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8 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space, New Way of Second Quantization of Fermions — Part I

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Abstract. Both algebras, Clifford and Grassmann, offer ″basis vectors″ for describing the internal degrees of freedom of fermions [5, 6, 12]. The oddness of the ″basis vectors″, transfered to the creation operators, which are tensor products of the finite number of ″basis vectors″ and the infinite number of momentum basis, and to their Hermitian conjugated partners annihilation operators, offers the second quantization of fermions without postulating the conditions proposed by Dirac [1–3], enabling the explanation of the Dirac's postulates. But while the Clifford fermions manifest the half integer spins — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the ″Grassmann fermions″ manifest the integer spins. In Part I properties of the creation and annihilation operators of integer spins ″Grassmann fermions″ are presented and the proposed equations of motion solved. The anticommutation relations of second quantized integer spin fermions are shown when applying on the vacuum state as well as when applying on the Hilbert space of the infinite number of ″Slater determinants″ with all the possibilities of empty and occupied ″fermion states″. In Part II the conditions are discussed under which the Clifford algebras offer the appearance of the second quantized fermions, enabling as well the appearance of families. In both parts, Part I and Part II, the relation between the Dirac way and our way of the second quantization of fermions is presented.

Povzetek. Avtorja obravnavata Cliffordovo in Grassmannovo algebro. Obe ponudita ″bazne vektorje″ za opis notranjega prostora fermionov [5, 6, 12]. ″Bazni vektorji″, ki antikomutirajo, poskrbijo za antikomutacijske lastnosti kreacijskih operatorjev, ki so tenzorski produkti končnega števila teh ″baznih vektorjev″ in neskončnega števila vektorjev običajnega prostora ter njihovih hermitsko konjugiranih anihilacijskih operatorjev. Antikomutatorji teh kreacijskih in anihilacijskih operatorjev izpolnjujejo vse pogoje, ki jih za drugo kvantizacijo fermionov postulira Dirac [1–3]. Predlagana pot avtorjev do druge kvantizacije stanj fermionskih polj pojasni Diracove postulate druge kvantizacije. Cliffordovi fermioni nosijo polceloštevilski spin — kar se ujema z opaženimi lastnostmi kvarkov in leptonov ter antikvarkov in antileptonov — "Grassmannovi fermioni" pa nosijo celoštevilski spin. Prvi del članka predstavi lastnosti kreacijskih in anihilacijskih operatorjev za ″Grassmannove fermione″, ko delujejo na vakuumsko stanje in tudi, ko delujejo na neskončno število "Slaterjevih determinant″ ″Grassmannovih fermionskih″ stanj vsemi možnostmi zasedenosti teh stanj. V drugem delu obravnavata avtorja pogoje, pri katerih Cliffordove algebre ponudijo opis fermionov v drugi kvantizaciji hkrati s pojavom družin fermionov. V obeh delih primerjata Diracovo pot z njuno potjo do druge kvantizacije fermionov.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Explanation of the Dirac postulates, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

8.1 Introduction

In a long series of works we, mainly one of us N.S.M.B. ([5–12, 15] and the references therein), have found phenomenological success with the model named by N.S.M.B the *spin-charge-family* theory, with fermions, the internal degrees of freedom of which is describable with the Clifford algebra of all linear combinations of products of γ^a's in $d = (13+1)$ (may be with d infinity), interacting with only gravity. The spins of fermions from higher dimensions, $d > (3+1)$, manifest in $d = (3+1)$ as charges of the *standard model*, gravity in higher dimensions manifest as the *standard model* gauge vector fields as well as the scalar Higgs and Yukawa couplings.

There are two anticommuting kinds of algebras, the Grassmann algebra and the Clifford algebra (of two independent subalgebras), expressible with each other. The Grassmann algebra, with elements θ^a, and their Hermitian conjugated partners $\frac{\partial}{\partial\theta^a}$ [12], can be used to describe the internal space of fermions with the integer spins and charges in the adjoint representations, the two Clifford algebras, we call their elements γ^a and $\tilde{\gamma}^a$, can each of them be used to describe half integer spins and charges in fundamental representations. The Grassmann algebra is equivalent to the two Clifford algebras and opposite.

The two papers explain how do the oddness of the internal space of fermions manifests in the single particle wave functions, relating the oddness of the wave functions to the corresponding creation and annihilation operators of the second quantized fermions, in the Grassmann case and in the Clifford case, explaining therefore the postulates of Dirac for the second quantized fermions. We also show that the requirement that the Clifford odd algebra represents the observed quarks and leptons and antiquarks and antileptons reduces the Clifford algebra for the factor of two, reducing at the same time the Grassmann algebra, disabling the possibility for the integer spin fermions.

In this paper it is demonstrated how do the Grassmann algebra — in Part I — and the two kinds of the Clifford algebras — in Part II — if used to describe the internal degrees of freedom of fermions, take care of the second quantization of fermions without postulating anticommutation relations [1–3]. Either the odd Grassmann algebra or the odd Clifford algebra offer namely the appearance of the creation operators, defined on the tensor products of the "basis vectors" of the internal space and of the momentum space basis. These creation operators, together with their Hermitian conjugated partners anihilation operators, inherit oddness from the "basis vectors" determined by the odd Grassmann or the odd Clifford algebras, fulfilling correspondingly, the anticommutation relations postulated by Dirac for the second quantized fermions, if they apply on the corresponding vacuum state, Eq. (8.7) (defined by the sum of products of all the annihilation times the corresponding Hermitian conjugated creation operators). Oddness of the "basis vectors", describing the internal space of fermions, guarantees the oddness of all the objects entering the tensor product.

In d-dimensional Grassmann space of anticommuting coordinates θ^a's, $i = (0, 1, 2, 3, 5, \cdots, d)$, there are 2^d "basis vectors", which are superposition of products of θ^a. One can arrange them into the odd and the even irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to θ^a's, $\frac{\partial}{\partial \theta_a}$'s, taken in Ref. [12] as, up to a sign, Hermitian conjugated to θ^a's, ($\theta^{a\dagger} = \eta^{aa} \frac{\partial}{\partial \theta_a}$, $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$), which form again 2^d "basis vectors". Again half of them odd and half of them even (the odd Hermitian conjugated to odd products of θ^a's, the even Hermitian conjugated to the even products of θ^a's). Grassmann space offers correspondingly $2 \cdot 2^d$ degrees of freedom.

There are two kinds of the Clifford "basis vectors", which are expressible with θ^a and $\frac{\partial}{\partial \theta_a}$: $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$ [6, 13, 14]. They are, up to η^{aa}, Hermitian operators. Each of these two kinds of the Clifford algebra objects has 2^d operators. "Basis vectors" of Clifford algebra have together again $2 \cdot 2^d$ degrees of freedom.

There is the *odd algebra* in all three cases, θ^a's, γ^a's, $\tilde{\gamma}^a$'s, which if used to generate the creation and annihilation operators for fermions, and correspondingly the single fermion states, leads to the Hilbert space of second quantized fermions obeying the anticommutation relations of Dirac [1] without postulating these relations: the anticommutation properties follow from the oddness of the "basis vectors" in any of these algebras.

Let us present steps which lead to the second quantized fermions:

i. The internal space of a fermion is described by either Clifford or Grassmann algebra of an odd Clifford character (superposition of an odd number of Clifford "coordinates" (operators) γ^a's or of an odd number of Clifford "coordinates" (operators) $\tilde{\gamma}^a$'s) or of an odd Grassmann character (superposition of an odd number of Grassmann "coordinates" (operators) θ^a's).

ii. The eigenvectors of all the (chosen) Cartan subalgebra members of the corresponding Lorentz algebra are used to define the "basis vectors" in the odd part of internal space of fermions. (The Cartan subalgebra is in all three cases chosen in the way to be in agreement with the ordinary choice.) The algebraic application of this "basis vectors" on the corresponding vacuum state (either Clifford $|\psi_{oc}>$, defined in Eq. (18) of Part II, or Grassmann $|\phi_{og}>$, Eq. (8.7), which is in the Grassmann case just the identity) generates the "basis states", describing the internal degrees of freedom of fermions. The members of the "basis vectors" manifest together with their Hermitian conjugated partners properties of creation and annihilation operators which anticommute, Eq. (8.11) in Part I and Eq. (18) in Part II, when applying on the corresponding vacuum state, due to the algebraic properties of the odd products of the algebra elements.

iii. The plane wave solutions of the corresponding Weyl equations (either Clifford, Eq. (23) or Grassmann, Eq. (8.21)) for free massless fermions are the tensor products of the superposition of the members of the "basis vectors" and of the momentum basis. The coefficients of the superposition correspondingly depend on a chosen momentum $\vec{p}$, with $|p^0| = |\vec{p}|$, for any of continuous many moments $\vec{p}$.

iv. The creation operators defined on the tensor products, $*_T$, of superposition of finite number of ″basis vectors″ defining the final internal space and of the infinite (continuous) momentum space, Eq. (24) in the Clifford case and Eq. (8.22) in the Grassmann case, have infinite basis.

v. Applied on the vacuum state these creation operators form anticommuting single fermion states of an odd Clifford/Grassmann character.

vi. The second quantized Hilbert space $\mathcal{H}$ consists of ″Slater determinants″ with no single particle state occupied (with no creation operators applying on the vacuum state), with one single particle state occupied (with one creation operator applying on the vacuum state), with two single particle states occupied (with two creation operator applying on the vacuum state), and so on. ″Slater determinants″ can as well be represented as the tensor product multiplication of all possible single particle states of any number.

vii. The creation operators together with their Hermitian conjugated partners annihilation operators fulfill, due to the oddness of the ″basis vectors″, while the momentum part commutes, the anticommutation relations, postulated by Dirac for second quantized fermion fields, not only when they apply on the vacuum state, but also when they apply on the Hilbert space $\mathcal{H}$, Eq. (39) in the Clifford case and Eq. (8.34) in the Grassmann case. In the Clifford case this happens only after ″freezing out″ half of the Clifford space, as it is shown in Part II, Sect. 2.2, what brings besides the correct anticommutation relations also the ″family″ quantum number to each irreducible representation of the Lorentz group of the remaining internal space.

The oddness of the creation operators forming the single fermion states of an odd character, transfers to the application of these creation operators on the Hilbert space of the second quantized fermions in the Clifford and in the Grassmann case.

viii. Correspondingly the creation and annihilation operators with the internal space described by either odd Clifford or odd Grassmann algebra, since fulfilling the anticommutation relations required for the second quantized fermions without postulates, explain the Dirac's postulates for the second quantized fermions.

In the subsection 8.1.1 of this section we discuss in a generalized way our assumption, that the oddness of the ″basis vectors″ in the internal space transfer to the corresponding creation and annihilation operators determining the second quantized single fermion states and correspondingly the Hilbert space of the second quantized fermions.

We present in Sect. 8.2 properties of the Grassmann odd (as well as, for our study of anticommuting ″Grassmann fermions″ not important, the Grassmann even) algebra and of the chosen ″basis vectors″ for even ($d = (2(2n+1), 4n)$, n is an integer) dimensional space-time, $d = (d-1)+1$, and illustrate anticommuting ″basis vectors″ on the case of $d = (5+1)$, Subsect. 8.2.1, chapter *A.b.*.

We define the action for the integer spin ″Grassmann fermions″ in Subsect. 8.2.2. Solutions of the corresponding equations of motion, which are the tensor products of finite number of ″basis vectors″ and of infinite number of basis in momentum space, define the creation operators depending on internal quantum numbers and on $\vec{p}$ in d-dimensional space-time. We illustrate the corresponding superposition of "basis vectors", solving the equation of motion in $d = (5 + 1)$ in chapter *B.a.*.

We present in Sect. 8.3 the Hilbert space $\mathcal{H}$ of the tensor multiplication of one fermion creation operators of all possible single particle states of an odd character and of any number, representing "Slater determinants" with no "Grassmann fermion" state occupied with "Grassmann fermions", with one "Grassmann fermion" state occupied, with two "Grassmann fermion" states occupied, up to the "Slater determinant" with all possible "Grassmann fermion" states of each of infinite number of momentum $\vec{p}$ occupied. The Hilbert space $\mathcal{H}$ is the tensor product $\prod_{\infty} \otimes_N$ of finite number of $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$, for (continues) infinite possibilities for $\vec{p}$.

On $\mathcal{H}$ the creation and annihilation operators manifest the anticommutation relations of second quantized "fermions" without any postulates. These second quantized "fermion" fields, manifesting in the Grassmann case an integer spin, offer in d-dimensional space, $d > (3 + 1)$, the description of the corresponding charges in adjoint representations. We follow in this paper to some extent Ref. [12].

In Subsect. 8.3.3 relation between the by Dirac postulated creation and annihilation operators and the creation and annihilation operators presented in this Part I — for integer spins "Grassmann fermions" — are discussed.

In Sect. 8.4 we comment on what we have learned from the second quantized "Grassmann fermion" fields with integer spin when internal degrees of freedom are described with Grassmann algebra and compare these recognitions with the recognitions, which the Clifford algebra is offering, discussions on which appear in Part II.

In Part II we present in equivalent sections properties of the two kinds of the Clifford algebras and discuss conditions under which odd products of odd elements (operators), γ^a and $\tilde{\gamma}^a$'s of the two Clifford algebras, demonstrate the anticommutation relations required for the second quantized fermion fields on the Hilbert space $\mathcal{H} = \prod_{\infty} \otimes_N \mathcal{H}_{\vec{p}}$, this time with the half integer spin, offering in d-dimensional space, $d > (3 + 1)$, the description of charges, as well as the appearance of families of fermions [12], both needed to describe the properties of the observed quarks and leptons and antiquarks and antileptons, appearing in families.

In Part II we discuss relations between the Dirac way of second quantization with postulates and our way using Clifford algebra.

This paper is a part of the project named the *spin-charge-family* theory of one of the authors (N.S.M.B.), so far offering the explanation for all the assumptions of the *standard model*, with the appearance of the scalar fields included.

The Clifford algebra offers in even d-dimensional spaces, $d \geq (13 + 1)$ indeed, the description of the internal degrees of freedom for the second quantized fermions with the half integer spins, explaining all the assumptions of the *standard model*: The appearance of charges of the observed quarks and leptons and their families, as well as the appearance of the corresponding gauge fields, the scalar fields, explaining the Higgs scalar and the Yukawa couplings, and in addition the appearance of the dark matter, of the matter/antimatter asymmetry, offering several predictions [5–11, 15, 16].

8.1.1 Our main assumption and definitions

In this subsection we clarify how does the main assumption of Part I and Part II, *the decision to describe the internal space of fermions with the "basis vectors" expressed with the superposition of odd products of the anticommuting members of the algebra*, either the Clifford one or the Grassmann one, acting algebraically, $*_A$, on the internal vacuum state $|\psi_o>$, relate to the creation and annihilation anticommuting operators of the second quantized fermion fields.

To appreciate the need for this kind of assumption, let us first have in mind that algebra with the product $*_A$ is only present in our work, usually not in other works, and thus has no well known physical meaning. It is at first a product by which you can multiply two internal wave functions B_i and B_j with each other,

$$C_k = B_i *_A B_j \,,$$
$$B_i *_A B_j = \mp B_j *_A B_i \,,$$

the sign $\mp$ depends on whether B_i and B_j are products of odd or even number of algebra elements: The sign is $-$ if both are (superposition of) odd products of algebra elements, in all other cases the sign is $+$.

Let $\mathbf{R}^{d-1}$ define the external spatial or momentum space. Then the tensor product $*_T$ extends the internal wave functions into the wave functions $\mathbf{C}_{\vec{p},\,i}$ defined in both spaces

$$\mathbf{C}_{\vec{p},\,i} = |\vec{p}> *_T |B_i>\,,$$

where again B_i represent the superposition of products of elements of the anticommuting algebras, in our case either θ^a or γ^a or $\tilde{\gamma}^a$, used in this paper.

We can make a choice of the orthogonal and normalized basis so that $<\mathbf{C}_{\vec{p},i}|\mathbf{C}_{\vec{p'},j}>= \delta(\vec{p}-\vec{p'})\,\delta_{ij}$. Let us point out that either B_i or $\mathbf{C}_{\vec{p},\,i}$ apply algebraically on the vacuum state, $B_i *_A |\psi_o>$ and $\mathbf{C}_{\vec{p},\,i} *_A |\psi_o>$.

Usually a product of single particle wave functions is not taken to have any physical meaning in as far as most physicists simply do not work with such products at all.

To give to the algebraic product, $*_A$, and to the tensor product, $*_T$, defined on the space of single particle wave functions, the physical meaning, we postulate the connection between the anticommuting/commuting properties of the "basis vectors", expressed with the odd/even products of the anticommuting algebra elements and the corresponding creation operators, creating second quantized single fermion/boson states

$$\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_A |\psi_o> = |\psi_{\vec{p},i}>\,,$$
$$\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_T |\psi_{\vec{p'},j}> = 0\,,$$
$$\text{if}\ \vec{p} = \vec{p'}\ \text{and}\ i = j\,,$$
$$\text{in all other cases} \quad \text{it follows}$$
$$\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_T \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p'},\,j}} *_A |\psi_o> = \mp\, \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p'},\,j}} *_T \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_A |\psi_o>\,,$$

with the sign $\pm$ depending on whether $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ have both an odd character, the sign is $-$, or not, then the sign is $+$.

To each creation operator $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ its Hermitian conjugated partner represents the annihilation operator $\hat{b}_{\mathbf{c}_{\vec{p}},i}$

$$
\begin{aligned}
\hat{b}_{\mathbf{c}_{\vec{p}},i} &= (\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i})^{\dagger}\,,\\
\text{with the} &\quad \text{property}\\
\hat{b}_{\mathbf{c}_{\vec{p}},i} *_A |\psi_o> &= 0\,,\\
\text{defining the} &\quad \text{vacuum state as}\\
|\psi_o>: &= \sum_i (B_i)^{\dagger} *_A B_i \,|\, I>
\end{aligned}
$$

where summation i runs over all different products of annihilation operator $\times$ its Hermitian conjugated creation operator, no matter for what $\vec{p}$, and $|\, I>$ represents the identity, $(B_i)^{\dagger}$ represents the Hermitian conjugated wave function to B_i.

Let the tensor multiplication $*_T$ denotes also the multiplication of any number of single particle states, and correspondingly of any number of creation operators.

What further means that to each single particle wave function we define the creation operator $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$, applying in a tensor product from the left hand side on the second quantized Hilbert space — consisting of all possible products of any number of the single particle wave functions — adding to the Hilbert space the single particle wave function created by this particular creation operator. In the case of the second quantized fermions, if this particular wave function with the quantum numbers and $\vec{p}$ of $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ is already among the single fermion wave functions of a particular product of fermion wave functions, the action of the creation operator gives zero, otherwise the number of the fermion wave functions increases for one. In the boson case the number of boson second quantized wave functions increases always for one.

If we apply with the annihilation operator $\hat{b}_{\mathbf{c}_{\vec{p}},i}$ on the second quantized Hilbert space, then the application gives a nonzero contribution only if the particular products of the single particle wave functions do include the wave function with the quantum number i and $\vec{p}$.

In a Slater determinant formalism the single particle wave functions define the empty or occupied places of any of infinite numbers of Slater determinants. The creation operator $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ applies on a particular Slater determinant from the left hand side. Jumping over occupied states to the place with its i and $\vec{p}$. If this state is occupied, the application gives in the fermion case zero, in the boson case the number of particles increase for one. The particular Slater determinant changes sign in the fermion case if $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ jumps over odd numbers of occupied states. In the boson case the sign of the Slater determinant does not change.

When annihilation operator $\hat{b}_{\mathbf{c}_{\vec{p}},i}$ applies on particular Slater determinant, it is jumping over occupied states to its own place. giving zero, if this space is empty and decreasing the number of occupied states of this space is occupied. The Slater determinant changes sign in the fermion case, if the number of occupied states before its own space is odd. In the boson case the sign does not change.

Let us stress that choosing antisymmetry or symmetry is a choice which we make when treating fermions or bosons, respectively, namely the choice of using oddness or evenness of basis vectors, that is the choice of using odd products or even products of algebra anticummuting elements.

To describe the second quantized fermion states we make a choice of the basis vectors, which are the superposition of the odd numbers of algebra elements, of both Clifford and Grassmann algebras.

The creation operators and their Hermitian conjugation partners annihilation operators therefore in the fermion case anticommute. The single fermion states, which are the application of the creation operators on the vacuum state $|\psi_o>$, manifest correspondingly as well the oddness. The vacuum state, defined as the sum over all different products of annihilation $\times$ the corresponding creation operators, have an even character.

Let us end up with the recognition:

One usually means antisymmetry when talking about Slater-determinants because otherwise one would not get determinants.

In the present paper [5–7, 13] the choice of the symmetrizing versus antisymmetrizing relates indeed the commutation versus anticommutation with respect to the a priori completely different product $*_A$, of anticommuting members of the Clifford or Grassmann algebra. The oddness or evenness of these products transfer to quantities to which these algebras extend.

8.2 Properties of Grassmann algebra in even dimensional spaces

In Grassmann d-dimensional space there are d anticommuting operators θ^a, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, .., d)$, and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, offering together $2 \cdot 2^d$ operators, the half of which are superposition of products of θ^a and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,, (a, b) = (0, 1, 2, 3, 5, \cdots, d)\,. \tag{8.1}$$

Defining [12]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,,$$
it follows
$$(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,. \tag{8.2}$$

The identity is the self adjoint member. The signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$ is assumed.

It appears useful to arrange 2^d products of θ^a into irreducible representations with respect to the Lorentz group with the generators [6]

$$\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,. \tag{8.3}$$

2^{d-1} members of the representations have an odd Grassmann character (those which are superposition of odd products of θ^a's). All the members of any particular odd irreducible representation follow from any starting member by the application of $\mathbf{S}^{ab}$'s.

If we exclude the self adjoint identity there is $(2^{d-1}-1)$ members of an even Grassmann character, they are even products of θ^a's. All the members of any particular even representation follow from any starting member by the application of $\mathbf{S}^{ab}$'s.

The Hermitian conjugated 2^{d-1} odd partners of odd representations of θ^a's and $(2^{d-1}-1)$ even partners of even representations of θ^a's are reachable from odd and even representations, respectively, by the application of Eq. (8.2).

It appears useful as well to make the choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra as follows

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,, \tag{8.4}$$

and choose the members of the irreducible representations of the Lorentz group to be the eigenvectors of all the members of the Cartan subalgebra of Eq. (8.4)

$$\begin{aligned}
\mathbf{S}^{ab}\, \frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) &= k\, \frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,\\
\mathbf{S}^{ab}\, \frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) &= 0\,,\\
&\text{or}\\
\mathbf{S}^{ab}\, \frac{1}{\sqrt{2}}\, \frac{i}{k}\theta^a\theta^b &= 0\,,
\end{aligned} \tag{8.5}$$

with $k^2 = \eta^{aa}\eta^{bb}$. The eigenvector $\frac{1}{\sqrt{2}}\,(\theta^0 \mp \theta^3)$ of $\mathbf{S}^{03}$ has the eigenvalue $k = \pm i$, the eigenvalues of all the other eigenvectors of the rest of the Cartan subalgebra members, Eq. (8.4), are $k = \pm 1$.

We choose the "basis vectors" to be products of odd nilpotents $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ and of even objects $\frac{i}{k}\theta^a\theta^b$, with eigenvalues $k = \pm i$ and 0, respectively.

Let us check how does $\mathbf{S}^{ac} = i(\theta^a \frac{\partial}{\partial \theta_c} - \theta^c \frac{\partial}{\partial \theta_a})$ transform the product of two "nilpotents" $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ and $\frac{1}{\sqrt{2}}\,(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d)$. Taking into account Eq. (8.3) one finds that $\mathcal{S}^{ac}\, \frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\, \frac{1}{\sqrt{2}}\,(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d) = -\frac{\eta^{aa}\eta^{cc}}{2k}\,(\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)$. $\mathcal{S}^{ac}$ transforms the product of two Grassmann odd eigenvectors of the Cartan subalgebra into the superposition of two Grassmann even eigenvectors.

"Basis vectors" have an odd or an even Grassmann character, if their products contain an odd or an even number of "nilpotents", $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, respectively. "Basis vectors" are normalized, up to a phase, in accordance with Eq. (8.38) of 8.5.

The Hermitian conjugated representations of (either an odd or an even) products of θ^a's can be obtained by taking into account Eq. (8.2) for each "nilpotent"

$$\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)^\dagger = \eta^{aa}\,\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_a} + \frac{\eta^{aa}}{-ik}\frac{\partial}{\partial\theta_b})\,,$$
$$(\frac{i}{k}\,\theta^a\theta^b)^\dagger = \frac{i}{k}\,\frac{\partial}{\partial\theta_a}\,\frac{\partial}{\partial\theta_b}\,. \tag{8.6}$$

Making a choice of the identity for the vacuum state,

$$|\phi_{og}> = |\,1>\,, \tag{8.7}$$

we see that algebraic products — we shall use a dot , $\cdot$, or without a dot for an algebraic product of eigenstates of the Cartan subalgebra forming "basis vectors" and $*_A$ for the algebraic product of "basis vectors" — of different θ^a's, if applied on such a vacuum state, give always nonzero contributions,

$$(\theta^0 \mp \theta^3)\cdot(\theta^1 \pm i\theta^2)\cdots(\theta^{d-1} \mp \theta^d)|\,1> \neq \text{zero},$$

(this is true also, if we substitute any of nilpotents $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ or all of them with the corresponding even operators $(\frac{i}{k}\,\theta^a\theta^b)$; in the case of odd Grassmann irreducible representations at least one nilpotent must remain). The Hermitian conjugated partners, Eq. (8.6), applied on $|\,1>$, give always zero

$$(\frac{\partial}{\partial\theta_0} \mp \frac{\partial}{\partial\theta_3})\cdot(\frac{\partial}{\partial\theta_1} \pm i\frac{\partial}{\partial\theta_2})\cdots(\frac{\partial}{\partial\theta_{d-1}} \pm i\frac{\partial}{\partial\theta_d})|\,1>= 0.$$

Let us notice the properties of the odd products θ^a's and of their Hermitian conjugated partners:

i. Superposition of products of different θ^a's, applied on the vacuum state $|\,1>$, give nonzero contribution. To create on the vacuum state the "fermion" states we make a choice of the "basis vectors" of the odd number of θ^a's, arranging them to be the eigenvectors of all the Cartan subalgebra elements, Eq. (8.4).

ii. The Hermitian conjugated partners of the "basis vectors", they are products of derivatives $\frac{\partial}{\partial\theta_a}$'s, give, when applied on the vacuum state $|\,1>$, Eq. (8.7), zero. Each annihilation operator annihilates the corresponding creation operator.

iii. The algebraic product, $*_A$, of a "basis vector" by itself gives zero, the algebraic anticommutator of any two "basis vectors" of an odd Grassmann character (superposition of an odd products of θ^a's) gives zero ("basis vectors" of the two decuplets in Table 8.1 and the "basis vector" of Eq. (8.13) $\frac{1}{2}(\theta^0 \mp \theta^3)$, for example, demonstrate this property).

iv. The algebraic application of any annihilation operator on the corresponding Hermitian conjugated "basis vector" gives identity, on all the rest of "basis vectors" gives zero. Correspondingly the algebraic anticommutators of the creation operators and their Hermitian conjugated partners, applied on the vacuum state, give identity, all the rest anticommutators of creation and annihilation operators applied on the vacuum state, give zero.

v. Correspondingly the "basis vectors" and their Hermitian conjugated partners, applied on the vacuum state $|\,1>$, Eq. (8.7), fulfill the properties of creation and annihilation operator, respectively, for the second quantized "fermions" on the level of one "fermion" state.

8.2.1 Grassmann "basis vectors"

We construct 2^{d-1} Grassmann odd "basis vectors" and $2^{d-1}-1$ (we skip self adjoint identity, which we use to describe the vacuum state $|1>$) Grassmann even "basis vectors" as superposition of odd and even products of θ^a's, respectively. Their Hermitian conjugated 2^{d-1} odd and $2^{d-1}-1$ even partners are, according to Eqs. (8.2, 8.6), determined by the corresponding superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, respectively [1].

A.a. Grassmann anticommuting "basis vectors" with integer spins

Let us choose in $d=2(2n+1)$-dimensional space-time, n is a positive integer, the starting Grassmann odd "basis vector" $\hat{b}^{\theta 1\dagger}_1$, which is the eigenvector of the Cartan subalgebra of Eqs. (8.4, 8.5) with the egenvalues $(+i,+1,+1,\cdots,+1)$, respectively, and has the Hermitian conjugated partner equal to $(\hat{b}^{\theta 1\dagger}_1)^\dagger = \hat{b}^{\theta 1}_1$,

$$\begin{aligned}\hat{b}^{\theta 1\dagger}_1 &:= \left(\frac{1}{\sqrt{2}}\right)^{\frac{d}{2}} (\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\\ &\cdots(\theta^{d-1}+i\theta^d)\,,\\ \hat{b}^{\theta 1}_1 &:= \left(\frac{1}{\sqrt{2}}\right)^{\frac{d}{2}} \left(\frac{\partial}{\partial\theta^{d-1}}-i\frac{\partial}{\partial\theta^d}\right)\cdots\left(\frac{\partial}{\partial\theta^0}-\frac{\partial}{\partial\theta^3}\right).\end{aligned} \tag{8.8}$$

In the case of $d=4n$, n is a positive integer, the corresponding starting Grassmann odd "basis vector" can be chosen as

$$\begin{aligned}\hat{b}^{\theta 1\dagger}_1 &:= \left(\frac{1}{\sqrt{2}}\right)^{\frac{d}{2}-1} (\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots\\ &\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,.\end{aligned} \tag{8.9}$$

All the rest of "basis vectors", belonging to the same irreducible representation of the Lorentz group, follow by the application of $\mathbf{S}^{ab}$'s.

We denote the members i of this starting irreducible representation k by $\hat{b}^{\theta k\dagger}_i$ and their Hermitian conjugated partners by $\hat{b}^{\theta k}_i$, with $k=1$.

"Basis vectors", belonging to different irreducible representations $k=2$, will be denoted by $\hat{b}^{\theta 2\dagger}_j$ and their Hermitian conjugated partners by $\hat{b}^{\theta 2}_j = (\hat{b}^{\theta k\dagger}_j)^\dagger$.

$\mathbf{S}^{ac}$'s, which do not belong to the Cartan subalgebra, transform step by step the two by two "nilpotents", no matter how many "nilpotents" are between the chosen two, up to a constant, as follows:

$\mathbf{S}^{ac}\frac{1}{\sqrt{2}}(\theta^a+\frac{\eta^{aa}}{ik}\theta^b)\cdots\frac{1}{\sqrt{2}}(\theta^c+\frac{\eta^{cc}}{ik'}\theta^d)\propto -\frac{\eta^{aa}\eta^{cc}}{2k}(\theta^a\theta^b+\frac{k}{k'}\theta^c\theta^d)\cdots$,

leaving at each step at least one "nilpotent" unchanged, so that the whole irreducible representation remains odd.

The superposition of $\mathbf{S}^{bd}$ and $i\mathbf{S}^{bc}$ transforms $-\frac{\eta^{aa}\eta^{cc}}{2k}(\theta^a\theta^b+\frac{k}{k'}\theta^c\theta^d)$ into $\frac{1}{\sqrt{2}}(\theta^a-\frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}(\theta^c-\frac{\eta^{cc}}{ik'}\theta^d)$, and not into $\frac{1}{\sqrt{2}}(\theta^a+\frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}(\theta^c-\frac{\eta^{cc}}{ik'}\theta^d)$ or into $\frac{1}{\sqrt{2}}(\theta^a-\frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}(\theta^c+\frac{\eta^{cc}}{ik'}\theta^d)$.

[1] Relations among operators and their Hermitian conjugated partners in both kinds of the Clifford algebra objects are more complicated than in the Grassmann case, where the Hermitian conjugated operators follow by taking into account Eq. (8.2). In the Clifford case $\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is proportional to $\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{i(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1+\frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

Therefore we can start another odd representation with the "basis vector" $\hat{b}^{\theta 2\dagger}_{1}$ as follows

$$\hat{b}^{\theta 2\dagger}_{1} := (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,$$
$$(\hat{b}^{\theta 2\dagger}_{1})^{\dagger} = \hat{b}^{\theta 1}_{2} := (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}} - \frac{\partial}{\partial\theta^{3}})\,. \tag{8.10}$$

The application of $\mathbf{S}^{ac}$'s determines the whole second irreducible representation $\hat{b}^{\theta 2\dagger}_{j}$.

One finds that each of these two irreducible representations has $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members, Ref. [12].

Taking into account Eq. (8.1), it follows that odd products of θ^a's anticommute and so do the odd products of $\frac{\partial}{\partial\theta_a}$'s.

Statement 1: The oddness of the products of θ^a's guarantees the anticommuting properties of all objects which include odd number of θ^a's.

One further sees that $\frac{\partial}{\partial\theta^a}\theta^b = \eta^{ab}$, while $\frac{\partial}{\partial\theta_a}|1> = 0$, and $\theta^a|1> = \theta^a|1>$. and $\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A +} =$ We can therefore conclude

$$\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A +}|\,1> = \delta_{ij}\,\delta^{kl}\,|\,1>\,,$$
$$\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l}_{j}\}_{*_A +}|\,1> = 0\,\cdot|\,1>\,,$$
$$\{\hat{b}^{\theta k\dagger}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A +}\,|\,1> = 0\,\cdot|\,1>\,,$$
$$\hat{b}^{\theta k}_{j}\,*_A\,|\,1> = 0\,\cdot|\,1>\,, \tag{8.11}$$

where $\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A +} = \hat{b}^{\theta k}_{i} *_A \hat{b}^{\theta l\dagger}_{j} + \hat{b}^{\theta l}_{j} *_A \hat{b}^{\theta k\dagger}_{i}$ is meant.

These anticommutation relations of the "basis vectors" of the odd Grassmann character, manifest on the level of the Grassmann algebra the anticommutation relations required by Dirac [1] for second quantized fermions.

The "Grassmann fermion basis states" can be obtained by the application of creation operators $\hat{b}^{\theta k\dagger}_{i}$ on the vacuum state $|1>$

$$|\phi^{k}_{o\,i}> = \hat{b}^{\theta k\dagger}_{i}\,|1>\,. \tag{8.12}$$

We use them to determine the internal space of "Grassmann fermions" in the tensor product $*_T$ of these "basis states" and of the momentum space, when looking for the anticommuting single particle "Grassmann states", which have, according to Eq. (8.5), an integer spin, and not half integer spin as it is the case for the so far observed fermions.

A.b. Illustration of anticommuting "basis vectors" in $d = (5+1)$*-dimensional space*

Let us illustrate properties of Grassmann odd representations for $d = (5+1)$-dimensional space.

Table 8.1 represents two decuplets, which are "egenvectors" of the Cartan subalgbra ($\mathbf{S}^{03}$, $\mathbf{S}^{12}$, $\mathbf{S}^{56}$), Eq. (8.4), of the Lorentz algebra $\mathbf{S}^{ab}$. The two decuplets represent two Grassmann odd irreducible representations of $SO(5,1)$.

One can read on the same table, from the first to the third and from the fourth to the sixth line of both decuplets, two Grassmann even triplet representations of $SO(3,1)$, if paying attention on the eigenvectors of $\mathbf{S}^{03}$ and $\mathbf{S}^{12}$ alone, while the eigenvector of $\mathbf{S}^{56}$ has, as a "spectator", the eigenvalue either $+1$ (the first triplet in both decuplets) or -1 (the second triplet in both decuplets). Each of the two decuplets contains also one "fourplet" with the "charge" $\mathbf{S}^{56}$ equal to zero ($(7^{th}, 8^{th}, 9^{th}, 10^{th})$ lines in each of the two decuplets (Table II in Ref. [6])).

Paying attention on the eigenvectors of $\mathbf{S}^{03}$ alone one recognizes as well even and odd representations of $SO(1,1)$: $\theta^0\theta^3$ and $\theta^0 \pm \theta^3$, respectively.

The Hermitian conjugated "basis vectors" follow by using Eq. (8.6) and is for the first "basis vector" of Table 8.1 equal to $(-)^2(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$. One correspondingly finds that when $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ applies on $(\frac{1}{\sqrt{2}})^3(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)$ the result is identity. Application of $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ on all the rest of "basis vectors" of the decuplet I as well as on all the "basis vectors" of the decuplet II gives zero. "Basis vectors" are orthonormalized with respect to Eq. (8.38). Let us notice that $\frac{\partial}{\partial\theta_a}$ on a "state" which is just an identity, $|1>$, gives zero, $\frac{\partial}{\partial\theta_a}|1>= 0$, while $\theta^a|1>$, or any superposition of products of θ^a's, applied on $|1>$, gives the "vector" back.

One easily sees that application of products of superposition of θ^a's on $|1>$ gives nonzero contribution, while application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on $|1>$ gives zero.

The two by $\mathbf{S}^{ab}$ decoupled Grassmann decuplets of Table 8.1 are the largest two irreducible representations of odd products of θ^a's. There are 12 additional Grassmann odd "vectors", arranged into irreducible representations of six singlets and six sixplets

$$
\begin{aligned}
&(\frac{1}{2}(\theta^0 \mp \theta^3), \frac{1}{2}(\theta^1 \pm i\theta^2), \frac{1}{2}(\theta^5 \pm i\theta^6),\\
&\frac{1}{2}(\theta^0 \mp \theta^3)\theta^1\theta^2\theta^5\theta^6, \frac{1}{2}(\theta^1 \pm i\theta^2)\theta^0\theta^3\theta^5\theta^6, \frac{1}{2}(\theta^5 \pm i\theta^6)\theta^0\theta^3\theta^1\theta^2). \qquad (8.13)
\end{aligned}
$$

The algebraic application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on the corresponding Hermitian conjugated partners, which are products of superposition of θ^a's, leads to the identity for either even or odd Grassmann character [2].

Besides 32 Grassmann odd eigenvectors of the Grassmann Cartan subalgebra, Eq. (8.4), there are $(32-1)$ Grassmann "basis vectors", which we arrange into irreducible representations, which are superposition of even products of θ^a's. The even self adjoint operator identity (which is indeed the normalized product of all the annihilation times $*_A$ creation operators) is used to represent the vacuum state.

It is not difficult to see that Grassmann "basis vectors" of an odd Grassmann character anticommute among themselves and so do odd products of superposition of $\frac{\partial}{\partial\theta^a}$'s, while equivalent even products commute.

The Grassmann odd algebra (as well as the two odd Clifford algebras) offers, due to the oddness of the internal space giving oddness as well to the elements of the tensor products of the internal space and of the momentum space, the description of the anticommuting second quantized fermion fields, as postulated by Dirac. But the Grassmann "fermions"

[2] We shall see in Part II that the vacuum states are in the Clifford case, similarly as in the Grassmann case, for both kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s, sums of products of the annihilation $\times$ its Hermitian conjugated creation operators, and correspondingly self adjoint operators, but they are not the identity.

I	i	decuplet of eigenvectors	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1	1	1
	2	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	1	1
	3	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1	1	1
	4	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1	1	-1
	5	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	1	-1
	6	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1	1	-1
	7	$(\frac{1}{\sqrt{2}})^2(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0	1	0
	8	$(\frac{1}{\sqrt{2}})^2(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0	1	0
	9	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	1	0
	10	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	1	0

II	i	decuplet of eigenvectors	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\gamma^{(5+1)}$	$\gamma^{(3+1)}$
	1	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1	-1	-1
	2	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	-1	-1
	3	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1	-1	-1
	4	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1	-1	1
	5	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	-1	1
	6	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1	-1	1
	7	$(\frac{1}{\sqrt{2}})^2(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0	-1	0
	8	$(\frac{1}{\sqrt{2}})^2(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0	-1	0
	9	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	-1	0
	10	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	-1	0

Table 8.1. The two decuplets, the odd eigenvectors of the Cartan subalgebra, Eq. (8.4), ($\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}$, for $SO(5,1)$) of the Lorentz algebra in Grassmann $(5+1)$-dimensional space, forming two irreducible representations, are presented. Table is partly taken from Ref. [12]. The "basis vectors" within each decuplet are reachable from any member by $\mathbf{S}^{ab}$'s and are decoupled from another decuplet. The two operators of handedness, $\Gamma^{((d-1)+1)}$ for $d=(6,4)$, are invariants of the Lorentz algebra, Eq. (8.40), $\Gamma^{(5+1)}$ for the whole decuplet, $\Gamma^{(3+1)}$ for the "triplets" and "fourplets".

carry the integer spins, while the observed fermions — quarks and leptons — carry half integer spin.

A.c. Grassmann commuting "basis vectors" with integer spins

Grassmann even "basis vectors" manifest the commutation relations, and not the anticommutation ones as it is the case for the Grassmann odd "basis vectors". Let us use in the Grassmann even case, that is the case of superposition of an even number of θ^a's in $d=2(2n+1)$, the notation $\hat{a}^{\theta k\dagger}_j$, again chosen to be eigenvectors of the Cartan subalgebra, Eq. (8.4), and let us start with one representative

$$\hat{a}^{\theta 1\dagger}_j := (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6) \cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,. \tag{8.14}$$

The rest of "basis vectors", belonging to the same Lorentz irreducible representation, follow by the application of $\mathbf{S}^{ab}$. The Hermitian conjugated partner of $\hat{a}^{\theta 1 \dagger}_1$ is $\hat{a}^{\theta 1}_1 = (\hat{a}^{\theta 1 \dagger}_1)^\dagger$

$$\hat{a}^{\theta 1}_1 := (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} \frac{\partial}{\partial \theta^d} \frac{\partial}{\partial \theta^{d-1}} (\frac{\partial}{\partial \theta^{d-3}} - i \frac{\partial}{\partial \theta^{d-2}}) \cdots (\frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3}) . \tag{8.15}$$

If $\hat{a}^{\theta k \dagger}_j$ represents a Grassmann even creation operator, with index k denoting different irreducible representations and index j denoting a particular member of the k^{th} irreducible representation, while $\hat{a}^{\theta k}_j$ represents its Hermitian conjugated partner, one obtains by taking into account Sect. 8.2, the relations

$$\begin{aligned}
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k' \dagger}_j\}_{*_A -} |1> &= \delta_{ij}\, \delta^{kk'}\, |1>, \\
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k'}_j\}_{*_A -} |1> &= 0 \cdot |1>, \\
\{\hat{a}^{\theta k \dagger}_i, \hat{a}^{\theta k' \dagger}_j\}_{*_A -} |1> &= 0 \cdot |1>, \\
\hat{a}^{\theta k}_i *_A |1> &= 0 \cdot |1>, \\
\hat{a}^{\theta k \dagger}_i *_A |1> &= |\phi^k_{e\, i}> .
\end{aligned} \tag{8.16}$$

Equivalently to the case of Grassmann odd "basis vectors" also here $\{\hat{a}^{\theta k}_i, \hat{a}^{\theta l \dagger}_j\}_{*_A -} = \hat{a}^{\theta k}_i *_A \hat{a}^{\theta l \dagger}_j - \hat{a}^{\theta l}_j *_A \hat{a}^{\theta k \dagger}_i$ is meant.

8.2.2 Action for free massless "Grassmann fermions" with integer spin [12]

In the Grassmann case the "basis vectors" of an odd Grassmann character, chosen to be the eigenvectors of the Cartan subalgebra of the Lorentz algebra in Grassmann space, Eq. (8.4), manifest the anticommutation relations of Eq. (8.11) on the algebraic level.

To compare the properties of creation and annihilation operators for "integer spin fermions", for which the internal degrees of freedom are described by the odd Grassmann algebra, with the creation and annihilation operators postulated by Dirac for the second quantized fermions depending on the quantum numbers of the internal space of fermions and on the momentum space, we need to define the tensor product $*_T$ of the odd "Grassmann basis states", described by the superposition of odd products of θ^a's (with the finite degrees of freedom) and of the momentum (or coordinate) space (with the infinite degrees of freedom), taking as the basis the tensor product of both spaces.

Statement 2: For deriving the anticommutation relations for the "Grassmann fermions", to be compared to anticommutation relations of the second quantized fermions, we need to define the tensor product of the Grassmann odd "basis vectors" and the momentum space

$$\mathbf{basis}_{(p^a, \theta^a)} = |p^a> *_T |\theta^a> . \tag{8.17}$$

We need even more, we need to find the Lorentz invariant action for, let say, free massless "Grassmann fermions" to define such a "basis", that would manifest the relation $|p^0| = |\vec{p}|$. We follow here the suggestion of one of us (N.S.M.B.) from Ref. [12].

$$\begin{aligned} \mathcal{A}_G &= \int d^dx\, d^d\theta\, \omega\{\phi^\dagger\, \gamma^0_G\, \frac{1}{2}\, \theta^a p_a \phi\} + h.c.\,, \\ \omega &= \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k)\,, \end{aligned} \tag{8.18}$$

with $\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial \theta_a})$, $(\gamma^a_G)^\dagger = \gamma^a_G$, for each $a = (0, 1, 2, 3, 5, \cdots, d)$. We use the integral over θ^a coordinates with the weight function ω from Eq. (8.38, 8.39). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator γ^0_G, which takes care of the Lorentz invariance. Namely

$$\begin{aligned} \mathbf{S}^{ab\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0})\, \mathbf{S}^{ab}\,, \\ \mathbf{S}^\dagger\,(1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0})\, \mathbf{S}^{-1}\,, \\ \mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab} + \mathbf{S}^{ab})}\,, \end{aligned} \tag{8.19}$$

while θ^a, $\frac{\partial}{\partial \theta_a}$ and p^a transform as Lorentz vectors.

The Lagrange density is up to the surface term equal to [3]

$$\begin{aligned} \mathcal{L}_G &= \frac{1}{2}\phi^\dagger\, \gamma^0_G (\theta^a - \frac{\partial}{\partial \theta_a})\, (\hat{p}_a \phi) \\ &= \frac{1}{4}\{\phi^\dagger\, \gamma^0_G\, (\theta^a - \frac{\partial}{\partial \theta_a})\, \hat{p}_a \phi - \\ &\quad (\hat{p}_a \phi^\dagger) \gamma^0_G (\theta^a - \frac{\partial}{\partial \theta_a}) \phi\}\,, \end{aligned} \tag{8.20}$$

leading to the equations of motion [4]

$$\frac{1}{2}\, \gamma^0_G\, (\theta^a - \frac{\partial}{\partial \theta_a})\, \hat{p}_a\, |\phi> \;= 0\,, \tag{8.21}$$

as well as the the "Klein-Gordon" equation,

$$(\theta^a - \frac{\partial}{\partial \theta_a})\, \hat{p}_a\, (\theta^b - \frac{\partial}{\partial \theta_b})\, \hat{p}_b\, |\phi> = 0 = \hat{p}_a \hat{p}^a\, |\phi>\,.$$

The eigenstates ϕ of equations of motion for free massless "Grassmann fermions", Eq. (8.21), can be found as the tensor product, Eq.(8.17) of the superposition of 2^{d-1} Grassmann odd "basis vectors" $\hat{b}^{\theta k\dagger}_i$ and the momentum space, represented by plane waves, applied on the vacuum state $|\,1>$. Let us remind

[3] Taking into account the relations $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$, $\gamma^0_G = -i\eta^{aa}\gamma^a\tilde{\gamma}^a$ the Lagrange density can be rewritten as $\mathcal{L}_G = -i\frac{1}{2}\phi^\dagger\, \gamma^0_G\, \tilde{\gamma}^a\, (\hat{p}_a \phi) = -i\frac{1}{4}\{\phi^\dagger\, \gamma^0_G\, \tilde{\gamma}^a\, \hat{p}_a \phi - \hat{p}_a \phi^\dagger\, \gamma^0_G\, \tilde{\gamma}^a\, \phi\}$.

[4] Varying the action with respect to $\phi^\dagger$ and ϕ it follows: $\frac{\partial \mathcal{L}_G}{\partial \phi^\dagger} - \hat{p}_a \frac{\partial \mathcal{L}_G}{\partial \hat{p}_a \phi^\dagger} = 0 = \frac{-i}{2}\gamma^0_G\, \tilde{\gamma}^a\, \hat{p}_a\, \phi$, and $\frac{\partial \mathcal{L}_G}{\partial \phi} - \hat{p}_a \frac{\partial \mathcal{L}_G}{\partial (\hat{p}_a \phi)} = 0 = \frac{i}{2}\hat{p}_a\, \phi^\dagger \gamma^0_G\, \tilde{\gamma}^a$.

that the "basis vectors" are the "eigenstates" of the Cartan subalgebra, Eq. (8.4), fulfilling (on the algebraic level) the anticommutation relations of Eq. (8.11). And since the oddness of the Grassmann odd "basis vectors" guarantees the oddness of the tensor products of the internal part of "Grassmann fermions" and of the plane waves, we expect the equivalent anticommutation relations also for the eigenstates of the Eq. (8.21), which define the single particle anticommuting states of "Grassmann fermions".

The coefficients, determining the superposition, depend on momentum p^a, $a = (0, 1, 2, 3, 5, \ldots, d)$, $|p^0| = |\vec{p}|$, of the plane wave solution $e^{-ip_a x^a}$.

Let us therefore define the new creation operators and the corresponding single particle "Grassmann fermion" states as the tensor product of two spaces, the Grassmann odd "basis vectors" and the momentum space basis

$$\begin{aligned}
\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}) &\stackrel{\text{def}}{=} \sum_i c^{ks}{}_i(\vec{p})\, \hat{b}^{\theta k\dagger}_i\,, \qquad |p^0| = |\vec{p}|\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}) &\stackrel{\text{def}}{=} \hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}) \cdot e^{-ip_a x^a}\,, \quad |p^0| = |\vec{p}|\,,\\
< x|\phi^{ks}_{tot}(\vec{p}) > &= \hat{\underline{\mathbf{b}}}^{\theta ks\dagger}_{tot}(\vec{p})\,|\,1>\,, \qquad |p^0| = |\vec{p}|\,,
\end{aligned} \tag{8.22}$$

with s representing different solutions of the equations of motion and k different irreducible representations of the Lorentz group, $\vec{p}$ denotes the chosen vector $(p^0, \vec{p})$ in momentum space.

One has further

$$|\phi^{ks}(x^0, \vec{x}) > = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \hat{\underline{\mathbf{b}}}^{\theta ks\dagger}(\vec{p})|_{|p^0|=|\vec{p}|}|\,1> \tag{8.23}$$

The orthogonalized states $|\phi^{ks}(\vec{p}) >$ fulfill the relation

$$\begin{aligned}
< \phi^{ks}(\vec{p})|\phi^{k's'}(\vec{p}') > &= \delta^{kk'}\, \delta_{ss'}\, \delta_{pp'}\,, \qquad |p^0| = |\vec{p}|\,,\\
< \phi^{k's'}(x^0, \vec{x}')|\phi^{ks}(x^0, \vec{x}) > &= \delta^{kk'}\, \delta_{ss'}\, \delta_{\vec{x}', \vec{x}}\,,
\end{aligned} \tag{8.24}$$

where we assumed the discretization of momenta $\vec{p}$ and coordinates $\vec{x}$.

In even dimensional spaces ($d = 2(2n+1)$ and $4n$) there are 2^{d-1} Grassmann odd superposition of "basis vectors", which belong to different irreducible representations, among them twice $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ of the kind presented in Eqs. (8.8, 8.9) and discussed in the chapter *A.b.* of the subsect. 8.2.1 and in Table 8.1 for a particular case $d = (5+1)$. The illustration for the superposition $\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ is presented, again for $d = (5+1)$, in chapter *B.a.*.

We introduced in Eq. (8.22) the creation operators $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ as the tensor product of the "basis vectors" of Grassmann algebra elements and the momentum basis. The Grassmann algebra elements transfer their oddness to the tensor products of these two basis. Correspondingly must $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ together with their Hermitian conjugated annihilation operators $(\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}))^\dagger = \hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})$ fulfill the the anticommutation relations equivalent to the anticommutation relations of

Eq. (8.11)

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')\}_{*_T +}|1> &= \delta^{kk'}\,\delta_{ss'}\delta(\vec{p}-\vec{p}\,')\,|1>\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'}_{tot}(\vec{p}\,')\}_{*_T +}|1> &= 0\,\cdot|1>\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')\}_{*_T +}|1> &= 0\,\cdot|1>\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})\,*_T\,|1> &= 0\,\cdot|1>\,,\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{8.25}$$

k labels different irreducible representations of Grassmann odd “basis vectors”, s labels different — orthogonal and normalized — solutions of equations of motion and $\vec{p}$ represent different momenta fulfilling the relation $(p^0)^2 = (\vec{p})^2$. Here we allow continuous momenta and take into account that

$$<\,1|\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})\,*_T\,\hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')|1> = \delta^{kk'}\delta^{ss'}\delta(\vec{p}-\vec{p}\,')\,, \tag{8.26}$$

in the case of continuous values of $\vec{p}$ in even d-dimensional space.

For each momentum $\vec{p}$ there are 2^{d-1} members of the odd Grassmann character, belonging to different irreducible representations. The plane wave solutions, belonging to different $\vec{p}$, are orthogonal, defining correspondingly ∞ many degrees of freedom for each of 2^{d-1} ”fermion” states, defined by $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$, when applying on the vacuum state $|1>$, Eq. (8.7).

With the choice of the Grassmann odd ”basis vectors” in the internal space of ”Grassmann fermions” and by extending these ”basis states” to momentum space to be able to solve the equations of motion, Eq. (8.21), we are able to define the creation operators $\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})$ of the odd Grassmann character, which together with their Hermitian conjugated partners annihilation operators, fulfill the anticommutation relations of Eq. (8.25), manifesting the properties of the second quantized fermion fields. Anticommutation properties of creation and annihilation operators are due to the odd Grassmann character of the ”basis vectors”.

To define the Hilbert space of all possible ”Slater determinants” of all possible occupied and empty fermion states and to discuss the application of $\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{k\, s\dagger}_{tot}(\vec{p})$ on ”Slater determinants”, let us see what the anticommutation relations, presented in Eq. (8.25), tell. We realize from Eq. (8.25) the properties

$$\begin{aligned}
\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s'}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{\theta k' s'}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\,,\\
\text{if at least one of}\,(k,s,\vec{p}) &\quad \text{distinguishes from}(k',s',\vec{p}\,')\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})|1> &= |1>\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})|1> &= 0\,,\\
|p^0| &= |\vec{p}|\,. \qquad (8.27)
\end{aligned}$$

From the above relations we recognize how do the creation and annihilation operators apply on "Slater determinants" of empty and occupied states, the later determined by $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$:

i. The creation operator $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ jumps over the creation operator defining the occupied state, which distinguish from the jumping creation one in at least one of $(k, s, \vec{p})$, changing sign of the "Slater determinant" every time, up to the last step when it comes to its own empty state, the one with its quantum numbers $(k, s, \vec{p})$, occupying this empty state, or if this state is already occupied, gives zero.

ii. The annihilation operator changes sign of the "Slater determinant" when ever jumping over the occupied state carrying different internal quantum numbers (k, s) or $\vec{p}$, unless it comes to the occupied state with its own $(k, s, \vec{p})$, emptying this state or, if this state is empty, gives zero.

We show in Part II that the Clifford odd "basis vectors" describe fermions with the half integer spin, offering as well the corresponding anticommutation relations, explaining Dirac's postulates for second quantized fermions.

We discuss in Sect. 8.3 the properties of the "Slater determinants" of the occupied and empty "Grassmann fermion states", created by $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$.

In Subsect. B.a. we present one solution of the equations of motion for free massless "Grassmann fermions".

B.a. Plane wave solutions of equations of motion, Eq. (8.21), *in* $d = (5+1)$-*dimensional space*

One of such plane wave massless solutions of the equations of motion in $d = (5+1)$-dimensional space for momentum $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, $p^0 = |p^0|$, is the superposition of "basis vectors", presented in Table 8.1 among the first three members of the first decuplet, $k = \text{I}$. This particular solution $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ carries the spin $\mathcal{S}^{12} = 1$ ("up") and the "charge"

$\mathcal{S}^{56} = 1$ (both from the point of view of $d = (3+1)$)

$$\begin{aligned}
&\hat{\underline{\mathbf{b}}}^{\theta 1 1\dagger}_{tot}(\vec{p}) := \beta\, (\frac{1}{\sqrt{2}})^2 \{ \frac{1}{\sqrt{2}}\, (\theta^0 - \theta^3)(\theta^1 + i\theta^2) \\
&- \frac{2(|p^0| - |p^3|)}{p^1 - ip^2}\, (\theta^0\theta^3 + i\theta^1\theta^2) \\
&- (\frac{(p^1 + ip^2)^2}{(|p^0| + |p^3|)^2})\, \frac{1}{\sqrt{2}} (\theta^0 + \theta^3)(\theta^1 - i\theta^2)\} \\
&\times (\theta^5 + i\theta^6)\, \cdot e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})}\,, \qquad |p^0| = |\vec{p}|\,,
\end{aligned}$$

β is the normalization factor. The notation $\hat{\underline{\mathbf{b}}}^{\theta 1 1\dagger}_{tot}(\vec{p})$ means that the creation operator represents the plane wave solution of the equations of motion, Eq. (8.21), for a particular $|p^0| = |\vec{p}|$.

Applied on the vacuum state the creation operator defines the second quantized single particle state of particular momentum $\vec{p}$. States, carrying different $\vec{p}$, are orthogonal (due to the orthogonality of the plane waves of different momenta and due to the orthogonality of $\hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ with respect to k and s, Eqs. (8.24, 8.26, 8.25)).

More solutions can be found in [12] and the references therein.

8.3 Hilbert space of anticommuting integer spin "Grassmann fermions"

The Grassmann odd creation operators $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$, with $|p^0| = |\vec{p}|$, are defined on the tensor products of 2^{d-1} "basis vectors", defining the internal space of integer spin "Grassmann fermions", and on infinite basis states of momentum space for each component of $\vec{p}$, chosen so that they solve for particular $(\vec{p})$ the equations of motion, Eq. (8.21). They fulfill together with their Hermitian conjugated annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ the anticommutation relations of Eq. (8.25).

These creation operators form the Hilbert space of "Slater determinants", defining for each "Slater determinant" places with either empty or occupied "Grassmann fermion" states.

Statement 3: Introducing the tensor product multiplication $*_T$ of any number of single "Grassmann fermion" states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ and with the identity included, applying on the vacuum state of any $(k, s, \vec{p})$), we generate the Hilbert space of the second quantized "Grassmann fermion" fields.

It is straightforward to recognize that the above definition of the Hilbert space is equivalent to the space of "Slater determinants" of all possible empty or occupied states of any momentum and any quantum numbers describing the internal space. The identity in this tensor product multiplication, for example, represents the "Slater determinant" of no single fermion state present.

The 2^{d-1} Grassmann odd creation operators of particular momentum $\vec{p}$, if applied on the vacuum state $|1>$, Eq. (8.7), define 2^{d-1} states. Since any state can be occupied or empty, the Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$ consists correspondingly of

$$N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-1}}\,. \tag{8.28}$$

"Slater determinants", namely the one with no occupied state, those with one occupied state, those with two occupied states, up to the one with all 2^{d-1} states occupied.

The total Hilbert space $\mathcal{H}$ of anticommuting integer spin "Grassmann fermions" consists of infinite many "Slater determinants" of particular $\vec{p}$, $\mathcal{H}_{\vec{p}}$, due to infinite many degrees of freedom in the momentum space

$$\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}} \,, \tag{8.29}$$

with the infinite number of degrees of freedom

$$N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-1}} \,. \tag{8.30}$$

8.3.1 "Slater determinants" of anticommuting integer spin "Grassmann fermions" of particular momentum $\vec{p}$

Let us write down explicitly these $2^{2^{d-1}}$ contributions to the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$, using the notation that $\mathbf{0}^{\mathbf{k}}_{\mathbf{s}\vec{\mathbf{p}}}$ represents the unoccupied state $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p})|1>$ (of the s^{th} solution of the equations of motion belonging to the k^{th} irreducible representation), while $\mathbf{1}^{\mathbf{k}}_{\mathbf{s}\vec{\mathbf{p}}}$ represents the corresponding occupied state.

The number operator is according to Eq. (8.11) and Eq. (8.27) equal to

$$\begin{aligned} N^{\theta k s}_{\vec{p}} &= \hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) \,, \\ N^{\theta k s}_{\vec{p}} *_T 0^{k}_{s\vec{p}} &= 0 \,, \quad N^{\theta k s}_{\vec{p}} *_T 1^{k}_{s\vec{p}} = 1 \,. \end{aligned} \tag{8.31}$$

Let us simplify the notation so that we count for $k = 1$ empty states as $\mathbf{0}_{\mathbf{r}\vec{\mathbf{p}}}$, and occupied states as $\mathbf{1}_{\mathbf{r}\vec{\mathbf{p}}}$, with $r = (1, \ldots, s^1_{max})$, for $k = 2$ we count $r = s^1_{max} + 1, \ldots, s^1_{max} + s^2_{max}$, up to $r = 2^{d-1}$. Correspondingly we can represent $\mathcal{H}_{\vec{p}}$ as follows

$$\begin{array}{ll} |\mathbf{0}_{1\vec{p}}, \mathbf{0}_{2\vec{p}}, \mathbf{0}_{3\vec{p}}, \ldots, \mathbf{0}_{2^{d-1}\vec{p}} > \,, & |\mathbf{1}_{1\vec{p}}, \mathbf{0}_{2\vec{p}}, \mathbf{0}_{3\vec{p}}, \ldots, \mathbf{0}_{2^{d-1}\vec{p}} >, \\ |\mathbf{0}_{1\vec{p}}, \mathbf{1}_{2\vec{p}}, \mathbf{0}_{3\vec{p}}, \ldots, \mathbf{0}_{2^{d-1}\vec{p}} > \,, & |\mathbf{0}_{1\vec{p}}, \mathbf{0}_{2\vec{p}}, \mathbf{1}_{3\vec{p}}, \ldots, \mathbf{0}_{2^{d-1}\vec{p}} >, \\ & \vdots \\ |\mathbf{1}_{1\vec{p}}, \mathbf{1}_{2\vec{p}}, \mathbf{0}_{3\vec{p}}, \ldots, \mathbf{0}_{2^{d-1}\vec{p}} > \,, & |\mathbf{1}_{1\vec{p}}, \mathbf{0}_{2\vec{p}}, \mathbf{1}_{3\vec{p}}, \ldots, \mathbf{0}_{2^{d-1}\vec{p}} >, \\ & \vdots \\ |\mathbf{1}_{1\vec{p}}, \mathbf{1}_{2\vec{p}}, \mathbf{1}_{3\vec{p}}, \ldots, \mathbf{1}_{2^{d-1}\vec{p}} > \,, & \end{array} \tag{8.32}$$

with a part with none of states occupied ($N_{r\vec{p}} = 0$ for all $r = 1, \ldots, 2^{d-1}$), with a part with only one of states occupied ($N_{r\vec{p}} = 1$ for a particular $r = 1, \ldots, 2^{d-1}$ while $N_{r'\vec{p}} = 0$ for all the others $r' \neq r$), with a part with only two of states occupied ($N_{r\vec{p}} = 1$ and $N_{r'\vec{p}} = 1$, where r and r' run from $1, \ldots, 2^{d-1}$), and so on. The last part has all the states occupied.

Taking into account Eq. (8.27) is not difficult to see that the creation operator $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$, when applied on this Hilbert space $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations for the second quantized "fermions".

$$\begin{aligned} \{\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p})\}_{*_T +} \mathcal{H}_{\vec{p}} &= \delta^{kk'}\, \delta_{ss'} \mathcal{H}_{\vec{p}}\,, \\ \{\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k' s'}_{tot}(\vec{p})\}_{*_T +}\, \mathcal{H}_{\vec{p}} &= 0 \cdot \mathcal{H}_{\vec{p}}\,, \\ \{\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p})\}_{*_T +}\, \mathcal{H}_{\vec{p}} &= 0 \cdot \mathcal{H}_{\vec{p}}\,. \end{aligned} \tag{8.33}$$

The proof for the above relations easily follows if taking into account that, when ever the creation or annihilation operator jumps over an odd products of occupied states, the sign changes. Then one sees that the contribution of the application of $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p})\, \mathcal{H}_{\vec{p}}$ has the opposite sign than the contribution of $\hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\, \mathcal{H}_{\vec{p}}$.

If the creation and annihilation operators are Hermitian conjugated to each other, the result of

$$\{\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) + \hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\}\mathcal{H}_{\vec{p}} = \mathcal{H}_{\vec{p}}$$

is the whole $\mathcal{H}_{\vec{p}}$ back. Each of the two summands operates on its own half of $\mathcal{H}_{\vec{p}}$. Jumping together over even number of occupied states $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ do not change the sign of particular "Slater determinant". (Let us add that $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ reduces for particular k and s the Hilbert space $\mathcal{H}_{\vec{p}}$ for a factor $\frac{1}{2}$, and so does $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$. The sum of both, applied on $\mathcal{H}_{\vec{p}}$, reproduces the whole $\mathcal{H}_{\vec{p}}$.)

8.3.2 "Slater determinants" of Hilbert space of anticommuting integer spin "fermions"

The total Hilbert space of anticommuting "fermions" is the infinite product of the Hilbert spaces of particular $\vec{p}$, $\mathcal{H} = \prod^{\infty}_{\vec{p}} \otimes_N \mathcal{H}_{\vec{p}}$, Eq. (8.29), represented by infinite numbers of "Slater determinants" $N_{\mathcal{H}} = \prod^{\infty}_{\vec{p}} 2^{2^{d-1}}$, Eq. (8.30). The notation $\otimes_N$ is to point out that creation operators $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$, which origin in superposition of odd number of θ^a's, keep their odd character also in the tensor products of the internal and momentum space, as well as in the "Slater determinants", in which creation operators determine the occupied states.

The application of creation operators $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ and their Hermitian conjugated annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ on the Hilbert space $\mathcal{H}$ has the property, manifested in Eq. (8.27), leading to the conclusion that the application of $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p'}) *_T \mathcal{H}$ is not zero if at least one of $(k, s, \vec{p})$ is not equal to $(k', s', \vec{p'})$, while $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p'}) *_T \mathcal{H} + \hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p'}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \mathcal{H} = 0$ for any

$(k, s, \vec{p})$ and any $(k', s', \vec{p'})$, what is not difficult to prove when taking into account Eq. (8.27).

One can easily show that the creation operators $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}\,')$ fulfill equivalent anticommutation on the whole Hilbert space of infinity many "Slater determinants" as they do on the Hilbert space $\mathcal{H}_{\vec{p}}$.

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\mathcal{H} &= \delta^{kk'}\, \delta_{ss'}\delta(\vec{p}-\vec{p}\,')\, \mathcal{H}\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} &= 0\, \cdot \mathcal{H}\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} &= 0\, \cdot \mathcal{H}\,.
\end{aligned} \tag{8.34}$$

Creation operators, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, operating on a vacuum state, as well as on the whole Hilbert space, define the second quantized fermion states.

8.3.3 Relations between creation operators $\hat{\underline{\mathbf{b}}}^{\theta\, k\, s\dagger}_{tot}(\vec{p})$ in the Grassmann odd algebra and the creation operators postulated by Dirac

Creation operators $\hat{\underline{\mathbf{b}}}^{\theta\, k\, s\dagger}_{tot}(\vec{p})$ define the second quantized "fermion" fields of integer spins.

Since the second quantized Dirac fermions have the half integer spin, the "Grassmann fermions", the internal degrees of which is described by the Grassmann odd algebra, have the integer spin. The comparison between the second quantized fields of Dirac and those presented in this Part I of the paper can only be done on a rather general level. We leave therefore the detailed comparison of the creation and annihilation operators for fermions with half integer spins between those postulated by Dirac and the ones following from the Clifford odd algebra presented in Part II to Subsect. 3.4 of Part II.

Here we discuss only the relations among appearance of the creation and annihilation operators offered by the Grassmann odd algebra and those postulated by Dirac. In both cases we treat only $d = (3{+}1)$-dimensional space, that is we solve the equations of motion for $p^a = (p^0, p^1, p^2, p^3)$ (in the case that $d > 4$ the rest of space demonstrates the charges in $d = (3+1)$, when $p^a = (p^0, p^1, p^2, p^3, 0, 0, \ldots, 0)$).

It is pointed out in what follows that both internal spaces — either the internal space postulated by Dirac or the internal space offered by the Grassmann algebra — are finite dimensional, as also the internal space offered by the Clifford algebra is finite dimensional.

In the Dirac case the second quantized states are in $d = (3+1)$ dimensions postulated as follows

$$\underline{\Psi}^{s\dagger}(x^0, \vec{x}) = \sum_{i, \vec{p}_k} \hat{\mathbf{a}}^\dagger_i(\vec{p}_k)\, u^s_i(\vec{p}_k)\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}\,. \tag{8.35}$$

$v^s_i(\vec{p}_k)$ $(= u^s_i\, e^{-i(p^0 x^0) - \varepsilon \vec{p}\cdot\vec{x}})$ are the two left handed ($\Gamma^{(3+1)} = -1$) and the two right handed ($\Gamma^{(3+1)} = 1$, Eq. (B.3)) two-component column matrices, representing the four solutions s of the Weyl equation for free massless fermions of particular momentum $|\vec{p}_k| = |p^0_k|$ ([2], Eqs. (20-49) - (20-51)), the factor $\varepsilon = \pm 1$ depends on the product of handedness and spin.

$\hat{\mathbf{a}}^\dagger_i(\vec{p}_k)$ are by Dirac postulated creation operators, which together with annihilation operators $\hat{\mathbf{a}}_i(\vec{p}_k)$, fulfill the anticommutation relations ([2], Eqs. (20-49) - (20-51)),

$$\begin{aligned}\{\hat{\mathbf{a}}^\dagger_i(\vec{p}_k), \hat{\mathbf{a}}^\dagger_j(\vec{p}_l)\}_{*_T+} &= 0 = \{\hat{\mathbf{a}}_i(\vec{p}_k), \hat{\mathbf{a}}_j(\vec{p}_l)\}_{*_T+}\,,\\ \{\hat{\mathbf{a}}_i(\vec{p}_k), \hat{\mathbf{a}}^\dagger_j(\vec{p}_l)\}_{*_T+} &= \delta_{ij}\delta_{\vec{p}_k\vec{p}_l}\,,\end{aligned} \tag{8.36}$$

in the case of discretized momenta for a fermion in a box. Creation operators and annihilation operators, $\hat{\mathbf{a}}^\dagger_i(\vec{p}_k)$ and $\hat{\mathbf{a}}_i(\vec{p}_k)$, are postulated to have on the Hilbert space of all "Slater determinants" these anticommutation properties.

To be able to relate the creation operators of Dirac $\hat{\mathbf{a}}^\dagger_i(\vec{p}_k)$ with $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}_k)$ from Eq. (8.34), let us remind the reader that $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}_k)$ is a superposition of basic vectors $\hat{b}^{\theta k\dagger}_i$ with the coefficients $c^{ks}{}_i(\vec{p})$, which depend on the momentum $\vec{p}$, Eq. (8.22) $(\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}) = \sum_i c^{ks}{}_i(\vec{p})\,\hat{b}^{\theta k\dagger}_i)$, so that $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}_k)$ $(= \sum_i c^{ks}{}_i(\vec{p})\,\hat{b}^{\theta k\dagger}_i\, e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})})$ solves the equations of motion for free massless "Grassmann fermions" for plane waves, while $|p^0| = |\vec{p}|$.

We treat in this subsection the Grassmann case in $(3+1)$-dimensional space only, without taking care on different irreducible representations k as well as on charges, in order to be able to relate the creation and annihilation operators in Grassmann space with the Dirac's ones. In this case the odd Grassmann creation operators are expressible with the "basic vectors", which are fourplets, presented in Table 8.1 on the 7^{th} up to the 10^{th} lines, the same on both decuplets, neglecting $\theta^5\theta^6$ contribution. (They have handedness in $d = (3+1)$ equal zero.)

Let us rewrite creation operators in the Dirac case so that their expressions resemble the expression for the creation operators

$$\hat{\underline{\mathbf{b}}}^{\theta s\dagger}_{tot}(\vec{p}_k) = \sum_i c^s{}_i(\vec{p})\,\hat{b}^{\theta\dagger}_i\, e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})},$$

leaving out the index of the irreducible representation.

$$\begin{aligned}\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k) &\stackrel{\text{def}}{=} \sum_i \hat{\mathbf{a}}^\dagger_i(\vec{p}_k)\,u^s_i(\vec{p}_k)\,e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})} \stackrel{\text{def}}{=} \sum_i \alpha^s_i(\vec{p}_k)\,\hat{a}^\dagger_i\, e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}\\ &\text{to be compared with}\\ \hat{\underline{\mathbf{b}}}^{\theta s\dagger}_{tot}(\vec{p}_k) &= \sum_i c^s{}_i(\vec{p})\,\hat{b}^{\theta\dagger}_i\, e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}\,.\end{aligned} \tag{8.37}$$

We define in the Dirac case two creation operators: $\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$ and $\hat{a}^\dagger_i$. Since $\underline{\Psi}^{s\dagger}(x^0,\vec{x}) = \sum_{\vec{p}_k} \hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$, Eq. (8.35), we realize that the two expressions

$$u^s_i(\vec{p}_k)\,\hat{\mathbf{a}}^\dagger_i(\vec{p}_k) \quad \text{and} \quad \alpha^s_i(\vec{p}_k)\,\hat{a}^\dagger_i$$

describe the same degrees of freedom.

These new creation operators $\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$ can not be related directly to $\hat{\underline{\mathbf{b}}}^{\theta s\dagger}_{tot}(\vec{p}_k)$, since the first ones describe the second quantized fields of the half integer spin fermions, while the later describe the second quantized integer spin "fermion" fields. However, both fulfill the anticomutation relations of Eq. (8.34).

The reader can notice that the creation operators $\hat{a}^{\dagger}_{i}$ do not depend on $\vec{p}$ as also $\hat{b}^{\theta\dagger}_{i}$ do not, both describing the internal degrees of freedom, while $\alpha^{s}_{i}(\vec{p}_{k})\,\hat{a}^{\dagger}_{i}$ and $\alpha^{s}_{i}(\vec{p}_{k})\,\hat{b}^{\theta\dagger}_{i}$ do.

The creation and annihilation operators of Dirac fulfill obviously the anticommutation relations of Eq. (8.34). To see this we only have to replace $\hat{\underline{\mathbf{b}}}^{\theta h\, s\dagger}_{tot}(\vec{p})$ by $\hat{\underline{\mathbf{a}}}^{h\, s\dagger}_{tot}(\vec{p})$ by taking into account relation of Eq. (8.37).

Creation and annihillation operators of the Dirac second quantized fermions with half integer spins are in Part II, in Subsect. III.D, related to the corresponding ones, offered by the Clifford algebra. Relating the creation and annihilation operators offered by the Clifford algebra objects with the Dirac's ones ensures us that the Clifford odd algebra explains the Dirac's postulates.

8.4 Conclusions

We learn in this Part I paper, that in d-dimensional space the superposition of odd products of θ^{a}'s exist, Eqs. (8.8, 8.10, 8.9), chosen to be the eigenvectors of the Cartan subalgebra, Eq. (8.5), which together with their Hermitian conjugated partners, odd products of $\frac{\partial}{\partial\theta_{a}}$'s, Eqs. (8.2, 8.8, 8.6), fulfill on the algebraic level on the vacuum state $|\phi_{o}>=|\,1>$, Eq. (8.25), the requirements for the anticommutation relations for the Dirac's fermions.

The creation operators defined on the tensor products of internal space of "Grassmann basis vectors" (of finite number of basis states) and of momentum space (with infinite number of basis states), arranged to be solutions of the equation of motion for free massless "Grassmann fermions", Eq. (8.21), form the infinite dimensional Hilbert space of "Slater determinants" of (continuous) infinite number of momenta, with $2^{2^{d-1}}$ possibilities for each momentum $\vec{p}$, Eq. (8.34)). These creation operators and their Hermitian conjugated partners fulfill on the Hilbert space the anticommutation relations postulated by Dirac for second quantized fermion fields.

We demonstrate the way of deriving second quantized integer fermion fields.

In the subsection 8.1.1 we clarify the relation between our description of the internal space of fermions with "basis vectors", manifesting oddness and transferring the oddness to the corresponding creation and annihilation operators of second quantized fermions, to the ordinary second quantized creation and annihilation operators from a slightly different point of view.

Since the creation and annihilation operators, which are superposition of odd products of θ^{a}'s and $\frac{\partial}{\partial\theta_{a}}$'s, respectively, anticommute algebraically when applying on the vacuum state, Eq. (8.11, 8.12) (while the corresponding even products of θ^{a}'s and $\frac{\partial}{\partial\theta_{a}}$'s commute, Eq. (8.16)), it follows that also creation operators, defined on tensor products of the finite number of "basis vectors" (describing the internal degrees of freedom of "Grassmann fermions") and on infinite basis of momentum space, together with their Hermitian conjugated partners annihilation operators, fulfill the anticommutation relations of Eq. (8.34). The use of the Grassmann odd algebra to describe the internal space of "Grassmann fermions" offers the anticommutation relations without postulating them: on the (simple) vacuum state as well as on the Hilbert space of infinite number of "Slater determinants" of all possible single particle states, empty or occupied, of the second quantized integer spin "fermion" fields. Correspondingly we second quantized "fermion fields" without postulating commutation relations of Dirac.

The internal "basis vectors" are chosen to be eigenvectors of the Cartan subalgebra operators in the way that the symmetry agrees with the properties of usual Dirac's creation and annihilation operators of second quantized fermions — in the Clifford case for half integer spin, while in the "Grassmann fermions" for the integer spins.

The "Grassmann fermions" carry the spin and charges, originated in $d \geq 5$, in the adjoint representations. "Grassmann fermions" offer no families, what means that there is no available operators, which would connect different irreducible representations of the Lorentz group (without breaking symmetries).

No elementary "Grassmann fermions" with the spins and charges in the adjoint representations have been observed, and since the observed quarks and leptons and anti-quarks and anti-leptons have half integer spins, charges in the fundamental representations and appear in families, it does not seem possible for the future observation of the integer spin elementary "Grassmann fermions", especially not since Eq. (19) in Part II demonstrates that the reduction of space in Clifford case, needed for the appearance of second quantized half integer fermions, reduces also the Grassmann space, leaving no place for second quantized "Grassmann fermions" with the integer spin.

In Part II two kinds of operators are studied; There are namely two kinds of the Clifford algebra objects, $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ and $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$, which anticommute, $\{\gamma^a, \tilde{\gamma}^a\}_+ = 0$ ($\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$), and offer correspondingly two kinds of independent representations.

Each of these two kinds of independent representations can be arranged into irreducible representations with respect to the two Lorentz generators — $S^{ab} = \frac{i}{4}\,(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and $\tilde{S}^{ab} = \frac{i}{4}\,(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$. All the Clifford irreducible representations of any of the two kinds of algebras are independent and disconnected.

The two Dirac's actions in d-dimensional space for free massless fermions ($\mathcal{A} = \int\, d^dx\, \frac{1}{2}\,(\psi^\dagger\gamma^0\,\gamma^a\, p_a\psi) + h.c.$ and $\tilde{\mathcal{A}} = \int\, d^dx\, \frac{1}{2}\,(\psi^\dagger\tilde{\gamma}^0\,\tilde{\gamma}^a\, p_a\psi) + h.c.$) lead to the equations of motion, which have the solutions in both kinds of algebras for an odd Clifford character (they are superposition of an odd products of γ^a's and $\tilde{\gamma}^a$'s, respectively), forming on the tensor product of finite number of "basis vectors" describing the internal space and of the infinite number of basis of momentum space, the creation and annihilation operators, which only "almost" anticommute, while the Grassmann odd creation and annihilation operators do anticommute. Although "vectors" of one irreducible representation of an odd Clifford algebra character, anticommute among themselves and so do their Hermitian conjugated partners in each of the two kinds of the Clifford algebras, the anticommutation relations among creation and annihilation operators in each of the two Clifford algebras separately, do not fulfill the requirement, that only the anticommutator of a creation operator and its Hermitian conjugated partner gives a nonzero contribution.

The decision, the postulate, Eq. (12), that only one kind of the Clifford algebra objects — we make a choice of γ^a — describes the internal space of fermions, while the second kind — $\tilde{\gamma}^a$ in this case — does not, and consequently determine "family" quantum numbers which distinguish among irreducible representations of S^{ab}, solves the problems:

a. Creation operators and their Hermitian conjugated partners, which are odd products of superpositions of γ^a, applied on the vacuum state, fulfill on the algebraic level the anticommutation relations, and the creation and annihilation operators creating the second quantized Clifford fermion fields fulfill all the requirements, which Dirac postulated for fermions.

b. Different irreducible representations with respect to S^{ab} carry now different "family" quantum numbers determined by $\frac{d}{2}$ commuting operators among $\tilde{S}^{ab}$.

c. The operators of the Lorentz algebra, which do not belong to the Cartan subalgebra, connect different irreducible representations of S^{ab}.

The above mentioned decision, Eq. (19) in Part II, obviously reduces the degrees of freedom of the odd (and even) Clifford algebra, while opening the possibility for the appearance of "families", as well as for the explanation for the Dirac's second quantization postulates. This decision, reducing as well the degrees of freedom of Grassmann algebra, disables the existence of the integer spin "fermions" as elementary particles.

Let us point out again at the end that when the internal part of the single particle wave function anticommute under the algebra product $*_A$, then this implies that the wave functions with such internal part anticommute under the extension of $*_A$ to the (full) single particle wave functions and so do anticommute the corresponding creation and annihilation operators what manifests also on the properties of the Hilbert space.

The anticommuting single fermion states manifest correspondingly the oddness already on the level of the first quantization.

8.5 APPENDIX: Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [6] of two functions of the Grassmann coordinates $<\mathbf{B}|\theta><\mathbf{C}|\theta>$, $<\mathbf{B}|\theta> = <\theta|\mathbf{B}>^{\dagger}$,

$$<\mathbf{b}|\theta> = \sum_{k=0}^{d} b_{a_1 \ldots a_k} \theta^{a_1} \cdots \theta^{a_k},$$

by requiring

$$\begin{aligned}
&\{d\theta^a, \theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a \theta^a = 1\,, \\
&\int d^d\theta \; \theta^0 \theta^1 \cdots \theta^d = 1\,, \\
&d^d\theta = d\theta^d \ldots d\theta^0\,, \quad \omega = \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k)\,,
\end{aligned} \tag{8.38}$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function [6] $\omega = \prod_{k=0}^{d}(\frac{\partial}{\partial\theta_k} + \theta^k)$ to define the scalar product in Grassmann space $<\mathbf{B}|\mathbf{C}>$

$$\begin{aligned} <\mathbf{B}|\mathbf{C}> &= \int d^d\theta^a \;\; \omega <\mathbf{B}|\theta><\theta|\mathbf{C}> \\ &= \sum_{k=0}^{d} \int b^*_{b_1\dots b_k} c_{b_1\dots b_k}\,. \end{aligned} \tag{8.39}$$

To define norms in Clifford space Eq. (8.38) can be used as well.

8.6 APPENDIX: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} = \alpha\varepsilon_{a_1 a_2\dots a_{d-1}\, a_d} S^{a_1 a_2}\cdot S^{a_3 a_4}\cdots S^{a_{d-1} a_d}\,, \tag{8.40}$$

with α, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case S^{ab} is defined in Eq. (8.3), while in the Clifford case Eq. (8.40) simplifies, if we take into account that $S^{ab}|_{a\neq b} = \frac{i}{2}\gamma^a\gamma^b$ and $\tilde{S}^{ab}|_{a\neq b} = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} := (i)^{d/2} \quad \prod_a \quad (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n\,. \tag{8.41}$$

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