(n,q)$ *through two distinct points in* $\mathrm{PG}(n,q)$ *is* $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

If $\mathcal{H}_\infty$ is any hyperplane of $\mathrm{PG}(n,q)$, then the *n-dimensional affine space over* $\mathrm{GF}(q)$ is $\mathrm{AG}(n,q) = \mathrm{PG}(n,q) \setminus \mathcal{H}_\infty$. The subspaces of $\mathrm{AG}(n,q)$ are the subspaces of $\mathrm{PG}(n,q)$ with the points of $\mathcal{H}_\infty$ deleted in each case. The hyperplane $\mathcal{H}_\infty$ is called the *hyperplane at infinity of* $\mathrm{AG}(n,q)$, and for $k = 0, 1, \ldots, n-2$ the k-dimensional subspaces in $\mathcal{H}_\infty$ are called the *k-spaces at infinity of* $\mathrm{AG}(n,q)$. Let $1 < d < n$ be an integer. Two d-spaces of $\mathrm{AG}(n,q)$ are called *parallel*, if the corresponding d-spaces of $\mathrm{PG}(n,q)$ intersect $\mathcal{H}_\infty$ in the same $(d-1)$-space. The parallelism is an equivalence relation on the set of d-spaces of $\mathrm{AG}(n,q)$. As a straightforward corollary of Proposition 2.2 we get the following.

Proposition 2.3. *In* $\mathrm{AG}(n,q)$ *each equivalence class of parallel d-spaces contains q^{n-d} subspaces.*

Projective and affine spaces provide examples of designs.

Example 2.4. Let $i < n$ be positive integers. The projective space $\mathrm{PG}(n,q)$ can be considered as a 2-design $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of $\mathrm{PG}(n,q)$, $\mathcal{B}$ is the set of i-spaces of $\mathrm{PG}(n,q)$ and I is the set theoretical inclusion. The parameters of D are $v = \frac{q^{n+1}-1}{q-1}$, $b = \begin{bmatrix} n+1 \\ i+1 \end{bmatrix}_q$, $k = \frac{q^{i+1}-1}{q-1}$, $r = \begin{bmatrix} n \\ i \end{bmatrix}_q$ and $\lambda = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q$.

Example 2.5. Let $i < n$ be positive integers. The affine space $\mathrm{AG}(n,q)$ can be considered as a 2-design $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of $\mathrm{AG}(n,q)$, $\mathcal{B}$ is the set of i-spaces of $\mathrm{AG}(n,q)$ and I is the set theoretical inclusion. The parameters of D are $v = q^n$, $b = q^{n-i}\begin{bmatrix} n \\ i \end{bmatrix}_q$, $k = q^i$, $r = \begin{bmatrix} n \\ i \end{bmatrix}_q$ and $\lambda = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q$.

In the rest of this paper Examples 2.4 and 2.5 will be denoted by $\mathrm{PG}^{(i)}(n,q)$ and by $\mathrm{AG}^{(i)}(n,q)$, respectively. We will use the terminology from geometry. An *i-spread*, $\mathcal{S}^i$, of $\mathrm{PG}(n,q)$ (or of $\mathrm{AG}(n,q)$) is a set of pairwise disjoint i-dimensional subspaces which gives a partition of the points of the geometry. In general, an *f-fold i-spread*, $\mathcal{S}^i_f$, is a set of i-dimensional subspaces such that every point of the geometry is contained in exactly f subspaces of $\mathcal{S}^i_f$. An *i-packing*, $\mathcal{P}^i$, of $\mathrm{PG}(n,q)$ (or of $\mathrm{AG}(n,q)$) is a set of spreads such that each i-dimensional subspace of the geometry is contained in exactly one of the spreads in $\mathcal{P}^i$, i.e., the spreads give a partition of the i-dimensional subspaces of the geometry. The i-spreads, f-fold i-spreads and i-packings induce a resolution class, an f-resolution class and a resolution in $\mathrm{PG}^{(i)}(n,q)$ (or in $\mathrm{AG}^{(i)}(n,q)$), respectively.

It is easy to construct spreads and packings in $\mathrm{AG}^{(i)}(n,q)$, because each parallel class of i-spaces is an i-spread. The situation is much more complicated in $\mathrm{PG}^{(i)}(n,q)$. There are only a few constructions of spreads. The following theorem summarizes the known existence conditions.

Theorem 2.6 ([22], Theorems 4.1 and 4.16)**.**

- *There exists an i-spread in $\mathrm{PG}^{(i)}(n,q)$ if and only if $(i+1)|(n+1)$.*
- *Suppose that i,l and n are positive integers such that $(l+1)|\gcd(i+1,n+1)$. Then there exists an f-fold i-spread in $\mathrm{PG}^{(i)}(n,q)$, where $f=(q^{i+1}-1)/(q^{l+1}-1)$.*

There exist several different 1-spreads (line spreads) in $\mathrm{PG}^{(1)}(3,q)$. We briefly mention two types. Let ℓ_1,ℓ_2 and ℓ_3 be three skew lines in $\mathrm{PG}(3,q)$. The set of the $q+1$ transversals of ℓ_1,ℓ_2 and ℓ_3 is called *regulus* and it is denoted by $\mathcal{R}(\ell_1,\ell_2,\ell_3)$. The classical construction of a line spread comes from a pencil of hyperbolic quadrics (see e.g. [20], Lemma 17.1.1) and it has the property that if it contains any three lines of a regulus $\mathcal{R}(\ell_1,\ell_2,\ell_3)$, then it contains each of the $q+1$ lines of $\mathcal{R}(\ell_1,\ell_2,\ell_3)$. This type of spread is called *regular*. A line spread in $\mathrm{PG}(3,q)$ is called *aregular*, if it contains no regulus. An example of an aregular spread can be found in [20], Lemma 17.3.3.

3 Factorizations arising from affine spaces

In this section, we investigate the spreads and packings of $\mathrm{AG}(n,q)$ and the corresponding factorizations of multigraphs. In each case we apply Lemma 2.1, so we identify the points of $\mathrm{AG}(n,q)$ with the vertices of the complete multigraph.

Theorem 3.1. *Let q be a prime power, $i<n$ be positive integers and $\lambda_i=\begin{bmatrix}n-1\\i-1\end{bmatrix}_q$. Then there exists a simple (q^i-1)-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$. $\mathcal{F}^i$ is decomposable if and only if there exists an f-fold $(i-1)$-spread in $\mathrm{PG}^{(i-1)}(n-1,q)$ for some $1\le f<\lambda_i$.*

Proof. Consider the n-dimensional affine space as $\mathrm{AG}(n,q)=\mathrm{PG}(n,q)\setminus\mathcal{H}_\infty$ where $\mathcal{H}_\infty$ is isomorphic to $\mathrm{PG}(n-1,q)$. Take the design $D=\mathrm{AG}^{(i)}(n,q)$ and apply Lemma 2.1. If Π^{i-1}_j is an $(i-1)$-space of $\mathcal{H}_\infty$, then the set of the q^{n-i} parallel affine i-spaces through Π^{i-1}_j is an i-spread of D. This spread induces a (q^i-1)-factor F^i_j for $j\in\{1,\dots,r\}$. If $\Pi^{i-1}_1,\Pi^{i-1}_2,\dots,\Pi^{i-1}_g$ are distinct $(i-1)$-spaces of $\mathcal{H}_\infty$ and they form an f-fold spread, then $f=(g(q^i-1))/(q^n-1)$, and the union of the corresponding (q^i-1)-factors F^i_j, for $j=1,2,\dots,g$, gives a (q^i-1)-factorization of fK_{q^n}. Distinct $(i-1)$-spaces of $\mathcal{H}_\infty$

obviously define distinct (q^i-1)-factors, so this factorization is simple. In particular if we consider all $(i-1)$-spaces of $\mathcal{H}_\infty$, then

$$g = \begin{bmatrix} n \\ i \end{bmatrix}_q, \; f = \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{q^i-1}{q^n-1} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q = \lambda_i,$$

hence the union of the corresponding factors gives a simple (q^i-1)-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$.

Suppose that $\mathcal{F}^i$ is decomposable, then there exist two positive integers μ_1 and μ_2 such that $\mu_1 + \mu_2 = \lambda_i$ and $\mathcal{F}^i$ can be written as the union $\mathcal{F}^i = \mathcal{F}_1 \cup \mathcal{F}_2$; $\mathcal{F}_1$ and $\mathcal{F}_2$ are (q^i-1)-factorizations of $\mu_1 K_{q^n}$ and $\mu_2 K_{q^n}$, respectively, having no (q^i-1)-factors in common, since $\mathcal{F}^i$ is simple. For $h = 1, 2$, the relation $\mu_h\binom{q^n}{2} = \binom{q^i}{2}q^{n-i}|\mathcal{F}_h|$ holds, hence $\mu_h(q^n-1) = (q^i-1)|\mathcal{F}_h|$. Without loss of generality we can set $\mathcal{F}_1 = \cup_{j=1}^{f_1} F_j^i$ with $f_1 = (\mu_1(q^n-1))/(q^i-1)$, and $\mathcal{F}_2 = \mathcal{F}^i \setminus \mathcal{F}_1$, $f_2 = |\mathcal{F}_2|$.

Let u_1 and u_2 be two affine points and let w be the point at infinity of the line u_1u_2. Since $\mathcal{F}_h$ is a factorization of $\mu_h K_{q^n}$, there are exactly μ_h factors of $\mathcal{F}_h$ containing the edge $[u_1, u_2]$, say $F_{j_1}^i, F_{j_2}^i, \ldots, F_{j_{\mu_h}}^i$. The edge $[u_1, u_2]$ belongs the $F_{j_s}^i$ if and only if $w \in \Pi_{j_s}^{i-1}$ for every $1 \le s \le \mu_h$. This happens if and only if $\cup_{j=1}^{f_h}\Pi_j^{i-1}$ contains each point of $\mathcal{H}_\infty$ exactly μ_h times, which means that $\cup_{j=1}^{f_h}\Pi_j^{i-1}$ is a μ_h-fold spread in $\mathcal{H}_\infty$, for every $h = 1, 2$. It is thus proved that if $\mathcal{F}^i$ is decomposable, then $\mathrm{PG}^{(i-1)}(n-1, q)$ posesses an f-fold spread for some $1 \le f < \lambda_i$.

Vice versa, suppose that there exists a μ_1-fold spread in $\mathrm{PG}^{(i-1)}(n-1, q)$ for some $1 \le \mu_1 < \lambda_i$. Let $\mathcal{F}_1 = \cup_{j=1}^{f_1} F_j^i$ be a μ_1-fold spread in $\mathcal{H}_\infty$. Then $|\mathcal{F}_1| = f_1 = \mu_1(q^n - 1)/(q^i-1)$. Let $\mathcal{T}$ be the set of all $(i-1)$-dimensional subspaces in $\mathcal{H}_\infty$ and let $\mathcal{F}_2 = \mathcal{T} \setminus \mathcal{F}_1$. Then $|\mathcal{T}| = \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q$, hence

$$|\mathcal{F}_2| = \begin{bmatrix} n \\ i \end{bmatrix}_q - \mu_1(q^n-1)/(q^i-1) = \left(\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q - \mu_1\right)\frac{q^n-1}{q^i-1},$$

so if $\mu_2 = \left[\begin{smallmatrix} n-1 \\ i-1 \end{smallmatrix}\right]_q - \mu_1$, then $\mathcal{F}_2$ is a μ_2-fold spread in $\mathcal{H}_\infty$ and $1 \le \mu_2 < \lambda_i$ holds.

As we have already seen, $\mathcal{F}_h$ defines a (q^i-1)-factorization of $\mu_h K_{q^n}$ for $h = 1, 2$. Then $\mathcal{F}^i = \mathcal{F}_1 \cup \mathcal{F}_2$, because $\mu_1 + \mu_2 = \lambda_i$. Hence the (q^i-1)-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$ is decomposable. □

Corollary 3.2. *If* $\gcd(i, n) > 1$ *then the* (q^i-1)*-factorization* $\mathcal{F}^i$ *of* $\lambda_i K_{q^n}$ *is decomposable.*

Proof. Let $1 < l+1$ be a divisor of $\gcd(i, n)$. Then it follows from Theorem 2.6 that there exists an $(q^i-1)/(q^{l+1}-1)$-fold spread in $\mathcal{H}_\infty$, so $\mathcal{F}^i$ is decomposable. □

To decide the decomposability of $\mathcal{F}^i$ in the cases $\gcd(i, n) = 1$ is a hard problem in general. We prove its indecomposability in the following important case.

Theorem 3.3. *The* $(q^{n-1}-1)$*-factorization* $\mathcal{F}^{n-1}$ *of* $(q^{n-1}-1)/(q-1)K_{q^n}$ *is indecomposable.*

Proof. It is enough to prove that if $\cup_{j=1}^{g}\Pi_j^{n-2}$ is an f-fold $(n-2)$-spread in $\mathcal{H}_\infty$, then $\cup_{j=1}^{g}\Pi_j^{n-2}$ consists of all $(n-2)$-dimensional subspaces of $\mathcal{H}_\infty$, because this implies $f = \lambda_{n-1}$, so the statement follows from Theorem 3.1.

Each Π_j^{n-2} contains exactly $(q^{n-1}-1)/(q-1)$ points, thus the standard double counting of the point-subspace pairs $p \in \Pi_j^{n-2}$ in $\mathcal{H}_\infty$ gives

$$g\frac{q^{n-1}-1}{q-1} = f\frac{q^n-1}{q-1},$$

hence

$$f = \frac{g(q^{n-1}-1)}{q^n-1}.$$

But $\gcd(q^n-1, q^{n-1}-1) = q-1$ and f is an integer, so $g \geq (q^n-1)/(q-1)$ which implies $g = (q^n-1)/(q-1)$, hence $f = \lambda_{n-1}$. □

In particular if $n = 2$, we get the following.

Corollary 3.4. *If q is a prime power then there exists a simple and indecomposable $(q-1)$-factorization of K_{q^2}.*

If $q = 2^r$ then each (q^i-1)-factor in $\mathcal{F}^i$ is the vertex-disjoint union of 2^{r-i} complete graphs on 2^i vertices. It is well-known that these graphs can be partitioned into one-factors in many ways (but not in all the ways, it was proved by Hartman and Rosa [19], that there is no cyclic one-factorization of K_{2^i} for $i \geq 3$), hence Theorem 3.1 implies several one-factorizations of $\lambda_i K_{2^r}$.

Each of the one-factorizations arising from $\mathcal{F}^i$ is simple, because distinct $(i-1)$-dimensional subspaces define distinct (q^i-1)-factors of $\mathcal{F}^i$, and the one-factors of $\lambda_i K_{q^n}$ arising from distinct (q^i-1)-factors of $\mathcal{F}^i$ are distinct, because they are the union of q^{n-i} one-factors on q^i vertices of a connected component.

There are both decomposable and indecomposable one-factorizations among these examples. We show it in the smallest case $q = 2$, $n = 3$. Let $\mathcal{F}^2$ be the 3-factorization of $3K_8$ induced by $\mathrm{AG}(3,2)$.

Let $\mathrm{PG}(3,2) = \mathrm{AG}(3,2) \cup \mathcal{H}_\infty$. Then $\mathcal{H}_\infty$ is isomorphic to the Fano plane. Let its points be $0, 1, 2, 3, 4, 5$ and 6 such that for $j = 0, 1, \ldots, 6$, the triples $L_j = (j, j+1, j+3)$ form the lines of the plane, where the addition is taken modulo 7. Now the 3-factors of $\mathcal{F}^2$ can be described in the following way. Let a be a fixed point in $\mathrm{AG}(3,2)$. Then L_j defines a 3-factor F_j^2 whose connected components are complete graphs $K_{2^i} = K_4$. Let $L_{j,a}$ be the complete graph containing a, and let $L_{j,\overline{a}}$ be the other component of F_j^2.

$\mathcal{H}_\infty$ defines one-factors and a one-factorization of K_8 in the following obvious way. The edge joining two points of $\mathrm{AG}(3,2)$, say b and c, belong to the one-factor G_s if and only if b, c and s are collinear points in $\mathrm{PG}(3,2)$. Then $\mathcal{G} = \cup_{s=0}^{6} G_s$ is a one-factorization of K_8.

We can define a decomposable one-factorization of $3K_8$ in the following way. Take $L_{j,a}$ and $L_{j,\overline{a}}$ and let $s \in L_j$ be any point. Then G_s gives a one-factor of $L_{j,a}$ and a one-factor of $L_{j,\overline{a}}$. Hence $\mathcal{G}_j = \cup_{s\in L_j} G_s$ is the union of three one-factors of $3K_8$, and $\mathcal{G}' = \cup_{j=0}^{6}\mathcal{G}_j$ is a one-factorization of $3K_8$.

In $\mathcal{H}_\infty$ there are three lines through the point s, hence $\mathcal{G}'$ contains each one-factor G_s three times. Thus $\mathcal{G}'$ is decomposable, because it is obviously the union of three copies of $\mathcal{G}$.

But we can define an indecomposable one-factorization, too. Let L_j be a line in $\mathcal{H}_\infty$, take $L_{j,a}$ and $L_{j,\overline{a}}$ and let M_j^1 be the one-factor which contains the following pairs of points in $\mathrm{AG}(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and b,c,j are collinear in $\mathrm{PG}(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and $b,c,j+1$ are collinear in $\mathrm{PG}(3,2)$.

Let M_j^2 be the one-factor which contains the following pairs of points in $\mathrm{AG}(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and $b,c,j+1$ are collinear in $\mathrm{PG}(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and $b,c,j+3$ are collinear in $\mathrm{PG}(3,2)$.

Finally let M_j^3 be the one-factor which contains the following pairs of points in $\mathrm{AG}(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and $b,c,j+3$ are collinear in $\mathrm{PG}(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and b,c,j are collinear in $\mathrm{PG}(3,2)$.

Then $\mathcal{M}_j = \cup_{t=1}^{3} M_j^t$ is a union of three one-factors of $3K_8$, and $\mathcal{M} = \cup_{j=0}^{6} \mathcal{M}_j$ is a one-factorization of $3K_8$.

Suppose that this one-factorization is decomposable. Then it contains a one-factorization $\mathcal{E}$ of K_8. $\mathcal{E}$ is the union of seven one-factors. We may assume without loss of generality, that M_0^1 belongs to $\mathcal{E}$. It contains an edge through a, let it be (a,b), and a pair (c,d) for which the lines ab and cd are parallel lines in $\mathrm{AG}(3,2)$. There are two more lines in the parallel class of ab, say ef and gh. It follows from the definition of the one-factors that exactly one of them contains the pairs (e,f) and (a,b), another one contains the pairs (e,f) and (c,d), and a third one contains the pairs (e,f) and (g,h). But $\mathcal{E}$ contains each pair exactly once, hence it must contain the one-factor containing the pairs (e,f) and (g,h). But this is a one-factor of type M_0^t, where $t \neq 1$. Hence $\mathcal{E}$ contains M_0^t where $t=2$ or 3. If we repeat the previous argument, we get that $\mathcal{E}$ must contain M_0^l for $1 \neq l \neq t$, too. Thus $\mathcal{E}$ is the union of triples of type M_j^t, $t=1,2,3$, but this is a contradiction, because $\mathcal{E}$ consists of seven one-factors.

4 Factorizations arising from projective spaces

There are two basic types of partitioning the point-set of finite projective spaces. Both types give factorizations of some multigraphs. In this section we discuss these constructions.

4.1 Spreads consisting of subspaces

It is easy to construct spreads in $\mathrm{PG}^{(i)}(n,q)$, Theorem 2.6 gives a necessary and sufficient existence condition. Packings are much more complicated objects. Only a few packings in $\mathrm{PG}^{(1)}(n,q)$ have been constructed so far. In each case of the known packings either n or q satisfies some conditions.

Theorem 4.1 (Beutelspacher, [6]). *Let $1<k$ be an integer and let $n=2^k-1$. Then there exists a packing in $\mathrm{PG}^{(1)}(n,q)$.*

Theorem 4.2 (Baker, [5]). *Let $1<k$ be an integer. Then there exists a packing in $\mathrm{PG}^{(1)}(2k-1,2)$.*

Applying the Basic Construction Lemma, we get the following existence theorems.

Corollary 4.3. *Let q be a prime power, $1<k$ be an integer and $v=\frac{q^{2^k}-1}{q-1}$. Then there exists a q-factorization of K_v induced by a line-packing in $\mathrm{PG}(2^k-1,q)$.*

Corollary 4.4. *Let $1 < k$ be an integer and $v = \frac{q^{2k}-1}{q-1}$. There exists a 2-factorization K_v induced by a line-packing in* $\mathrm{PG}(2k-1, 2)$.

If $k = 2$ then Corollary 4.4 gives a solution of *Kirkman's fifteen schoolgirls problem*, which was first posed in 1850 (for the history of the problem we refer to [7]), while Corollary 4.3 gives a solution of the generalised problem in the case of $(q^2+1)(q+1)$ schoolgirls.

The complete classification of packings in $\mathrm{PG}^{(i)}(n, q)$ is known only in the case $i = 1$, $n = 3$ and $q = 2$. There are 240 projectively distinct packings of lines in $\mathrm{PG}(3, 2)$ (see [20], Subsection 17.5).

If $\gcd(q + 1, 3) = 3$, then there is a construction of aregular spreads in $\mathrm{PG}^{(1)}(3, q)$ due to Bruen and Hirschfeld [11] which is completly different from the constructions of Theorems 4.1 and 4.2. It is based on the geometric properties of twisted cubics.

A normal rational curve of order 3 in $\mathrm{PG}(3, q)$ is called *twisted cubic*. It is known that a twisted cubic is projectively equivalent to the set of points $\{(t^3 : t^2 : t : 1) : t \in \mathrm{GF}(q)\} \cup \{(1 : 0 : 0 : 0)\}$. In [20] it was shown that there exist aregular spreads given by a twisted cubic. For a detailed description of twisted cubics and the proofs of the following theorems we refer to [20], Section 21.

Theorem 4.5. *Let G_q be the group of projectivities in $\mathrm{PG}(3, q)$ fixing a twisted cubic $\mathcal{C}$. Then*

- $G_q \cong \mathrm{PGL}(2, q)$ *and it acts triply transitively on the points of $\mathcal{C}$.*
- *If $q \geq 5$ then the number of twisted cubics in $\mathrm{PG}(3, q)$ is $q^5(q^4 - 1)(q^3 - 1)$.*

Theorem 4.6. *Let $\mathcal{C}$ be a twisted cubic in $\mathrm{PG}(3, q)$. If $\gcd(q + 1, 3) = 3$, then there exists a spread in $\mathrm{PG}^{(1)}(3, q)$ induced by $\mathcal{C}$.*

Using the spreads associated to twisted cubics and the Basic Construction Lemma, we get the following multigraph factorization.

Theorem 4.7. *Let $q \geq 5$ be a prime power, $\lambda = q^5(q^4-1)(q-1)$ and $v = q^3 + q^2 + q + 1$. If $\gcd(q + 1, 3) = 3$, then there exists a simple q-factorization of λK_v induced by the set of twisted cubics in $\mathrm{PG}(3, q)$.*

Proof. Let C be the set of twisted cubics in $\mathrm{PG}(3, q)$. For $\mathcal{C} \in \mathsf{C}$ let $\mathcal{L}_{\mathcal{C}}$ be the spread in $\mathrm{PG}^{(1)}(3, q)$ induced by $\mathcal{C}$. If ℓ is a line and c_ℓ denotes the number of twisted cubics $\mathcal{C}$ with the property that ℓ belongs to $\mathcal{L}_{\mathcal{C}}$, then it follows from Theorem 4.5 that c_ℓ does not depend on ℓ. Hence

$$\begin{aligned} c_\ell &= \frac{|\{\text{twisted cubics in } \mathrm{PG}(3,q)\}| \times |\{\text{lines in a spread of } \mathrm{PG}(3,q)\}|}{|\{\text{lines in } \mathrm{PG}(3,q)\}|} \\ &= \frac{q^5(q^4-1)(q^3-1) \times (q^2+1)}{(q^2+1)(q^2+q+1)} = q^5(q^4-1)(q-1). \end{aligned}$$

Thus C induces a $|\mathsf{C}|$-fold spread in $\mathrm{PG}^{(1)}(3, q)$. Each spread $\mathcal{L}_{\mathcal{C}}$ induces a q-factor in K_v, therefore the Basic Construction Lemma gives that $\bigcup_{\mathcal{C} \in \mathsf{C}} \mathcal{L}_{\mathcal{C}}$ is a q-factorization of λK_v. Any two distinct twisted cubics define different spreads, hence the factorization is simple by definition. □

4.2 Constructions from subgeometries

If the order of the base field is not prime, then projective spaces can be partitioned by subgeometries. Let $1 < k$ be an integer. Since $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(q^k)$, so $\mathrm{PG}(n,q)$ is naturally embedded into $\mathrm{PG}(n,q^k)$ if the coordinate system is fixed. Any $\mathrm{PG}(n,q)$ embedded into $\mathrm{PG}(n,q^k)$ is called a *subgeometry*. Using cyclic projectivities one can prove that any $\mathrm{PG}(n,q^k)$ can be partitioned by subgeometries $\mathrm{PG}(n,q)$. For a detailed description of cyclic projectivities, subgeometries, and the proofs of the following three theorems we refer to [22], Section 4.

Theorem 4.8 ([22], Lemma 4.20)**.** *Let $s(n,q,q^k)$ denote the number of subgeometries $\mathrm{PG}(n,q)$ in $\mathrm{PG}(n,q^k)$. Then*

$$s(n,q,q^k) = q^{\binom{n+1}{2}(k-1)} \prod_{i=2}^{n+1} \frac{q^{ki}-1}{q^i-1}.$$

Theorem 4.9 ([22], Theorem 4.29)**.** $\mathrm{PG}(n,q^k)$ *can be partitioned into* $\theta(n,q,q^k) = \frac{(q^{k(n+1)}-1)(q-1)}{(q^k-1)(q^{n+1}-1)}$ *disjoint subgeometries* $\mathrm{PG}(n,q)$ *if and only if* $\gcd(k,n+1)=1$.

Theorem 4.10 ([22], Theorem 4.35)**.** *Suppose that* $\gcd(k,n+1)=1$. *Let $p_0(n,q,q^k)$ denote the number of projectivities which act cyclically on a $\mathrm{PG}(n,q)$ of $\mathrm{PG}(n,q^k)$ such that determine different partitions. Then*

$$p_0(n,q,q^k) = q^{k\binom{n+1}{2}} \frac{\prod_{i=1}^{n}(q^{ki}-1)}{n+1}.$$

Any given subgeometry $\mathrm{PG}(n,q)$ is contained in

$$\rho_0(n,q) = q^{\binom{n+1}{2}} \frac{\prod_{i=1}^{n}(q^{i}-1)}{n+1}$$

of these partitions.

We can consider the partitions of the point-set of $\mathrm{PG}(n,q^k)$ by subgeometries $\mathrm{PG}(n,q)$.

Each partition of $\mathrm{PG}(n,q^k)$ into subgeometries $\mathrm{PG}(n,q)$ defines a $\left(\frac{q(q^n-1)}{q-1}\right)$-factor of K_v, with $v = \frac{q^{k(n+1)}-1}{q^k-1}$. Each projectivity which acts cyclically on a $\mathrm{PG}(n,q)$ defines a $\left(\frac{q(q^n-1)}{q-1}\right)$-factorizations of the corresponding complete multigraph.

Theorem 4.11. *Let q be a prime power, $1 < k$ and n be positive integers for which $\gcd(k,n+1)=1$ holds. Let $\lambda = \frac{q^{\binom{n+1}{2}k}(q^k-1)(q^n-1)}{q^{k-1}(n+1)(q-1)} \prod_{i=1}^{n-1}(q^{ki}-1)$ and $v = \frac{q^{k(n+1)}-1}{q^k-1}$. Then there exist a simple $\left(q\frac{q^n-1}{q-1}\right)$-factorization of λK_v induced by the set of those projectivities which act cyclically on a $\mathrm{PG}(n,q)$ of $\mathrm{PG}(n,q^k)$ such that they determine different partitions.*

Proof. It follows from Theorem 4.8 that the number S_e of subgeometries $\mathrm{PG}(n,q)$ through two points of $\mathrm{PG}(n,q^k)$ is

$$S_e = \frac{s(n,q,q^k) \times |\{\text{points in } \mathrm{PG}(n,q)\}| \times (|\{\text{points in } \mathrm{PG}(n,q)\}| - 1)}{|\{\text{points in } \mathrm{PG}(n,q^k)\}| \times (|\{\text{points in } \mathrm{PG}(n,q^k)\}| - 1)}$$
$$= \frac{q^{\binom{n+1}{2}(k-1)}(q^k-1)}{q^{k-1}(q-1)} \prod_{i=1}^{n-1} \frac{q^{ki}-1}{q^i-1}.$$

Each cyclic projectivity determines different partitions, hence it determines different factors. Thus $\lambda = S_e \times \rho_0(n,q)$. □

We cannot decide the decomposability of the factorization construted in the previous theorem in general, but we can prove the existence of indecomposable factorizations in some cases. To do this we need the following result from number theory.

Lemma 4.12 ([22], Lemma 4.24)**.** *If r, s and x are positive integers with $x > 1$, then $\frac{(x^{rs}-1)(x-1)}{(x^r-1)(x^s-1)}$ is an integer if and only if $\gcd(r,s) = 1$.*

We apply it in a particular case.

Proposition 4.13. *Let q be a prime power, $1 < k$ and n be positive integers for which $\gcd(k,n+1) = 1$ and $\gcd(k,n) \neq 1$ hold. Let $d = \gcd\left(\frac{q^{kn}-1}{q^k-1}, \frac{q^n-1}{q-1}\right)$, $v = \frac{q^{k(n+1)}-1}{q^k-1}$ and $m = q\frac{q^n-1}{q-1}$. Suppose that $\mathcal{F}$ is an m-factorization of λK_v for some λ such that each factor is the disjoint union of $\theta(n,q,q^k)$ complete graphs on $(q^{n+1}-1)/(q-1)$ vertices. If f denotes the number of m-factors in $\mathcal{F}$ then $\frac{q^n-1}{d(q-1)}$ divides λ and $q^{k-1}\frac{q^{kn}-1}{d(q^k-1)}$ divides f.*

Proof. The standard double counting gives

$$\lambda \times \binom{v}{2} = \binom{m+1}{2} \times \theta(n,q,q^k) \times f,$$

thus $\lambda \times q^{k-1}\frac{q^{kn}-1}{d(q^k-1)} = f \times \frac{q^n-1}{d(q-1)}$. Because of Lemma 4.12, $\frac{q^n-1}{d(q-1)}$ divides λ, hence $q^{k-1}\frac{q^{kn}-1}{d(q^k-1)}$ divides f. □

As a direct corollary of the previous proposition we get the following result about the indecomposibility of the factorizations constructed in Theorem 4.11 .

Theorem 4.14. *Let q be a prime power, $1 < k$ and n be positive integers for which $\gcd(k,n+1) = 1$ and $\gcd(k,n) \neq 1$ hold. Let $d = \gcd\left(\frac{q^{kn}-1}{q^k-1}, \frac{q^n-1}{q-1}\right)$, $v = \frac{q^{k(n+1)}-1}{q^k-1}$ and $m = q\frac{q^n-1}{q-1}$. Then there exist a simple and indecomposable m-factorization of λK_v, where $\lambda = t\frac{q^n-1}{d(q-1)}$ for some t in $\{1,\dots,d\frac{q^{\binom{n+1}{2}k}(q^k-1)}{q^{k-1}(n+1)}\prod_{i=1}^{n-1}(q^{ki}-1)\}$.*

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References

[1] P. Adams, D. Bryant and B. Maenhaut, Cube factorizations of complete graphs, *J. Combin. Des.* **12** (2004), 381–388.

[2] B. Alspach, The Oberwolfach problem, in: C. J. Colbourn and J. H. Dinitz (eds), *CRC Handbook of Combinatorial Designs,* CRC Press, Boca Raton, FL, 1996, 394–395.

[3] D. Archdeacon and J. H. Dinitz, Constructing indecomposable 1-factorizations of the complete multigraph, *Discrete Math.* **92** (1991), 9–19.

[4] A. H. Baartmans and W. D. Wallis, Indecomposable factorizations of multigraphs, *Discrete Math.* **78** (1989), 37–43.

[5] R. D. Baker, Partitioning the planes of $\mathrm{AG}_{2m}(2)$ into 2-designs, *Discrete Math.* **15** (1976), 205–211.

[6] A. Beutelspacher, On parallelisms in finite projective spaces, *Geom. Dedicata* **3** (1974), 35–40.

[7] N. L. Biggs, T. P. Kirkman, mathematician, *Bull. London Math. Soc.* **13** (1981), 97–120.

[8] S. Bonvicini, G. Mazzuoccolo and G. Rinaldi, On 2-factorizations of the complete graph: from the k-pyramidal to the universal property, *J. Combin. Des.* **17** (2009), 211–228.

[9] R. C. Bose, A note on the resolvability of balanced incomplete block designs, *Sankhya* **6** (1942), 105–110.

[10] R. H. Bruck, Construction problems in finite projective spaces, in: A. Barlotti (ed.), *Finite geometric structures and their applications*, Edizioni Cremonese, Rome, 1973, 105–188.

[11] A. A. Bruen and J. W. P. Hirschfeld, Applications of line geometry over finite fields, I: The twisted cubic, *Geom. Dedicata* **6** (1977), 495–509.

[12] D. Bryant, D. Horsley, B. Maenhaut and B. R. Smith, Cycle decompositions of complete multigraphs, *J. Combin. Des.* **19** (2011), 42–69.

[13] M. Buratti and G. Rinaldi, 1-rotational k-factorizations of the complete graph and new solutions to the Oberwolfach problem, *J. Combin. Des.* **16** (2008), 87–100.

[14] P. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, London Mathematical Society Student Texts 22, Cambridge University Press, Cambridge, 1991.

[15] W. Chu, New constructions of simple and indecomposable 1-factorizations of complete multigraphs, *J. Statist. Plann. Inference* **94** (2001), 181–196.

[16] C. J. Colbourn, M. J. Colbourn and A. Rosa, Indecomposable 1-factorizations of the complete multigraph, *J. Austral. Math. Soc. Ser. A* **39** (1985), 334–343.

[17] G. L. Ebert, Partitioning projective geometries into caps, *Canad. J. Math.* **37** (1985), 1163–1175.

[18] M. N. Ferencak and A. J. W. Hilton, Regular multigraphs and their semiregular factorizations, *Congr. Numer.* **209** (2011), 149–159.

[19] A. Hartman and A. Rosa, Cyclic one-factorization of the complete graph, *European. J. Combin.* **6** (1985), 45–48.

[20] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, Clarendon Press, Oxford, 1985.

[21] J. W. P. Hirschfeld J. A. Thas, *General Galois geometries*, Clarendon Press, Oxford, 1991.

[22] J. W. P. Hirschfeld, *Projective geometries over finite fields*, second ed., Clarendon Press, Oxford, 1998.

[23] B. C. Kestenband, Projective geometries that are disjoint unions of caps, *Canad. J. Math.* **32** (1980), 1299–1305.

[24] G. B. Khosrovshahi, Ch. Maysoori and B. Tayfeh-Rezaie, A note on 3-factorizations of K_{10}, *J. Combin. Des.* **9** (2001), 379–383.

[25] Gy. Kiss, One-factorizations of complete multigraphs and quadrics in $\mathrm{PG}(n, q)$, *J. Combin. Des.* **10** (2002), 139–143.

[26] G. Korchmáros, A. Siciliano and A. Sonnino, 1-factorizations of complete multigraphs arising from finite geometry, *J. Combin. Theory Ser. A* **93** (2001), 385–390.

[27] B. R. Smith, Cycle decompositions of complete multigraphs, *J. Combin. Des.* **18** (2010), 85–93.

[28] A. Sonnino, One-factorizations of complete multigraphs arising from maximal $(k; n)$-arcs in $\mathrm{PG}(2, 2^h)$, *Discrete Math.* **231** (2001), 447–451.

[29] W. D. Wallis, *One-factorizations*, Mathematics and its Applications, 390, Kluwer Academic Publishers Group, Dordrecht, 1997.

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Regular embeddings of cycles with multiple edges revisited

Dedicated to Dragan Marušič on the occasion of his 60th birthday

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Abstract

Regular embeddings of cycles with multiple edges have been reappearing in the literature for quite some time, both in and outside topological graph theory. The present paper aims to draw a complete picture of these maps by providing a detailed description, classification, and enumeration of regular embeddings of cycles with multiple edges on both orientable and non-orientable surfaces. Most of the results have been known in one form or another, but here they are presented from a unique viewpoint based on finite group theory. Our approach brings additional information about both the maps and their automorphism groups, and also gives extra insight into their relationships.

Keywords: Regular embedding, multiple edge, Hölder's Theorem, Möbius map.

Math. Subj. Class.: 20B25, 05C10

1 Introduction

Classification of all regular embeddings of a given graph on orientable or non-orientable surfaces has been addressed by many researchers in topological graph theory. On the abstract level, the classification problem was solved by Gardiner et al. where graphs which underlie a regular map were characterised by means of a condition requiring the existence of certain subgroups in the graph automorphism group [7]. This condition allows one to identify all existing map automorphism groups within the graph automorphism group and subsequently to determine all regular embeddings of the graph that have a specified subgroup as its automorphism group. Nevertheless, practical application of the condition depends on understanding the structure of the automorphism group of the graph and therefore has serious limitations. At present, a complete classification is known for only a few infinite classes of graphs, most notably, for complete graphs [8, 9, 27], complete bipartite graphs [11, 22], hypercubes [4, 14, 21], and for several others basic classes of graphs (see for example [6, 29]).

In this paper we focus on regular maps whose underlying graph is a cycle with multiple edges; for brevity we call such maps *multicyclic*. Multicyclic maps can be regarded as combinations of two well understood families of maps: spherical embeddings of cycles and dipole maps. The study of these maps has a fairly long history which exceeds the context of the proper topological graph theory [1, 3, 5, 17]. In early papers, these maps typically occur in the dual form as maps where each face meets precisely two others; in [26] such maps are called *bicontactual*. An important special case, regular maps with two faces, was extensively discussed by Coxeter and Moser in their celebrated book [5], mentioning much older works of Brahana [2] and Threlfall [24]. In full generality, bicontactual regular maps were first considered in 1985 by Wilson [26]. Using a geometric approach depending on tracing map diagrams Wilson derived a classification of bicontactual maps on both orientable and non-orientable surfaces. Unfortunately, his result does not immediately translate via surface duality to regular embeddings of cycles with multiple edges, and even the basic information such as the orientability character or the number of regular

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embeddings for a given length and multiplicity of a cycle is hard to extract.

In 1989, Wilson [27, Theorem 3] proved that a regular embedding of any graph with multiple edges is either a totally branched covering over a regular embedding of the corresponding simple graph, or a *cantankerous* map, a regular map with edge-multiplicity 2 where every 2-cycle is orientation-reversing. He went on to show that (among other things) an even cycle has a regular embedding with every multiplicity while an odd cycle has a regular embedding with every odd multiplicity but with no even multiplicity.

Cantankerous maps, under a more appropriate term *Möbius regular maps*, were again studied by Li and Širáň in [15, 16] within the context of maps with an unfaithful action of the automorphism group on the vertex set. With the help of general results about unfaithful maps they produced a classification of all multicyclic regular maps on both orientable and non-orientable surfaces [15, Propositions 7 and 8]. In particular, they proved that the doubled n-cycle admits a Möbius regular embedding if and only if n is divisible by 3. Nevertheless, neither the enumeration of isomorphism classes of the maps nor orientable regularity were treated in their papers.

To complete the history of classification of multicyclic regular maps we should mention an unpublished work of Škoviera and Zlatoš [23] where a general framework for the study of regular embeddings of graphs with multiplicity m based on $\mathbb{Z}_m$-valuations was developed. In a subsequent work [22], this theory was applied to deriving a classification of orientably regular embeddings of multicycles. Although the latter classification enables enumeration, no information about the automorphism groups of the maps was given.

The present paper intends to complete the picture of multicyclic regular maps by providing a detailed description, classification, and enumeration of multicyclic maps on both orientable and non-orientable surfaces along with additional information about the automorphism groups of maps and relationships between them. In contrast to the majority of previous papers on this topic we deal with both *orientably regular maps* (where map automorphisms are necessarily orientation-preserving) and with *regular maps* (where map automorphisms include reflections and the maps may also lie on non-orientable surfaces). Our interest in these maps is substantiated by the fact that large classes of regular maps have multicyclic quotients and that multicyclic maps can often serve as important extremal examples [1, 3, 17].

The following two theorems, stated in a simplified enumerative form, are our main results. More detailed statements can be found in the subsequent sections. Throughout the paper $C_n^{(m)}$ denotes the graph resulting from the cycle C_n of length n by replacing each edge with m parallel edges.

Theorem 1.1. *Let p be the number of orientably regular embeddings of the graph $C_n^{(m)}$ where $n \geq 3$ and $m \geq 2$, and let $\mu(m)$ denote the number of solutions of the congruence $e^2 \equiv 1 \pmod{m}$. Then*

(i) $p = 0$ *if n is odd and m is even,*

(ii) $p = 1$ *if both n and m are odd,*

(iii) $p = \mu(m)$ *if n is even and m is odd,*

(iv) $p = 2\mu(m)$ *if $n \equiv 0 \pmod{4}$ and m is even,*

(v) $p = 2\mu(2m)$ *if $n \equiv 2 \pmod{4}$ and m is even.*

Furthermore, all these embeddings are reflexible.

Theorem 1.2. *Let q be the number of non-orientable regular embeddings of the graph $C_n^{(m)}$ where $n \geq 3$ and $m \geq 2$. Then*

(i) $q = 1$ *if both n and m are odd, in which case the antipodal double cover of the map is an orientably regular embedding of $C_{2n}^{(m)}$ corresponding to the solution $e = -1$ of the congruence $e^2 \equiv 1 \pmod{m}$ listed in item* (iii) *of Theorem 1.1,*

(ii) $q = 1$ *if $n \equiv 0 \pmod 3$ and $m = 2$, in which case the map is a Möbius map, and*

(iii) $q = 0$ *in all other cases.*

Theorem 1.2 has an interesting corollary which strengthens a result of Wilson [26] and shows that there exist no regular maps with nilpotent automorphism group on non-orientable surfaces of genus greater than 1. In contrast, nilpotent regular maps on orientable surfaces of arbitrarily large genus are abundant; for further information see [18].

Theorem 1.3. *There exist no non-orientable regular maps whose automorphism group is nilpotent except the dihedral embeddings of bouquets of 2^n circles into the projective plane and their duals, embeddings of cycles of length 2^n, for $n \geq 1$.*

2 Orientably regular embeddings of $C_n^{(m)}$

It is well-known that every d-valent orientably regular map $\mathcal{M}$ can be identified with a triple $(G;\, x, y)$ where G is a finite group and $\{x, y\}$ is a generating set for G with $x^d = y^2 = 1$; see, for example [10, 21]. Such a triple is called an *algebraic orientably regular map*. Elements of the group represent darts of $\mathcal{M}$, that is, edges endowed with an orientation. The right translation $g \mapsto gx$, $g \in G$, by the generator x corresponds to the *rotation* of the map, the permutation that cyclically permutes darts directed away from vertices consistently with the orientation of the surface. The translation $g \mapsto gy$, $g \in G$, by the generator y corresponds to the *dart-reversing involution*, which switches the direction of each dart to the opposite direction. The vertices, edges, and faces of $\mathcal{M}$ are in a one-to-one correspondence with the left cosets of $\langle x\rangle$, $\langle y\rangle$, and $\langle xy\rangle$, respectively, and the incidence between the objects corresponds to non-empty intersection of cosets.

A *map homomorphism* $(G;\, x, y) \to (G';\, x', y')$ between algebraic maps $(G;\, x, y)$ and $(G';\, x', y')$ is a group homomorphism $G \to G'$ that takes x to x' and y to y'. In topological terms, a map homomorphism corresponds to an orientation-preserving covering projection of maps, possibly branched over vertices, face-centres, and free ends of semiedges (where the branching index must be 2). It follows that two algebraic maps $(G;\, x, y)$ and $(G;\, x', y')$ represent isomorphic orientably regular maps if and only if there is an automorphism of G taking x to x' and y to y'. Each automorphism of the map $\mathcal{M} = (G;\, x, y)$ corresponds to a left translation $g \mapsto ag$, $g \in G$, where a is a fixed element of G. In particular, the group $\mathrm{Aut}^+(\mathcal{M})$ of all orientation preserving automorphisms of $\mathcal{M}$ is isomorphic to G. The map automorphism corresponding to the generator x generates a cyclic vertex-stabiliser in the automorphism group of $(G;\, x, y)$, while y generates the edge-stabiliser, which is necessarily of order two. An orientably regular map $\mathcal{M} = (G;\, x, y)$ is *reflexible* if it is isomorphic to its mirror image $\mathcal{M}^{-1} = (G;\, x^{-1}, y)$; otherwise $(G;\, x, y)$ is *chiral*.

In what follows, we often identify the group G that underlies an algebraic map $\mathcal{M} = (G;\, x, y)$ with its left regular representation, and it should be easy for the reader to see from the context which notion is in use.

Before proceeding to the classification of orientably regular embeddings of cycles with multiple edges we present a general result about orientably regular embeddings of graphs with multiple edges. For a non-trivial simple graph X let $X^{(m)}$ denote the graph arising from X by replacing each edge with m parallel edges. To avoid trivial cases, the multiplicity m of every graph $X^{(m)}$ will always be at least 2. We show that every orientably regular embedding of $X^{(m)}$ determines two orientably regular maps, an orientably regular embedding of X and a regular embedding of the *dipole graph* D_m with multiplicity m, which consists of two vertices and m parallel edges joining them. In this context it may be useful to recall a result from [19] and [18] that every orientably regular embedding of D_m is isomorphic to a map $\mathcal{D}(m,e)$ arising from the metacyclic group $G(m,e)$ given by the presentation

$$G(m,e) = \langle x, y \mid x^m = y^2 = 1, yxy = x^e\rangle.$$

where $e^2 \equiv 1 \pmod m$. Moreover, two dipole maps $\mathcal{D}(m,e)$ and $\mathcal{D}(m,e')$ are isomorphic if and only if $e \equiv e' \pmod m$.

We are now ready for the result about the structure of orientably regular maps with multiple edges.

Theorem 2.1. *Let $\mathcal{M} = (G; x, y)$ be a regular map of valency d with underlying graph $X^{(m)}$ of order at least 2. Set $A = \langle x^{d/m}\rangle$ and $B = \langle x^{d/m}, y\rangle$. Then:*

(i) *The group A is a normal subgroup of G, the map $\mathcal{M}/A = (G/A; xA, yA)$ is a regular embedding of X, and the natural projection $\mathcal{M} \to \mathcal{M}/A$ is a map homomorphism bijective on the vertices.*

(ii) *$\mathcal{M}' = (B; x^{d/m}, y)$ is a dipole map isomorphic to $\mathcal{D}(m,e)$ for some integer e such that $e^2 \equiv 1 \pmod m$.*

(iii) *$G = \langle x, y \mid x^{km} = y^2 = 1, yx^k y = x^{ek}, \dots\rangle$, where $e^2 \equiv 1 \pmod m$ and $k = d/m$ is the valency of X. In particular, the multiplicity m of $\mathcal{M}$ is the largest positive divisor q of d such that $\langle x^{d/q}\rangle \trianglelefteq G$.*

(iv) *If $\mathcal{M}$ is not bipartite, then $e \equiv 1 \pmod m$.*

Proof. By our assumption, $\mathcal{M}$ contains neither loops nor semiedges. Since any two vertices of $\mathcal{M}$ are joined by m parallel edges and G acts regularly on the darts of $\mathcal{M}$, the subgroup $A = \langle x^k\rangle$ fixes two vertices and acts regularly on the set of edges joining them. Applying the regularity again, A fixes all the vertices of $\mathcal{M}$ pointwise. In particular, A is a normal subgroup of G. The natural projection $\mathcal{M} = (G; x, y) \to (G/A; xA, yA) = \mathcal{M}/A$ is a map homomorphism which is bijective on the vertices. It follows that the underlying graph of $\mathcal{M}/A$ is X. This proves (i).

By definition, y transposes a pair of adjacent vertices. It follows that $\mathcal{M}' = (B; x^k, y)$ is an orientably regular map with two vertices and m parallel edges. Since B contains the cyclic group A as a subgroup of index 2, B is a metacyclic group with presentation

$$B = \langle x^k, y \mid (x^k)^m = y^2 = 1, (x^k)^y = x^{ek}\rangle$$

where $e^2 \equiv 1 \pmod m$ (see [19]). It follows that G has a presentation as stated in (iii). In particular, we see that the multiplicity m of $\mathcal{M}$ is the largest positive divisor q of d such that $\langle x^{d/q}\rangle \trianglelefteq G$. This proves (ii) and (iii).

To finish the proof, assume that $\mathcal{M}$ is non-bipartite. Thus there exists a relation

$$w(x,y) = x^{a_1} y x^{a_2} y \dots x^{a_{2r+1}} y = 1$$

where y appears an odd number of times, say $2r+1$ times. Then

$$x^{d/m} = x^{d/m}w(x,y) = w(x,y)(x^{d/m})^{e^{2r+1}} = x^{de/m},$$

and hence $e \equiv 1 \pmod{m}$, as required. □

Now we proceed to multicyclic regular maps. We start by introducing a family of orientably regular maps $\mathcal{C}(n,m;e,f)$ as follows. Let $\mathcal{C}(n,m;e,f) = (G;\, x,y)$ where $G = G(n,m;e,f)$ is a group given by the presentation

$$G(n,m;e,f) = \langle x, y \mid x^{2m} = y^2 = 1, y^{-1}x^2y = (x^2)^e, (xy)^n = (x^2)^f\rangle. \tag{2.1}$$

The parameters m and n are positive integers, $n \geq 3$, and $e, f \in \mathbb{Z}_m$.

We now show that every orientably regular embedding of $C_n^{(m)}$ is isomorphic to one of the maps $\mathcal{C}(n,m;e,f)$ for suitable integers e and f, and classify these maps up to isomorphism.

Theorem 2.2. *The graph $C_n^{(m)}$, with $n \geq 3$ and $m \geq 2$, has an orientably regular embedding for each n and m, unless n is odd and m is even. Every such embedding is reflexible and is isomorphic to one of the maps $\mathcal{C}(n,m;\, e,f)$ where e and f are as follows:*

(i) *If both n and m are odd, then $e = 1$ and $f \equiv (n+m)/2 \pmod{m}$. In particular, there is only one orientably regular embedding in this case.*

(ii) *If $n \equiv 0 \pmod 4$ and m is odd, then $e^2 \equiv 1 \pmod{m}$ and $f \equiv (e+1)n/4 \pmod{m}$.*

(iii) *If $n \equiv 2 \pmod 4$ and m is odd, then $e^2 \equiv 1 \pmod{m}$ and $f \equiv ((e+1)n+2m)/4 \pmod{m}$ for even e, and $e^2 \equiv 1 \pmod{2m}$ and $f \equiv (e+1)n/4 \pmod{m}$ for odd e.*

(iv) *If $n \equiv 0 \pmod 4$ and m is even, then $e^2 \equiv 1 \pmod{m}$ and $f \equiv (e+1)n/4 \pmod{m}$ or $f \equiv ((e+1)n+2m)/4 \pmod{m}$.*

(v) *If $n \equiv 2 \pmod 4$ and m is even, then $e^2 \equiv 1 \pmod{2m}$ and $f \equiv (e+1)n/4 \pmod{m}$ or $f \equiv ((e+1)n+2m)/4 \pmod{m}$.*

Two such embeddings $\mathcal{C}(n,m;e,f)$ and $\mathcal{C}(n,m;e',f')$ are isomorphic if and only if $e \equiv e' \pmod{m}$ and $f \equiv f' \pmod{m}$.

Reflexible orientably regular embeddings of $C_n^{(m)}$ were previously classified in [15, Proposition 7] leaving the possibility for the existence chiral maps open. It follows from Theorem 2.2 that no chiral embeddings of $C_n^{(m)}$ exist and therefore the two families coincide.

Our proof of Theorem 2.2 uses a classical result of Hölder concerning the structure of metacyclic groups (see Zassenhaus [28, p. 99]).

Theorem 2.3 (Hölder's Theorem)**.** *Every extension of a cyclic group of order $m \geq 2$ by a cyclic group of order $n \geq 2$ is determined by two integers e and f satisfying the congruences*

$$e^n \equiv 1 \pmod{m} \quad \textit{and} \quad f(e-1) \equiv 0 \pmod{m},$$

and is isomorphic to the group $G(e, f)$ *with presentation*

$$G(e,f) = \langle a, b \mid a^m = 1, b^n = a^f, b^{-1}ab = a^e \rangle.$$

Furthermore, the extension determined by e *and* f *is equivalent to that determined by* e' *and* f' *if and only if* $e \equiv e' \pmod{m}$ *and* $f \equiv f' \pmod{m}$.

Proof of Theorem 2.2. Let $\mathcal{M} = (G; x, y)$ be an orientably regular embedding of the graph $C_n^{(m)}$. By Theorem 2.1(i), $A = \langle x^2 \rangle \trianglelefteq G$ and $\mathcal{M}/A = (G/A; xA, yA)$ is an orientably regular embedding of the simple cycle C_n, where

$$G/A \cong \langle \bar{x}, \bar{y} \mid \bar{x}^2 = \bar{y}^2 = (\bar{x}\bar{y})^n = 1 \rangle.$$

By Theorem 2.1(iii), the group G has the presentation (2.1) for some $e, f \in \mathbb{Z}_m$ where

$$e^2 \equiv 1 \pmod{m}. \tag{2.2}$$

Set $a = x^2$ and $b = xy$. Then $K = \langle a, b \rangle$ is a metacyclic group with presentation

$$\langle a, b \mid a^m = 1, b^n = a^f, a^b = a^e \rangle. \tag{2.3}$$

We apply Hölder's Theorem to conclude that e and f satisfy the congruences

$$f(e-1) \equiv 0 \pmod{m} \tag{2.4}$$

and

$$e^n \equiv 1 \pmod{m}. \tag{2.5}$$

From the presentation of G we deduce that

$$a^y = a^e \quad \text{and} \quad b^y = (xy)^y = yx = yx^2x^{-1} = x^{2e}yx^{-1} = a^e b^{-1}.$$

For brevity, denote $s = \sum_{i=0}^{n-1} e^i$. Then

$$a^f \overset{(2.4)}{=} a^{ef} = (a^f)^y = (b^n)^y = (b^y)^n = (a^e b^{-1})^n = a^s b^{-n} = a^{s-f},$$

whence

$$2f \equiv s \pmod{m}. \tag{2.6}$$

In the above proof we see that $G = K \rtimes \langle y \rangle$, and hence G has an alternative presentation

$$G = \langle x, y \mid a = x^2, b = xy, a^m = 1, b^n = a^f, a^b = a^e, y^2 = 1, a^y = a^e, b^y = a^e b^{-1} \rangle. \tag{2.7}$$

Conversely, given a group G defined by (2.1) (or equivalently by (2.7)) with the parameters n, m, e, and f satisfying (2.2), (2.4), (2.5) and (2.6), we see that $|G| = |K \rtimes \langle y \rangle| = 2|K|$, and from Hölder's Theorem we get that $|G| = 2mn$. By Theorem 2.1, the map $(G; x, y)$ corresponds to an orientably regular embedding of $C_n^{(m)}$.

Recall that two embeddings $\mathcal{C}(n, m; e, f) = (G(n, m; e, f); x, y)$ and $\mathcal{C}(n, m; e', f') = (G(n, m; e', f'); x', y')$ are isomorphic if and only if the assignment $x \mapsto x'$, $y \mapsto y'$

extends to a group isomorphism. Routine calculations show that this occurs if and only if $e \equiv e' \pmod{m}$ and $f \equiv f' \pmod{m}$. For the maps $(G(n,m;e,f);\, x,y)$ and $(G(n,m;e,f);\, x^{-1},y)$ the latter condition is clearly satisfied, which immediately implies that each of the maps $\mathcal{C}(n,m;e,f)$ is reflexible.

To obtain more details on these embeddings we need to solve the system of congruences (2.2), (2.4), (2.5) and (2.6). First notice that if n is odd and m is even, then e is odd. According to (2.2) we have $e^2 \equiv 1 \pmod{m}$, and therefore $s = \sum_{i=0}^{n-1} e^i = (n-1)(e+1)/2+1$. However, $4|(n-1)(e+1)$, so s is odd, violating (2.6). In other words, if n is odd and m is even, $C_n^{(m)}$ does not admit any orientably regular embedding. Our discussion now splits into five cases dealing with the remaining conditions on n and m, each corresponding to an item of Theorem 2.2.

Case (i)**.** Both n and m are odd.

Theorem 2.1 (iv) yields that $e \equiv 1 \pmod{m}$. By substituting $e = 1$ into (2.6) we get $2f \equiv n \pmod{m}$, which implies that $2f \equiv (n+m) \pmod{m}$ and consequently $2(f-(n+m)/2) \equiv 0 \pmod{m}$. Since m is odd, we infer that $f \equiv (n+m)/2 \pmod{m}$, and Case (i) is done.

Now we deal with the cases where n is even. Using (2.2) we get $s = \sum_{i=0}^{n-1} e^i = (e+1)n/2$. Substituting for s into (2.6) we obtain

$$2f \equiv (e+1)n/2 \pmod{m}. \tag{2.8}$$

Case (ii)**.** $n \equiv 0 \pmod{4}$ and m is odd.

Since $n \equiv 0 \pmod{4}$, from (2.8) we get $f \equiv (e+1)n/4 \pmod{m}$. By substituting f into (2.4) we obtain

$$f(e-1) \equiv (e^2-1)n/4 \equiv 0 \pmod{m}. \tag{2.9}$$

Therefore e and f satisfy (2.4), and Case (ii) is done.

Case (iii)**.** $n \equiv 2 \pmod{4}$ and m is odd.

If $e+1$ is even, then $f \equiv (e+1)n/4 \pmod{m}$. By substituting f into (2.4) we obtain $n(e^2-1)/4 \equiv 0 \pmod{m}$. Since $n \equiv 2 \pmod{4}$, we get $e^2-1 \equiv 0 \pmod{2}$. If we combine this with (2.2), we get $e^2 \equiv 1 \pmod{2m}$.

If $e+1$ is odd, then $(e+1)n/2$ is odd, and hence $(e+1)n/2+m$ is even. We may rewrite (2.8) in the form $2f \equiv (e+1)n/2+m \pmod{m}$, and obtain $f \equiv ((e+1)n+2m)/4 \pmod{m}$. By substituting f into (2.4) we further get

$$n(e^2-1)/2 + m(e-1) \equiv 0 \pmod{2m}.$$

Since m is odd, by the Chinese Remainder Theorem, this is equivalent to

$$\begin{cases} n(e^2-1)/2 \equiv 0 \pmod{m}, & \text{(2.10a)} \\ n(e^2-1)/2 + m(e-1) \equiv 0 \pmod{2}. & \text{(2.10b)} \end{cases}$$

By applying (2.2) we may conclude that (2.10a) holds. Since $n/2$, m, $e+1$, and $e-1$ are all odd, we see that (2.10b) holds, too. Hence (2.4) is satisfied by e and f exactly when the conditions in the statement are satisfied. This completes Case (iii).

Case (iv). $n \equiv 0 \pmod 4$ and m is even.

In this case (2.8) has two solutions $f = (e+1)n/4$ and $f = (e+1)n/4 + m/2$ in $\mathbb{Z}_m$. If we insert them into (2.4), we see that (2.4) is satisfied, and Case (iv) is complete.

Case (v). $n \equiv 2 \pmod 4$ and m is even.

(2.8) has two solutions in $\mathbb{Z}_m$, namely $f = (e+1)n/4$ or $f = (e+1)n/4 + m/2$. It remains to show $e^2 \equiv 1 \pmod{2m}$. By substitution of f into (2.4) we get

$$(e^2 - 1)n/4 \equiv 0 \pmod m. \tag{2.11}$$

By the assumption, we may set $m = 2^r m_0$ where $r \geq 1$ and m_0 is odd. A combination of (2.2) and (2.11) yields the system

$$\begin{cases} e^2 - 1 & \equiv 0 \pmod{2^r m_0}, \\ e^2 - 1 & \equiv 0 \pmod{2^{r+1}(m_0/h)}, \end{cases}$$

where $h = \gcd(m, n/2)$. By the assumption, $n/2$ is odd, so $h = \gcd(m_0, n/2)$. By the Chinese Remainder Theorem, the above system is equivalent to the system

$$\begin{cases} e^2 - 1 & \equiv 0 \pmod{2^{r+1}}, \\ e^2 - 1 & \equiv 0 \pmod{m_0}. \end{cases}$$

We now apply the Chinese Remainder Theorem once again and get $e^2 \equiv 1 \pmod{2m}$. This completes Case (v) as well as the proof of Theorem 2.2. □

The next corollary determines the basic parameters of the maps $\mathcal{C}(n, m; e, f)$. Recall that the *type* of a regular or orientably regular map $\mathcal{M}$ is the symbol $\{p, q\}$ where p is the face-size and q is the vertex-valency of $\mathcal{M}$.

Corollary 2.4. *The map* $\mathcal{C}(n, m; e, f)$ *has type* $\{nm/h, 2m\}$ *and its genus is* $n(m-1)/2 - (h-1)$, *where* $h = \gcd(f, m)$.

Proof. To determine the type and the genus of $\mathcal{C}(n, m; e, f)$ we need to determine the order of the element xy. Since $(xy)^n = (x^2)^f$ and x^2 has order m, we see that the order of xy is nm/h where $h = \gcd(f, m)$. It follows that the map has type $\{nm/h, 2m\}$. Since $|G(n, m; e, f)| = 2mn$, the numbers of vertices, edges, and faces of $\mathcal{C}(n, m; e, f)$ are n, mn, and $2h$, respectively. Therefore, by the Euler-Poincaré Formula, the map has genus $n(m-1)/2 - (h-1)$, as claimed. □

Remark 2.5. Let $\lambda(g)$ denote the order of a largest group of conformal automorphisms of a compact Riemann surface of genus g. Accola [1] and MacLachlan [17] independently proved that $8(g+1) \leq \lambda(g) \leq 84(g-1)$ for $g \geq 2$ and there are infinitely many integers $g \geq 2$ for which the equality $\lambda(g) = 8(g+1)$ holds. If we take $n = 4$, $e = -1$, and $f = 0$ in Corollary 2.4, we get that the genus of $\mathcal{C}(4, m; -1, 0)$ is $g = m - 1$ with the automorphism group G of order $|G| = 8(g+1)$, the lower bound of $\lambda(g)$.

3 Non-orientable regular embeddings of $C_n^{(m)}$

As in the orientable case, regular maps on non-orientable surfaces can be represented in a purely algebraic manner [12]. Every regular map $\mathcal{M}$ on a closed surface, and hence every regular map on a non-orientable surface, may be identified with a quadruple $(G;\, l, r, t)$ where G is a finite group and $\{l, r, t\}$ is a generating set for G with $l^2 = r^2 = t^2 = (lt)^2 = 1$, where the elements l, r, t and lt are all nontrivial. Such a quadruple is called an *algebraic regular map*. Elements of G represent flags of $\mathcal{M}$, pairwise incident triples of the form (v, e, f) where v is a vertex, e is an edge and f is a face of $\mathcal{M}$. The right translations of G by l, r, and t correspond to the *longitudinal*, *rotary*, and the *transversal* involution of $\mathcal{M}$, respectively. The longitudinal involution fixes e and f of each flag (v, e, f) while interchanging the end-vertices of e. The rotary involution fixes v and f of (v, e, f) while interchanging the two edges sharing the same corner of f at v. The transversal involution fixes v and e of (v, e, f) while interchanging the two faces incident with e. The vertices, edges, and faces of $\mathcal{M}$ are in a one-to-one correspondence with the letf cosets of the subgroups $\langle r, t\rangle$, $\langle l, t\rangle$, and $\langle l, r\rangle$, and the incidence between the objects corresponds to non-empty intersection of cosets.

Two algebraic maps $(G;\, l, r, t)$ and $(G;\, l', r', t')$ represent isomorphic regular maps if and only if there is an automorphism of G taking l to l', r to r', and t to t'. Each automorphism of the map $\mathcal{M} = (G;\, l, r, t)$ corresponds to a left translation $g \mapsto ag$, $g \in G$, where a is a fixed element of G. In particular, the group $\mathrm{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ is isomorphic to G.

The underlying surface of a regular map $(G;\, l, r, t)$ need not be non-orientable, nevertheless, the criterion of orientability is easy: a regular map $(G;\, l, r, t)$ is orientable if and only if the *even-word subgroup* $G^+ = \langle rt, tl\rangle$ has index 2 in G. Thus, if $\mathcal{M} = (G;\, l, r, t)$ is non-orientable, then $G^+ = G$, and the triple $\tilde{\mathcal{M}} = (G;\, x, y)$ with $x = rt$ and $y = tl$ represents an orientably regular map such that $\mathrm{Aut}^+(\tilde{\mathcal{M}}) \cong G \cong \mathrm{Aut}(\mathcal{M})$. The orientably regular map $\tilde{\mathcal{M}}$ is known as the *antipodal double cover* over $\mathcal{M}$; conversely, $\mathcal{M}$ is said to be a *halved non-orientable quotient* of $\tilde{\mathcal{M}}$. An orientably regular map is called *antipodal* if it admits a halved non-orientable quotient.

Observe that the involution $t \in G$ plays the role of a reflection of $\tilde{\mathcal{M}}$, since $x^t = x^{-1}$ and $y^t = y^{-1} = y$. In general, an *inner reflection* of an orientably regular map $\mathcal{N} = (G;\, x, y)$ is any element $g \in G$ satisfying the following conditions:

$$\begin{cases} g^2 = 1, & \text{(3.1a)} \\ g^{-1}xg = x^{-1}, & \text{(3.1b)} \\ g^{-1}yg = y. & \text{(3.1c)} \end{cases}$$

Orientably regular maps admitting inner reflections are called *algebraically antipodal*. It is proved in [20, Theorem 7.5] that an algebraically antipodal orientably regular map is antipodal with the exception of spherical dipole maps $\mathcal{D}(m, -1)$ where m is odd, their duals, and regular maps with a single vertex and valency at most 2. More precisely, if $\mathcal{N} = (G;\, x, y)$ is an orientably regular map and g is an inner reflection of $\mathcal{N}$, then with the exception of the maps just mentioned, the map $\mathcal{M} = \mathcal{N}_g = (G;\, xg, yg, g)$ is a halved quotient of $\mathcal{N}$.

Although the antipodal double cover over a non-orientable regular map is uniquely determined, the same is not true for halved quotients: an antipodal regular map may have different halved quotients corresponding to different inner reflections [25]. However, conditions (3.1a)–(3.1c) imply that if g_1 and g_2 are two inner reflections, then there exists a central involution z such that $g_2 = zg_1$. In particular, the number of inner reflections of an antipodal map equals the number of central involutions (including the identity).

Before moving on to the classification of non-orientable multicyclic regular maps it will be useful to recall that non-orientable regular maps with multiple edges occur in two varieties: either every pair of parallel edges forms an orientation-preserving cycle or there exists a pair of parallel edges forming an orientation-reversing cycle. By regularity, in the latter case every edge must be involved in such a cycle. Following [16], we call such maps *Möbius regular maps*. Möbius maps were earlier investigated by Wilson [27] under the name *cantankerous maps*. By using geometric arguments Wilson showed that the multiplicity of such a map must be 2 (see [27, p. 265] and also [16, Lemma 6]). For the sake of completeness we include a proof of this fact based on the determination of all non-orientable regular embeddings of dipoles.

Observe that the dipole D_2 has exactly one non-orientable embedding, which is regular and its supporting surface is the projective plane. This embedding is isomorphic to the map $(H;\, l, r, t)$ where H is the dihedral group of order 8 with presentation

$$H = \langle l, r, t \mid l^2 = r^2 = t^2 = (lt)^2 = 1, (rt)^2 = 1, (lrt)^2 = t\rangle. \tag{3.2}$$

The next lemma shows that there is no other non-orientable regular embedding of any dipole.

Lemma 3.1. *There are no non-orientable regular embeddings of the dipole D_m except the unique embedding of D_2 in the projective plane isomorphic to the map defined by the presentation* (3.2).

Proof. It is clear that the dipole D_2 has a unique embedding in the projective plane and that the embedding is regular. Now let $\mathcal{M} = (H;\, l, r, t)$ be a non-orientable regular embedding of D_m with $m \geq 2$. Since D_m has just two vertices, the subgroup $D = \langle r, t\rangle$ has index two in H. Clearly, D is dihedral of order $2m$. If we set $a = rt$, then $\langle a\rangle \trianglelefteq D$ and $D = \langle a, t\rangle$. Since $D \trianglelefteq H$ and D is dihedral, we get

$$lal = a^i t^j, \quad \text{where} \quad i \in \mathbb{Z}_m \text{ and } j \in \mathbb{Z}_2. \tag{3.3}$$

Suppose that $j = 0$. From (3.3) we then deduce that $a = l^2 a l^2 = a^{i^2}$, proving that $i^2 \equiv 1 \pmod{m}$. It follows that H has presentation

$$\langle a, t, l \mid a^m = l^2 = t^2 = (lt)^2 = 1, tat = a^{-1}, lal = a^i\rangle.$$

However, it is straightforward to verify that the even-word subgroup $H^+ = \langle rt, tl\rangle = \langle a, lt\rangle$ has index two in H, contradicting the assumption that $\mathcal{M}$ is non-orientable. Therefore $j = 1$. It follows that the element $a^i t^j = a^i t$ is an involution, and hence a is an involution as well. In particular, $m = 2$.

Now suppose that $i = 0$. Then (3.3) reduces to

$$lrlt \overset{lt=tl}{=} lrtl \overset{a=rt}{=} lal \overset{(3.3)}{=} t,$$

implying that $r = 1$, which is impossible. Therefore $i = 1$ and (3.3) reduces to the relation $lal = at$. Thus $\mathcal{M}$ is isomorphic to the previously defined embedding of D_2 into the projective plane, and the proof is complete. □

The following result appears in [16] and [27]. We provide a purely algebraic proof.

Theorem 3.2. *Let X be a simple graph of order at least 2 and of valency d. Then a regular embedding $\mathcal{M} = (G; l, r, t)$ of the graph $X^{(m)}$ with multiplicity m is a Möbius regular map if and only if $m = 2$ and the generators l, r and t satisfy the identity*

$$(l(rt)^d)^2 = t. \tag{3.4}$$

Proof. Note that the action of the automorphism group G on the flags of $\mathcal{M}$ induces an action on the vertices and an action on the edges of $\mathcal{M}$. We may therefore assume that the subgroup $\langle r, t\rangle$ of G fixes a vertex u, and the subgroup $\langle l, t\rangle$ fixes an edge joining the vertex u and an adjacent vertex v. Let H be the subgroup of G fixing the set $\{u, v\}$. Then H may be regarded as the automorphism group of a regular embedding $\mathcal{H}$ of the dipole D_m whose vertices are u and v and whose edges are the edges between u and v. The underlying graph structure implies that d is the smallest positive integer k such that the element $(rt)^k$ fixes both u and v. Therefore, $\mathcal{H} = (H; (rt)^d, l, t)$.

If $\mathcal{M}$ is a Möbius regular map, then the regularity of $\mathcal{M}$ implies that $\mathcal{H}$ is also a Möbius regular map. By Lemma 3.1, $m = 2$ and the identity (3.4) holds. Conversely, if $m = 2$ and $\mathcal{M}$ satisfies the identity (3.4), then $\mathcal{H}$ is a non-orientable embedding of the 2-cycle C_2. The definition of $\mathcal{H}$ implies that $\mathcal{M}$ contains an orientation reversing cycle, so $\mathcal{M}$ is Möbius regular map. □

To formulate our classification theorem for non-orientable multicyclic regular maps we define two families of maps. First, let $\mathcal{C}(m, n)$ denote the non-orientable regular map $(H(m, n); l, r, t)$ with $H(m, n)$ being the group with presentation

$$\langle l, r, t \mid l^2 = r^2 = t^2 = (tl)^2 = (rt)^{2m} = (rl)^{2n} = (rtrl)^2 = 1, (rt)^m(rl)^n = r\rangle, \tag{3.5}$$

where $n \geq 3$ and $m \geq 1$ are odd integers. It is not difficult to see that the antipodal double cover of $\mathcal{C}(m, n)$ is the multicyclic orientably regular map $\mathcal{C}(2n, m, -1, 0)$.

Second, let $\mathcal{M}(n)$ denote the non-orientable regular map $(H(n); l, r, t)$ with $H(n)$ being the group with presentation

$$\langle l, r, t \mid l^2 = r^2 = t^2 = (tl)^2 = (rt)^4 = (rl)^n = 1, (l(tr)^2)^2 = t\rangle, \tag{3.6}$$

where $n \equiv 0 \pmod 3$. By Theorem 3.2, the relation $(l(tr)^2)^2 = t$ in the above definition forces $\mathcal{M}(n)$ to be a Möbius map.

The following theorem is, except for the enumeration part, due to Li and Širáň [15, Proposition 8]. Our proof is based on the classification of orientably regular embeddings of $C_n^{(m)}$ presented in the previous section and on Theorem 3.2 about Möbius maps proved above. The original proof of Li and Širáň employed the analysis of regular maps with an unfaithful action of the map automorphism group on vertices.

Theorem 3.3. *Let $\mathcal{M}$ be a non-orientable regular embedding of an m-fold n-cycle $C_n^{(m)}$. Then either*

(i) *m and n are both odd, and $\mathcal{M}$ is isomorphic to the map $\mathcal{C}(m, n)$, or*

(ii) *$m = 2$, $n \equiv 0 \pmod 3$, and $\mathcal{M}$ is isomorphic to the Möbius map $\mathcal{M}(n)$.*

Moreover, for each pair (n,m) *of admissible integers there is a unique non-orientable regular embedding of* $C_n^{(m)}$.

Proof. Let $\mathcal{M} = (G;\, l,r,t)$ be a non-orientable regular embedding of $C_n^{(m)}$, and let $\tilde{\mathcal{M}} = (G;\, x,y)$ be the antipodal double cover of $\mathcal{M}$, where $x = rt$ and $y = tl$. We distinguish two cases.

Case (i)**.** Every 2-cycle of $\mathcal{M}$ is orientation-preserving.

Since the antipodal cover is a smooth cover, the valency of $\mathcal{M}$ is preserved, and each set of m parallel edges in $\mathcal{M}$ lifts to a set of m parallel edges in $\tilde{\mathcal{M}}$. So $\tilde{\mathcal{M}}$ is an orientably regular embedding of $C_{2n}^{(m)}$. By Theorem 2.2, $\tilde{\mathcal{M}} = (G;\, x,y)$, where

$$G = G(2n,m,e,f) = \langle x,y \mid x^{2m} = y^2 = 1, (x^2)^y = (x^2)^e, (xy)^{2n} = (x^2)^f \rangle, \quad (3.7)$$

where the parameters $2n$, m, e, and f satisfy the numerical conditions stated in Theorem 2.2. Let $K = \langle x^2 \rangle$. Then $K \trianglelefteq G$ and $\tilde{\mathcal{M}}/K \cong \mathcal{C}_{2n}$, where $\mathcal{C}_{2n}$ denotes the dihedral map of type $\{2n, 2\}$ on the sphere. The inner reflection t of $\tilde{\mathcal{M}}$ projects onto the inner reflection $\bar{t} = (\bar{x}\bar{y})^n$ of $\mathcal{C}_{2n}$. It follows that $t = x^{2k}(xy)^n$ for some $k \in \mathbb{Z}_m$. Using the commuting rule

$$(x^2)^y = x^{2e} \quad (3.8)$$

we get

$$x^{-1} \overset{(3.1b)}{=} (x^{2k}(xy)^n)^{-1}x(x^{2k}(xy)^n) \overset{(3.8)}{=} (xy)^{-n}x(xy)^n, \quad (3.9)$$

which implies

$$x^{-2} \overset{(3.9)}{=} (xy)^{-n}x^2(xy)^n \overset{(3.8)}{=} \begin{cases} x^2 & \text{if } n \text{ is even} \quad (3.10a) \\ x^{2e} & \text{if } n \text{ is odd.} \quad (3.10b) \end{cases}$$

Suppose that n is even. From (3.10a) we deduce that $x^4 = 1$, so $m = 2$. Theorem 2.2 now implies that $e = 1$ and therefore

$$x^{-1} \overset{(3.9)}{=} (xy)^{-n}x(xy)^n = (yxx^{-2})^n x(xy)^n \overset{(3.8)}{=} x^{-2n+2}(yx)^{2n}x^{-1}. \quad (3.11)$$

Since $(xy)^{2n} = x^{2f}$, we have $(yx)^{2n} = y^{-1}((xy)^{2n})y = (x^{2f})^y = x^{2ef} \overset{(2.4)}{=} x^{2f}$. If we combine this with (3.11), we get $x^{-2n+2f+1} = 1$. Consequently, $-2n+2f+1 \equiv 0 \pmod 4$, which cannot hold. It follows that n must be odd. By (3.10b), $e = m-1$. We have

$$1 \overset{(3.1a)}{=} t^2 = x^{2k}(xy)^n x^{2k}(xy)^n \overset{(3.8)}{=} (xy)^{2n} \overset{(3.7)}{=} x^{2f},$$

so $f = 0$. Moreover,

$$\begin{aligned} 1 &\overset{(3.1c)}{=} (x^{2k}(xy)^n)^{-1}y(x^{2k}(xy)^n)y = (yx^{-1})^n x^{-2k}y(x^{2k}(xy)^n)y \\ &= (yxx^{-2})^n x^{-4k}y(xy)^n y \overset{(3.8)}{=} x^{4k}(yxx^{-2})^n(yx)^n \overset{(3.8)}{=} x^{4k+2}(yx)^{2n} \\ &\overset{(3.7)}{=} x^{4k+2}x^{2f} = x^{4k+2}. \end{aligned}$$

Since the order of x^2 is m, we get $2k+1 \equiv 0 \pmod{m}$, and hence $k = (m-1)/2$. It can easily be verified that if both m and n are odd, then $x^{m-1}(xy)^n$ is the unique inner reflection of the map $\mathcal{C}(2n, m; m-1, 0)$ that gives rise to a non-orientable regular embedding of $C_n^{(m)}$. Let $t = x^{m-1}(xy)^n$, $r = xt$ and $l = yt$. Then, upon substitution, the group $G(2n, m; m-1, 0)$ receives the presentation (3.5), and hence $\mathcal{M} \cong \mathcal{C}(m, n)$.

Case (ii)**.** There exists an orientation-reversing 2-cycle in $\mathcal{M}$.

By Theorem 3.2, $m = 2$, every 2-cycle is orientation-reversing, and $\mathcal{M} = (G; l, r, t)$ is a Möbius regular map. It follows that the antipodal double cover $\tilde{\mathcal{M}} = (G; rt, tl)$ of $\mathcal{M}$ is an orientably regular embedding of the lexicographic product $C_n[\bar{K}_2]$, where each vertex u of $\mathcal{M}$ lifts to two vertices u_0 and u_1 which are antipodal points of $\tilde{\mathcal{M}}$. Without loss of generality we may assume that the generator $x = rt$ fixes a certain vertex u_0 and that the generator y fixes an edge u_0v_0 incident with v_0. Set $a = x^2$, $b = a^y$, and $K = \langle a, b\rangle$. Then K acts regularly on the darts (u_i, v_j) where $i, j \in \mathbb{Z}_2$. By regularity, $K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We first show that $K \trianglelefteq G$. It is evident that

$$a^x = a, \quad a^y = b \quad \text{and} \quad b^y = a. \tag{3.12}$$

Notice that since u_0 and u_1 are antipodal points, x fixes both u_0 and u_1. Similarly, yxy fixes both v_0 and v_1. Using regularity again we see that yx^2y interchanges u_0 and u_1. Therefore, for any dart of the form (u_i, v_j) we get

$$b^x(u_i, v_j) = x^{-1}yx^2yx(u_i, v_j) = x^{-1}yx^2y(u_i, w_k) = x^{-1}(u_{i+1}, w_h) = (u_{i+1}, v_s),$$

where $k, h, s \in \mathbb{Z}_2$ and w_0 and w_1 are vertices adjacent to u_0 but distinct from v_0 and v_1. Thus $b^x \in K$ and hence $K \trianglelefteq G$.

Next we show that $b^x = ab$. Since $\mathcal{M}$ is a halved quotient of $\tilde{\mathcal{M}}$ in which every pair of antipodal vertices u_0 and u_1 is identified to a single vertex u, there exists an inner reflection g which identifies the antipodal pairs. Notice that $\tilde{\mathcal{M}}/K \cong \mathcal{C}_n$, where $\mathcal{C}_n$ is the dihedral map of type $\{n, 2\}$. By regularity, $g \in K$. From (3.1b) we see that $g \neq 1$ and $g \neq a$. If we had $g = b$, then from (3.1c) we would derive that $b^y = b$, which is a contradiction with the assumption $b^y = a$. Therefore $g = ab = x^2yx^2y$ and consequently

$$x^g = x^2(yx^2y)x(x^2)(yx^2y) \overset{(3.1b)}{=} x^{-1}.$$

Since $[x^2, yx^2y] = 1$, we get

$$b^x = x^{-1}(yx^2y)x = x^{-2}yx^2y = x^2yx^2y = ab. \tag{3.13}$$

For convenience, set $z = xy$. Then $z^n \in K$, which implies that $z^n = a^ib^j$ for some $i, j \in \mathbb{Z}_2$. From (3.12) and (3.13) we derive that

$$a^z = b, \quad b^z = ab, \quad \text{and} \quad (ab)^z = a, \tag{3.14}$$

so $a^ib^j = z^n = (z^n)^z = (a^ib^j)^z \overset{(3.14)}{=} b^i(ab)^j = a^jb^{i+j}$, and hence $a^{i-j} = b^i$. Since $\langle a\rangle \cap \langle b\rangle = 1$, we see that $i = 0$ and $j = 0$. Therefore, $z^n = 1$. Observe that the action of z on K defined by (3.14) induces the permutation $(1)(a, b, ab)$, which implies that $n \equiv 0 \pmod{3}$. Moreover, since

$$z^y = (xy)^y = yx = yx^{-1}x^2 = (xy)^{-1}x^2 = z^{-1}a, \tag{3.15}$$

we get $G = \langle a, b, z, y\rangle = (K \rtimes \langle z\rangle) \rtimes \langle y\rangle$. Therefore G is defined by the presentation

$$\langle x, y \mid a = x^2, b = a^y, x^4 = y^2 = [a, b] = (xy)^n = 1, b^x = ab\rangle. \qquad (3.16)$$

Conversely, It is straightforward to verify the group given by (3.16) with $n \equiv 0 \pmod 3$ gives rise to an orientably regular embedding of the graph $C_n[\bar{K}_2]$.

To complete the proof it remains to show that $t = ab$ is a unique inner reflection giving rise to a halved quotient with a multicyclic underlying graph. Let $\mathrm{Ic}(G)$ denote the subgroup of G generated by all its central involutions. From the previous part of the proof we know that G/K is isomorphic to the dihedral group of order $2n$. If $g \in \mathrm{Ic}(G)$, then $gK \in Z(G/K)$. Since

$$Z(G/K) = \begin{cases} 1, & n \quad \text{is odd,} \\ \langle \bar{z}^{n/2}\rangle, & n \quad \text{is even,} \end{cases}$$

there exist elements $i, j, k \in \mathbb{Z}_2$ such that $g = a^i b^j$ if n is odd, and $g = a^i b^j (z^{n/2})^k$ if n is even.

If n is odd, then $a^i b^j = g = g^y = (a^i b^j)^y = a^j b^i$. Since $\langle a\rangle \cap \langle b\rangle = 1$, we have $i = j$. Moreover, we have $a^i b^i = g = g^z = (a^i b^i)^z = a^i b^{2i} = a^i$, so $i = 0$ and hence $\mathrm{Ic}(G) = 1$.

Now we assume n is even. Recall that $n \equiv 0 \pmod 3$. We deduce from (3.14) that $[a, z^{n/2}] = [b, z^{n/2}] = 1$, and from (3.15) that $[y, z^{n/2}] = 1$. So $z^{n/2} \in \mathrm{Ic}(G)$. Applying similar techniques we may deduce that if an element $h = a^i b^j$ belongs to $\mathrm{Ic}(G)$, then $h = 1$. Therefore, $\mathrm{Ic}(G) = \langle z^{n/2}\rangle$.

Summing up, we have proved that if n is odd, there is a unique inner reflection ab, and if n is even, there are two inner reflections ab and $abz^{n/2}$. However, the latter inner reflection does not produce an embedding of $C_n^{(m)}$. If we set $t = ab$, $r = xt$, and $l = yt$, then the presentation (3.16) transforms to the presentation (3.5). In either case, the antipodal regular covering that projects onto a non-orientable regular embedding of $C_n^{(m)}$ is unique, so for each pair (n, m) of admissible integers there is a unique non-orientable regular embedding of $C_n^{(m)}$, as claimed. □

As a corollary to the main theorem of this section we present a strengthening of a result due to Wilson [26] about non-orientable regular maps whose number of edges is a power of 2. Our result features two infinite classes of projective-planar regular maps arising as halved quotients of dihedral spherical maps: a unique embedding of the cycle C_n of length $n \geq 1$ in the projective plane which is a halved quotient of the map $\mathcal{C}_{2n}$, and its dual, the *balanced embedding* of the bouquet of n loops in the projective plane which is the halved quotient of the dipole map $\mathcal{D}(2n, -1)$.

Theorem 3.4. *If $\mathcal{M}$ is a regular map with nilpotent automorphism group, then $\mathrm{Aut}(\mathcal{M})$ is a 2-group. Furthermore, if $\mathcal{M}$ is non-orientable, then $\mathcal{M}$ is either the balanced embedding of the bouquet of 2^n loops into the projective plane for some $n \geq 0$ or its dual, a projective-planar embedding of the cycle C_{2^n}.*

Proof. Let $\mathcal{M} = (G; l, r, t)$ be a regular map where G is nilpotent. Then G can be expressed as a direct product $H \times K$ where H is a 2-group and K has odd order. The elements l, r, and t belong to H, because they are involutions. It follows that $H \cong G$ and $K = 1$. In other words, G is a 2-group.

Now assume that $\mathcal{M} = (G;\, l, r, t)$ is non-orientable. Since G is a 2-group, $\mathcal{M}$ has 2^n edges for some $n \geq 0$ and hence $|G| = 2^{n+2}$. We proceed by induction on n to show that $\mathcal{M}$ is either the balanced embedding of the bouquet of 2^n loops into the projective plane with $n \geq 0$ or its dual.

An easy check of non-orientable regular maps with at most four edges shows that the claim is true for $n \leq 2$. For the induction step assume that the statement holds for some $n \geq 2$ and let $\mathcal{M} = (G;\, l, r, t)$ be a non-orientable regular map with 2^{n+1} edges, so that $|G| = 2^{n+3}$. Since G is a 2-group, it has a non-trivial centre, and hence it contains a central involution $z \neq 1$. Clearly, $\langle z \rangle \trianglelefteq G$, so we may construct the quotient map $\bar{\mathcal{M}} = (\bar{G}; \bar{l}, \bar{r}, \bar{t})$ where $\bar{G} = G/\langle z \rangle$ and $\bar{l}$, $\bar{r}$, and $\bar{t}$ are the images of r, l, and t, respectively. By the induction hypothesis, $\bar{\mathcal{M}}$ is either an embedding of the cycle C_{2^n} into the projective plane or its dual. We may clearly assume that $\bar{\mathcal{M}}$ is an embedded cycle. Then $\bar{G}$ has a presentation

$$\bar{G} = \langle \bar{l}, \bar{r}, \bar{t} \mid \bar{l}^2 = \bar{r}^2 = \bar{t}^2 = (\bar{l}\bar{t})^2 = 1, (\bar{r}\bar{t})^2 = (\bar{r}\bar{l})^{2^n} = 1 \rangle.$$

Since G is a cyclic central extension of $\bar{G}$ by $\langle z \rangle$, we get

$$G = \langle l, r, t \mid l^2 = r^2 = t^2 = (lt)^2 = 1, (rt)^2 = z^i, (rl)^{2^n} = z^j \rangle,$$

for some $i, j \in \mathbb{Z}_2$. If $i = 0$, then $j = 1$, and $\mathcal{M}$ is an embedding of $C_{2^{n+1}}$ into the projective plane. If $i = 1$, then since $lt(rt)^2lt = z = (rt)^2$, the underlying graph of $\mathcal{M}$ is a cycle with multiplicity 2, which violates Theorem 3.3. This establishes the induction step, and the proof is complete. □

Remark 3.5. Breda d'Azevedo, Nedela, and Širáň [3] showed that for any integer $p \equiv 7 \pmod{12}$ with $p > 7$ the groups

$$G_{j,l} = \langle r, s \mid x^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1 \rangle,$$

where $j > l \geq 3$ and $(j-1)(l-1) = p + 1$ give rise to infinitely many non-orientable regular maps of Euler characteristic $-p$. If p is a prime, this family forms a complete set of regular maps on the non-orientable surface of Euler characteristic $-p$. After setting $l = m$ and $j = n$ it becomes clear that these maps are identical with regular embeddings of $C_n^{(m)}$ defined by (3.5).

Remark 3.6. Malnič, Nedela, and Škoviera [18] proved that if the automorphism group of an orientably regular map $\mathcal{M}$ is nilpotent, then $\mathcal{M}$ can be decomposed into a direct product of two orientably regular maps, an orientably regular map whose automorphism group is a 2-group and a semistar of odd valency [18, Theorem 3.2]. Since the automorphism group of every non-orientable regular map is also the automorphism group of its antipodal double cover, it follows from Theorem 3.4 that no orientably regular maps with nilpotent automorphism groups are antipodal, except the dihedral maps $\{2^n, 2\}$ on the sphere and their duals.

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References

[1] R.D.M. Accola, On the number of automorphisms of a closed Riemann surface, *Trans. Amer. Math. Soc.* **131** (1968), 398–408.

[2] H.R. Brahana, Regular maps and their groups, *Amer. J. Math.* **41** (1927), 268–284.

[3] A. Breda d'Azevedo, R. Nedela and J. Širáň, Classification of regular maps of negative prime Euler characteristic, *Trans. Amer. Math. Soc.* **357** (2005), 4175–4190.

[4] D.A. Catalano, M. D. E. Conder, S.-F. Du, Y. S. Kwon, R. Nedela and S. Wilson, Classification of regular embeddings of n-dimensional cubes, *J. Algebraic Combin.*, **33** (2011), 215–238.

[5] H.M.S. Coxeter and W.O.J. Moser, *Genarators and Relations for Discrete Groups*, Berlin, New York, Springer-Verlag, 1972.

[6] S.-F. Du, J. H. Kwak and R. Nedela, Regular embeddings of complete multipartite graphs, *European J. Combin.* **26** (2005), 505–519.

[7] A. Gardiner, R. Nedela, J. Širáň and M. Škoviera, Characterisation of graphs which underlie regular maps on closed surfaces, *J. London Math. Soc.* **59** (1999), 100–108.

[8] L.D. James, Imbeddings of the complete graph, *Ars Combin.* 16-B (1983), 57–72.

[9] L.D. James and G.A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory Ser. B* **39** (1985), 353–367.

[10] G.A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* **37** (1978), 273–307.

[11] G.A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, *Proc. London Math. Soc.* **101** (2010), 427–453.

[12] G.A. Jones and J.S. Thornton, Operations on maps, and outer automorphisms, *J. Combin. Theory Ser. B* **35** (1983), 93–103 .

[13] Y.S. Kwon, New regular embeddings of n-cubes Q_n, *J. Graph Theory* **46** (2004), 297–312.

[14] Y.S. Kwon and R. Nedela, Non-existence of nonorientable regular embeddings of n-dimensional cubes, *Discrete Math.* **307** (3-5) (2007), 511–516.

[15] C.H. Li and J. Širáň, Regular maps whose groups do not act faithfully on vertices, edges, or faces, *European J. Combin.* **26** (2005), 521–541.

[16] C.H. Li and J. Širáň, Möbius regular maps, *J. Combin. Theory Ser. B* **97** (2007), 57–73.

[17] C. MacLachlan, A bound for the number of automorphisms of a compact Riemann surface, *J. London Math. Soc.* **44** (1969), 265–272.

[18] A. Malnič, R. Nedela, and M. Škoviera, Regular maps with nilpotent automorphism groups, *European J. Combin.* **33** (2012), 1974-1986.

[19] R. Nedela and M. Škoviera, Regular maps of canonical double coverings of graphs. *J. Combin. Theory Ser. B* **67** (1996), 249–277.

[20] R. Nedela and M. Škoviera, Regular maps from voltage assignments and exponent groups, *European J. Combin.* **18** (1997), 807–823.

[21] R. Nedela and M. Škoviera, Exponents of orientable maps, *Proc. London Math. Soc.* **75** (1997), 1–31.

[22] R. Nedela, M. Škoviera and A. Zlatoš, *Regular embeddings of cycles with multiple edges*, manuscript (2001).

[23] M. Škoviera and A. Zlatoš, *Regular maps with multiple edges on orientable surfaces*, manuscript (2000).

[24] W. Threlfall, Gruppenbilder, *Abh. sächs. Akad. Wiss. Math.-phys. Kl.* **41** (1932), 1–59.

[25] S.E. Wilson, Non-orientable regular maps, *Ars Combin.* **5** (1978), 213–218.

[26] S.E. Wilson, Bicontactual regular maps, *Pacific J. Math.* **120** (1985), 437–451.

[27] S.E. Wilson, Cantankerous maps and rotary embeddings of K_n, *J. Combin. Theory Ser. B* **47** (3)(1989), 262–279.

[28] H. Zassenhaus, *The Theory of Groups*, Chelsea Publishing Co., New York, 1949.

[29] J.-Y. Zhang and S.-F. Du, On the orientable regular embeddings of complete multipartite graphs, *European J. Combin.* **33** (2012), 1303–1312.

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Strongly regular m-Cayley circulant graphs and digraphs

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Abstract

The first part of this paper is a survey about strongly regular graphs and digraphs admitting a semiregular cyclic group of automorphisms. In the second part, some new types of such digraphs, called uniform and almost uniform, are studied. By using partial sum families, the form of the parameters is determined and some directed strongly regular graphs derived from these partial sum families with previously unknown parameters are obtained.

Keywords: m-Cayley, circulant, strongly regular graphs, strongly regular digraphs, uniform partial sum families, almost-uniform partial sum families.

Math. Subj. Class.: 05Cxx, 05C20, 05C25, 05C50, 05E30

1 Introduction

Strongly regular graphs, which we define below, were introduced by Bose [4] in 1963. They constitute a very important class of graphs; in fact, they are one of the most basic association schemes, more specifically, they are the ones with two classes.

Definition 1.1. A graph X without loops of valency k and order v is called a strongly regular graph with parameters v, k, λ, μ (for short, (v, k, λ, μ)-SRG) if any two adjacent vertices have exactly λ common neighbours and any two distinct non-adjacent vertices have exactly μ common neighbours.

A SRG graph X is said to be trivial if X or its complement is a disjoint union of complete graphs.

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Dedicated to the memory of my mother.

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It is well known that if a non-trivial SRG is not a conference graph, that is, if the parameters are not of the form $k = (v-1)/2, \mu = (v-1)/4, \lambda = (v-5)/4$, then the eigenvalues of an adjacency matrix are integer numbers, and consequently $\Delta = \sqrt{\beta^2 + 4\gamma}$, where $\beta = \lambda - \mu$ and $\gamma = k - \mu$, is also an integer.

The problem of studying what SRGs have nice cyclic automorphism groups has received a considerable attention over the past decades.

A group of automorphisms of a graph is said to be regular if it acts transitively on the set of vertices and all the stabilizers are trivial. A graph is called circulant if it admits a cyclic regular group of automorphisms.

By a classical result of W.G. Bridges and R.A. Mena [5] it is known that the Paley graphs are the only non-trivial circulant strongly regular graphs.

Given integers $m \geq 1$ and $n \geq 2$, an automorphism group of a graph is called (m,n)-*semiregular* if it has m orbits of length n and no other orbit, and the action is regular on each orbit. An m-Cayley graph X is a graph admitting an (m,n)-semiregular group H of automorphisms. When H is abelian, we say that X is *m-Abelian*. If H is generated by an automorphism ρ (that is to say, when H is a cyclic group) and $m = 1$ (respectively, $m = 2$) we say that X is *n-circulant* (respectively, *n-bicirculant*). Sometimes, when a graph is m-Cayley over a cyclic group, we will just say that the graph is 'm-Cayley circulant', although that this terminology does not mean that the graph admits a regular group of automorphisms and should not be confused with the definition of circulant graph. Every m-Cayley graph X can be represented, following the terminology established by A. Malnič et al. in [18], by an $m \times m$ array of subsets of H in the following way. Let $U_0, \ldots, U_{m-1}$ be the m orbits of H, and for each i let $u_i \in U_i$. For each i and j, let $S_{i,j}$ be defined by $S_{i,j} = \{\rho \in H \mid u_i \to \rho(u_j)\}$. The family $(S_{i,j})$ is called the *symbol* of $\mathcal{G}$ relative to $(H; u_0, \ldots, u_{m-1})$.

The notion of strongly regular graph was generalized for directed graphs by Duval in [6]. A directed graph X without loops, of order v in which every vertex has both in-degree and out-degree k is called a *directed strongly regular graph with parameters* (v,k,μ,λ,t) (for short, (v,k,μ,λ,t)-DSRG or simply DSRG if we do not specify the parameters) whenever for any vertex u of X there are t undirected edges having u as an endvertex and for every two different vertices u and w of X the number of paths $p(u,w)$ of length 2 starting at u and ending at w depends only on whether uw is an arc of X or not. In particular,

$$p(u,w) = \begin{cases} t & \text{if } u = w \\ \lambda & \text{if } u \neq w \text{ and } uw \in A(X) \\ \mu & \text{if } u \neq w \text{ and } uw \notin A(X) \end{cases}$$

(where $A(X)$ denotes the arc set of X). A directed strongly regular graph will also be refereed to as a *strongly regular digraph*.

DSRGs have received a considerable attention in the literature (see, for instance, [7], [8], [9], [11], [12], [13], [14]).

The following relation between the parameters of a DSRG is obvious:

$$k(k-\beta) = \mu v + \gamma, \tag{1.1}$$

where $\beta = \lambda - \mu$ and $\gamma = t - \mu$.

It is well known that $\beta^2 + 4\gamma$ is a square, unless

$$k = t = (v-1)/2, \mu = (v-1)/4, \lambda = (v-5)/4, \tag{1.2}$$

in which case the graph is undirected and is a conference graph, or

$$k = (v-1)/2, \mu = (v+1)/4, \lambda = (v-3)/4, t = 0. \tag{1.3}$$

We define

$$\Delta = \sqrt{\beta^2 + 4\gamma},$$

which is an integer if the parameters of the digraph are not as indicated above.

A DSRG X is called trivial if $k = t$ and X is trivial as an undirected SRG.

Next, we will review the main results in this paper. In the first part of this paper we will make a review of the study of strongly regular graphs and digraphs which are m-Cayley over a cyclic group (m-Cayley circulant SRGs and DSRGs). In Section 2 we will focus on the undirected graphs, and in Section 3 on the more general case of directed graphs.

In the second part we will present some new results on two structures that produce circulant m-Cayley DSRGs. We study in Sections 4,5 and 6 the first structure, uniform partial sum families, which was proposed in the last section of [2], and we obtain there the general form of the parameters for the circulant case and give an sporadic uniform partial sum family which originates a DSRG with parameters previously undecided. In Section 7, we study almost uniform partial sum families, which are a generalization of uniform partial sum families, and obtain again the form of the parameters for the circulant case. Finally, we use almost uniform partial sum families to obtain three DSRG with parameters previously undecided.

2 Semiregular SRGs over cyclic groups

In this section we will focuss on undirected graphs. In particular, we will review some results on strongly regular graphs admitting cyclic semiregular groups of automorphisms.

Probably, the systematic study of circulant SRGs was began by D. Marušič in his seminal paper [21]. There, he obtained the form of the parameters of such graphs for the bicirculant case.

Proposition 2.1. *If a non-trivial $(2n, k, \lambda, \mu)$ bicirculant strongly regular graph exists with prime n then, up to complementation, the parameters of the graph are*

$$v = 2n = 4s^2 + 4s + 2, k = 2s^2 + s, \lambda = s^2 - 1, \mu = s^2.$$

He determined also some properties of the elements of the symbol. He denoted $S_{0,0}$, $S_{1,1}$ and $S_{0,1}$ by R, S and T, respectively, and he proved that the non-zero elements of the cyclic group C_n are the disjoint union of R and S, and that $|R| = |S| = s^2 + s$ and $|T| = s^2$. He found also examples of such graphs for $s = 1$ and $s = 2$.

He considered also non-trivial tricirculant SRGs over a cyclic group C_n of prime order and proved that, for such a graph or its complement, the elements of the symbol $S_{0,0}, S_{1,1}$ and $S_{2,2}$ form a partition of the set of non-zero elements of C_n.

His results were generalized by De Resmini and Jungnickel in [22]. They proved that the form of the parameters of a non-trivial bicirculant SRG over a cyclic group of order n was the one indicated in Proposition 2.1 if n is odd or not divisible by Δ. They found also one such graph for $s = 4$. Unfortunately, the technique that they used to construct the graph, which is based on the existence of certain difference families over cyclic groups, was not fruitful to get examples with $s \geq 5$.

This was further generalized by K.H. Leung and S.L. Ma in [17]. They found the general form of the parameters of non-trivial bicirculant SRGs, as stated in the following proposition, where $c = |S_{0,1}|, d = |S_{0,0}|$:

Proposition 2.2. *Up to complementation, the parameters for any non-trivial bicirculant SRG over a cyclic group of order n are the following:*

1. $(n; c, d; \lambda, \mu) = (2m^2 + 2m + 1; m^2, m^2 + m; m^2 - 1, m^2)$ *where* $m \geq 1$.
2. $(n; c, d; \lambda, \mu) = (2m^2; m^2, m^2 - m; m^2 - m; m^2 - m)$ *where* $m \geq 2$.
3. $(n; c, d; \lambda, \mu) = (2m^2; m^2, m^2 + m; m^2 + m; m^2 + m)$ *where* $m \geq 3$.
4. $(n; c, d; \lambda, \mu) = (2m^2; m^2 \pm m, m^2; m^2 \pm m; m^2 \pm m)$ *where* $m \geq 2$.

Besides the graphs found by Marušič and by De Resmini and Jungnickel, Leung and Ma found one example for $m = 2$ in family 2 and one example for $m = 2$ in family 4 of the previous proposition. They also proved the non-existence of bicirculants SRG over cyclic groups of order $2m^2$ where $m = p^r u, p$ is a prime congruent to 3 modulo 4, p and u are relatively prime and $u^2 < p^r$.

More bicirculant SRGs were found by A. Malnič et al. in [19]. Concretely, they obtained bicirculant SRGs with parameters of the same form as in Proposition 2.1 for $s = 3, 4$ and 5. The graphs that they found were the first known pairs of complementary bicirculant SRGs which are vertex-transitive but not edge-transitive.

K. Kutnar et al. studied tricirculant SRGs in [16]. They proved that, under certain appropriate conditions, the elements on the symbol of a tricirculant SRG over a cyclic group C_n satisfy that $S_{0,0}, S_{1,1}, S_{2,2}$ form a partition of $C_n - \{0\}$ and the parameters of the graph can be determined. They denoted $S_{0,0}, S_{1,1}, S_{2,2}, S_{0,1}, S_{1,2}$ and $S_{2,0}$ by A, B, C, R, S and T, respectively (the other elements in the symbol can be easily determined from these ones), and they denoted by $TCay(C_n; A, B, C; R, S, T)$ the associated tricirculant. In the next proposition, for a subset $D \subseteq C_n$, we set $\overline{D} = C_n \setminus (S \cup \{0\})$.

Proposition 2.3. *Let $X = TCay(C_n; (A, B, C; R, S, T)$ be a non-trivial $(3n, k, \lambda, \mu)$-strongly regular tricirculant, where $|A| + |B| + |C| \leq |\overline{A}| + |\overline{B}| + |\overline{C}|$. Assume $A \cup B \cup C \neq \varnothing$. If n is a prime or n is coprime to 6Δ then there exists an integer s such that the following two statements hold.*

1. *If the cardinalities of A, B and C are not all equal, then*

$$(3n, k, \lambda, \mu) = (3(12s^2 + 9s + 2), (4s + 1)(3s + 1), s(4s + 3), s(4s + 1)).$$

   If in addition $|A| = |C| \neq |B|$ (which is equivalent to $|R| = |S| \neq |T|$), then

$$|A| = 2s(1 + 2s), |B| = (4s + 1)(s + 1), |R| = s(4s + 1) \text{ and } |T| = (1 + 2s)^2.$$

2. *If $|A| = |B| = |C|$ then*

$$(3n, k, \lambda, \mu) = (3(3s^2 - 3s + 1), s(3s - 1), s^2 + s - 1, s^2).$$

   In this case $|A| = s(s - 1)$ and $|R| = |S| = |T| = s^2$.

When $|A| = |C| \neq |B|$ they proved that, for $s = -1$, exactly one tricirculant SRG exists up to isomorphisms, and for $s = 1$ no such graph exists.

When $|A| = |B| = |C|$ they proved that for $s = 2$ exactly one tricirculant SRG exists up to isomorphisms, and for $s = 3$ exactly four exist, and that for $s = -2$ no such graph exists.

3 Semiregular DSRGs over cyclic groups

DSRGs admitting a semiregular group of automorphisms were studied by Martínez and Araluze in [20] by using the concept of partial sum family. In the next definition, the third identity holds in the group ring $\mathbb{Z}[H]$. As usual, we will identify a subset of H with the sum in $\mathbb{Z}[H]$ of its elements.

Definition 3.1. Let H be a group of order n and let m be an integer with $m \geq 1$. A family $\mathfrak{S} = \{S_{i,j}\}$, with $0 \leq i, j < m$, of subsets of H is a $(m, n, k, \mu, \lambda, t)$-partial sum family (for short, $(m, n, k, \mu, \lambda, t)$-PSF, or simply PSF if we do not specify the parameters) if it satisfies:

1. for every i it holds that $0 \notin S_{i,i}$, where 0 is the identity of H.
2. for every i it holds that $\sum_{j=0}^{m-1} |S_{i,j}| = \sum_{j=0}^{m-1} |S_{j,i}| = k$
3. for every i and j it holds that $\sum_{l=0}^{m-1} S_{l,j} S_{i,l} = \delta_{i,j}\gamma\{0\} + \beta S_{i,j} + \mu H$, where $\delta_{i,j}$ is the Kronecker delta, and where $\gamma = t - \mu$ and $\beta = \lambda - \mu$.

The symbol notation, presented in the introduction for undirected graphs, is valid also for directed graphs. Martínez and Araluze proved that the existence of a $(m, n, k, \mu, \lambda, t)$-partial sum family over a group H is equivalent to the existence of a (mn, k, μ, λ, t)-DSRG which admits a group of automorphisms isomorphic to H acting semiregularly and with m orbits (in fact, the elements of the PSF form the symbol of the digraph).

They found 17 new DSRGs by using partial sum families. 14 of them had cyclic groups as groups of automorphisms, which are the kind of digraphs that we are considering in this paper. More specifically, they found:

Four PSFs with parameters $(2, 15, 13, 6, 5, 8), (2, 17, 14, 6, 5, 12), (2, 17, 15, 7, 6, 9), (2, 20, 17, 8, 6, 11)$ which produce bicirculant digraphs.

Five PSFs with parameters $(3, 9, 10, 4, 3, 6), (3, 11, 11, 4, 3, 4), (3, 13, 10, 3, 1, 6), (3, 14, 14, 5, 4, 5), (3, 15, 14, 4, 5, 6)$ which produce tricirculant digraphs.

Four PSFs with parameters $(4, 7, 7, 2, 1, 2), (4, 8, 9, 3, 1, 6), (4, 8, 10, 3, 3, 7), (4, 11, 10, 2, 3, 4)$ which produce tetracirculant digraphs.

One PSF with parameters $(5, 7, 8, 2, 1, 4)$ which produce a pentacirculant digraph.

Finally, Araluze et al. found in [2] the form of the possible parameters of bicirculant DSRGs. They called the partial sum families partial sum quadruples in this case when $m = 2$.

Proposition 3.2. *Let $\mathfrak{S}$ be a non-trivial PSQ in a cyclic group H. Then, up to complementation, the parameters of $\mathfrak{S}$ are of the following form, where $U = k - t$ and s, f are positive integers, and where $\mathbf{q} = |S_{1,0}|$ and $\mathbf{r} = |S_{0,0}|$:*

1. $n = s(2f + 1), \mathbf{q} = sf, \mathbf{r} = sf, k = 2sf, \mu = sf, \lambda = s(f - 1), t = sf.$
2. $n = s(2f + 1), \mathbf{q} = s(f + 1), \mathbf{r} = sf, k = s(2f + 1), \mu = s(f + 1), \lambda = sf, t = s(f + 1).$
3. $n = 4s, \mathbf{q} = 2s, \mathbf{r} = 2s - 1, k = 4s - 1, \mu = s, \lambda = 3s - 2, t = 3s - 1.$
4. $n = 2s^2 + 2s + 1 + 2U, \mathbf{q} = s^2 + U, \mathbf{r} = s^2 + s + U, k = 2s^2 + s + 2U, \mu = s^2 + U, \lambda = s^2 - 1 + U$ *and* $t = 2s^2 + s + U.$
5. $n = 2s^2 + 2U, \mathbf{q} = s^2 + U, \mathbf{r} = s^2 + s + U, k = 2s^2 + s + 2U, \mu = s^2 + s + U, \lambda = s^2 + s + U$ *and* $t = 2s^2 + s + U$*, where* $2s$ *divides* $s^2 + U$.

6. $n = 2s^2 + 2U, \mathbf{q} = s^2 + U, \mathbf{r} = s^2 - s + U, k = 2s^2 - s + 2U, \mu = s^2 - s + U, \lambda = s^2 - s + U$ *and* $t = 2s^2 - s + U$, *where* $2s$ *divides* $s^2 + U$.
7. $n = 2s^2 + 2U, \mathbf{q} = s^2 \pm s + U, \mathbf{r} = s^2 + U, k = 2s^2 \pm s + 2U, \mu = s^2 \pm s + U, \lambda = s^2 \pm s + U$ *and* $t = 2s^2 \pm s + U$, *where* s *divides* U.
8. $n = 4s^2, \mathbf{q} = 2s^2 \pm 2s, \mathbf{r} = 2s^2 \pm s, k = 4s^2 \pm 3s, \mu = (s \pm 1)(2s \pm 1), \lambda = (s \pm 1)(2s \pm 1)$ *and* $t = s^2 + (s \pm 1)(2s \pm 1)$

Families 4,5,6 and 7 correspond with the ones found by Leung and Ma in 2.2 (in fact, they correspond to the particular case $U = 0$). Araluze et al. proved that PSQs with parameters as in families 1,2 and 3 exists for every s and f. They found several sporadic examples for the other families, two of which produced DSRGs not isomorphic to any known ones with that parameters.

Finally, we will mention a result about bicirculant association schemes. Let us recall first the definition of association scheme.

An association scheme of class s is a pair $\mathfrak{X} = (X, \mathfrak{R})$, where $\mathfrak{R} = \{R_0, \dots, R_s\}$ is a partition of X^2 that satisfies:

1. $R_0 = \{(x, x) \mid x \in X\}$.
2. For every $i = 0, \dots, n$, we have $\{(y, x) \mid (x, y) \in R_i\} \in \mathfrak{R}$.
3. For every $i, j, k = 0, \dots, n$, there exists a non-negative integer $p^k_{i,j}$ such that $|\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}| = p^k_{i,j}$ whenever $(x, y) \in R_k$.

The cardinality of X is called the order of the association scheme. The R_i are called the basic relations of the association scheme, and the (X, R_i) are called the basic digraphs. It is an easy consequence of part 3 of the definition that all the basic digraphs (X, R_i) are regular. The degree of R_i is defined as the degree of (X, R_i). The association schemes of class one are called trivial.

The linear span $\mathbb{C}[\mathfrak{R}]$ of the adjacency matrices of the R_i is called the Bose-Mesner algebra of the association scheme. The vector space $\mathbb{C}^X$ is a left $\mathbb{C}[\mathfrak{R}]$-module in a natural way. Since $\mathbb{C}[\mathfrak{R}]$ is a semisimple algebra, the mentioned vector space $\mathbb{C}^X$ decomposes as a direct sum of irreducible $\mathbb{C}[\mathfrak{R}]$-submodules, which are called the irreducible representations of $\mathfrak{X}$.

An association scheme $\mathfrak{X} = (X, \{R_0, \dots, R_s\})$ is said to be primitive if all the R_i are strongly connected.

An automorphism of an association scheme $\mathfrak{X} = (X, \{R_0, \dots, R_s\})$ is a bijection of X that is an automorphism of all the digraphs corresponding to the relations R_i. An association scheme $\mathfrak{X} = (X, \{R_0, \dots, R_s\})$ is said to be bicirculant if there exists a cyclic group of automorphisms of $\mathfrak{X}$ which acts semiregularly on X and has 2 orbits.

I. Kovács et al. obtained in [15] the following result:

Proposition 3.3. *Let* $\mathfrak{X}$ *be a primitive bicirculant scheme on* $2p^e$ *points,* $p > 2$ *a prime. Then*

1. $\mathfrak{X}$ *is of class at most two; and*
2. *if the class is exactly 2, then* $2p^e = (2s + 1)^2 + 1$ *for some natural number* s*, and the degrees of basic digraphs of* $\mathfrak{X}$ *are* $1, s(2s+1), (s+1)(2s+1)$*, and the multiplicities of the irreducible representations of* $\mathfrak{X}$ *in its standard module are* $1, p^e, p^e - 1$.

Although this result of I. Kovács et al. is mainly applied to the study of primitive permutation groups, it has an interpretation in terms of the kind of objects in which we are interested in this paper, because association schemes of class two are DSRGs, so that they proved that non-trivial primitive bicirculant schemes of certain orders are in fact DSRGs, and they obtained additional restrictions on that orders.

4 Uniform partial sum families

Uniform partial sum families were introduced in [2] as a kind of digraphs which generalizes what happens in family 4 of Proposition 3.2 and in part 2 of Proposition 2.3.

Definition 4.1. Let H be a group of order $n \geq 2$ and let $m \geq 1$ be an integer. Then an $(m, n, k, \mu, \lambda, t)$-partial sum family $\mathfrak{S} = \{S_{i,j}\}$, with $0 \leq i, j < m$, of subsets of H is uniform if it satisfies the following conditions:

1. The cardinalities of the 'diagonal' blocks $S_{i,i}$ are all equal.
2. The cardinalities of the 'non-diagonal' blocks $S_{i,j}$, $i \neq j$, are all equal.
3. The 'diagonal' blocks $\{S_{i,i} : i \in \mathbb{Z}_m\}$ form a partition of $H - \{0\}$, where 0 is the identity element of the group.

The following two propositions give infinite families, found in [2], of uniform PSFs:

Proposition 4.2. *If H is a group of order $ef+1$ and $A_0, \ldots, A_{e-1}$ is a partition of $H - \{0\}$ and $|A_i| = f$ for every i, then $\mathfrak{S} = \{S_{i,j}\}$, where $S_{i,j} = A_j$, is a uniform $(e, ef+1, ef, f, f-1, f)$-PSF in H.*

Proposition 4.3. *If H is a group of order $ef+1$ and $A_0, \ldots, A_{e-1}$ is a partition of $H - \{0\}$ with $|A_i| = f$ for every i, then $\mathfrak{S} = \{S_{i,j}\}$, where*

$$S_{i,j} = \begin{cases} A_i & \text{if } i = j, \\ \{0\} \cup A_j & \text{otherwise,} \end{cases}$$

is a uniform $(e, ef+1, e(f+1)-1, f+1, e+f-2, e+f-1)$-PSF in H.

We let $A = |S_{0,0}|$, and $B = |S_{0,1}|$. Thus, $|S_{i,i}| = A$ for every i, and $|S_{i,j}| = B$ for every distinct i and j.

Lemma 4.4. $B = (2k - \beta \pm \Delta)/(2m)$ *and* $A = k + (1-m)B$.

Proof. Using the second part of Definition 3.1 we obtain

$$A + (m-1)B = k,$$

and applying the trivial character in the third part of the same definition with $i = 0, j = 1$ we obtain

$$2AB + (m-2)B^2 = \beta B + \mu n.$$

Using (1.1) with the DSRG associated to the PSF we get

$$k(k-\beta) = \mu m n + \gamma.$$

Now, the result follows from the three previous identities and the definition of Δ. □

5 Form of the parameters

Now we will derive the parameters of uniform PSFs when the group H is cyclic. We will need first a lemma.

Lemma 5.1. *Let $\mathfrak{S}$ be a PSF in an abelian group H, and let χ be a non-trivial character of H. Then,*

$$\sum_{l=0}^{m-1} \chi(S_{l,l}) = m(\beta - \Delta)/2 + i\Delta \textit{ for some } i \in \{0, ..., m\}.$$

Proof. By using Definition 3.1, we have that the matrix $A_\chi = (\chi(S_{i,j}))$ satisfies $A_\chi^2 = \beta A_\chi + \gamma I_m$, and therefore its trace $\sum_{l=0}^{m-1} \chi(S_{l,l})$ is the sum of m roots of the polynomial $x^2 - \beta x - \gamma$. Since those roots are $\frac{1}{2}(\beta \pm \Delta)$, the result follows. □

Proposition 5.2. *If an m-circulant $(m, n, k, \mu, \lambda, t)$-uniform partial sum family exists with $m \geq 3$, then the parameters have one of the following forms:*

1.

$$n = sm - rm + r^2m + 1, k = sm + r^2m - r, t = r^2m - r + s,$$

$$\lambda = rm + s + r^2 - 2r - 1, \mu = s + r^2$$

$$A = s - r + r^2, B = s + r^2$$

with r an integer and s a non-negative integer.

2.

$$n = \frac{2i(i-1)mr^2}{s}, k = \frac{-2r^2m^2i + 2m^2r^2i^2 - sri - s + smri}{sm}$$

$$t = \frac{-2sr^2i^2 - 4sri + 2sr^2mi - 2s + 4m^2r^2i^2 - 4m^2r^2i + 4smri + s^2}{2m^2s},$$

$$\lambda = \frac{4m^2r^2i^2 - 4m^2r^2i - 2smr + s^2 + 2rm^2s - 4sm}{2m^2s}$$

$$\mu = \frac{(s - 2rm + 2rmi)(s + 2rmi)}{2m^2s}$$

$$A = \frac{-2r^2mi + 2mr^2i^2 - s}{sm}, B = \frac{ri(s - 2rm + 2rmi)}{sm}$$

with r an integer, s a non-negative integer and $i \in \{2, ..., m\}$, and where $2m$ divides s.

Proof. By the previous lemma, for every non-trivial caracter χ of H it holds that

$$\sum_{l=0}^{m-1} \chi(S_{l,l}) = m(\beta - \Delta)/2 + i\Delta \text{ for some } i \in \{0, ..., m\}.$$

By part 3 of Definition 4.1 we have that the previous sum equals -1, and hence we deduce that

$$\beta = ((m - 2i)\Delta - 2)/m, \tag{5.1}$$

where Δ is a non-negative integer. Now, from the definition of Δ we obtain

$$\gamma = -\frac{(\Delta i + 1)(\Delta i + 1 - \Delta m)}{m^2}. \tag{5.2}$$

Putting $U = k - t$, we deduce from (5.2) that

$$k = U + \mu - \frac{(\Delta i + 1)(\Delta i + 1 - \Delta m)}{m^2}. \tag{5.3}$$

From (1.1),(5.1),(5.2) and (5.3) we get

$$\begin{aligned} n = &\left(Um^2 + m - 1 + \mu m^2 - 2\Delta i + m\Delta i + \Delta m - \Delta m^2 + \Delta^2 im - \Delta^2 i^2\right) \\ &\left(Um^2 - 1 + m + \mu m^2 - 2\Delta i + \Delta m + m\Delta i + \Delta^2 im - \Delta^2 i^2\right)/(m^5\mu) \end{aligned} \tag{5.4}$$

Let us suppose that the plus sign holds in Lemma 4.4. Since $A = (n - 1)/m$, we obtain from that lemma that

$$n = \frac{m\Delta i + m - m^2\Delta i + Um^2 - 1 + \mu m^2 - 2\Delta i + \Delta m + \Delta^2 im - \Delta^2 i^2}{m^2}. \tag{5.5}$$

From (5.4) and (5.5) we find μ as

$$\begin{aligned} \mu = &\frac{1}{2m^2(-1+m)}(-m^2\Delta i + 2\Delta^2 mi - 2\Delta^2 i^2 + 4m\Delta i - 4\Delta i - 2 - m^2 + 3m \\ &- 2m^2\Delta - Um^3 + 2m\Delta + 2Um^2 + m^3\Delta i + \Delta^2 i^2 m - \Delta^2 m^2 i \\ &\pm m((Um^2 + 3m\Delta i + \Delta^2 im - 2\Delta i - \Delta^2 i^2 + m - 1 - m^2\Delta i)^2 \\ &+ 4im^2\Delta^2(m-1)(-1+i))^{1/2}) \end{aligned} \tag{5.6}$$

Since β is an integer, $i = 0$ is not possible (see (5.1) and use that $m \geq 3$). Let us suppose that $i = 1$.

Then,

$$\mu = -\frac{Um^2 - 1 + m - 2\Delta + 2\Delta m - \Delta m^2 - \Delta^2 + \Delta^2 m}{m^2} \tag{5.7}$$

or

$$\mu = \frac{Um^2 - 1 + m - 2\Delta + 2\Delta m - \Delta^2 + \Delta^2 m}{m^2(-1+m)}. \tag{5.8}$$

Let us suppose that (5.7) holds. Then we get from (5.4) that $n = 0$, which gives a contradiction.

Therefore, (5.8) must hold, and using this and (5.1),(5.2),(5.3) and (5.4) we obtain that the parameters are

$$n = \frac{mU}{m-1} - \Delta + \frac{(\Delta+1)^2}{m}, \tag{5.9}$$

$$k = \frac{Um^2 - \Delta - \Delta^2 + \Delta^2 m + \Delta m}{m(m-1)},$$

$$t = \frac{-\Delta^2 + \Delta^2 m - \Delta + \Delta m + Um}{m(m-1)},$$

$$\lambda = \frac{Um^2 - 1 + 3m - 2\Delta + 4\Delta m - \Delta^2 + \Delta^2 m - 3\Delta m^2 + \Delta m^3 - 2m^2}{m^2(m-1)},$$

$$\mu = \frac{U}{m-1} + \frac{(\Delta+1)^2}{m^2}, \tag{5.10}$$

$$A = \frac{Um^2 - \Delta m^2 + 3\Delta m + 2m + \Delta^2 m - 2\Delta - 1 - \Delta^2 - m^2}{m^2(m-1)},$$

$$B = \frac{Um^2 - 1 + m - 2\Delta + 2\Delta m - \Delta^2 + \Delta^2 m}{m^2(m-1)}.$$

From (5.2), we have that $m|(\Delta+1)^2$, and then we deduce from (5.9) that $m-1|mU$. Since $m-1$ and m are coprime, then $m-1|U$. Now, from (5.10) we deduce that $m|\Delta+1$. Putting $U = s(m-1)$ and $\Delta = rm-1$ we obtain parameters as in part 1 of the proposition.

Now, let us suppose that $i > 1$.

Since μ is an integer, the square root in (5.6) is also an integer, and

$$\begin{aligned}\mu = {} & \frac{1}{2m^2(-1+m)}(-m^2\Delta i + 2\Delta^2 mi - 2\Delta^2 i^2 + 4m\Delta i - 4\Delta i - 2 - m^2 + 3m \\ & - 2m^2\Delta - Um^3 + 2m\Delta + 2Um^2 + m^3\Delta i + \Delta^2 i^2 m - \Delta^2 m^2 i \\ & \pm m(Um^2 + 3m\Delta i + \Delta^2 im - 2\Delta i - \Delta^2 i^2 + m - 1 - m^2\Delta i + s))\end{aligned} \tag{5.11}$$

where s is a non-negative integer and

$$\begin{aligned}U = {} & (-4m^3\Delta^2 i + 4\Delta^2 im^2 - 4m^2\Delta^2 i^2 + 4m^3\Delta^2 i^2 - s^2 - 6sm\Delta i \\ & - 2s\Delta^2 im + 4s\Delta i + 2s\Delta^2 i^2 - 2sm + 2s + 2sm^2\Delta i)(2sm^2)\end{aligned}$$

Hence,

$$\mu = -1/2\,\frac{\left(-s - 2m\Delta i + 2m^2\Delta i\right)\left(-s + 2m\Delta - 2m\Delta i - 2m^2\Delta + 2m^2\Delta i\right)}{sm^2(-1+m)} \tag{5.12}$$

or

$$\mu = 1/2\,\frac{(s - 2m\Delta + 2m\Delta i)(s + 2m\Delta i)}{m^2 s} \tag{5.13}$$

Let us suppose that (5.12) holds. Then we get from (5.4) that $n = -s/(2m(-1+m))$, and this contradicts that n must be positive.

Therefore, (5.13) must hold, and using this and (5.1),(5.2),(5.3) and (5.4) we obtain, by putting $\Delta = r$, that the parameters are as in part 2 of the statement. From the form of A we deduce that m divides s, and from the form of μ we conclude that 2 divides s. From this two facts and the expression for A we obtain that $2m$ divides s.

Finally, if the minus sign holds in Lemma 4.4, the parameters have the same form as when the plus sign holds by putting $\Delta = -r$, so that r can have both positive and negative values. □

By Proposition 4.2, m-circulant uniform PSFs as stated in part 1 of Proposition 5.2 always exist for $r = 0$ and, by Proposition 4.3, m-circulant uniform PSFs as stated in part 1 of Proposition 5.2 always exist for $r = 1$.

6 The tricirculant case

Now we will consider the case when $m = 3$. In this case, we will call the PSF tricirculant.

Proposition 6.1. *If a tri-circulant $(3, n, k, \mu, \lambda, t)$-uniform partial sum family exists, then either*

1.

$$n = 3r^2 - 3r + 1 + 3s, k = 3r^2 - r + 3s, \mu = r^2 + s, \lambda = r^2 + r - 1 + s,$$
$$t = 3r^2 - r + s, A = r^2 - r + s \text{ and } B = r^2 + s. \tag{6.1}$$

2.

$$n = 9r^2 + 6r + 1, k = 9r^2 + 10r + 2, \mu = 3r^2 + 5r + 2, \lambda = 3r^2 + 4r + 1,$$
$$t = 5r^2 + 6r + 2, A = 3r^2 + 2r \text{ and } B = 3r^2 + 4r + 1. \tag{6.2}$$

3.

$$n = 90r^2 - 60r + 10, k = 90r^2 - 40r + 3, \mu = 30r^2 - 5r, \lambda = 30r^2 - 10r + 1,$$
$$t = 80r^2 - 40r + 6, A = 30r^2 - 20r + 3 \text{ and } B = 30r^2 - 10r, \tag{6.3}$$

with r an integer and s a non-negative integer.

Proof. Using the same notations as in the proof of Proposition 5.2, we have $2 + 2i\Delta \equiv 0 \pmod 3$, and hence $i = 1$ or $i = 2$. If $i = 1$, by following the proof of that proposition, we can see that the parameters are as in (6.1). Let us suppose that $i = 2$. We will assume that the plus sign holds in Lemma 4.4, because when the minus sign holds, it is easy to prove that the parameters have the same form, changing the sign of r. We have from (5.11) that $(9U + 2(\Delta - 1)^2)^2 + 144\Delta^2 = ((9U + 2(\Delta - 1)^2) + s)^2$ for some positive integer s, and hence $2(9U + 2(\Delta - 1)^2)s + s^2 = 144\Delta^2$. Since $U \geq 0$, we deduce that $s \leq 39$ and, since 6 must divide s, then the possible values that s can take are $6, 12, 18, 24, 30, 36$. Now, having into account that $\Delta \equiv 1 \pmod 3$ and the expression for n in Proposition 5.2 we have that for $s = 18, 36$ we obtain a contradiction with the fact that n is an integer, and for

$s = 6, 24$ we obtain a contradiction with the fact that 3 divides $n - 1$. For $s = 12$, by putting $\Delta = 1 + 3r$ we obtain parameters as in part (6.2), and for $s = 30$, by putting $\Delta = 1 + 3u$ we have $U = \frac{2u^2+8u-7}{5}$ and hence $u \equiv -2 \pmod 5$. Putting now $u = -2 + 5r$ we obtain parameters as in (6.3). □

We can eliminate some values of r in family (6.3). In the next proposition, Gröbner bases are used (for definitions and results on Gröbner bases, we refer the reader to [3]).

Proposition 6.2. *If a tricirculant uniform PSF exists with parameters as in family (6.3) of the previous proposition, then* $r \equiv 1 \pmod 2$.

Proof. If χ is any non-trivial character of H then we obtain from Definition 3.1, by applying the character χ, that

$$\sum_{l=0}^{2} \chi(S_{i,l})\chi(S_{l,j}) - \delta_{i,j}\gamma - \beta\chi(S_{i,j}) = 0 \text{ for every } i, j. \tag{6.4}$$

By putting the indeterminate $T_{i,j}$ instead of $\chi(S_{i,j})$ and calculating a Gröbner basis for the polynomials obtained from the lefthandsides in (6.4) with respect to the lexicographic order with

$$T_{0,0} > T_{1,1} > T_{2,2} > T_{0,1} > T_{1,0} > T_{0,2} > T_{2,0} > T_{1,2} > T_{2,1} > r,$$

we conclude that

$$\begin{aligned} &\chi(S_{1,1})^2 - \chi(S_{1,1}) - 6 + 35r + \chi(S_{1,2})\chi(S_{2,1}) \\ &- 50r^2 + \chi(S_{0,1})\chi(S_{1,0}) + 5\chi(S_{1,1})r = 0. \end{aligned} \tag{6.5}$$

Now we take the character χ that takes the generator of the cyclic group H to $\theta^{n/2}$, where θ is a primitive n-th root of unity. Since A is an odd number and B is an even number, we have that the $\chi(S_{i,j})$ are integer numbers such that $\chi(S_{i,j})$ is odd if $i = j$ and even in other case. Now, we obtain from (6.5) that

$$(\chi(S_{1,1}) - 1)(\chi(S_{1,1}) + r) \equiv 2(r^2 + 1) \pmod 4.$$

Suppose that r is even. Then the above implies that $\chi(S_{1,1}) \equiv 3 \pmod 4$. We show below that this leads to a contradiction, hence r must be indeed odd.

By the choice of the character χ we get for every $i \in \{0, 1, 2\}$,

$$\chi(S_{i,i}) = -|S_{i,i}| + 2|S_{i,i} \cap \ker(\chi)| = -30r^2 + 20r - 3 + 2|S_{i,i} \cap \ker(\chi)|.$$

The group $\ker(\chi)$ is of order $n/2 = 45r^2 - 30r + 5$, which is odd. Using this and that $\cup_{i=0}^{2}(S_{i,i} \cap \ker(\chi)) = \ker(\chi) - \{0\}$, we see that at least one of the numbers $|S_{i,i} \cap \ker(\chi)|$ must be even. We may set the indices at the beginning so that $|S_{1,1} \cap \ker(\chi)|$ is even, and hence above we find $\chi(S_{1,1}) \equiv 1 \pmod 4$, a contradiction. □

Apart from the infinite families of uniform PSFs showed in Section 4, we will analyze some sporadic examples for the tricirculant case.

When $s = 0$ in the first family in Proposition 6.1, that is, when the graph is undirected, Kutnar et al. found in [16] PSFs for $r = 1, 2$ and 3.

For $r = -1, s = 2$, a PSF with the parameters and structure indicated in Proposition 6.1 was found by the author and A. Araluze in [20].

We have found now for $r = 2, s = 2$, by using techniques of combinatorial optimization, a uniform tricirculant PSF which generates a DSRG with new parameters, which appear as an undecided case in Hobart and Brouwer's table [10]. This PSF is

$$S_{0,0} = (3,4,7,9), S_{0,1} = (0,4,7,8,11,12), S_{0,2} = (0,1,2,4,6,10),$$

$$S_{1,0} = (1,2,5,6,8,9), S_{1,1} = (1,5,6,12), S_{1,2} = (2,3,5,6,7,8),$$

$$S_{2,0} = (0,1,3,7,9,11), S_{2,1} = (0,5,6,7,8,11), S_{2,2} = (2,8,10,11).$$

7 Almost-uniform partial sum families

In this section we will present a kind of partial sum families that generalizes the uniform ones.

Definition 7.1. Let H be a group of order $n \geq 2, H'$ a subgroup of H of order n' and let $m \geq 1$ be an integer. Then an $(m, n, k, \mu, \lambda, t)$-partial sum family $\mathfrak{S} = \{S_{i,j}\}$, with $0 \leq i, j < m$, with the $S_{i,j}$ subsets of H, is almost-uniform with respect to H' if it satisfies the following conditions:

1. The cardinalities of the 'diagonal' blocks $S_{i,i}$ are all equal.
2. The cardinalities of the 'non-diagonal' blocks $S_{i,j}$, $i \neq j$, are all equal.
3. The 'diagonal' blocks $\{S_{i,i} : i \in \mathbb{Z}_m\}$ form a partition of $H - H'$.

Remark 7.2. Observe that in the particular case when $H' = \{e\}$ almost uniform PSFs are just uniform PSFs. In what follows, we will study the case when H' is a proper subgroup of H.

Let us analyze the form of the parameters of m-circulant $(m, n, k, \mu, \lambda, t)$-almost-uniform partial sum families over a cyclic group H with respect to a proper and non-trivial subgroup H':

Proposition 7.3. *If an $(m, n, k, \mu, \lambda, t)$-almost-uniform partial sum family exists with $m \geq 3$ with respect to a proper and non-trivial subgroup H' of the cyclic group H, then the parameters have one of the following forms:*

1.

$$n = (im - m + i)\, imr^2 + (1 - i)\, mr + ms,$$

$$k = (im - m + i)\, imr^2 + (-i + m - im)\, r + ms,$$

$$\mu = (im - m + i)\, ir^2 - ri + s,$$

$$\lambda = (im - m + i)\, ir^2 + (m - 3\, i)\, r + s,$$

$$t = -ri + i^2r^2m + s, A = (im - m + i)\, ir^2 + (1 - 2i)\, r + s,$$

$$\textit{and } B = (im - m + i)\, ir^2 + (1 - i)\, r + s.$$

2.

$$n = \left(j^2 - j\right) r^2 + (-j - 1)\, r + sm + \frac{r^2 j^2}{m},$$

$$k = \left(j^2 - j\right) r^2 - rj + sm + \frac{rj\,(-1 + rj)}{m},$$

$$\mu = s + \frac{rj\,(rjm - rm + rj - m)}{m^2},$$

$$\lambda = s + r + \frac{rj\,(-3\,m - rm + rjm + rj)}{m^2},$$

$$t = s + \frac{rj\,(-1 + rj)}{m},$$

$$A = s + \frac{rj\,(rmj - 2\,m - mr + rj)}{m^2},$$

$$B = s + \frac{rj\,(-m - mr + rmj + rj)}{m^2},$$

where m *divides* rj.

3.

$$n = \frac{2p\,(p - 1)\, mr^2}{s}, k = -\frac{r\,\left(2\,r\,m^2p - 2\,r\,m^2p^2 + si + sp - smp\right)}{sm},$$

$$t = -(-2\,sr^2mi + 2\,sr^2i^2 + 4\,sr^2ip - 2\,sr^2mp + 2\,sr^2p^2 + 4\,m^2r^2p - 4\,m^2r^2p^2 + 2\,srm - 4\,srmp - s^2)/(2m^2s),$$

$$\lambda = -1/2\,\frac{4\,m^2r^2p - 4\,r^2m^2p^2 + 2\,srm - s^2 - 2\,rm^2s + 4\,srmi}{m^2s},$$

$$\mu = 1/2\,\frac{(s + 2\,rmp)\,(s - 2\,rm + 2\,rmp)}{m^2s},$$

$$A = -\frac{r\,\left(-2\,mrp^2 + 2\,mrp + si\right)}{ms}, B = \frac{rp\,(s - 2\,mr + 2\,mrp)}{ms}.$$

Proof. We have $\dot\bigcup_i S_{i,i} = H - H'$. If χ is a character of H which is non-trivial on H', then $\sum_i \chi(S_{i,i}) = 0$. If χ' is trivial on H' but not on H, then $\sum_i \chi'(S_{i,i}) = -|H'|$. Since, by Lemma 5.1, the difference of both sums of characters must be divisible by Δ, we have that $|H'| = i\Delta$ with $i \in \{1, ..., m\}$. Since, again by Lemma 5.1, $\sum_l \chi(S_{l,l}) = m(\beta - \Delta)/2 + j\Delta$ with $j \in \{0, ...m\}$, we have

$$\beta = (m - 2j)\Delta/m, \tag{7.1}$$

where Δ is a non-negative integer. We have

$$\gamma = (jm - j^2)\Delta^2/m^2. \tag{7.2}$$

Now,

$$\gamma = t - \mu = (jm - j^2)\Delta^2/m^2, k = U + \mu + (jm - j^2)\Delta^2/m^2 (\text{where } U = k - t),$$
$$\beta = (m - 2j)\Delta/m. \tag{7.3}$$

From these equalities and (1.1) we get

$$\begin{aligned} n =&(Um^2 + \mu m^2 + \Delta mj + \Delta^2 mj - \Delta^2 j^2)(Um^2 + \mu m^2 + \Delta mj \\ &- \Delta m^2 + j\Delta^2 m - j^2\Delta^2)/(m^5\mu) \end{aligned} \tag{7.4}$$

Let us suppose that the plus sign holds in Lemma 4.4 Since the PSF is almost-uniform we have $n - i\Delta = mA$, and therefore

$$n = \frac{Um^2 + \mu\, m^2 + \Delta\, mj - j^2\Delta^2 + j\Delta^2 m - \Delta\, m^2 j + i\Delta\, m^2}{m^2} \tag{7.5}$$

From (7.4) and (7.5) we deduce

$$\begin{aligned} \mu = &(-\Delta\, m^2 - m^3 U + mj^2\Delta^2 - m^3 i\Delta + 2\, Um^2 - m^2 j\Delta^2 + m^3\Delta\, j - \Delta\, m^2 j \\ &+ 2\,\Delta\, mj - 2\, j^2\Delta^2 + 2\, j\Delta^2 m \pm m((Um^2 - \Delta\, m - 2\,\Delta\, mi + i\Delta\, m^2 + 3\,\Delta\, mj \\ &- j^2\Delta^2 + j\Delta^2 m - \Delta\, m^2 j)^2 + 4\, m^2\Delta^2(i - j + 1)(i - j)(m - 1))^{1/2}) \\ &/(2m^2(m - 1)). \end{aligned}$$

Let us suppose that $j = i$. Then,

$$\mu = -\frac{Um^2 + \Delta\, mi + i\Delta^2 m - i^2\Delta^2}{m^2} \tag{7.6}$$

or

$$\mu = \frac{Um^2 + \Delta\, mi - \Delta\, m^2 + i\Delta^2 m - i^2\Delta^2}{m^2\,(m - 1)} \tag{7.7}$$

If (7.6) holds, then we get from (7.4) that $n = 0$, which is a contradiction.
Thus, (7.7) must hold, and we obtain

$$n = \frac{Um^2 + \Delta\, mi - \Delta\, m + i\Delta^2 m - i^2\Delta^2}{m\,(m - 1)} \tag{7.8}$$

$$k = \frac{Um^2 + \Delta\, i - \Delta\, m + i\Delta^2 m - i^2\Delta^2}{m\,(m-1)} \tag{7.9}$$

$$t = \frac{i\Delta^2 m - i^2\Delta^2 + Um + \Delta\, i - \Delta\, m}{m\,(m-1)} \tag{7.10}$$

$$\lambda = \frac{Um^2 + 3\,\Delta\, mi - 2\,\Delta\, m^2 + i\Delta^2 m - i^2\Delta^2 + \Delta\, m^3 - 2\,\Delta\, m^2 i}{m^2\,(m-1)} \tag{7.11}$$

$$\mu = \frac{Um^2 + \Delta\, mi - \Delta\, m^2 + i\Delta^2 m - i^2\Delta^2}{m^2\,(m-1)} \tag{7.12}$$

$$A = \frac{Um^2 + 2\,\Delta\, mi - \Delta\, m + i\Delta^2 m - i^2\Delta^2 - \Delta\, m^2 i}{m^2\,(m-1)} \tag{7.13}$$

$$B = \frac{Um^2 + \Delta mi - \Delta m + i\Delta^2 m - i^2\Delta^2}{m^2\,(m-1)} \tag{7.14}$$

From (7.12) we have $i^2\Delta^2 \equiv 0 \pmod{m}$ and, from (7.9), $-i^2\Delta^2 + i\Delta \equiv 0 \pmod{m}$. Therefore, $i\Delta \equiv 0 \pmod{m}$.

From (7.13), $(im - i^2)\Delta^2 + (2i-1)m\Delta \equiv 0 \pmod{m^2}$. Using this and the previous congruence, we obtain $m\Delta \equiv 0 \pmod{m^2}$, and hence

$$\Delta \equiv 0 \pmod{m}. \tag{7.15}$$

From (7.12),

$$U \equiv (i^2 - i)\Delta^2 + (1-i)\Delta \pmod{m-1}. \tag{7.16}$$

Now, using the two previous congruences and putting

$$\Delta = rm \text{ and } U = i(i-1)r^2m^2 + (1-i)rm + s(m-1),$$

we obtain parameters as in part 1 of the proposition.

Now, let us suppose that $j = i + 1$.

In this case, by reasoning in a similar way as before, we obtain

$$n = \frac{Um^2 - \Delta\, m^2 + j\Delta^2 m + \Delta\, m + \Delta\, mj - j^2\Delta^2}{m\,(m-1)} \tag{7.17}$$

$$k = \frac{Um^2 + j\Delta + j\Delta^2 m - j^2\Delta^2}{m\,(m-1)} \tag{7.18}$$

$$\mu = \frac{Um^2 + \Delta\, mj + j\Delta^2 m - j^2\Delta^2}{m^2\,(m-1)} \tag{7.19}$$

$$\lambda = \frac{Um^2 + 3\,\Delta\, mj + j\Delta^2 m - j^2\Delta^2 - \Delta\, m^2 + \Delta\, m^3 - 2\,\Delta\, m^2 j}{m^2\,(m-1)} \tag{7.20}$$

$$t = \frac{j\Delta^2 m - j^2\Delta^2 + Um + j\Delta}{m\,(m-1)} \tag{7.21}$$

$$A = \frac{Um^2 + j\Delta^2 m + 2\,\Delta\, mj - j^2\Delta^2 - \Delta\, m^2 j}{m^2\,(m-1)} \tag{7.22}$$

$$B = \frac{Um^2 + \Delta\, mj + j\Delta^2 m - j^2\Delta^2}{m^2\,(m-1)} \tag{7.23}$$

From (7.19) we have $\Delta^2 j^2 \equiv 0 \pmod m$, and from (7.18), $\Delta j - \Delta^2 j^2 \equiv 0 \pmod m$, and hence $\Delta j \equiv 0 \pmod m$.

From (7.19), we have $U + \Delta j + \Delta^2 j - \Delta^2 j^2 \equiv 0 \pmod{m-1}$.

Putting $U = -\Delta j - \Delta^2 j + \Delta^2 j^2 + s(m-1)$ and $\Delta = r$, we obtain parameters as in part 2 of the proposition.

When $j \neq i$ and $j \neq i+1$ we find the parameters following the line of the proof of Proposition 5.2 and putting $p = j - i$ and $r = \Delta$.

If the minus sign holds in Lemma 4.4, we find analogous families of parameters by putting $r = -\Delta$, so that r can be positive or negative. □

We have found, by using methods of combinatorial optimization, 14 examples of PSFs corresponding to family 1, which are listed in the appendix. For three of the examples, corresponding to $m = 3, i = 1, r = -1, s = 3, m = 3, i = 1, r = -1, s = 4$ and to $m = 5, i = 1, r = 1, s = 2$, the parameters appear as undecided cases in Hobart and Brouwer's table [10]. We wonder if such PSFs exist for $m = 3, i = 1, r = 1$ or $r = -1$ and every positive integer s.

8 Appendix

Family 1:

For $m = 3$:

1. $i = 1, r = -1, s = 1$:

   $(((1), (0,1,5), (2,3,5)), ((0,3,5), (3), (0,1,4)),$
   $((1,3,4), (0,1,2), (5)))$

2. $i = 1, r = -1, s = 2$:

   $(((5,8), (0,4,6,7), (0,1,3,7)), ((1,2,3,5), (2,4), (0,5,6,7)),$
   $((2,3,6,8), (0,3,4,7), (1,7)))$

3. $i = 1, r = -1, s = 3$:

   $(((5,9,11), (1,2,4,9,10), (0,1,3,8,9)),$
   $((3,5,8,9,11), (1,2,10), (0,1,3,7,9)),$
   $((0,2,3,4,11), (4,5,7,8,9), (3,6,7)))$

4. $i = 1, r = -1, s = 4$:

   $(((6,9,12,13), (0,1,3,4,11,12), (6,7,10,11,13,14)),$
   $((1,4,5,8,10,14), (1,7,11,14), (0,2,5,6,9,11)),$
   $((1,2,3,4,7,9), (0,7,8,9,10,13), (2,3,4,8)))$

5. $i = 1, r = 1, s = 1$:

   $(((5), (3,4), (2,3)), ((2,3), (1), (0,5)), ((0,3), (1,4), (3)))$

6. $i = 1, r = 1, s = 2$:

   $(((7,8),(0,1,2),(2,3,4)),((3,7,8),(1,5),(2,3,7)),$
   $((0,5,7),(0,2,7),(2,4)))$

7. $i = 1, r = 1, s = 3$:

   $(((3,9,10),(0,5,6,7),(6,7,8,9)),((0,3,9,10),(5,6,7),(6,7,8,9)),$
   $((2,3,5,8),(0,5,10,11),(1,2,11)))$

8. $i = 1, r = 1, s = 4$:

   $(((6,8,12,14),(3,5,9,11,12),(5,7,11,13,14)),$
   $((1,3,4,5,7),(1,2,4,13),(0,2,3,4,6)),$
   $((1,4,8,10,12),(1,5,7,9,13),(3,7,9,11)))$

For $m = 4$:

1. $i = 1, r = -1, s = 1$:

   $(((4,6,7),(4,5),(4,7),(0,1)),((3,4),(1,2,4),(4,5),(1,6)),$
   $((1,4),(2,3),(4,5,6),(2,3)),((0,1),(2,7),(5,6),(2,3,4)))$

2. $i = 1, r = 1, s = 1$:

   $(((5),(4,7),(3,6),(4,7)),((1,2),(1),(0,7),(0,1)),$
   $((1,2),(0,1),(7),(0,1)),((1,4),(0,3),(2,7),(3)))$

3. $i = 1, r = 1, s = 2$:

   $(((1,2),(3,4,5),(4,5,6),(2,6,10)),$
   $((2,4,9),(5,7),(1,6,8),(2,6,10)),$
   $((6,7,8),(9,10,11),(10,11),(0,4,8)),$
   $((0,2,7),(3,5,10),(4,6,11),(4,8)))$

4. $i = 1, r = 1, s = 3$:

   $(((3,13,14),(8,9,10,11),(1,2,3,12),(0,9,10,11)),$
   $((4,5,7,10),(1,2,15),(3,8,9,10),(0,1,2,7)),$
   $((1,2,4,7),(12,13,14,15),(5,6,7),(4,13,14,15)),$
   $((0,3,13,14),(8,9,10,11),(1,2,3,12),(9,10,11)))$

For $m = 5$:

1. $i = 1, r = 1, s = 1$:

   $(((5),(0,7),(1,2),(6,7),(3,6)),((3,8),(3),(4,5),(0,9),(6,9)),$
   $((4,9),(6,9),(1),(5,6),(2,5)),((3,8),(0,3),(4,5),(9),(6,9)),$
   $((4,9),(1,4),(5,6),(0,1),(7)))$

2. $i = 1, r = 1, s = 2$:

   $(((2,13),(0,2,4),(7,9,11),(0,2,4),(10,12,14)),$
   $((3,13,14),(1,5),(7,8,12),(0,1,5),(0,10,11)),$
   $((1,5,6),(3,7,8),(10,14),(3,7,8),(2,3,13)),$
   $((2,6,13),(0,4,8),(0,7,11),(4,8),(3,10,14)),$
   $((3,10,14),(1,5,12),(4,8,12),(1,5,12),(7,11)))$

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References

[1] A. Araluze, K. Kutnar, L. Martínez and D. Marušič, Edge Connectivity in Difference Graphs and some new constructions of Partial Sum Families, *European J. Combin.* **32** (2011), 352–360.

[2] A. Araluze, I. Kovács, K. Kutnar, L. Martínez, D. Marušič, Partial sum quadruples and bi-Abelian digraphs, *J. Combin. Theory Ser. A* **119** (2012), 1811–1831.

[3] T. Becker, V. Weispfenning and H. Kredel, *Gröbner bases, a computational approach to commutative algebra, Graduate texts in mathematics, vol. 141*, Springer, New York (1993).

[4] R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* **13** (1963), 389–419.

[5] W. G. Bridges and R. A. Mena, Rational circulants with rational spectra and cyclic strongly regular graphs, *Ars Combin.* **8** (1979), 143–161.

[6] A. Duval, A directed graph version of strongly regular graphs, *J. Combin. Theory Ser. A* **47** (1988), 71–100.

[7] A. Duval and D. Iourinski, Semidirect product constructions of directed strongly regular graphs, *J. Combin. Theory Ser. A* **104** (2003), 157–167.

[8] F. Fiedler, M. H. Klin and M. Muzychuk, Small vertex-transitive directed strongly regular graphs, *Discrete Math.* **255** (2002), 87–115.

[9] S. A. Hobart and T. J. Shaw, A note on a family of directed strongly regular graphs, *European J. Combin.* **20** (1999), 819–820.

[10] S. A. Hobart, *Parameters of directed strongly regular graphs*, `http://homepages.cwi.nl/\textasciitildeaeb/math/dsrg/dsrg.html`

[11] L. K. Jørgensen, Directed strongly regular graphs with $\mu = \lambda$, *Discrete Math.* **231** (2001), 289–293.

[12] L. K. Jørgensen, Non-existence of directed strongly regular graphs, *Discrete Math.* **264** (2003), 111–126.

[13] L. K. Jørgensen, Rank of adjacency matrices of directed (strongly) regular graphs, *Linear Algebra Appl.* **407** (2005), 233–241.

[14] M. Klin, A. Munemasa, M. Muzychuk and P. -H. Zieschang, Directed strongly regular graphs obtained from coherent algebras, *Linear Algebra Appl.* **377** (2004), 83–109.

[15] I. Kovács, D. Marušič, M. Muzychuk, Primitive bicirculant association schemes and a generalization of Wielandt's theorem, *Trans. Amer. Math. Soc.* **362** (2010), 3203–3221.

[16] K. Kutnar, D. Marušič, Š. Miklavič, P. Šparl, Strongly regular tri-Cayley graphs, *European J. Combin.* **30** (2009), 822–832.

[17] K.H. Leung, S.L.Ma, Partial difference triples, *J. Algebraic Combin.* **2** (1993), 397–409.

[18] A. Malnič, D. Marušič, P. Šparl and B. Frelih, Symmetry structure of bicirculants, *Discrete Math.* **307** (2007), 409–414.

[19] A. Malnič, D. Marušič and P. Šparl, On strongly regular bicirculants, *European J. Combin.* **28** (2007), 891–900.

[20] L. Martínez and A. Araluze, New tools for the construction of directed strongly regular graphs: difference digraphs and partial sum families, *J. Combin. Theory Ser. B* **100** (2010), 720–728.

[21] D. Marušič, Strongly regular bicirculants and tricirculants, *Ars Combin.* **25 C** (1988), 11–15.

[22] M. J. de Resmini and D. Jungnickel, Strongly regular semi-Cayley graphs, *J. Algebraic Combin.* **1** (1992), 171–195.

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On automorphism groups of graph truncations

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Abstract

It is well known that the Petersen graph, the Coxeter graph, as well as the graphs obtained from these two graphs by replacing each vertex with a triangle, are trivalent vertex-transitive graphs without Hamilton cycles, and are indeed the only known connected vertex-transitive graphs of valency at least two without Hamilton cycles. It is known by many that the replacement of a vertex with a triangle in a trivalent vertex-transitive graph results in a vertex-transitive graph if and only if the original graph is also arc-transitive. In this paper, we generalize this notion to t-regular graphs Γ and replace each vertex with a complete graph K_t on t vertices. We determine necessary and sufficient conditions for $\mathscr{T}(\Gamma)$ to be hamiltonian, show $\mathrm{Aut}(\mathscr{T}(\Gamma)) \cong \mathrm{Aut}(\Gamma)$, as well as show that if Γ is vertex-transitive, then $\mathscr{T}(\Gamma)$ is vertex-transitive if and only if Γ is arc-transitive. Finally, in the case where t is prime we determine necessary and sufficient conditions for $\mathscr{T}(\Gamma)$ to be isomorphic to a Cayley graph as well as an additional necessary and sufficient condition for $\mathscr{T}(\Gamma)$ to be vertex-transitive.

Keywords: Truncation, automorphism group, Cayley graph, Hamiltonian.

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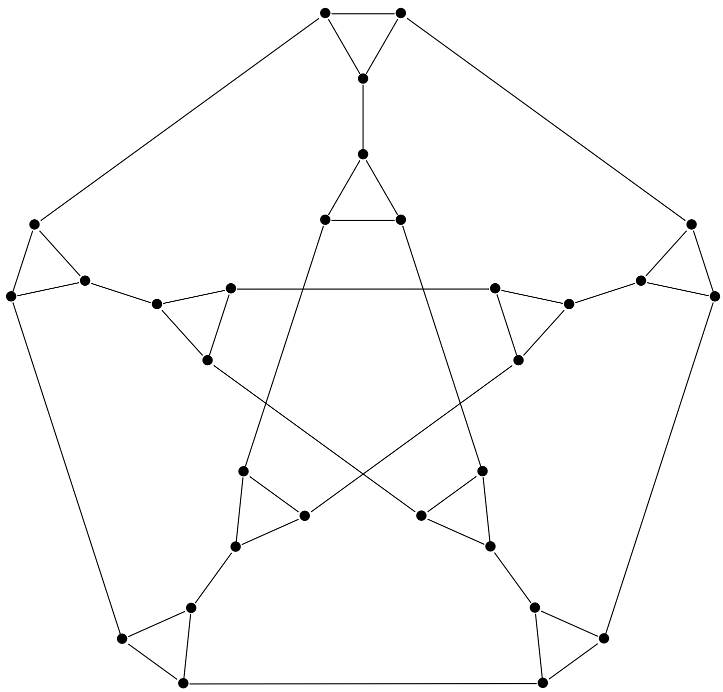

Figure 1: The truncation of the Petersen graph

1 Introduction

In 1969 Lovász posed the problem below (this statement is written exactly as Lovász wrote it).

Problem 1.1 (Lovász, 1969)**.** *Let us construct a finite, connected, undirected graph which is symmetric and has no simple path containing all elements. A graph is called symmetric, if for any two vertices x, y it has an automorphism mapping x into y.*

Usually this problem is stated as the conjecture that every vertex-transitive graph contains a Hamilton path (here "vertex-transitive" and "symmetric" are synonyms). Typically though it is usually the case that this conjecture is verified by showing that a a particular vertex-transitive graph contains a Hamilton cycle. Much work has been done in attempting to verify this conjecture — see [5] for some recent information regarding progress on this conjecture. The Petersen graph is vertex-transitive but does not contain a Hamilton cycle (see for example [3, Theorem 1.5.1]), while Tutte [9] first showed that the Coxeter graph is not hamiltonian, with an additional proof by Biggs [1]. The graphs obtained from the Petersen graph and the Coxeter graph by replacing each vertex with a triangle — called the **truncation** — are also vertex-transitive graphs that do not contain a Hamilton cycle. These four graphs are the only known connected vertex-transitive graphs, other than K_2, that do not have a Hamilton cycle. The truncation of the Petersen graph is shown in Figure 1.

It turns out that the truncation of the truncation of the Petersen and Coxeter graphs are not vertex-transitive. It is known by many that the truncation of a trivalent graph Γ

is vertex-transitive if and only if $\mathrm{Aut}(\Gamma)$ is also transitive on the edges of Γ, or **edge-transitive**, although neither of the previously stated facts are proven in the literature. We will generalize the notion of truncation to vertex-transitive graphs of valencies other than 3. Note that as a triangle can be viewed as either a cycle of length 3 or a complete graph K_3, there are two natural generalizations of the idea of truncation. Namely, one can "replace" each vertex with a cycle or with a complete graph, or even with an arbitrary graph. These have been studied for example in [2, 6, 11] We note that in [11], the graph obtained by replacing each vertex with a complete graph is called a clique-inserted graph, and that replacing each vertex with a cycle is motivated by map truncation. For a vertex $v \in V(\Gamma)$, we denote the valency of v in Γ by $\mathrm{val}(v)$.

Definition 1.2. Let Γ be a graph. The **truncation** $\mathscr{T}(\Gamma)$ of Γ is obtained from Γ by replacing each vertex v of Γ with a clique on $\mathrm{val}(v)$ vertices, denoted T_v, and whenever $uv \in E(\Gamma)$, then one vertex of T_v is adjacent to one vertex of T_u and no vertex of T_v is adjacent to more than one vertex outside of T_v.

Note that if $uv \in E(\Gamma)$, we do not specify which vertex of T_u is adjacent to T_v. Obviously, different choices of such vertices will result in different graphs, but all such choices result in isomorphic graphs as each T_u and T_v is a complete graph and no vertex of T_v is adjacent to more than one vertex outside of T_v. Also, as each vertex in T_v is adjacent to $\mathrm{val}(v) - 1$ vertices inside T_v and exactly one vertex outside T_v, a vertex $u \in T_v$ has valency $\mathrm{val}(v)$ in the truncation $\mathscr{T}(\Gamma)$. In particular, the truncation of a t-regular graph is still t-regular.

We should point out that there is an equivalent definition of graph truncation introduced in [7]. For a graph Γ, let $S(\Gamma)$ be the graph obtained from Γ by subdividing each edge via the insertion of a single vertex, and $L(\Gamma)$ be the line graph of Γ. Then $\mathscr{T}(\Gamma) = L(S(\Gamma))$.

In this paper we show $\mathscr{T}(\Gamma)$ contains a Hamilton cycle if and only if Γ has a spanning eulerian subgraph (Theorem 2.1). We then show that for a graph Γ with minimal valency at least 3 the automorphism group of its truncation is isomorphic to $\mathrm{Aut}(\Gamma)$ (Theorem 3.6), and subsequently that a connected vertex-transitive graph with minimal valency at least 3 has a vertex-transitive truncation if and only if it is **arc-transitive** (Theorem 3.7), or transitive on the set of directed edges or arcs of Γ. We remark that this is consistent with the statements earlier that a trivalent vertex-transitive graph has vertex-transitive truncation if and only if it is edge-transitive as a vertex- and edge-transitive graph of odd valency is necessarily arc-transitive [10, 7.53]. Finally, for a vertex-transitive graph Γ of prime valency, we also determine necessary and sufficient conditions for $\mathscr{T}(\Gamma)$ be a Cayley graph provided that $\mathscr{T}(\Gamma)$ is vertex-transitive (Theorem 3.8), as well as provide an alternative characterization of when $\mathscr{T}(\Gamma)$ is vertex-transitive (Theorem 3.10). We begin with necessary and sufficient conditions for $\mathscr{T}(\Gamma)$ to be Hamiltonian.

2 Hamiltonicity of Graph Truncations

Theorem 2.1. *If Γ is a graph, then $\mathscr{T}(\Gamma)$ contains a Hamilton cycle if and only if Γ contains a connected spanning eulerian subgraph.*

Proof. First suppose that Γ has a connected spanning eulerian subgraph Δ, and let $v_0v_1 \cdots v_rv_0$ be an Euler tour of Δ, where we traverse the tour so that the edge v_iv_{i+1} is traversed from v_i to v_{i+1}. Given that the edge v_iv_{i+1} is traversed from v_i to v_{i+1}, let $u_{i,0}$ and $u_{i+1,1}$ denote the vertices of T_{v_i} and $T_{v_{i+1}}$, respectively, that are adjacent. For each $x \in V(\Gamma)$, we let x_m be the largest nonnegative integer for which $v_{x_m} = x$, set $Y_x = \{i < x_m : v_i = x\}$, and $Z_x = \{u_{i,0}, u_{i,1} : i \in Y_x\}$. For each $1 \leq i \leq r$ construct a path P_i as follows: Setting $x = v_i$, we let $P_i = u_{i,0}$ if $i < x_m$, while if $i = x_m$, we let Q_i be a Hamilton path from $u_{i,1}$ to $u_{i,0}$ in $\mathscr{T}(\Gamma)[T_x - Z_x]$, the subgraph of $\mathscr{T}(\Gamma)$ induced by $T_x - Z_x$. Let P_i be the path obtained from Q_i by removing the initial vertex $u_{i,1}$ of Q_i. We observe that Q_i and consequently P_i certainly exist as $\mathscr{T}(\Gamma)[T_x - Z_x]$ is a clique. Then

$$u_{0,0}u_{1,1}P_1u_{1,0}u_{2,1}P_2u_{2,0} \dots u_{r,1}P_ru_{r,0}u_{0,1}P_0u_{0,0}$$

is a Hamilton cycle in $\mathscr{T}(\Gamma)$. Intuitively, we travel along the Euler tour until the last time we visit a T_v, at which point we traverse all the previously unvisited vertices of T_v.

Conversely, suppose that $H = v_0v_1 \dots v_n$ is a Hamilton cycle in $\mathscr{T}(\Gamma)$ (so $v_n = v_0$). For each $0 \leq i \leq n$, let $v_i \in T_{x_i}$, $x_i \in V(\Gamma)$. Let $E' = \{x_ix_{i+1} : T_{x_i} \neq T_{x_{i+1}}\}$, so that E' is simply the set of edges of H that connect different inserted cliques. Then the edges of E' form a spanning connected subgraph of Γ as H is a Hamilton cycle in $\mathscr{T}(\Gamma)$. Additionally, with the exception of T_{x_0} and $T_{x_n} = T_{x_0}$, each time one traverses H and enters a T_x, one must exit that T_x. We conclude that every vertex of the graph formed by the edges of E' has even valency and so this graph is eulerian. □

In the case of trivalent graphs, as the only spanning eulerian subgraph of a trivalent graph is necessarily a Hamilton cycle, we have the following result.

Corollary 2.2. *The truncation of a trivalent graph* Γ *is hamiltonian if and only if* Γ *is hamiltonian.*

As the Petersen graph and the Coxeter graph are both trivalent graphs that are not hamiltonian, the following result is evident.

Corollary 2.3. *The truncations of the Petersen graph and the Coxeter graph are not hamiltonian.*

3 Vertex-transitive Graph Truncations

While the truncation of *any* vertex-transitive trivalent graph that is not hamiltonian will give a trivalent graph that is not hamiltonian, the truncation of such a graph *need not be vertex-transitive*. Indeed, the truncations of the truncations of the Petersen and Coexeter graphs are not hamiltonian, but it turns out that they are not vertex-transitive. We now investigate when the truncation of a vertex-transitive graph is vertex-transitive. We begin by studying the relationship between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\mathscr{T}(\Gamma))$ for any graph Γ.

Definition 3.1. Let Γ be a graph. We call the set $\mathcal{T} = \{T_v : v \in V(\Gamma)\}$ the **fundamental vertex partition** of $V(\mathscr{T}(\Gamma)$. There is also a **fundamental edge partition** of $E(\mathscr{T}(\Gamma))$ with two cells, where one cell consists of those edges within a T_v, $v \in V(\Gamma)$ (the **clique edges**), and the other cell consisting of those edges between two inserted cliques (the **original edges**).

Lemma 3.2. *Let* Γ *be a graph with each vertex of valency at least* 3*. Then the fundamental vertex and edge partitions of* $\mathscr{T}(\Gamma)$ *are invariant under* $\mathrm{Aut}(\mathscr{T}(\Gamma))$.

Proof. We need only show that the fundamental vertex partition of $\mathscr{T}(\Gamma)$ is invariant under $\mathrm{Aut}(\mathscr{T}(\Gamma))$, as this implies that the fundamental edge partition is invariant under $\mathrm{Aut}(\mathscr{T}(\Gamma))$. An edge xy with $x \in V(T_v)$ and $y \in V(T_u)$, $u \neq v$, cannot belong to a triangle because y is the only neighbor of x not in $V(T_v)$ and x is the only neighbor of y not in $V(T_u)$. On the other hand, every edge xy with x, y in the same $V(T_v)$ belongs to a triangle because T_v is a clique and $t \geq 3$. This then implies that $\mathrm{Aut}(\mathscr{T}(\Gamma))$ permutes the sets in the partition $\mathcal{T} = \{V(T_v) : v \in V(\Gamma)\}$. □

We now introduce standard permutation group theoretic terms related to the fundamental vertex partition.

Definition 3.3. Let $G \leq S_n$ be transitive and act on $\mathbb{Z}_n$. Let $B \subseteq \mathbb{Z}_n$. If $g(B) = B$ or $g(B) \cap B = \emptyset$ for every $g \in G$, then B is a **block** of G. If B is a block, then $g(B)$ is also a block for all $g \in G$, and $\{g(B) : g \in G\}$ is a G**-invariant partition**. Of course, singleton sets and $\mathbb{Z}_n$ are blocks for every transitive group $G \leq S_n$, and the corresponding G-invariant partitions are **trivial**. If G has a nontrivial G-invariant partition, then G is **imprimitive**, and is **primitive** otherwise. Finally, for a G-invariant partition $\mathcal{B}$, we denote by $\mathrm{fix}_G(\mathcal{B})$ the subgroup of G which fixes each block of $\mathcal{B}$ setwise. That is, $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$.

We remark that if $\mathrm{Aut}(\mathscr{T}(\Gamma))$ is transitive, then the fundamental vertex partition is an $\mathrm{Aut}(\mathscr{T}(\Gamma))$-invariant partition.

Now observe that if Γ is a cycle of length n, then $\mathscr{T}(\Gamma)$ is a cycle of length $2n$. Hence $\mathrm{Aut}(\mathscr{T}(\Gamma)) = D_{2n}$, the dihedral group of order $4n$. Henceforth, we will assume that every vertex has valency at least 3, in which case $\mathscr{T}(\Gamma)$ always contains a triangle.

Theorem 3.4. *Let Γ be a graph where each vertex of Γ has valency at least 3. Then $\mathrm{Aut}(\mathscr{T}(\Gamma))$ acts faithfully on the fundamental vertex partition $\mathcal{T}$ and is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma)$.*

Proof. That $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{T}$ was established in Lemma 3.2. Let K be the kernel of the action of $\mathrm{Aut}(\mathscr{T}(\Gamma))$ on $\mathcal{T}$ (if $\mathrm{Aut}(\mathscr{T}(\Gamma))$ is transitive, then $K = \mathrm{fix}_{\mathrm{Aut}(\mathscr{T}(\Gamma))}(\mathcal{T})$). We claim that $K = 1$. Indeed, if $K \neq 1$ with $1 \neq \gamma \in K$, then let $T_v \in \mathcal{T}$ such that $K|_{T_v} \neq 1$. Then there exist distinct $x, y \in T_v$ such that $\gamma(x) = y$. Now, x is adjacent to some vertex $z \in T_u$, $u \neq v$, and so $\gamma(x)\gamma(z)$ is also an edge from T_v to T_u. However, there is only one edge from a vertex of T_v to a vertex of T_u in $\mathscr{T}(\Gamma)$, a contradiction. Thus $K = 1$.

To finish the result, we need only show that if $\gamma \in \mathrm{Aut}(\mathscr{T}(\Gamma))$, then $\bar{\gamma} \in \mathrm{Aut}(\Gamma)$, where $\bar{\gamma}$ is the induced action of γ on $\mathcal{T}$. So suppose that $uv \in E(\Gamma)$. Then some vertex of T_u is adjacent to some vertex of T_v, and as $\gamma \in \mathrm{Aut}(\mathscr{T}(\Gamma))$, some vertex of $\gamma(T_u) = T_{\bar{\gamma}(u)}$ is adjacent to some vertex of $\gamma(T_v) = T_{\bar{\gamma}(v)}$. But this occurs if and only if $\bar{\gamma}(u)\bar{\gamma}(v) \in E(\Gamma)$. □

Corollary 3.5. *The truncations of the truncations of the Petersen and Coxeter graphs are not vertex-transitive.*

Proof. Let Γ be the Petersen or Coexeter graph. If $\mathscr{T}(\mathscr{T}(\Gamma))$ is vertex-transitive, then 9 divides $|\mathrm{Aut}(\mathscr{T}(\mathscr{T}(\Gamma)))|$ as $|V(\mathscr{T}(\mathscr{T}(\Gamma)))| = 9|V(\Gamma)|$ and the size of an orbit or a group divides the order of the group. By Theorem 3.4 applied twice, we see that 9 divides $|\mathrm{Aut}(\Gamma)|$. However, the automorphism group of the Petersen graph has order 120 as it is isomorphic to S_5 (see for example [3, Theorem 1.4.6]) while the automorphism group of the Coexeter graph has order 336 as it is $\mathrm{PGL}(2,7)$ (see for example [1]), neither of which are divisible by 9. □

Theorem 3.6. *If* Γ *is a graph with each vertex of valency at least* 3*, then* $\mathrm{Aut}(\mathscr{T}(\Gamma)) \cong \mathrm{Aut}(\Gamma)$.

Proof. In view of Theorem 3.4, it suffices to show that each element of $\mathrm{Aut}(\Gamma)$ induces an element of $\mathrm{Aut}(\mathscr{T}(\Gamma))$, and that different elements of $\mathrm{Aut}(\Gamma)$ induce different elements of $\mathrm{Aut}(\mathscr{T}(\Gamma))$.

Let $\gamma \in \mathrm{Aut}(\Gamma)$. Let $x \in V(\mathscr{T}(\Gamma))$, and $v \in V(\Gamma)$ with $x \in T_v$. Then there exists a unique $y \in V(\mathscr{T})$ not contained in T_v with $xy \in E(\mathscr{T}(\Gamma))$. Let $u \in V(\Gamma)$ such that $y \in T_u$. Then $\gamma(u)\gamma(v) \in E(\Gamma)$, and so there exists vertices $x' \in T_{\gamma(v)}$ and $y' \in T_{\gamma(u)}$ such that $x'y' \in E(\mathscr{T}(\Gamma))$. Define $\bar{\gamma} : V(\mathscr{T}(\Gamma)) \mapsto (\mathscr{T}(\Gamma))$ by $\bar{\gamma}(x) = x'$. Note that as the original edges of $\mathscr{T}(\Gamma)$ form a perfect matching, $\bar{\gamma}$ is a well-defined bijection. Additionally, by definition, $\bar{\gamma}$ maps the original edges of $\mathscr{T}(\Gamma)$ to the original edges of $\mathscr{T}(\Gamma)$. As $\bar{\gamma}$ also map the fundamental vertex partition of $\mathscr{T}(\Gamma)$ to itself, it maps the clique edges of $\mathscr{T}(\Gamma)$ to the clique edges of $\mathscr{T}(\Gamma)$. Thus $\bar{\gamma} \in \mathrm{Aut}(\mathscr{T}(\Gamma)$. Finally, as the induced action of $\bar{\gamma}$ on $\mathcal{T}$ is γ and $\mathrm{Aut}(\mathscr{T}(\Gamma)$ is faithful on $\mathcal{T}$ by Lemma 3.2, different elements of $\mathrm{Aut}(\Gamma)$ induce different elements of $\mathrm{Aut}(\mathscr{T}(\Gamma)$. □

For a transitive group G acting on $\mathbb{Z}_n$, we denote the **stabilizer in** G **of** v by $\mathrm{Stab}_G(v)$. Then $\mathrm{Stab}_G(v) = \{g \in G : g(v) = v\}$. Concerning the statement of the following result, a vertex- and edge-transitive graph of odd valency is necessarily arc-transitive [10, 7.53].

Theorem 3.7. *If* Γ *is a connected vertex-transitive graph with each vertex of valency* $t \geq 3$*, then* $\mathscr{T}(\Gamma)$ *is vertex-transitive if and only if* Γ *is arc-transitive. Additionally,* $\mathscr{T}(\mathscr{T}(\Gamma))$ *is not vertex-transitive.*

Proof. Before proceeding, some general observations about $\mathscr{T}(\Gamma)$ are in order. As $\mathscr{T}(\Gamma)$ is regular of valency t, $\mathscr{T}(\Gamma)[T_v]$ is regular of valency $t-1$. As $|V(T_v)| = t$, we see that there are exactly t edges with one end in T_v and the other end not in T_v, and of course each vertex of T_v is incident with exactly one such edge. Additionally, between some vertex of T_v and some vertex of T_u there is either exactly one edge if $uv \in E(\Gamma)$ or no edges if $uv \notin E(\Gamma)$. Thus, each edge with one endpoint in T_v and the other endpoint outside of T_v uniquely determines a vertex in T_v and uniquely determines a T_u in which the other endpoint of the edge is a vertex. Conversely, each vertex x of T_v uniquely determines an edge with x as an endpoint and one endpoint not in T_v.

Suppose that Γ is arc-transitive, with $v \in V(\Gamma)$. Then $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(v)$ is transitive on the neighbors $N_\Gamma(v)$ of v. Set $N_\Gamma(v) = \{u_1, \ldots, u_t\}$ and let $\gamma_{i,j} \in \mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(v)$ such that $\gamma_i(u_i) = u_j$. As $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathscr{T}(\Gamma))$ by Theorem 3.6 and $\mathrm{Aut}(\mathscr{T}(\Gamma))$ acts faithfully on $\mathcal{T} = \{T_v : v \in V(\Gamma)\}$ by Theorem 3.4, there exists a unique $\hat{\gamma}_{i,j} \in \mathrm{Aut}(\mathscr{T}(\Gamma))$ such that the action of $\hat{\gamma}_{i,j}$ on $\mathcal{T}$ is $\gamma_{i,j}$. As the action of $\mathrm{Aut}(\mathscr{T}(\Gamma))$ on $\mathcal{T}$ is $\mathrm{Aut}(\Gamma)$ which is transitive, in order to show that $\mathrm{Aut}(\mathscr{T}(\Gamma))$ is transitive it suffices to show that $\{\delta \in \mathrm{Aut}(\mathscr{T}(\Gamma)) : \delta(T_v) = T_v\}$ is transitive on T_v. Let $x, y \in T_v$. Then there exist $1 \leq i, j \leq t$ such that $xv_{u_i}, yv_{u_j} \in E(\mathscr{T}(\Gamma))$, where $v_{u_i} \in T_{u_i}$ and $v_{u_j} \in T_{u_j}$. Then $\hat{\gamma}_{i,j}(T_{u_i}) = T_{u_j}$ and $\hat{\gamma}_{i,j}(T_v) = T_v$. As each vertex of T_v is incident with exactly one edge whose other endpoint is not in T_v and $\hat{\gamma}_{i,j} \in \mathrm{Aut}(\mathscr{T}(\Gamma))$, we have that $\hat{\gamma}_{i,j}(xv_{u_i})$ is the unique edge of $\mathscr{T}(\Gamma)$ with one endpoint in T_v and the other in T_{u_j}. That is, $\hat{\gamma}_{i,j}(xv_{u_i}) = yv_{u_j}$. As

$\hat{\gamma}_{i,j}(T_v) = T_v$, we conclude that $\hat{\gamma}_{i,j}(x) = y$. Thus $\{\delta \in \mathrm{Aut}(\mathscr{T}(\Gamma)) : \delta(T_v) = T_v\}$ is transitive on T_v and the result follows.

Conversely, suppose that $\mathscr{T}(\Gamma)$ is vertex-transitive. It suffices to show that the stabilizer in $\mathrm{Aut}(\Gamma)$ of $v \in V(\Gamma)$ is transitive on its neighbors. Observe that $\mathcal{T} = \{T_v : v \in V(\Gamma)\}$ is an $\mathrm{Aut}(\mathscr{T}(\Gamma))$-invariant partition and $H_v = \{h \in \mathrm{Aut}(\mathscr{T}(\Gamma)) : h(T_v) = T_v\}$, $v \in V(\Gamma)$, is transitive on T_v. Hence, if $x, y \in V(T_v)$, then there exists $\gamma_{x,y} \in \mathrm{Aut}(\mathscr{T}(\Gamma))$ such that $\gamma_{x,y}(x) = y$. For each vertex x of T_v, we denote the uniquely determined edge with x as an endpoint and with the other endpoint not in T_v by $e_x = xz_x$. We let $v_x \in V(\Gamma)$ be such that $z_x \in V(T_{v_x})$, and observe that if $x, y \in V(T_v)$ with $x \neq y$, then $T_{v_x} \neq T_{v_y}$. Each edge xz_x with $x \in V(T_v)$ then induces an edge $vv_x \in E(\Gamma)$, and such edges are pairwise distinct. More specifically, there are exactly t edges $vv_x \in E(\Gamma)$ induced by edges of the form xz_x with $x \in V(T_v)$. This then implies that the neighbors in Γ of v are $\{v_x : x \in V(T_v)\}$. Finally, observe that $\gamma_{x,y}(xz_x)$ is an edge with one endpoint $y \in V(T_v)$ and $\gamma_{x,y}(z_x) \notin V(T_v)$. Denoting by $\bar{\gamma}_{x,y}$ the automorphism of $V(\Gamma)$ induced by the action of $\gamma_{x,y}$ on $\mathcal{T}$ (with each T_a identified with the vertex a), we see that $\bar{\gamma}_{x,y}(vv_x) = vv_y$, and the stabilizer of $v \in V(\Gamma)$ in $\mathrm{Aut}(\Gamma)$ is transitive on the neighbors of v and so Γ is arc-transitive.

It now only remains to show that $\mathscr{T}(\mathscr{T}(\Gamma))$ is not vertex-transitive. In view of our earlier arguments, it suffices to show that if Γ is edge-transitive, then $\mathscr{T}(\Gamma)$ is not edge-transitive. If Γ is edge-transitive, then $\mathrm{Aut}(\mathscr{T}(\Gamma))$ is transitive, and $\mathrm{Aut}(\mathscr{T}(\Gamma))$ admits $\mathcal{T}$ as an $\mathrm{Aut}(\Gamma)$-invariant partition. But $\mathscr{T}(\Gamma)$ contains edges with both endpoints in T_v and edges with one endpoint in T_v and one endpoint outside of T_v. As $\mathcal{T}$ is an $\mathrm{Aut}(\Gamma)$-invariant partition, no automorphism will map an edge of the former type to an edge of the latter type. □

We now restrict our attention to graphs with prime valency. We remark that in the following result, the restriction to prime valency is only used to establish sufficiency.

Theorem 3.8. *If Γ is a connected arc-transitive graph of prime valency $t \geq 3$ and order n, then $\mathscr{T}(\Gamma)$ is isomorphic to a Cayley graph if and only if $\mathrm{Aut}(\Gamma)$ contains a transitive group of order nt.*

Proof. As a vertex-transitive graph is isomorphic to a Cayley graph if and only if its automorphism group contains a regular subgroup [8], $\mathscr{T}(\Gamma)$ being a Cayley graph implies that $\mathrm{Aut}(\mathscr{T}(\Gamma))$ contains a transitive group R of order nt which is a isomorphic to a transitive subgroup of $\mathrm{Aut}(\Gamma)$ of order nt by Theorem 3.4.

Conversely suppose there exists $R \leq \mathrm{Aut}(\Gamma)$ such that R is transitive on $V(\Gamma)$ and has order nt. It suffices to show that for fixed $v \in V(\Gamma)$, the subgroup H of R fixing the set $V(T_v)$ is transitive on $V(T_v)$, as then R is transitive and as $|R| = |V(\mathscr{T}(\Gamma))|$, we have that R is regular.

Now, $\mathcal{T}$ is an $\mathrm{Aut}(\mathscr{T}(\Gamma))$-invariant partition, and the action of R on $\mathcal{T}$, which we denote by $R/\mathcal{T}$, is transitive. Then $H/\mathcal{T}$ fixes the vertex $v \in V(\Gamma)$, and so $|H/\mathcal{T}| = t$. Let $\tau \in H$ be of prime order t. Then every orbit of τ has prime order t or has order 1. If τ in its action on T_v is a t-cycle, then H is transitive on $V(T_v)$ and the result follows. Otherwise, τ is the identity in its action on $V(T_v)$. As Γ is connected and τ has prime order $t \neq 1$, there exists $u \in V(\Gamma)$ such that the action of τ on T_u is the identity, and some vertex y of T_u is adjacent to a vertex $z \in V(T_w)$, $w \neq v$, and z is not fixed by τ. Applying τ to the edge yz, we see that y is adjacent to t vertices not in T_v and to $t-1$ vertices contained in T_v. Then the valency of y is $2t-1$, a contradiction. □

Corollary 3.9. *The truncations of neither the Petersen graph nor the Coxeter graph are isomorphic to Cayley graphs.*

Proof. As the automorphism groups of the Petersen graph and the Coxeter graph are S_5 and $\mathrm{PGL}(2,7)$ of orders 120 and 336, respectively, according to Theorem 3.8 the truncations of these graphs are Cayley if and only if their automorphism groups contain subgroups of index 4. Each of these automorphism groups though contain simple groups of index 2 (A_5 and $\mathrm{PSL}(2,7)$ respectively), and as neither of them are direct products, their only normal proper nontrivial subgroups are A_5 and $\mathrm{PSL}(2,7)$, respectively. Any subgroup of either of index 4 would then give an embedding into S_4 by [4, Corollary 4.9], which is solvable. □

Theorem 3.10. *If* Γ *is a connected vertex-transitive graph of prime valency* p*, then* $\mathscr{T}(\Gamma)$ *is vertex-transitive if and only if* $\mathrm{Aut}(\Gamma)$ *contains an element of order* p *with a fixed point.*

Proof. Suppose that $\mathscr{T}(\Gamma)$ is vertex-transitive. Then $\mathcal{T}$ is an $\mathrm{Aut}(\mathscr{T}(\Gamma))$-invariant partition, and so $\mathrm{Stab}_{\mathrm{Aut}(\mathscr{T}(\Gamma))}(T_v)|_{T_v}$ is transitive on T_v. As T_v has order p, it follows that $\mathrm{Stab}_{\mathrm{Aut}(\mathscr{T}(\Gamma))}(T_v)|_{T_v}$ contains an element of order p. Let $\gamma \in \mathrm{Stab}_{\mathrm{Aut}(\mathscr{T}(\Gamma))}(T_v)$ such that $\gamma|_{T_v}$ has order p. Without loss of generality, we assume that γ has order p (although we will no longer necessarily have that $\gamma|_{T_v}$ has order p — for our purposes we only need an element of order p that fixes some T_v). By Theorem 3.4, $\mathrm{Aut}(\mathscr{T}(\Gamma))$ acts faithfully on $\mathcal{T}$, and so $\gamma/\mathcal{T} \neq 1$. Then $\gamma/\mathcal{T}$ has order p, fixes the point v, and by Theorem 3.4 the permutation $\gamma/\mathcal{T}$ is contained in $\mathrm{Aut}(\Gamma)$.

Suppose that $\mathrm{Aut}(\Gamma)$ contains an element γ of order p with a fixed point. As Γ is connected, some fixed point u of γ is adjacent in Γ to some point u that is not fixed by γ. This follows as there is certainly vertices $x, y \in V(\Gamma)$ with x fixed by γ and y not fixed by γ. We then let u be the first vertex of an xy-path in Γ that is not fixed by γ, and v its predecessor on the chosen xy-path. As γ preserves adjacency, all elements in the (nontrivial) orbit of u are neighbors of v. Since γ is of prime order p, the orbit of u is of length p, and since p is the valency of the graph, the orbit of u contains all the neighbors of v. Thus, $\mathrm{Aut}(\Gamma)$ acts transitively on the arcs emanating from v and transitively on the vertices of Γ, and is therefore arc-transitive. The result then follows by Theorem 3.7. □

There are a few questions which remain unanswered. First, is it true that Theorem 3.8 holds when the valency t is not prime? Similarly, does Theorem 3.10 hold when the valency is not prime? More specifically, if Γ has valency t is it the case that $\mathscr{T}(\Gamma)$ is vertex-transitive if and only if $\mathrm{Aut}(\Gamma)$ contains a subgroup of order t with a fixed point (i.e. every element fixes the same point). Finally, what exactly is the group $\mathrm{Stab}_{\mathrm{Aut}(\mathscr{T}(\Gamma))}(T_v)$ in its action on T_v? Of course, as an abstract group it is isomorphic to $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(v)$, but what is the action?

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References

[1] N. Biggs, Three remarkable graphs, *Canad. J. Math.* **25** (1973), 397–411.

[2] G. Exoo and R. Jajcay, Recursive constructions of small regular graphs of given degree and girth, *Discrete Math.* **312** (2012), 2612–2619, doi:10.1016/j.disc.2011.10.021.

[3] D. A. Holton and J. Sheehan, *The Petersen graph*, volume 7 of *Australian Mathematical Society Lecture Series*, Cambridge University Press, Cambridge, 1993, doi:10.1017/CBO9780511662058.

[4] T. W. Hungerford, *Algebra*, volume 73 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1980, reprint of the 1974 original.

[5] K. Kutnar and D. Marušič, Hamilton cycles and paths in vertex-transitive graphs—current directions, *Discrete Math.* **309** (2009), 5491–5500, doi:10.1016/j.disc.2009.02.017.

[6] A. Malnič, T. Pisanski and A. Žitnik, The clone cover, *Ars Math. Contemp.* **8** (2015), 95–113.

[7] T. Pisanski and T. W. Tucker, Growth in repeated truncations of maps, *Atti Sem. Mat. Fis. Univ. Modena* **49** (2001), 167–176, dedicated to the memory of Professor M. Pezzana (Italian).

[8] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438.

[9] W. T. Tutte, A non-Hamiltonian graph, *Canad. Math. Bull.* **3** (1960), 1–5.

[10] W. T. Tutte, *Connectivity in graphs*, Mathematical Expositions, No. 15, University of Toronto Press, Toronto, Ont., 1966.

[11] F. Zhang, Y.-C. Chen and Z. Chen, Clique-inserted-graphs and spectral dynamics of clique-inserting, *J. Math. Anal. Appl.* **349** (2009), 211–225, doi:10.1016/j.jmaa.2008.08.036.

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The equalization scheme of the residual voluntary health insurance in Slovenia

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Abstract

Residual voluntary health insurance in Slovenia covers the difference between the (recognised) value of the health service and the part of this value that is payed by the compulsory health insurance. From the inception of compulsory health insurance in 1992, residual voluntary health insurance has open enrolment. From 2006 community rating applied, as well as an equalization scheme with which the differences in health services expenses, arising from the different structures of the insurees of the single insurance undertakings regarding age and gender, shall be equalized. The equalization scheme of the residual voluntary health insurance in Slovenia is presented, along with a detailed explanation of the formulae required for the computation.

Keywords: residual voluntary health insurance, equalization scheme, claims equalization, risk equalization.

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1 Introduction

In Slovenia, healthcare financing from public sources is organized through compulsory health insurance as a healthcare system of Bismarkian type. The Health Care and Health Insurance Act (HCHIA) [7] regulates both the public compulsory health insurance as well as the private voluntary health insurance.

Although the basket of health benefits for an insuree of the compulsory health insurance is extensive, for a particular health service, the extent of its financing in charge of the compulsory health insurance may, in line with Article 23 of HCHIA, for the majority of

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adult insurees vary from 100% to as low as 10% of the (recognized) value of the health service; the payment of the difference or the residual amount up to 100% of the health service value being the obligation of the insuree that received the health service.

Covering the difference or the residual mentioned above is possible through voluntary health insurance of residual type, called residual voluntary health insurance. (The direct translation from Slovenian would be ”complementary”, but since the term ”complementary health insurance” has many different meanings we rather use the word ”residual”.) Residual health insurance is the most expanded voluntary health insurance in Slovenia, with a dissemination of over 95% among the insurees of the compulsory health insurance who are liable for residual payments. Nowadays, residual health insurance in Slovenia is offered by three insurance undertakings.

Among all types of voluntary health insurances determined in HCHIA, only the residual health insurance is fully regulated. From its introduction in 1992, open enrollment is its main characteristic. Since 2006, when the reform of residual health insurance applied, community rating is prescribed, with discounts limited to 3%, and late entry loadings of 3% per year, the total premium rise amounting to at most 80%.

To restrict risk selection in residual insurance, easily achieved through contracting mainly young persons, the legislator decided to introduce the *equalization scheme of the residual health insurance*, with which the differences in health services expenses, arising from the different structures of the insurees of the single insurance undertakings regarding age and gender, shall be equalized. It is a fact that the purpose of the introduction of the equalization scheme is not given at a legislative level.

The equalization scheme is implemented as a system of transfer of equalization amounts among insurance undertakings. It is calculated ex-post from data included in the reports on the performance of residual insurance for the past equalization period that the insurance undertakings transmit to the ministry, competent for health. Each report contains, for each of the seven age groups (up to 25 years, 25-35, 35-45, 45-55, 55-65, 65-75, above 75 years), for each gender and for each of the three months of the referential equalization period, the number of insurees of the individual insurance undertaking and the health services expenses for the referential period that where accounted up to one month after the referential period, where the health services expenses include the amounts of account claims arising from residual insurance coverage and the amounts accounted as compensations to the health service provider for the transmition of data needed for the functioning of residual health insurance and determined in HCHIA. Also, the report includes, for each age group and for each gender, for each of the three months of each of the three preceding equalization periods, the health services expenses for that preceding equalization period, that where accounted in the three months after the first month of the referential equalization period.

Armstrong, Paolucci and Van de Ven [2] emphasized that a distinction needs to be made between the term ”risk equalization” and the term ”claims equalization”, since besides the ex-ante nature of the first and ex-post of the latter, a major difference is that with the latter insurers have only limited incentives for efficiency as they retain only a limited financial responsibility because both the risk profiles and the claims costs per risk group are equalised, while the former is a mechanism that is used to ensure that risk solidarity principles apply, to prevent competition from occurring on the basis of risk selection, and to foster competition based on costs and quality of care. In this sense the equalization scheme of residual voluntary health insurance in Slovenia belongs to the class of claims equalization. A glossary of terms can be found in [1]. The benefits of risk equalization in health insurance markets are discussed in [5, 1]. Stam gives in [4] an insight to testing the effectiveness of risk equalization models in health insurance.

The exact wording of the legislative regulation regarding the equalization scheme of residual voluntary health insurance in Slovenia, that is, a selection of the respective articles of HCHIA, can be found in the Appendix.

The equalizations of differences shall be settled for each equalization period. In Section 2 we give a detailed presentation of all the needed calculations for the ministry, competent for health, to bring a decision about the equalization.

The introduction of an equalization scheme in the specific extremely regulated market of residual voluntary health insurance in Slovenia should be necessarily accompanied by explicit provisions for protection of competition. The lack of such provisions and implications are discussed in [6].

2 Computation of the equalization amount

The computation of the equalization amount is described below.

We use the following notation:

i … serial number of the insurance undertaking, included in the equalization in the referential equalization period;

j … serial number of the referential equalization period;

k … relative number of the preceding equalization period with respect to the referential equalization period $(k = 0, 1, 2, 3)$; $k = 0$ stands for the referential period;

r … age group of insurees;

$\mathrm{N}_r^{i,j}$ … number of insurees of the insurance undertaking i, that in the referential equalization period j belong to the age group r; this number is calculated as the average of the numbers of insurees on the first day of every month of the referential equalization period j;

$\mathrm{N}_r^{i,j,k}$ … number of insurees of the insurance undertaking i that, according to the data on the day of reporting for the referential equalization period j, in the preceding equalization period $(j - k)$ belonged to the the age group r. The following equalities hold,

$$\mathrm{N}_r^{i,j,k} = \mathrm{N}_r^{i,j-k,0} = \mathrm{N}_r^{i,j-k},$$

since the insurance undertakings ought to have up-to-date data of its insurance contracts.

For further computations the amounts of health service expenses transmitted from insurance undertakings to the ministry, competent for health, in the reports on the performance of residual insurance for the past equalization period, are essential. We denote

$\mathrm{AE}_r^{i,j,k}$ ... amount of health service expenses of the insurance undertaking i within the referential equalization period j relative to the preceding equalization period $(j-k)$ and for the age group r. For $k=0$ we denote $\mathrm{AE}_r^{i,j,0}$ as $\mathrm{AE}_r^{i,j}$.

The average amount of health service expenses of each insurance undertaking i is calculated for every age group j according to the formula

$$\overline{\mathrm{AE}}_r^{i,j} = \frac{\mathrm{AE}_r^{i,j,0}}{\mathrm{N}_r^{i,j,0}} = \frac{\mathrm{AE}_r^{i,j}}{\mathrm{N}_r^{i,j}},$$

where

$\overline{\mathrm{AE}}_r^{i,j}$ ... the average amount of health service expenses of the insurance undertaking i within the referential equalization period j and for the age group r.

The average amount of health service expenses of all insurance undertakings, included in the computation, is calculated for each age group according to the formula

$$\overline{\mathrm{AE}}_r^{\bullet,j} = \frac{\sum\limits_m \mathrm{AE}_r^{m,j}}{\sum\limits_m \mathrm{N}_r^{m,j}},$$

where

$\overline{\mathrm{AE}}_r^{\bullet,j}$ ... the average amount of health service expenses of all insurance undertakings included in the computation, within the referential equalization period j and for the age group r.

A common rounding policy should be applied.

The standardized number of insurees in an age group of an individual insurance undertaking is calculated for every age group using the formula

$$\mathrm{SN}_r^{i,j} = \sum_m \mathrm{N}_r^{m,j} \cdot \frac{\sum\limits_s \mathrm{N}_s^{i,j}}{\sum\limits_{m,s} \mathrm{N}_s^{m,j}},$$

where

$\mathrm{SN}_r^{i,j}$ ... the standardized number of insurees for the insurance undertaking i within the referential equalization period j and for the age group r.

The standardized amount of health service expenses of an individual insurance undertaking is calculated within an age group r having more than 2000 persons for every equalization period (referential and preceding) according to the formula

$$\mathrm{SAE}_r^{i,j} = \mathrm{SN}_r^{i,j} \cdot \overline{\mathrm{AE}}_r^{i,j},$$

where

$\mathrm{SAE}_r^{i,j}$ ... the standardized amount of health service expenses for the insurance undertaking i within the referential equalization period j and for the age group r.

In case the age group r of an individual insurance undertaking contains less than 2000 persons the standardized amount of health service expenses for this age group of this insurance undertaking is given by the formula

$$\mathrm{SAE}_r^{i,j} = \mathrm{SN}_r^{i,j} \cdot \overline{\mathrm{AE}}_r^{\bullet,j}.$$

The basic equalization amount for the insurance undertaking for the referential equalization period j shall be calculated as the difference between the sum of amounts of expenses for health services in residual insurance of the insurance undertaking in all age groups, and the sum of standardized amounts of expenses for health services in residual insurance of the same insurance undertaking for all age groups. Hence

$$\mathrm{BEA}^{i,j} = \sum_r \mathrm{AE}_r^{i,j} - \sum_r \mathrm{SAE}_r^{i,j},$$

where

$\mathrm{BEA}^{i,j}$ ... the basic equalization amount for the insurance undertaking i for the equalization period j.

Let us determine the equalization amount of an insurance undertaking before taking in consideration the (possible) transfered amount (in case in the preceding equalization period the treshold was not attained).

The calculation of the equalization amount of an individual insurance undertaking is based on the comparison of the sum of the positive basic equalization amounts,

$$\mathrm{BEA}^{\geq,j} = \sum_{\substack{i \\ \mathrm{BEA}^{i,j} \geq 0}} \mathrm{BEA}^{i,j}$$

where

$\mathrm{BEA}^{\geq,j}$ ... the sum of nonnegative basic equalization amounts for the referential equalization period j, and the sum of negative basic equalization amounts,

$$\mathrm{BEA}^{<,j} = \sum_{\substack{i \\ \mathrm{BEA}^{i,j}<0}} \mathrm{BEA}^{i,j}$$

where

$\mathrm{BEA}^{<,j}$ ... the sum of negative basic equalization amounts for the referential equalization period j.

If the conditions of Article 62.h of HCHIA are met, that is, if the equalization treshold is not attained, a transfer of equalization amounts in the subsequent equalization periods is carried. So for the insurance undertaking i and the equalization period j, the computing procedure determines first the equalization amount before taking into consideration the transfer , denoted $\mathrm{EAB}^{i,j}$, where

$$\mathrm{EAB}^{i,j} = \begin{cases} \mathrm{BEA}^{i,j}, & \text{if } (\mathrm{BEA}^{i,j} \geq 0) \text{ and } (\mathrm{BEA}^{\geq,j} \leq -\mathrm{BEA}^{<,j}); \\ \mathrm{BEA}^{i,j}, & \text{if } (\mathrm{BEA}^{i,j} < 0) \text{ and } (\mathrm{BEA}^{\geq,j} > -\mathrm{BEA}^{<,j}); \\ \mathrm{BEA}^{i,j} \cdot \dfrac{(-\mathrm{BEA}^{<,j})}{\mathrm{BEA}^{\geq,j}}, & \text{if } (\mathrm{BEA}^{i,j} \geq 0) \text{ and } (\mathrm{BEA}^{\geq,j} > -\mathrm{BEA}^{<,j}); \\ \mathrm{BEA}^{i,j} \cdot \dfrac{\mathrm{BEA}^{\geq,j}}{(-\mathrm{BEA}^{<,j})}, & \text{if } (\mathrm{BEA}^{i,j} < 0) \text{ and } (\mathrm{BEA}^{\geq,j} \leq -\mathrm{BEA}^{<,j}). \end{cases}$$

We remark that for the k-th preceding equalization periods, $k = 1, 2, 3$, the ministry, competent for health – taking into account the data from the reports on the performance of residual insurance for the referential equalization period about the health services expenses for the k-th preceding equalization period, that were accounted in the period of three months after the first month of the referential equalization period – once again executes the computation of the equalization amount before taking into consideration the transfer, that is, the updated $\mathrm{EAB}^{i,j-k}$, and adds up to $\mathrm{EAB}^{i,j}$ (the equalization amount before taking into account the transfer, obtained by taking into consideration only the referential equalization period) the difference between the updated $\mathrm{EAB}^{i,j-k}$ and the previously computed $\mathrm{EAB}^{i,j-k}$ in the then-referential equalization period $(j - k)$.

The equalization amount of the insurance undertaking is equal to the sum of the equalization amount of this insurance undertaking before the transfer (for the referential equalization period) and the transfered equalization amount (from the preceding equalization period), that is,

$$\mathrm{EA}^{i,j} = \mathrm{EAB}^{i,j} + \mathrm{TEA}^{i,j-1},$$

where

$\mathrm{EA}^{i,j}$ ... the equalization amount of the insurance undertaking i for the equalization period j;

$\mathrm{TEA}^{i,j-1}$ ... the transfered equalization amount of the insurance undertaking i from the equalization period $(j - 1)$ to the equalization period j, calculated as described below.

We first consider the threshold attainment testing.

In case the sum of all positive equalization amounts for the referential equalization period does not attain the equalization threshold for the referential equalization period, being equal to one and a half percent of the health services expenses, the equalization for the referential equalization period is not performed, and the equalization amounts are transfered to the subsequent referential equalization period.

The equalization threshold for the referential equalization period j is calculated as follows:

$$\mathrm{THR}^{\bullet,j} = 1,5 \cdot \sum_{\substack{i,r \\ k=0,1,2,3}} \mathrm{AE}_r^{i,j,k}.$$

The equalization in the referential equalization period is performed if the following condition is met:

$$\sum_{\substack{i \\ \mathrm{EA}^{i,j} \geq 0}} \mathrm{EA}^{i,j} \geq \mathrm{THR}^{\bullet,j}.$$

In the referential equalization period the insurance undertaking is a payer in the equalization if its equalization amount for this equalization period is negative, otherwise the insurance undertaking is a receiver in the equalization. In this case the transfered equalization amount of the insurance undertaking i from the referential equalization period j to the subsequent referential equalization period $(j+1)$ equals zero, that is, $\mathrm{TEA}^{i,j-1} = 0$.

If the condition for performing equalization from the preceding paragraph is not met, then the equalization is not performed, the transfered equalization amount of the insurance undertaking i from the referential equalization period j to the subsequent referential equalization period (j+1) is equal to the equalization amount of the insurance undertaking i for the referential equalization period j, that is, $\mathrm{TEA}^{i,j} = \mathrm{EA}^{i,j}$.

3 Appendix - The legislative wording

The equalization scheme of the residual voluntary health insurance as a method of risk equalization is regulated within the Health Care and Health Insurance Act (HCHIA) [7]. A selected list of provisions follows (translation from Slovenian by Liliane Strmčnik).

> **Article 62.d**
> The insurance undertakings providing residual insurance shall be included in the equalization scheme of residual insurance, with which the differences in health services expenses, arising from the different structures of the insurees of the single insurance undertakings regarding age and gender, shall be equalized. Health services expenses shall include the amounts of account claims arising from residual insurance coverage and the amounts accounted as compensations for the ensured data from point 7 of the second paragraph of Article 62 of this Act. The amount of these compensations shall be agreed between the insurance undertakings and the providers of health services as a percentage of the amounts of the gross account of claims, at the most up to 0,75 percent.
> The insurance undertakings providing residual insurance shall keep account records on

the health services expenses for each insuree. Health services providers shall be bound to transmit the insurance undertakings all the necessary data.
The insurance undertakings providing residual insurance shall participate in the calculation of equalization amounts and in the equalization of differences. The insurance undertakings starting to perform residual insurance shall be exempted from participating in the calculation of equalization amounts and in the equalization of differences for the period of the first twelve months of operation in the field of residual insurances.

Article 62.e

The basic equalization amount for the insurance undertaking shall be calculated as the difference between the amount of expenses for health services in residual insurance of the insurance undertaking and the standard amount of expenses for health services in residual insurance of the same insurance undertaking. Should the first amount of expenses be lower than the second, the insurance undertaking shall be payer in the equalization otherwise it shall be receiver in the equalization.
The equalization amount shall be calculated for the insurance undertaking by preserving its basic equalization amount or proportionally reducing it in such a way that the sum of the equalization amounts of the payers as well as the sum of the equalization amounts of the receivers shall be equal to the lower of the sum of the basic equalization amounts of the receivers and the sum of the basic equalization amounts of the payers.
The amount of health services expenses in residual insurance of the insurance undertaking shall be the sum of health services expenses arising from the coverage of residual insurance, increased for the amount of compensations for the ensured data from the preceding article. Into account shall be taken the health services expenses that shall have been accounted up to the last day inclusive of the month after the close of the equalization period.
The standardized amount of health services expenses in residual insurance shall be the sum of standardized amounts of expenses according to age groups in the insurance undertaking. The standardized amount of health services expenses in residual insurance for an age class in the insurance undertaking equals the product of:
1. the average amount of health services expenses for an insuree in the insurance undertaking within this age class or, if the age class in the insurance undertaking comprehends less than 2.000 persons, the average amount of health services expenses for an insuree in all insurance undertakings within this age class;
2. the number of insurees in residual insurance within this age class in all insurance undertakings providing residual insurance and participating in the calculation of equalization amounts, and
3. the quotient of the number of insurees in residual insurance in the insurance undertaking and the number of all insurees in residual insurance in all insurance undertakings providing residual insurance and participating in the calculation of the equalization amounts.
The average amount of health services expenses for an insuree in an insurance undertaking within an age class from point 1 of the preceding paragraph shall be determined so that the sum of all health services expenses of the insurance undertaking within the age class is divided by the number of insurees in the insurance undertaking within the same age class.
The number of insurees in residual insurance in the insurance undertaking shall be the average number of insurees on the first day of each month within the equalization period, for which on that same day the insurance undertaking shall bear the responsibility of payment for the health services expenses from residual insurance.
Age groups by gender and age shall start, separately for men and women, at the age of 15 and follow one another within the span of ten years up to the age of 75. The last

age groups, separated by gender and age, shall be represented by the insurees aged 75 and more. The insurees who have not yet attained the age of 15 belong to the first two groups.
The age of the insuree in residual insurance shall be reckoned as the age attained within the calendar year. The equalization period shall be a period of three successive calendar months. The first equalization period within a calendar year shall start with the first day of the calendar year.

Article 62.f

The payer insurance undertaking shall be obliged to pay the equalization amount to the receiver insurance undertakings.
The ministry competent for health shall bring a decision about the equalization. Unless otherwise stipulated by this Act, for the decision procedure of the ministry, competent for health, the provisions of the Act regulating the general administrative procedure shall be applied.
The equalizations of differences shall be settled for each equalization period. For the calculation of the equalization of differences in health services expenses between insurance undertakings, the latter shall be bound to transmit to the ministry, competent for health, the reports on the performance of residual insurance for the past equalization period within 20 days after the end of the month following an individual equalization period.
Upon receipt of the reports from the preceding paragraph, the ministry, competent for health, shall hand over these reports to each of the insurance undertakings within 8 days. The insurance undertaking may give a statement on these reports within 8 days upon receipt of the reports. Should the insurance undertaking refer to documentary evidence in its statement, it should enclose these documents to the statement. Should the insurance undertaking not enclose the documentary evidence to its statement, the ministry competent for health shall take into account only the evidence enclosed to the statement for making its decision.
The ministry competent for health shall decide about the equalization in a written order on equalization without an appointed day within at most 15 days upon expiry of the terms from the preceding paragraph. In its written order the Ministry, competent for health, shall state which insurance undertakings are the payers and which are the receivers in the equalization scheme and the amounts that the payer insurance undertakings shall pay on the accounts of the receiver insurance undertakings. The accounts of the insurance undertakings shall be stated in the written order.
The payer insurance undertaking shall pay the equalization amount within 8 days upon receipt of the written order on equalization. In case of delay in payment, the insurance undertaking shall be obliged to pay legal interests on arrears according to the Legal Penalty Rate Act (Official Gazette RS, no. 56/03).
For the time from the expiry of the equalization period until the issue of the written order on equalization, interest shall accrue in the amount of average pondered interest rate on the inter-bank money market. The ministry competent for health shall decide about the obligation of interest payment in the written order on equalization.
The ministry, competent for health, may call for a debate if it estimate it necessary for the clarification or for the ascertainment of the decisive facts. In this case the insurance undertakings may give their statements in oral form as well, at the debate.
The insurance undertaking in a state member of the European Union, acting within the frame of the European Union and being authorized, in accordance with the Insurance Act, to perform insurance business on the territory of the Republic of Slovenia either through a branch office or directly, shall appoint, for the duration of providing residual insurance business, a mandatory for the delivery of the written orders of the ministry

competent for health. The insurance undertaking shall notify the ministry, competent for health, on the data about its authorized mandatory for delivery. For the time during which the insurance undertaking has no authorized mandatory for delivery or before notification of his data to the ministry competent for health, the written orders shall be delivered by publication on the notice board of the ministry, competent for health.
The detailed instructions on the contents of the report shall be stipulated by the minister, competent for health.
The detailed instructions for accounting monitoring and statement of business events regarding the performance of equalization shall be stipulated by the Insurance Supervision Agency.

...

Article 62.h
Equalization of differences in expenses for health services shall not be performed for the equalization periods in which the sum of the amounts for equalization that the payer insurance undertakings are obliged to pay is lower than one and a half percent of the amounts of gross calculated claims, increased for the amount of compensation for ensuring the data from point 7 of the second paragraph of Article 62 of this Act. The ministry, competent for health shall state, in a written order, that equalization shall not be performed. Should equalization not be performed in a particular accounting period, the equalization amount shall be transferred to the next accounting period, added to the sum from the prepreceding sentence and taken into account for equalization.

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References

[1] J. Armstrong, F. Paolucci, H. McLeod, W.P.M.M van de Ven, Risk equalization in voluntary health insurance markets: A three country comparison, *Health Policy* **98** (2010), 39–49.

[2] J. Armstrong, F. Paolucci, W.P.M.M van de Ven, Editorial: Risk equalization in voluntary health insurance markets, *Health Policy* **98** (2010), 1–2.

[3] Ivan Gracar, personal communication from 2005 on.

[4] P. J. A. Stam, *Testing the effectiveness of risk equalization models in health insurance - A new method and its application*, PhD Thesis, Erasmus University, Rotterdam, 2007.

[5] W. P. M. M Van de Ven, R.P. Ellis, Risk adjustment in competitive health plan markets, in: *Handbook of health economics*, vol. 1, ed. A.J. Culyer and J.P. Newhouse, Elsevier Science BV, Amsterdam, 2000, 755-845.

[6] B. Zgrablić, The equalization scheme of residual voluntary health insurance in Slovenia and protection of competition, *in preparation*.

[7] Health Care and Health Insurance Act, *Official Gazette of the Republic of Slovenia* no. 9/1992 and further (in Slovenian).

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[1] First A. Author, Second B. Author and Third C. Author, Article title, *Journal Title* **121** (1982), 1–100.

[2] First A. Author, Book title, third ed., Publisher, New York, 1982.

[3] First A. Author and Second B. Author, Chapter in an edited book, in: First Editor, Second Editor (eds.), *Book Title*, Publisher, Amsterdam, 1999, 232–345.

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