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On the mutually independent Hamiltonian cycles in faulty hypercubes*

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Abstract

Two ordered Hamiltonian paths in the n-dimensional hypercube Q_n are said to be independent if i-th vertices of the paths are distinct for every $1 \leq i \leq 2^n$. Similarly, two s-starting Hamiltonian cycles are independent if i-th vertices of the cycle are distinct for every $2 \leq i \leq 2^n$. A set S of Hamiltonian paths and s-starting Hamiltonian cycles are mutually independent if every two paths or cycles, respectively, from S are independent. We show that for every set F of f edges and $n-f$ pairs of adjacent vertices w_i and b_i, there are $n-f$ mutually independent Hamiltonian paths with endvertices w_i, b_i and avoiding edges of F in Q_n. We also show that Q_n contains $n-f$ fault-free mutually independent s-starting Hamiltonian cycles, for every set of $f \leq n-2$ faulty edges in Q_n and every vertex s. This improves previously known results on the numbers of mutually independent Hamiltonian paths and cycles in the hypercube with faulty edges.

Keywords: hypercube, Hamiltonian path, Hamiltonian cycle, faulty edges, interconnection network

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1 Introduction

A parallel computer network is often modeled as an undirected graph in which the vertices correspond to processors and the edges correspond to communication links between the processors. Graphs which represent topological structure of parallel computer networks are required to posses elegant properties such as small degree and diameter, high connectivity, recursive structure, symmetry, etc. Moreover, one of the major concerns of the parallel network design is its robustness, i.e. tolerance to the occurence of faults. Failures could happen in hardware, software or even because of missing transmitted packet. In this paper we study a fault tolerance of the hypercube, one of the most popular architectures which has all above mentioned properties.

The *n-dimensional hypercube* Q_n is a (bipartite) graph with all binary vectors of length n as vertices, and with edges between every two vertices that differ in exactly one coordinate. Connection failures in computer network correspond to *faulty edges* in the underlying graph. It is important that network stays highly connected even if several connection failures appear. For this reason, mutually independent Hamiltonian paths/cycles of Q_n with arbitrarily chosen f faulty edges are studied.

In this paper, n always denotes a positive integer and $[n]$ denotes the set $\{1, 2, \ldots, n\}$. A path in the graph G is a sequence $P = (v_1, v_2, \ldots, v_k)$ of distinct vertices such that every two consecutive vertices are adjacent. For a path $P = (v_1, v_2, \ldots, v_k)$ we say that v_1 and v_k are the *endvertices* of P, and that P is a v_1v_k-path, which is denoted by $P[v_1, v_k]$. A path in G is *Hamiltonian* if it contains all vertices of G. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively, and let $m = |V(G)|$. Two Hamiltonian paths $P_1 = (u_1, u_2, \ldots, u_m)$ and $P_2 = (v_1, v_2, \ldots, v_m)$ of G are *independent* if $u_i \neq v_i$ for all $i \in [m]$. A set S of Hamiltonian paths of G is *mutually independent* if every two paths from S are independent. A study of such paths is motivated by the problem of simultaneous transmitting packets along these path such that they never meet in the same vertex.

A cycle is a sequence $C = (v_1, v_2, \ldots, v_k)$ of $k \geq 3$ distinct vertices such that every two consecutive vertices, including the first and the last vertex of the sequence are adjacent. We say that the cycle $C = (v_1, v_2, \ldots, v_k)$ is *v_1-starting* to emphasize the first vertex v_1 and we denote it by $C[v_1]$. A cycle C in a graph G is *Hamiltonian*, if it contains all vertices of G. Two v-starting Hamiltonian cycles $C_1 = (v, u_2, \ldots, u_m)$ and $C_2 = (v, v_2, \ldots, v_m)$ are *independent* if $v_i \neq u_i$ for all $2 \leq i \leq m$. A set S of v-starting Hamiltonian cycles of G is *mutually independent* if every two cycles from S are independent. A study of mutually independent v-starting Hamiltonian cycles is motivated by the problem of transferring different pieces of a given message from one node to all recipients simultaneously such that they never meet in the same node.

In 2005, Sun, Lin, Huang and Hsu [10] proved that for any vertex s, the n-dimensional hypercube Q_n contains $n-1$ mutually independent s-starting Hamiltonian cycles if $n = 2, 3$; and n mutually independent s-starting Hamiltonian cycles if $n \geq 4$. They also proved that for any set of $n-1$ distinct pairs of adjacent vertices, Q_n contains $n-1$ mutually independent Hamiltonian paths with these pairs of vertices as endvertices. In 2006, Hsieh and Yu [4] claimed that the n-dimensional hypercube Q_n with at most $f \leq n-2$ faulty edges contains a set of $n-1-f$ mutually independent Hamiltonian paths and a set of $n-1-f$ mutually independent s-starting Hamiltonian cycles for any vertex s. However, in 2007, Kueng, Lin, Liang, Tan and Hsu [6] noticed a flaw in their proof and published the correction. In 2009 Hsieh and Weng [3] proved that for $n \geq 3$, Q_n with at most $f \leq n-2$ faulty edges contains a set of $n-1-f$ mutually independent Hamiltonian paths between any two vertices of different parity. In 2010 Shih, Tan and Hsu [9] studied mutually independent paths of different length in Q_n.

In this paper, we improve previous known results by showing that Q_n contains a set of $n-f$ mutually independent Hamiltonian paths, see Theorem 13. We also prove that Q_n with at most $f \leq n-2$ faulty edges contains a set of $n-f$ mutually independent s-starting Hamiltonian cycles for any vertex s, see Theorem 15. This is the optimal result since s may be incident with f faulty edges.

2 Preliminaries

In this section we define notations and summarize previously known results that we use.

The distance of two edges $e_1, e_2 \in E(G)$ is the minimal distance between a vertex of e_1 and a vertex of e_2. Let us say that the edge $v_iv_j \in E(G)$ is *directed*, if we fix the order of its vertices by (v_i, v_j). We say that a cycle $C = (v_1, v_2, \ldots, v_k)$ is *directed* if v_iv_{i+1} are directed edges in C for all $i \in [k]$ (where $v_{k+1} = v_1$).

Let Q_n be the n-dimensional hypercube. For a vertex $v \in V(Q_n)$, let v^i be the neighbor of v that differs from v exactly in the i-th coordinate. We say that the edge vv^i is *i-directional*. Furthermore, for an edge $e = uv$ we denote $e^i = u^iv^i$. The *antipodal* vertex to a vertex v differs from v in all coordinates, and is denoted by $\overline{v}$. Note that the hypercube Q_n is an n-regular graph with 2^n vertices.

Two vertices of Q_n are of the *same parity* if both of them have even (odd) number of 1's. We say the vertex is white (black) if it has even (odd) number of 1's. Note that vertices of each parity form bipartite classes of Q_n. Consequently, u and v have the same parity if and only if $d(u,v)$ is even. We say that the edges u_iu_{i+1} and u_ju_{j+1} of a directed path or cycle $(u_1, u_2, \ldots, u_n)$ have the same *parity* if u_i and u_j are of the same parity; that is, $i - j \equiv 0 \pmod 2$.

For $d \in [n]$ and $i \in \{0,1\}$ let $Q^{d;i}_{n-1}$ be the subgraph of Q_n induced by the vertices with i on the d-th coordinate. Notice that $Q^{d;i}_{n-1}$ is isomorphic to Q_{n-1}. In other words, by removing all edges of the direction d, the hypercube Q_n splits into two (induced) subgraphs $Q^{d;0}_{n-1}$, $Q^{d;1}_{n-1}$ isomorphic to Q_{n-1}. We say that Q_n is split along the direction d into subcubes $Q^{d;0}_{n-1}$ and $Q^{d;1}_{n-1}$. Let us write Q^{i}_{n-1} instead of $Q^{d;i}_{n-1}$ if the direction d is clear from the context. Furthermore, we generalize this concept as follows. For $\{d_1, d_2, \ldots, d_p\} \subseteq [n]$ and $(i_1, i_2, \ldots, i_p) \in \{0,1\}^p$ let $Q^{(d_1,d_2,\ldots,d_p);(i_1,i_2,\ldots,i_p)}_{n-p}$ be the subgraph of Q_n induced by all the vertices whose d_1-th, d_2-th, $\ldots$, d_p-th coordinate equals to $i_1, i_2, \ldots, i_p$, respectively. Let us write simply $Q^{i_1i_2\cdots i_p}_{n-p}$ for $Q^{(d_1,d_2,\ldots,d_p);(i_1,i_2,\ldots,i_p)}_{n-p}$ when $(d_1, d_2, \ldots, d_p)$ is clear from the context.

The *cartesian product* $G \,\square\, H$ of two graphs G and H is the graph with the vertex set

$$V(G \,\square\, H) = \{(u, v);\ u \in V(G), v \in V(H)\},$$

and the edge set

$$E(G \,\square\, H) = \{(u_1, v_1)(u_2, v_2);\ u_1u_2 \in E(G) \text{ and } v_1 = v_2,\ \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(H)\}.$$

Note that $Q_n \,\square\, Q_m$ is isomorphic to Q_{n+m}. For $u \in V(Q_n)$ and $v \in V(Q_m)$ let (u, v) represent the vertex of Q_{n+m} with u on the first n coordinates and v on the last m coordinates.

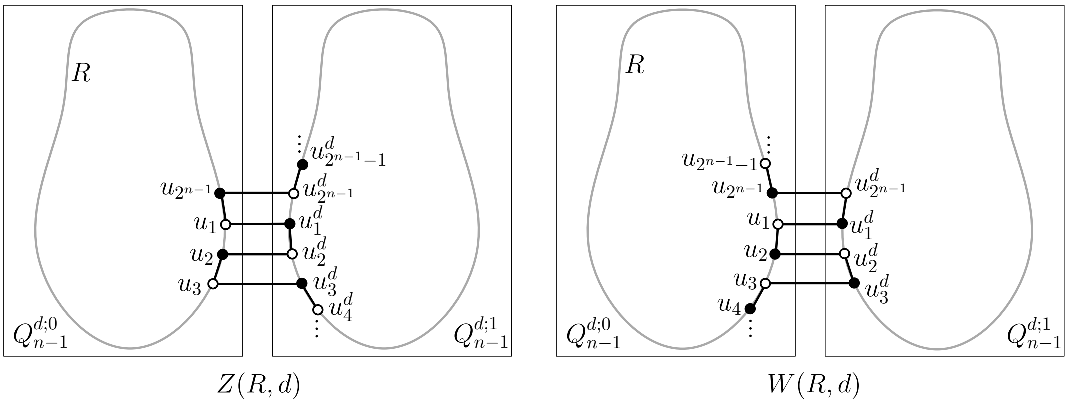

Figure 1: The directed zigzag Hamiltonian cycle $Z(R, d)$ and the directed zigzag Hamiltonian cycle $W(R, d)$ of Q_n.

Now we define a useful concept of zigzag paths and cycles. Let $R = (u_1, u_2, \ldots, u_{2^{n-1}})$ be a directed Hamiltonian path (cycle) in $Q^{d;i}_{n-1}$ for some $i \in \{0, 1\}$ and $d \in [n]$. Then, we say that

$$Z(R, d) = (u_1, u^d_1, u^d_2, u_2, \ldots, u^d_{2^{n-1}}, u_{2^{n-1}})$$

and

$$W(R, d) = (u^d_1, u_1, u_2, u^d_2, \ldots, u_{2^{n-1}}, u^d_{2^{n-1}})$$

are directed *zigzag* Hamiltonian paths (cycles) in Q_n. See Figure 1 for an illustration. Zigzag cycles have the following property.

Proposition 1. *Let $R = (u_1, \ldots, u_{2^{n-1}})$ be a Hamiltonian cycle in $Q^{d:i}_{n-1}$ for some $d \in [n]$ and $b \in \{0, 1\}$. Then for every distinct $0 \le i, j < 2^{n-2}$, the subpaths $P_1[u_{2i+1}, u_{2i}]$ and $P_2[u_{2j+1}, u^d_{2j+1}]$ of $Z(R, d)$ and $W(R, d)$, respectively, are independent Hamiltonian paths in Q_n.*

Proof. By the definition of $Z(R, d)$ and $W(R, d)$, we have

$$P_1[u_{2i+1}, u_{2i}] = (u_{2i+1}, u^d_{2i+1}, u^d_{2i+2}, u_{2i+2}, u_{2i+3}, \ldots, u^d_{2i}, u_{2i}),$$
$$P_2[u_{2j+1}, u^d_{2j+1}] = (u_{2j+1}, u_{2j+2}, u^d_{2j+2}, u^d_{2j+3}, u_{2j+3}, \ldots, u^d_{2j}, u^d_{2j+1}),$$

where the indices are taken cyclically; that is, $u_{2^{n-1}+1} = u_1$. Observe that the k-th vertices, $1 \leq k \leq 2^n$, of P_1 and P_2 are in distinct subcubes if k is even. If $k \equiv 1 \pmod 3$, they are in the form of u_{2i+s} and u_{2j+s} for some s. If $k \equiv 3 \pmod 3$, they are in the form u^d_{2i+s} and u^d_{2j+s} for some s. Thus, since i and j are distinct, the k-th vertices of P_1 and P_2 are distinct for every $1 \leq k \leq 2^n$. □

Now, we list the results that we need. It is well known that the hypercube Q_n is Hamiltonian for every $n \geq 2$. It is also Hamiltonian laceable [2]; that is, there is a Hamiltonian path between every two vertices of opposite parity. Even if some faulty edges appear in Q_n, the hypercube Q_n stays Hamiltonian laceable.

Proposition 2 (Tsai et al. [11]). *Let $F \subseteq E(Q_n)$, $n \geq 2$ and $|F| \leq n-2$. Then, there exists a Hamiltonian path in $Q_n - F$ between every two vertices of opposite parity.*

We also need several basic results on Hamiltonian cycles and paths in the hypercube with some removed vertices. The following proposition describes the case of one removed vertex.

Proposition 3 (Lewinter and Widulski [7]). *For $n \geq 2$ and every three distinct vertices $u_1, u_2, v \in V(Q_n)$ such that u_1, u_2 have the same parity opposite to the parity of $v \in V(Q_n)$, the graph $Q_n - \{v\}$ has a Hamiltonian u_1u_2-path P.*

A similar result holds for the case of two removed vertices.

Proposition 4 (Sun et al. [10]). *The graph $Q_n - \{u, v\}$, $n \geq 4$ is Hamiltonian laceable for every two vertices u and v of opposite parity.*

A set $M \subseteq E(G)$ of pairwise non-adjacent edges is called a *matching*. A matching M is *perfect* if every vertex of G is covered by M. Kreweras [5] conjectured that every perfect matching of the hypercube Q_n, where $n \geq 2$, can be extended to a Hamiltonian cycle. Fink [1] affirmatively answered this conjecture by proving a stronger result for the complete graph on the vertices of Q_n, denoted by $K(Q_n)$.

Theorem 5 (Fink [1]). *For every perfect matching M of $K(Q_n)$ where $n \geq 2$, there exists a perfect matching N of Q_n such that $M \cup N$ forms a Hamiltonian cycle of $K(Q_n)$.*

We say that k edges $e_1, e_2, \ldots, e_k \in E(Q_n)$ are *rigid* if they have distinct directions. Note that necessarily $k \leq n$. For a set S of edges of Q_n, we say that S *saturates* a vertex v if some edge of S is incident with the vertex v. Otherwise, v is said to be *unsaturated* by S. Furthermore, we say that a vertex v of Q_n is *blocked* by S if all neighbors of v are saturated by S and v is not saturated by S.

Theorem 6 (Limaye and Sarvate [8]). *If a matching $M \subseteq E(Q_n)$ of size $n \geq 2$ does not extend to a perfect matching in Q_n, then there is an unsaturated vertex v whose neighborhood is saturated by M.*

The previous results on mutually independent Hamiltonian paths and cycles in the hypercube are as follows.

Theorem 7 (Sun et al. [10]). *For any $s \in V(Q_n)$, the hypercube Q_n contains $n-1$ mutually independent s-starting Hamiltonian cycles if $2 \leq n \leq 3$, and n mutually independent s-starting Hamiltonian cycles if $n \geq 4$.*

Lemma 8 (Sun et al. [10]). *Let $w_1, w_2, \ldots, w_{n-1}$ be vertices of the same parity in Q_n, $n \geq 2$ and let $\{w_1b_1, w_2b_2, \ldots, w_{n-1}b_{n-1}\} \subseteq E(Q_n)$ be a matching in Q_n. Then, Q_n contains $n-1$ mutually independent Hamiltonian paths $P_1[w_1, b_1], P_2[w_2, b_2], \ldots, P_{n-1}[w_{n-1}, b_{n-1}]$.*

Lemma 9 (Kueng et al. [6]). *Let $F \subseteq E(Q_n)$, $n \geq 3$, $f = |F| \leq n-2$ and $w_1, w_2, \ldots, w_k$ be vertices of the same parity in Q_n, $k \leq n-1-f$. Let $\{w_1b_1, w_2b_2, \ldots, w_kb_k\} \subseteq E(Q_n)$ be a matching in Q_n. Then, $Q_n - F$ contains k mutually independent Hamiltonian paths $P_1[w_1, b_1], P_2[w_2, b_2], \ldots, P_k[w_k, b_k]$.*

We use Theorem 6, Theorem 7 and Lemma 8 to improve the result by Kueng, Lin, Liang, Tan and Hsu stated in Theorem 10. In the next section, we prove Theorem 13 which is improvement of Lemma 8 then we apply it in Section 4 (Theorem 15) to improve the following result.

Theorem 10 (Kueng et al. [6]). *Let $F \subseteq E(Q_n)$, $n \geq 4$, $f = |F| \leq n-2$, and $s \in V(Q_n)$. Then, $Q_n - F$ has $n-1-f$ mutually independent s-starting Hamiltonian cycles.*

3 Independent Hamiltonian paths in hypercubes

We start with an improvement in a special case that follows from Theorem 5.

Lemma 11. *Let $w_1, w_2, \ldots, w_k$ be vertices of the same parity in Q_n, $n \geq 2$. If $w_1b_1, w_2b_2, \ldots, w_kb_k$ are edges of a perfect matching M of Q_n, then Q_n has k mutually independent Hamiltonian paths $P_1[w_1, b_1], P_2[w_2, b_2], \ldots, P_k[w_k, b_k]$.*

Proof. By Theorem 5, there is a Hamiltonian cycle C containing the edges $w_1b_1, w_2b_2, \ldots, w_kb_k$. Moreover, edges $w_1b_1, w_2b_2, \ldots, w_kb_k$ have the same parity on C as they are included in the perfect matching M. If we disconnect the cycle C between vertices w_i and b_i, we obtain a Hamiltonian path $P_i[w_i, b_i]$ of Q_n. As $P_1, P_2, \ldots, P_k \subset C$ and $w_1b_1, w_2b_2, \ldots, w_kb_k$ have the same parity on C, P_i and P_j are independent for every distinct $i, j \in [k]$. □

We need the next proposition to prove Theorem 13.

Proposition 12. *Let $\mathcal{P}_i$ be a set of mutually independent Hamiltonian paths in $Q_{n-1}^{d;i}$ for $i = 0, 1$ and some direction d. Then, the set $\{Z(P, d); P \in \mathcal{P}_0 \cup \mathcal{P}_1\}$ is a set of mutually independent Hamiltonian paths in Q_n.*

Proof. Let $P_1, P_2 \in \mathcal{P}_0$. Then, observe that $Z(P_1, d)$, $Z(P_2, d)$ are mutually independent Hamiltonian paths in Q_n. Indeed, since every t-th vertex v of P_1 and t-th vertex u of P_2 are distinct, we infer that v^d and u^d are distinct and so $Z(P_1, d)$ and $Z(P_2, d)$ are independent in Q_n. A similar argument holds if $P_1, P_2 \in \mathcal{P}_1$.

Now, let $P_i \in \mathcal{P}_i$ for $i = 0, 1$. Then, the claim obviously holds as t-th vertices of $Z(P_0, d)$ and $Z(P_1, d)$ are in distinct parts $Q_{n-1}^{d;0}$ and $Q_{n-1}^{d;1}$ for all $t \in [2^n]$. □

The following theorem improves Lemma 8 by one additional independent Hamiltonian path.

Theorem 13. *Let $w_1, w_2, \ldots, w_n$ be vertices of the same parity in Q_n and let $M = \{w_1b_1, w_2b_2, \ldots, w_nb_n\} \subseteq E(Q_n)$ be a matching of Q_n $(n \geq 2)$. Then, Q_n has n mutually independent Hamiltonian paths $P_1[w_1, b_1], P_2[w_2, b_2], \ldots, P_n[w_n, b_n]$.*

Proof. We prove that Q_n contains n mutually independent Hamiltonian paths $P_i[w_i, b_i]$ for $i \in [n]$ by induction on the dimension n. The base of induction for Q_2 trivially holds since Q_2 contains two mutually independent Hamiltonian paths whose first and last vertices are vertices of two independent edges of M, respectively. Now, we assume that the statement holds for Q_{n-1} and we prove it for Q_n, $n \geq 3$. We consider three cases regarding M.

Case 1: *The matching M extends to a perfect matching.* Then, Q_n has n mutually independent Hamiltonian paths by Lemma 11.

In the remaining two cases, we assume due to Theorem 6 that some vertex v is blocked by M.

Case 2: *M is not rigid.* We proceed similarly as in the proof of Lemma 8 from Sun et. al. [10]. Since M is not rigid, there exists a direction d such that M contains no d-directional edge. We split Q_n along the direction d and we obtain two subcubes Q_{n-1}^0 and Q_{n-1}^1. Since there exists $v \in V(Q_n)$ blocked by M, for some $i \in \{0, 1\}$ the subcube Q_{n-1}^i contains one edge w_kb_k of M where $k \in [n]$ and the subcube Q_{n-1}^{1-i} contains all the other edges of M. By induction, there is one Hamiltonian path $P_k[w_k, b_k]$ in Q_{n-1}^i and $n-1$ mutually independent Hamiltonian paths $P_l[w_l, b_l]$ in Q_{n-1}^{1-i} for $l \in [n] \setminus \{k\}$. We extend all these Hamiltonian paths P_j to Hamiltonian zigzag paths $Z(P_j, d)$ in Q_n, which are mutually independent by Proposition 12.

Case 3: *M is rigid.* First, in case $n = 3$, there is only one possibility up to isomorphism that the vertex v is blocked by a set of three rigid edges. In this case the example of mutually independent Hamiltonian paths are

$$
\begin{aligned}
P_1[w_1, b_1] &= (w_1, b, w_3, b_3, v, b_2, w_2, b_1),\\
P_2[w_2, b_2] &= (w_2, b_1, w_1, b, w_3, b_3, v, b_2),\\
P_3[w_3, b_3] &= (w_3, b_2, v, b_1, w_2, b, w_1, b_3).
\end{aligned}
$$

as illustrated in Figure 2.

Suppose now $n \geq 4$. We can assume $b_i = w_i^i$ for every $i \in [n]$ as M is a set of n rigid edges. Our aim is the following: We split Q_n along an arbitrary direction $k \in [n]$

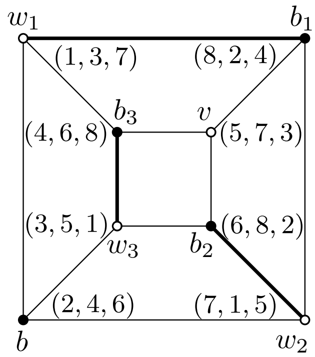

Figure 2: Q_3 with three mutually independent Hamiltonian paths. Each vertex u of Q_3 is associated with a triple (k_1, k_2, k_3) which says that u is the k_i-th vertex in the i-th Hamiltonian path $P_i[w_i, b_i]$.

into subcubes Q^0_{n-1} and Q^1_{n-1}. Notice that one of the subcubes Q^0_{n-1}, Q^1_{n-1} contains one edge of the matching M and the other subcube contains all the remaining edges except the one which is of direction k. Without loss of generality, we may assume that the vertex v is black and $v \in V(Q^0_{n-1})$. Then, Q^0_{n-1} contains $n-2$ edges of M, and Q^1_{n-1} contains $e_j = w_j b_j \in M$ for some $j \in [n] \setminus \{k\}$ such that $w_j = v^k$. Notice that b_k is adjacent to w_j. The vertices of the edges e_j, $w_j b_k$ in Q^1_{n-1} are neighbors of the vertices of edges e^k_j, vw_k in Q^0_{n-1}, respectively. Note that the edge $w^k_j w_k$ is incident with v, as $v = w^k_j$. See Figure 3 for an illustration.

In the rest of the proof we proceed as follows: We find an v-starting Hamiltonian cycle $C^0 = (v, v_2, \ldots, v_{2^{n-1}})$ of Q^0_{n-1} such that C^0 contains $M \setminus \{e_j, e_k\} \cup \{e^k_j\}$, the edges of $M \setminus \{e_j, e_k\}$ have the same parity on C^0 and $v_2 = w_k$, $v_{2^{n-1}} = w_i$. Then, the cycle $C = Z(C^0, k)$ is a Hamiltonian cycle of Q_n containing M. Furthermore, the edges of

$$M \setminus \{e_k\} = \{w_1b_1, \ldots, w_{k-1}b_{k-1}, w_{k+1}b_{k+1}, \ldots, w_nb_n\}$$

have the same parity on C. Then, the paths

$$P_1[w_1, b_1], \ldots, P_{k-1}[w_{k-1}, b_{k-1}], P_{k+1}[w_{k+1}, b_{k+1}], \ldots, P_n[w_n, b_n]$$

on C are mutually independent Hamiltonian paths of Q_n. Finally, for the differently directed edge $e_k = w_k b_k$ on the cycle C we find a Hamiltonian path $P_k[w_k, b_k]$ that is mutually independent with all the other already constructed Hamiltonian paths of Q_n.

Now, let us find an v-starting Hamiltonian cycle C^0 of Q^0_{n-1} such that C^0 contains $M \setminus \{e_j, e_k\} \cup \{e^k_j\}$ and the edges of $M \setminus \{e_j, e_k\}$ have the same parity on C^0. Note that $e^k_j = w^k_j b^k_j$ is an j-directional edge in Q^0_{n-1} and it is incident with e_i for some $i \in [n] \setminus \{j, k\}$. We split Q^0_{n-1} along the direction j into subcubes Q^{00}_{n-2} and Q^{01}_{n-2}. One of the subcubes Q^{00}_{n-2} and Q^{01}_{n-2} contains the edges of

$$M' = M \setminus \{e_i, e_j, e_k\}$$

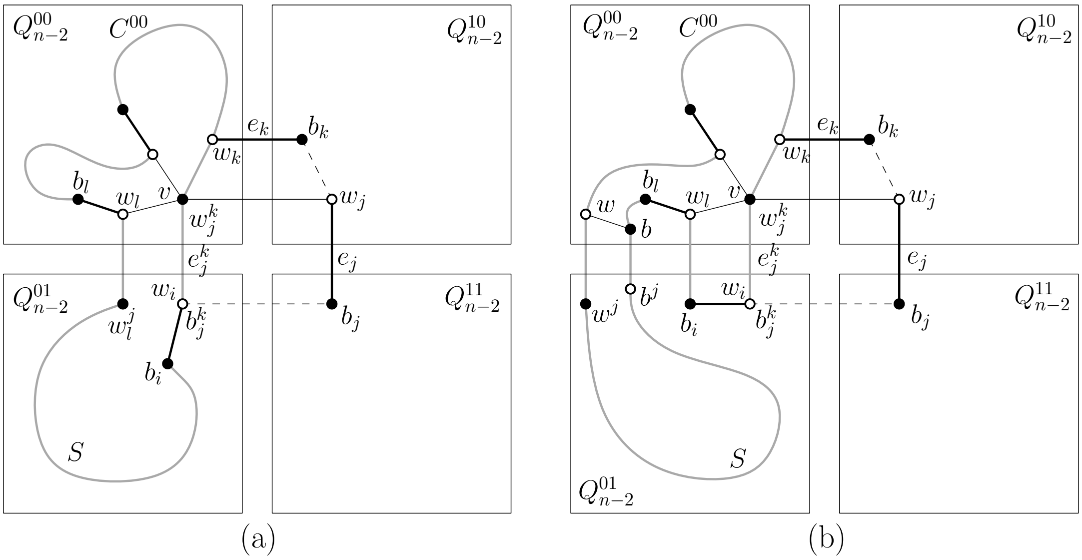

Figure 3: (a) The construction of a Hamiltonian cycle C^0 of Q^0_{n-1} if $w^j_l \neq b_i$. (b) The construction of a Hamiltonian cycle C^0 of Q^0_{n-1} if $w^j_l = b_i$. The edges of M are bold.

and the other contains the edge e_i. Without loss of generality we assume Q^{00}_{n-2} contains M' and therefore it also contains the vertex v, see Figure 3. The set of edges M' is a matching of Q^{00}_{n-2} such that Q^{00}_{n-2} has no vertex u with neighborhood saturated by M' since Q^{00}_{n-2} contains $n-3$ edges of M'. We extend M' to a perfect matching R of Q^{00}_{n-2} by Theorem 6. Note that R contains the edge vw_k. Then, we apply Theorem 5 and find a Hamiltonian cycle C^{00} of Q^{00}_{n-2} containing M' as edges of the same parity.

Let w_l be the neighbor of the vertex v on the Hamiltonian cycle C^{00} other than w_k. Now, we find a Hamiltonian cycle C^0 of Q^0_{n-1}. To do so, we distinguish the following two cases regarding whether w^j_l and b_i coincide.

Subcase 3.1: $w^j_l \neq b_i$. See Figure 3(a) for an illustration. By Proposition 3, $Q^{01}_{n-2} - \{w_i\}$ contains a Hamiltonian path $S[w^j_l, b_i]$. Let $P[w_k, w_l]$ be the path from w_k to w_l on the Hamiltonian cycle C^{00} in Q^{00}_{n-2}. Then, the desired v-starting Hamiltonian cycle C^0 of Q^0_{n-1} is

$$C^0 = (v, P, S, w_i).$$

Subcase 3.2: $w^j_l = b_i$. See Figure 3(b) for an illustration. We choose two adjacent vertices w and b on the Hamiltonian cycle C^{00} of Q^{00}_{n-2} such that $wb \notin M$ and b is a black vertex distinct from v. Observe that we can always choose such w and b since $n \geq 4$. Note that $\{w^j, b^j\} \cap \{w_i, b_i\} = \emptyset$. Let $P[w_k, w_l]$ be the directed path from w_k to w_l on the Hamiltonian cycle C^{00}, and without loss of generality we may assume that the vertex b follows the vertex w on the path P. Let $R_1[w_k, w], R_2[b, w_l]$ be the subpaths of the path P.

Subcase 3.2.1: $n = 4$. The v-starting Hamiltonian cycle C^0 of Q_3^0 is

$$C^0 = (v, R_1, w^j, b^j, R_2, b_i, w_i).$$

Subcase 3.2.2: $n = 5$. Note that we could choose wb among four edges of C^{00} that are not part of the matching M and are not incident with v. Observe that only one configuration of two pairs of adjacent vertices $w^j b^j$ and $w_i b_i$ in Q_3^{01} up to isomorphism is possible so that there is no Hamiltonian path $S[w^j, b^j]$ in $Q_3^{01} - \{w_i, b_i\}$. Thus, we choose wb such that this configuration is avoided. Then, the desired v-starting Hamiltonian cycle C^0 of Q_{n-1}^0 is

$$C^0 = (v, R_1, S, R_2, b_i, w_i).$$

Subcase 3.2.3: $n > 5$. We find a Hamiltonian path $S[w^j, b^j]$ in $Q_{n-2}^{01} - \{w_i, b_i\}$ by Proposition 4 and the desired v-starting Hamiltonian cycle C^0 of Q_{n-1}^0 is

$$C^0 = (v, R_1, S, R_2, b_i, w_i).$$

This establishes Subcase 3.2.

Finally, it remains to find a Hamiltonian path $P_k[w_k, b_k]$ of Q_n that is mutually independent with already constructed Hamiltonian paths $P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_n$ of Q_n. So, let P_k be the Hamiltonian path of Q_n induced by Hamiltonian cycle $W(C^0, k)$ of Q_n. The Hamiltonian path P_r and P_k are independent for every $r \in [n] \setminus \{k\}$ by Proposition 1, as P_r are Hamiltonian paths induced by $Z(C^0, k)$ and b_k, w_k and w_r, b_r are consecutive pairs of vertices on $Z(C^0, k)$. □

4 Independent Hamiltonian cycles in faulty Q_n

The following lemma is used as a base of induction in the proof of Theorem 15.

Lemma 14. *Let $F \subseteq E(Q_4)$, $f = |F| \leq 2$, $s \in V(Q_4)$. Then, $Q_4 - F$ has $4 - f$ mutually independent s-starting Hamiltonian cycles.*

Proof. Let $s = \mathbf{0}$ be the starting vertex. We distinguish three cases regarding the number of faulty edges f.

Case 1: $F = \emptyset$. It holds by Theorem 7.

Case 2: $F = \{e\}$. The proof of this case is straightforward. For a given vertex $s = \mathbf{0}$ and any faulty edge e, we show that there exist three s-starting mutually independent Hamiltonian cycles. Automorphisms which preserve the vertex s, are called *s-preserving*. They can be presented as permutations between dimensions. Clearly, s-preserving automorphisms preserve distances to s. Furthermore, note that for every two edges e, g with the same distance to s there exists an s-preserving automorphism that maps e to g. Observe on Figure 4 that the edges sv_9, v_5v_{13}, v_4v_{12}, v_8v_{16} are at distance 0, 1, 2, 3 from the vertex s,

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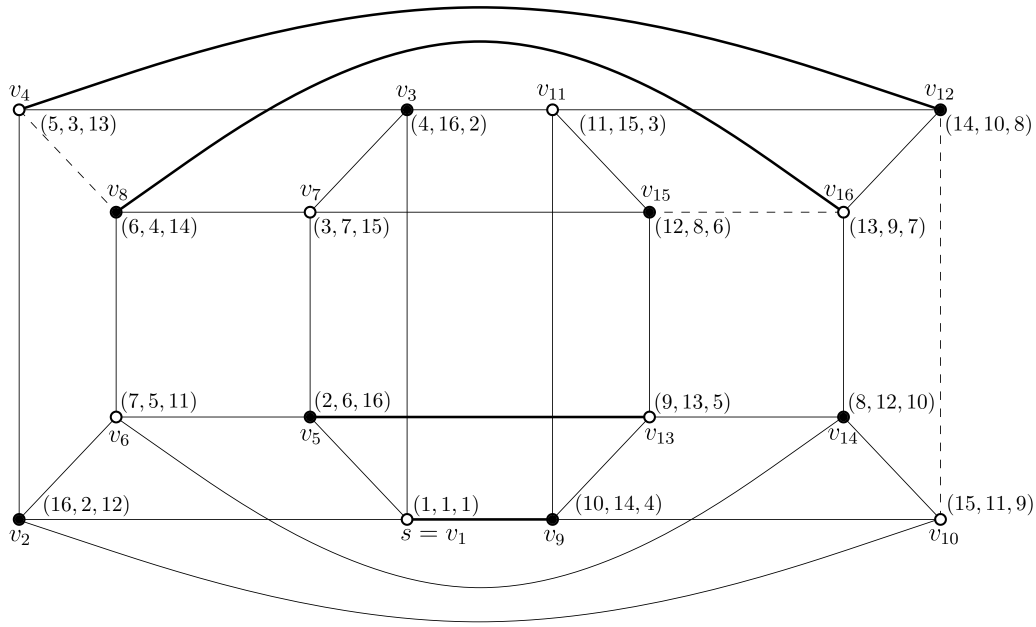

Figure 4: Three mutually independent s-starting Hamiltonian cycles C_1, C_2, C_3 of Q_4. Each vertex u of Q_4 is associated with a triple (k_1, k_2, k_3) which says that u is the k_i-th vertex in the Hamiltonian cycle C_i.

respectively. Thus, there exists an s-preserving automorphism of Q_4 such that the faulty edge e is mapped to one of the these edges. After applying such automorphism in Q_4, the s-starting mutually independent cycles are

$$\begin{aligned} C_1 &= (s, v_5, v_7, v_3, v_4, v_8, v_6, v_{14}, v_{13}, v_9, v_{11}, v_{15}, v_{16}, v_{12}, v_{10}, v_2), \\ C_2 &= (s, v_2, v_4, v_8, v_6, v_5, v_7, v_{15}, v_{16}, v_{12}, v_{10}, v_{14}, v_{13}, v_9, v_{11}, v_3), \\ C_3 &= (s, v_3, v_{11}, v_9, v_{13}, v_{15}, v_{16}, v_{12}, v_{10}, v_{14}, v_6, v_2, v_4, v_8, v_7, v_5). \end{aligned} \quad (1)$$

Note that they are all avoiding the edges sv_9, v_5v_{13}, v_4v_{12}, v_8v_{16}. For an illustration see Figure 4.

Case 3: $F = \{e_1, e_2\}$. First consider the following remark for Q_3. There is a Hamiltonian cycle that contains the first edge and avoids the second edge for any two edges of Q_3 by Proposition 2. Furthermore, Q_3 has two independent Hamiltonian cycles

$$\begin{aligned} C_1 &= (s, x_1, y_1, x_2, y_3, t, y_2, x_3), \\ C_2 &= (s, x_3, y_3, t, y_2, x_1, y_1, x_2) \end{aligned}$$

as on Figure 5 and they are unique up to isomorphism. Notice that the edge y_1x_2 has the same direction on both cycles. By some s-preserving automorphism of Q_3, the edge y_1x_2 can move to any y_ix_j edge for $i, j = 1, 2, 3$. Similarly, y_3t can move to y_1t or y_2t by some s-preserving automorphism of Q_3.

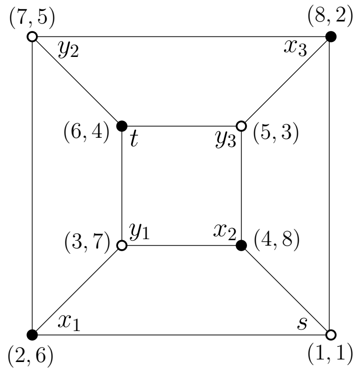

Figure 5: Two independent s-starting Hamiltonian cycles $C_1 = (s, x_1, y_1, x_2, y_3, t, y_2, x_3)$ and $C_2 = (s, x_3, y_3, t, y_2, x_1, y_1, x_2)$ of Q_3^0. Each vertex u of Q_3^0 is associated with a tuple (k_1, k_2) which says that u is the k_1-th vertex in C_1 and k_2-th vertex in C_2.

We split Q_4 along some direction d into Q_3^0 and Q_3^1. We assume that $s \in V(Q_3^0)$ and vertices of Q_3^0 are denoted as in Figure 5. Now, we distinguish the following cases regarding the position of e_1 and e_2:

Subcase 3.1: *Both e_1, e_2 are incident with s.* Then, we may assume that $e_1 = sx_1$ and $e_2 = sx_2$. In Q_3^1 we find Hamiltonian paths $P[s^d, x_1^d]$ and $R[x_2^d, s^d]$. Observe that $H_1 = (s, P, C_1 \setminus \{s\})$ and $H_2 = (C_2 \setminus \{sx_2\}, R)$ are s-starting Hamiltonian cycles of Q_4. Furthermore, all except the 9-th vertices of H_1, H_2 are in distinct subcubes C_3^0, C_3^1 and the 9-th vertices of H_1, H_2 are x_2^d, x_1^d, respectively. Hence H_1, H_2 are independent.

For the purpose of clarity in the following cases we denote $e_1 = a_1b_1$ and $e_2 = a_2b_2$.

Subcase 3.2: $e_1, e_2 \in E(Q_3^1)$. If both e_1 and e_2 are incident with s^d, then we may assume $e_1 = s^d x_1^d$ and $e_2 = s^d x_2^d$. In Q_3^1 we find Hamiltonian paths $P[s^d, x_1^d]$ $R[x_2^d, s^d]$ which avoids e_1 and e_2 by Proposition 2. Then, we can argue as in the previous case that $H_1 = (s, P, C_1 \setminus \{s\})$ and $H_2 = (C_2 \setminus \{sx_2\}, R)$ are independent s-starting Hamiltonian cycles of Q_4.

So, we can assume that e_2 is not incident with s^d and hence, we can assume that $e_2 = y_1^d x_2^d$ or $e_2 = t^d y_3^d$. Let H be a Hamiltonian cycle in Q_3^1 that contains e_2 and avoids e_1. Then, $(P_1, H \setminus \{e_2\}, P_2)$ and $(R_1, H \setminus \{e_2\}, R_2)$ are independent s-starting Hamiltonian cycles in Q_4, where $P_1[s, a_2^d]$, $P_2[b_2^d, x_3]$ are subpaths of C_1 and $R_1[s, a_2^d]$, $R_2[b_2^d, x_2]$ are subpaths of C_2.

Subcase 3.3: *Either e_1 or e_2 is incident with s.* We can assume e_1 is incident with s. Let d be the direction of e_1, then e_2 can be in $Q_3^{d;0}$, $Q_3^{d;1}$ or it can be of direction d. If e_2 is in Q_3^0, then we can assume $e_2 = y_1x_2$ or $e_2 = ty_3$. In Q_3^1 we take a Hamiltonian cycle H that contains e_2^d. Then, observe that $(P_1, H \setminus \{e_2^d\}, P_2)$ and $(R_1, H \setminus \{e_2^d\}, R_2)$ are independent s-starting Hamiltonian cycles, where $P_1[s, a_2]$, $P_2[b_2, x_3]$ are subpaths of C_1 and $R_1[s, a_2]$, $R_2[b_2, x_2]$ are subpaths of C_2. If $e_2 \in E(Q_3^1)$, we take a Hamiltonian cycle H of Q_3^1 that avoids e_2 and contains an edge $(y_1x_2)^d$, or $(ty_3)^d$. Then, $(P_1, H \setminus \{(ty_3)^d\}, P_2)$

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and $(R_1, H \setminus \{(ty_3)^d\}, R_2)$ are independent s-starting Hamiltonian cycles, where $P_1[s,t]$, $P_2[y_3, x_3]$ are subpaths of C_1 and $R_1[s,t]$, $R_2[y_3, x_2]$ are subpaths of C_2. Finally, if e_2 is of direction d, then y_1x_2 or ty_3 is not incident with e_2. Let us assume ty_3 is not incident with e_2. We take a Hamiltonian cycle H that contains $(ty_3)^d$ in Q_3^1. Then, $(P_1, H \setminus \{(ty_3)^d\}, P_2)$ and $(R_1, H \setminus \{(ty_3)^d\}, R_2)$ are independent s-starting Hamiltonian cycles, where $P_1[s,t]$, $P_2[y_3, x_3]$ are subpaths of C_1, $R_1[s,t]$ and $R_2[y_3, x_2]$ are subpaths of C_2.

Subcase 3.4: Finally, excluding the previous cases, the direction d keeps e_1 in Q_3^0 and e_2 in Q_3^1. As e_1 is not incident with s, we can assume $e_1 = y_1x_2$ or $e_1 = ty_3$. Again, we take a Hamiltonian cycle H in Q_3^1 that contains e_1^d. We may assume H avoids e_2 unless $e_1^d = e_2$. Now $(P_1, H \setminus \{e_1^d\}, P_2)$ and $(R_1, H \setminus \{e_1^d\}, R_2)$ are independent s-starting Hamiltonian cycles, where $P_1[s, a_1]$, $P_2[b_1, x_3]$ are subpaths of C_1 and $R_1[s, a_1]$, $R_2[b_1, x_2]$ are subpaths of C_2. □

The following theorem improves Theorem 10 by one additional Hamiltonian cycle. For simplicity, let us denote $\mathbf{0} = \{0\}^n$ and $\mathbf{1} = \{1\}^n$ in Q_n.

Theorem 15. *Let $F \subseteq E(Q_n)$, $n \geq 4$, $f = |F| \leq n-2$, and $s \in V(Q_n)$. Then, $Q_n - F$ has $n - f$ mutually independent s-starting Hamiltonian cycles.*

Proof. If Q_n has no faulty edges, i.e. $f = 0$, then Q_n has n mutually independent s-starting Hamiltonian cycles by Theorem 7. So, we assume $f \geq 1$.

We proceed by induction on n. For $n = 4$ the statement holds by Lemma 14. Let us assume that the statement holds for $n-1$, and we will prove it for $n \geq 5$. By symmetry, we may assume $s = \mathbf{0} \in V(Q_n)$. Furthermore, let $D_F = \{d \in [n]; \exists vv^d \in F\}$ be the set of directions of faulty edges in Q_n. In the following we need one additional definition. Assume that $C_1, C_2, \ldots, C_{n-f}$ are mutually independent $v_{i,1}$-starting Hamiltonian cycles in Q_m for $n - f \leq m < n$ and $C_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,2^m})$. Then, for $u = (u_1, u_2, \ldots, u_{n-m}) \in V(Q_{n-m})$ let $C_1^u, C_2^u, \ldots, C_{n-f}^u$ be the Hamiltonian cycles in $Q_m^{(d_1, d_2, \ldots, d_{n-m});u}$, where $d_1 < \cdots < d_{n-m}$ and $d_1, \ldots, d_{n-m} \in D_F$. Let us denote

$$S_k^u = \{(v_{i,k}, u) \in V(Q_m^u); i \in [n-f]\};$$

that is, S_k^u is the set of k-th vertices of $C_1^u, C_2^u, \ldots, C_{n-f}^u$.

First, we consider the case of one faulty edge. See Figure 6(a) for an illustration. We split Q_n along the direction d of the faulty edge into subcubes Q_{n-1}^0, Q_{n-1}^1. By induction, there are $n-1$ mutually independent s-starting Hamiltonian cycles $C_1^0, C_2^0, \ldots, C_{n-1}^0$ in $Q_{n-1}^0 - F$. As $2^{n-1} - 2 - 2(n-1) > 0$ for $n \geq 5$, we can find an integer $1 < k < 2^{n-1}$ such that none of the vertices of $S_k^0 \cup S_{k+1}^0$ is incident with the faulty edge.

We map the vertices of S_k^0, S_{k+1}^0 along the direction d into Q_{n-1}^1 and obtain S_k^1, S_{k+1}^1; which are sets of distinct $n-1$ pairs of adjacent vertices $v_{i,k}^d$, $v_{i,k+1}^d$ in Q_{n-1}^1. In Q_{n-1}^1 there are no faulty edges, so by Theorem 13, in $Q_{n-1}^1 - F$ there are $n-1$ mutually independent Hamiltonian paths

$$U_1[v_{1,k}^d, v_{1,k+1}^d], U_2[v_{2,k}^d, v_{2,k+1}^d], \ldots, U_{n-1}[v_{n-1,k}^d, v_{n-1,k+1}^d].$$

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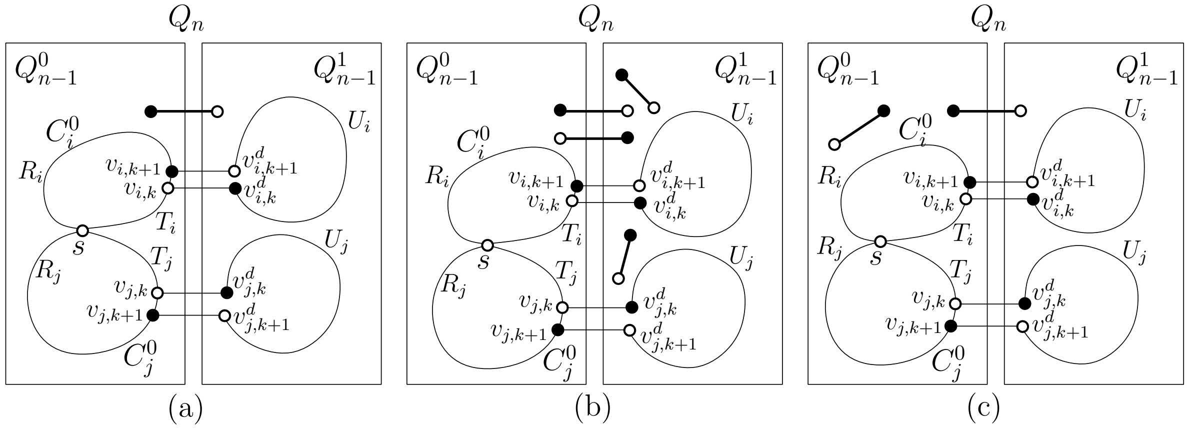

Figure 6: The construction of a set of s-starting mutually independent Hamiltonian cycles in Q_n with: (a) one faulty edge, (b) at least two faulty edges of the same direction d, (c) a faulty edge of direction d and at least one faulty edge in $Q^{d;0}_{n-1}$.

Then, for every $i \in [n-f]$,

$$C_i = (T_i, U_i, R_i)$$

is an s-starting Hamiltonian cycle in Q_n where $T_i[s, v_{i,k}], R_i[v_{i,k+1}, v_{i,2^{n-1}}]$ are subpaths of the cycle C^0_i. Moreover, the cycles $C_1, C_2, \dots, C_{n-1}$ are mutually independent.

Next, if there are two or more faulty edges (i.e. $f \geq 2$), we distinguish three cases.

Case 1: *F is not rigid.* Then, there exists a direction $d \in D_F$ containing at least two faulty edges. We split Q_n along the direction d into Q^0_{n-1}, Q^1_{n-1}. Let f_2 be the number of faulty edges of direction d, and let f_0, f_1 be the number of faulty edges in Q^0_{n-1}, Q^1_{n-1}, respectively; so $f_0 + f_1 + f_2 = f$. By induction, we can find $n-1-f_0$ mutually independent Hamiltonian cycles $C^0_1, C^0_2, \dots, C^0_{n-1-f_0}$ in $Q^0_{n-1} - F$. We take the first $n-f$ cycles $C^0_1, C^0_2, \dots, C^0_{n-f}$. We choose k such that $1 < k < 2^{n-1}$ and none of the vertices of S^0_k, S^0_{k+1} is incident with any faulty edge of direction d. Such k exists as $2^{n-1} - 2 - 2f_2(n-f) > 0$ for all $n \geq 5$.

We map the vertices of S^0_k, S^0_{k+1} along the direction d into Q^1_{n-1} and we obtain S^1_k, S^1_{k+1}; which are sets of $n-f$ pairs of adjacent vertices $v^d_{i,k}$, $v^d_{i,k+1}$ in Q^1_{n-1}. Since $n-f \leq n-2-f_1$, there exist $n-f$ mutually independent Hamiltonian paths

$$U_1[v^d_{1,k}, v^d_{1,k+1}], U_2[v^d_{2,k}, v^d_{2,k+1}], \dots, U_{n-f}[v^d_{n-f,k}, v^d_{n-f,k+1}]$$

of $Q^1_{n-1} - F$ by Lemma 9. Then, for every $i \in [n-f]$,

$$C_i = (T_i, U_i, R_i)$$

is an s-starting Hamiltonian cycle in $Q_n - F$ where $T_i[s, v_{i,k}], R_i[v_{i,k+1}, v_{i,2^{n-1}}]$ are the subpaths of the cycle C^0_i. Moreover, the cycles $C_1, C_2, \dots, C_{n-f}$ are mutually independent. See Figure 6(b) for an illustration.

Case 2: *F is rigid and there exists a direction $d \in D_F$ such that the subcube $Q^{d;0}_{n-1}$ contains at least one faulty edge.* We split Q_n along the direction d into Q^0_{n-1}, Q^1_{n-1}. Let f_0, f_1 be the number of faulty edges in Q^0_{n-1}, Q^1_{n-1}, respectively; so $0 < f_0 < f$ and $f_0 + f_1 + 1 = f$. We proceed similarly as in Case 1. By induction, there are $n - 1 - f_0$ mutually independent Hamiltonian cycles $C^0_1, C^0_2, \ldots, C^0_{n-1-f_0}$ in $Q^0_{n-1} - F$. We take the first $n - f$ cycles $C^0_1, C^0_2, \ldots, C^0_{n-f}$ and choose k such that $1 < k < 2^{n-1}$ and none of the vertices of $S^0_k \cup S^0_{k+1}$ is incident with the faulty edge of direction d. We always find such k as $2^{n-1} - 2 - 2(n - f) > 0$ for $n \geq 5$.

We map the vertices of S^0_k, S^0_{k+1} along the direction d into Q^1_{n-1} and we obtain S^1_k, S^1_{k+1}; which are sets of $n-f$ pairs of adjacent vertices $v^d_{i,k}$, $v^d_{i,k+1}$ in Q^1_{n-1}. Since $n-f \leq n-2-f_1$, we can find $n - f$ mutually independent Hamiltonian paths

$$U_1[v^d_{1,k}, v^d_{1,k+1}], U_2[v^d_{2,k}, v^d_{2,k+1}], \ldots, U_{n-f}[v^d_{n-f,k}, v^d_{n-f,k+1}]$$

of $Q^1_{n-1} - F$ by Lemma 9. Then, for every $i \in [n - f]$,

$$C_i = (T_i, U_i, R_i),$$

is an s-starting Hamiltonian cycle in $Q_n - F$ where $T_i[s, v_{i,k}]$, $R_i[v_{i,k+1}, v_{i,2^{n-1}}]$ are subpaths of the cycle C^0_i. Moreover, the cycles $C_1, C_2, \ldots, C_{n-f}$ are mutually independent. See Figure 6(c) for an illustration.

Case 3: *F is rigid and for every $d \in D_F$, the subcube $Q^{d;0}_{n-1}$ has no faulty edge.* We can consider Q_n as a Cartesian product $Q_n = Q_{n-f+1} \,\Box\, Q_{f-1}$ such that the coordinates of Q_{f-1} are obtained by projection of the coordinates of Q_n on $D_F \setminus \{z\}$ for some $z \in D_F$. Let e_z denote the faulty edge of direction z. Let us define $Z_F = (d_1, d_2, \ldots, d_{f-1})$ for $d_1, \ldots, d_{f-1} \in D_F \setminus \{z\}$ and $d_1 < \cdots < d_{f-1}$. For the purpose of clarity let us denote $r = 2^{f-1}$ and $q = 2^{n-f+1}$. Furthermore, let $H = (u_1, u_2, \ldots, u_r)$ be an arbitrary Hamiltonian cycle of Q_{f-1} such that $u_1 = \mathbf{0}$. Let t_j denote the direction of the edge $u_j u_{j+1}$. Recall that $Q^{Z_F;u_j}_{n-f+1}$ are subcubes of Q_n for every $j \in [r]$ and $s \in V(Q^{\mathbf{0}}_{n-f+1})$. Since there exists no direction $d \in D_F$ such that $Q^{d;0}_{n-1}$ has a faulty edge, one faulty edge is in $Q^{\mathbf{1}}_{n-f+1}$ and all the others are incident with precisely one vertex from $Q^{\mathbf{1}}_{n-f+1}$. By Theorem 7, we can find $n-f$ mutually independent s-starting Hamiltonian cycles $C^{\mathbf{0}}_1, C^{\mathbf{0}}_2, \ldots, C^{\mathbf{0}}_{n-f}$ in $Q^{\mathbf{0}}_{n-f+1}$. Let $C^{\mathbf{0}}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,q})$.

Regarding the number of faulty edges, we distinguish two cases.

Subcase 3.1: $f \geq 3$. See Figure 7 for an illustration of case, when $f = 3$. Since $f \geq 3$, the vertex u_2 is never antipodal to the vertex $u_1 = \mathbf{0}$ in Q_{f-1}, i.e. $u_2 \neq \mathbf{1}$. Hence $Q^{u_2}_{n-f+1}$ has no faulty edge and there is no faulty edge of direction t_1 incident with a vertex from $Q^{u_1}_{n-f+1}$.

We choose k such that $1 < k < q$ and map the vertices $S^{\mathbf{0}}_k$, $S^{\mathbf{0}}_{k+1}$ along the direction t_1 into $Q^{u_2}_{n-f+1}$. We obtain vertices $S^{u_2}_k$, $S^{u_2}_{k+1}$ which are sets of $n-f$ distinct pairs of adjacent vertices $v^{t_1}_{i,k}$, $v^{t_1}_{i,k+1}$ in $Q^{u_2}_{n-f+1}$. The subcube $Q^{u_2}_{n-f+1}$ is of dimension $n-f+1$ and has a set of $n - f$ edges $N = \{v^{t_1}_{i,k} v^{t_1}_{i,k+1}; i \in [n - f]\}$. We extend N into the perfect matching M of $Q^{u_2}_{n-f+1}$ and find a Hamiltonian cycle $G_2 = (w_1, w_2, \ldots, w_q)$ containing the edges of M by Theorem 5. Note that the edges of N have the same parity on G_2 as they are included in the perfect matching M.

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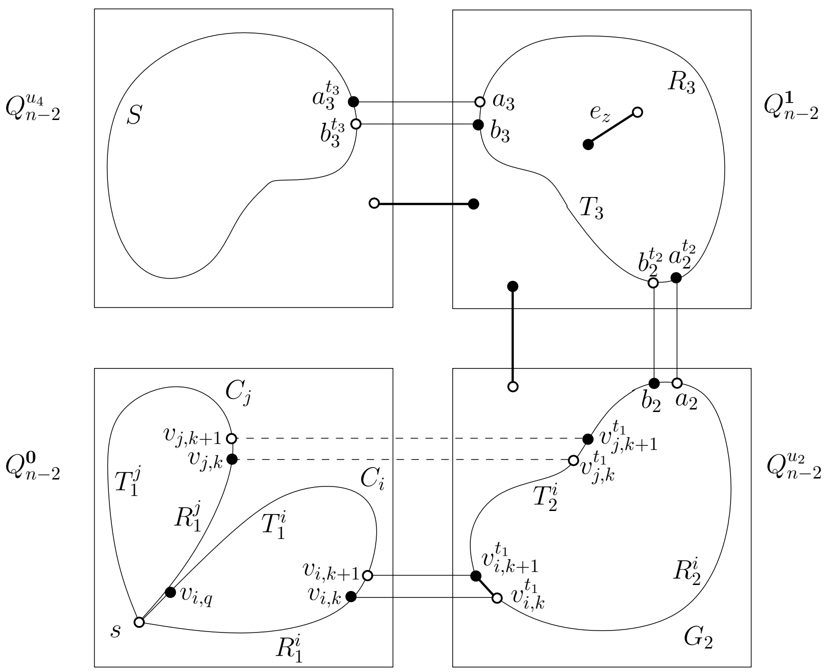

Figure 7: The construction of $n-3$ mutually independent Hamiltonian cycles in Q_n for $n \geq 5$, when the faulty edges are rigid and for every direction, $d \in D_F$, the subcube $Q^{d;0}_{n-1}$ has no faulty edge (The example of Subcase 3.1).

Now, we choose an edge a_2b_2 on G_2 such that a_2b_2 is not incident with the faulty edge of direction t_2 (if such faulty edge exists) and distinct from N. Note that we can always choose such a_2b_2 as $n-f+2 < 2^n$ for $f \geq 3$ and $n \geq 5$. Let us assume that a_2b_2 and $v^{t_1}_{i,k}v^{t_1}_{i,k+1}$ have different parity on G_2. We map a_2 and b_2 along the direction t_2 into $Q^{u_3}_{n-f+1}$. By Proposition 2 we find a Hamiltonian path $G_3[a_2^{t_2}, b_2^{t_2}]$ in $Q^{u_3}_{n-f+1}$ which avoids e_z (if $e_z \in E(Q^{u_3}_{n-f+1})$). We proceed similarly for every $j = 3, 4, \ldots, r$. We choose consecutive vertices a_j, b_j on G_j such that a_j, b_j, $a^{t_{j-1}}_{j-1}$, $b^{t_{j-1}}_{j-1}$ are distinct. We map a_j and b_j along the direction t_j into $Q^{u_{j+1}}_{n-f+1}$ and by Proposition 2 we find a Hamiltonian path $G_{j+1}[a^{t_{j-1}}_{j-1}, b^{t_{j-1}}_{j-1}]$ in $Q^{u_{j+1}}_{n-f+1}$ that avoids e_z (if $e_z \in E(Q^{u_{j+1}}_{n-f+1})$). Then, for all $i \in [n-f]$

$$C_i = (R^i_1, R^i_2, R_3, \ldots, R_{r-1}, S, T_{r-1}, \ldots, T_3, T^i_2, T^i_1)$$

are mutually independent Hamiltonian cycles, where $R^i_1[s, v_{i,k}]$, $T^i_1[v_{i,k+1}, v_{i,q}]$ are subpaths of $C^{\mathbf{0}}_i$, $R^i_2[v^{t_1}_{i,k}, a_2]$, $T^i_2[b_2, v^{t_1}_{i,k+1}]$ are subpaths of G_2, $S[a^{t_{r-1}}_{r-1}, b^{t_{r-1}}_{r-1}]$ is a subpath of G_r and $R_j[a^{t_{j-1}}_{j-1}, a_j]$, $T_j[b_j, b^{t_{j-1}}_{j-1}]$ are subpaths of G_j for every $j = 3, 4, \ldots, r-1$.

Subcase 3.2: $f = 2$. We further distinguish this case regarding the dimension of the hypercube Q_n.

Subcase 3.2.1: $n \geq 6$. Let d_1, d_2 be the directions of faulty edges e_1, e_2, respectively and let us denote $q = 2^{n-2}$. We split Q_n along d_1 and d_2 into $Q^{(d_1,d_2);00}_{n-2}$, $Q^{(d_1,d_2);10}_{n-2}$,

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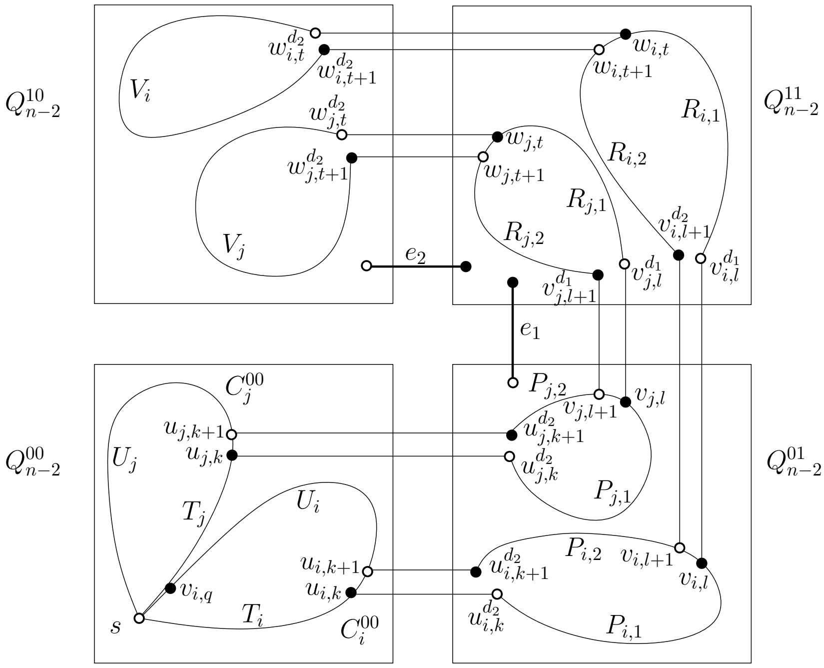

Figure 8: The construction of $n-2$ mutually independent Hamiltonian cycles in Q_n for $n \geq 6$, when the faulty edges f_1, f_2 are rigid and for every direction, $d \in \{d_1, d_2\}$, the subcube $Q^{d;0}_{n-1}$ has no faulty edge.

$Q^{(d_1,d_2);11}_{n-2}$, $Q^{(d_1,d_2);01}_{n-2}$. We find $n-2$ mutually independent s-starting Hamiltonian cycles $C^{00}_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,q})$ in Q^{00}_{n-2} by Theorem 7. See Figure 8 for an illustration. We choose k such that $1 < k < q$ and map the vertices of S^{00}_k, S^{00}_{k+1} along d_2 into Q^{01}_{n-2}. We obtain S^{01}_k, S^{01}_{k+1}; which are sets of $n-2$ pairs of adjacent vertices $u^{d_2}_{i,k}$, $u^{d_2}_{i,k+1}$ in Q^{01}_{n-2}. We can find $n-2$ mutually independent Hamiltonian paths

$$P_1[u^{d_2}_{1,k}, u^{d_2}_{1,k+1}], \ldots, P_{n-2}[u^{d_2}_{n-2,k}, u^{d_2}_{n-2,k+1}]$$

of Q^{01}_{n-2} by Theorem 13. Then, $C^{01}_i = P_i \cup \{u^{d_2}_{i,k} u^{d_2}_{i,k+1}\}$ is a Hamiltonian cycle of Q^{01}_{n-2} for every $i \in [n-2]$. Let us denote $C^{01}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,q})$. We choose l such that $1 \leq l < q$ and none of the vertices $S^{01}_l \cup S^{01}_{l+1}$ is incident with the faulty edge e_1. We can always find such l as $2^{n-2} - 1 - 2(n-2) > 0$ for $n \geq 6$. We map the vertices of S^{01}_l, S^{01}_{l+1} along d_1 into Q^{11}_{n-2} and obtain S^{11}_l, S^{11}_{l+1}; which are sets of $n-2$ pairs of adjacent vertices $v^{d_1}_{i,l}$, $v^{d_1}_{i,l+1}$ in Q^{11}_{n-2}. We can find $n-2$ mutually independent Hamiltonian paths

$$R_1[v^{d_1}_{1,l}, v^{d_1}_{1,l+1}], \ldots, R_{n-2}[v^{d_1}_{n-2,l}, v^{d_1}_{n-2,l+1}]$$

of Q^{11}_{n-2} by Theorem 13. Then, $C^{11}_i = R_i \cup \{v^{d_1}_{i,l} v^{d_1}_{i,l+1}\}$ is a Hamiltonian cycle of Q^{11}_{n-2} for every $i \in [n-2]$. Let us denote $C^{11}_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,q})$. We choose t such that $1 \leq t < q$ and and none of the vertices $S^{11}_t \cup S^{11}_{t+1}$ is incident with the faulty edge e_2. We can always

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find such t as $2^{n-2} - 1 - 2(n-2) > 0$ for $n \geq 6$. We map the vertices of S_t^{11}, S_{t+1}^{11} along d_2 into Q_{n-2}^{10} and obtain S_t^{10}, S_{t+1}^{10}; which are sets of $n-2$ pairs of adjacent vertices $w_{i,t}^{d_2}$, $w_{i,t+1}^{d_2}$ in Q_{n-2}^{10}. We can find $n-2$ mutually independent Hamiltonian paths

$$V_1[w_{1,t}^{d_2}, w_{1,t+1}^{d_2}], \ldots, V_{n-2}[w_{n-2,t}^{d_2}, w_{n-2,t+1}^{d_2}]$$

of Q_{n-2}^{10} by Theorem 13. Then, for every $i \in [n-2]$,

$$C_i = (T_i, P_{i,1}, R_{i,1}, V_i, R_{i,2}, P_{i,2}, U_i)$$

is an s-starting Hamiltonian cycle in $Q_n - \{e_1, e_2\}$ where $T_i[s, u_{i,k}]$, $U_i[u_{i,k+1}, u_{i,q}]$ are subpaths of C_i^{00}, $P_{i,1}[v_{i,1}, v_{i,l}]$, $P_{i,2}[v_{i,l+1}, v_{i,q}]$ are subpaths of P_i and $R_{i,1}[w_{i,1}, w_{i,t}]$, $R_{i,2}[w_{i,t+1}, w_{i,q}]$ are subpaths of R_i. Moreover, the cycles $C_1, C_2, \ldots, C_{n-2}$ are mutually independent.

Subcase 3.2.2: $n = 5$. Let us denote $F = \{e_1, e_2\}$. We split Q_5 along the direction d of the faulty edge e_1. Then, Q_4^0 contains the vertex s and Q_4^1 contains e_2. In Q_4^0 we choose three mutually independent s-starting Hamiltonian cycles C_1, C_2, C_3 defined by (1), see Figure 4. From independent directed edges $v_4v_8, v_{15}v_{16}, v_{12}v_{10}$ of C_1, C_2, C_3, we choose the edge $e = ab$ such that e is not incident with e_1. In Q_4^1 we find a hamiltonian path $P[a^d, b^d]$ which avoids e_2. We obtain three mutually independent s-starting fault-free Hamiltonian cycles

$$\begin{aligned} C_1 &= (R_1, P, P_1), \\ C_2 &= (R_2, P, P_2), \\ C_3 &= (R_3, P, P_3), \end{aligned}$$

where $R_1[s, a]$, $P_1[b, v_2]$ are subpaths of C_1, $R_2[s, a]$, $P_2[b, v_3]$ are subpaths of C_2 and $R_3[s, a]$, $P_3[b, v_5]$ are subpaths of C_3.

□

5 Conclusion

In this paper we study the problem of mutually independent Hamiltonian paths and s-starting Hamiltonian cycles of n-dimensional hypercube Q_n. We prove that there are $k \leq 2^{n-1}$ mutually independent Hamiltonian paths $P_1[w_1, b_1], P_2[w_2, b_2], \ldots, P_k[w_k, b_k]$ for a matching $M = \{w_1b_1, w_2b_2, \ldots, w_kb_k\} \subseteq E(Q_n)$ if M is extendable to a perfect matching. We prove that there are n mutually independent Hamiltonian paths $P_i[w_i, b_i]$ for any matching $M = \{w_1b_1, w_2b_2, \ldots, w_nb_n\} \subseteq E(Q_n)$ in Q_n which improves previously known result by one additional Hamiltonian path. We also prove that there are $n - f$ mutually independent s-starting Hamiltonian cycles in $Q_n - F$, where F is a set of $f \leq n-2$ arbitrary faulty edges and s is an arbitrary vertex. This improves previously known result by one additional s-starting Hamiltonian cycle. Moreover, it is the optimal result as faulty edges may be all incident with the vertex s.

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