IMFM
Institute of Mathematics, Physics and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series
Vol. 48 (2010), 1134
ISSN 2232-2094

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Ljubljana, December 14, 2010

Preprint series, IMFM, ISSN 2232-2094, no. 1134, December 14, 2010

Retracts of products of chordal graphs

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Abstract

In this paper, we characterize the graphs G that are retracts of Cartesian products of chordal graphs. We show that they are exactly the weakly modular graphs that do not contain $K_{2,3}$, k-wheels W_k, and k-wheels minus one spoke W_k^- ($k \geq 4$) as induced subgraphs. We also show that these graphs G are exactly the cage-amalgamation graphs introduced by Brešar and Tepeh Horvat (2009); this solves the open question raised by these authors. Finally, we prove that replacing all products of cliques of G by products of "solid" simplices, we obtain a polyhedral cell complex which, endowed with an intrinsic Euclidean metric, is a CAT(0) space. This generalizes similar results about median graphs as retracts of hypercubes (products of edges) and median graphs as 1-skeletons of CAT(0) cubical complexes.

1 Introduction

Median graphs constitute one of the most important classes of graphs investigated in metric graph theory and occur in different areas of discrete mathematics, metric geometry, and computer science. Median graphs and related median structures (median algebras and median complexes) have many nice properties and admit numerous characterizations. All median structures are intimately related to hypercubes: median graphs are isometric subgraphs of hypercubes; in fact, by a classical result of Bandelt [1] they are the retracts of hypercubes into which they embed isometrically. It was also shown by Isbell [23] and van de Vel [31] that every finite median graph G can be obtained by successive applications of gated amalgamations from hypercubes, thus showing that the only prime median graph is the two–vertex complete graph K_2 (a graph with at least two vertices is said to be prime if it is neither a Cartesian product nor a gated amalgam of smaller graphs). A related construction of median graphs via convex expansions is given in [25, 26]. Median graphs also have a remarkable algebraic structure, which is induced by the ternary operation on the vertex set that assigns to each triplet of vertices the unique median vertex, and their algebra can be characterized using

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Preprint series, IMFM, ISSN 2232-2094, no. 1134, December 14, 2010

four natural axioms [7, 23] among all discrete ternary algebras. Finally, it was shown in [16, 27] that the cubical complexes derived from median graphs by replacing graphic cubes by solid cubes are exactly the CAT(0) cubical complexes. Thus, due to a result of Gromov [20], median complexes can be characterized as simply connected cubical complexes with triangle-free links of vertices. For more detailed information about median structures, the interested reader can consult the survey [6] and the books [18, 22, 26, 32].

This structure theory of graphs based on two fundamental operations, viz., Cartesian multiplication and gated amalgamation, was further elaborated for more general classes of graphs. Some of the results for median graphs have been extended to quasi-median graphs introduced by Mulder [26] and further studied in [8, 10, 33]: quasi-median graphs are precisely the weakly modular graphs not containing induced $K_{2,3}$ and $K_4 - e$; they can also be characterized as the retracts of Hamming graphs (Cartesian products of complete graphs) and can be obtained from complete graphs by Cartesian products and gated amalgamations. More recently, Bandelt and Chepoi [3, 4, 5] presented a similar decomposition scheme of weakly median graphs and characterized the prime graphs with respect to this decomposition: the hyperoctahedra and their subgraphs, the 5-wheel W_5, and the 2-connected plane triangulations in which all inner vertices have degrees ≥ 6. Using these results and a result of Chastand [13, 14], they further showed that weakly median graphs are the retracts of the Cartesian products of their primes and presented an axiomatic characterization of underlying weakly median algebras. The extensive research on generalizations of median graphs leads to a general framework for the study of classes of graphs, closed for Cartesian products and gated amalgamations, proposed in [9, 13, 14].

In this paper, we continue this line of research and characterize the graphs G which are retracts of Cartesian products of chordal graphs. We show that they are exactly the weakly modular graphs which do not contain $K_{2,3}$, k-wheels W_k, and k-wheels minus one edge W_k^- ($k \geq 4$) as induced subgraphs. We establish that these graphs G are exactly the cage-amalgamation graphs introduced by Brešar and Tepeh Horvat [11], i.e. the graphs which can be obtained via successive gated amalgamations from Cartesian products of chordal graphs; this solves the open question raised in [11]. Finally, we show that replacing all products of cliques of G by products of "solid" simplices, we will obtain a polyhedral cell complex which, endowed with an intrinsic l_2-metric, is a CAT(0) space. This generalizes similar results about median graphs as retracts of hypercubes (products of edges) and median graphs as 1-skeletons of CAT(0) cubical complexes.

2 Preliminaries and results

All graphs $G = (V, E)$ occurring here are undirected, connected, and without loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths, that is, of all vertices (metrically) *between* u and v: $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any pair of its vertices. An induced subgraph H of a graph G is said to be *gated* if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x' (i.e., $x' \in I(x,y)$). The smallest convex (or gated, respectively) subgraph containing a given subgraph S is the *convex hull* (or *gated hull,* respectively) of S. A graph $G=(V,E)$ is *isometrically embeddable* into a graph $H=(W,F)$ if there exists a mapping $\varphi : V \rightarrow W$ such that $d_H(\varphi(u),\varphi(v)) = d_G(u,v)$ for all vertices $u,v \in V$. A *retraction* φ of H is an idempotent nonexpansive mapping of H into itself, that is, $\varphi^2 = \varphi : W \rightarrow W$ with $d(\varphi(x),\varphi(y)) \leq d(x,y)$ for all $x,y \in W$. The subgraph of H induced by the image of H under φ is referred to as a *retract* of H.

A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G. The *Cartesian product* [22] $G = G_1 \Box \ldots \Box G_n$ of n graphs $G_1,\ldots,G_n$ has the n-tuples $(x_1,\ldots,x_n)$ as its vertices (with vertex x_i from G_i) and an edge between two vertices $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ if and only if, for some i, the vertices x_i and y_i are adjacent in G_i, and $x_j = y_j$ for the remaining $j \neq i$. Obviously, $d_G(u,v) = \sum_{i=1}^{n} d_{G_i}(u_i,v_i)$ for any two vertices $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots,v_n)$ of G. In regard to a decomposition scheme involving multiplication and amalgamation, a graph with at least two vertices is said to be *prime* if it is neither a Cartesian product nor a gated amalgam of smaller graphs. For instance, the only prime median graph is the two-vertex complete graph K_2 [23, 31] and the prime quasi-median graphs are exactly the complete graphs [8, 23].

A graph G is *weakly modular* [2, 15] if its distance function d satisfies the following triangle and quadrangle conditions (see Figure 1):

Triangle condition: for any three vertices u,v,w with $1 = d(v,w) < d(u,v) = d(u,w)$ there exists a common neighbor x of v and w such that $d(u,x) = d(u,v) - 1$.

Quadrangle condition: for any four vertices u,v,w,z with $d(v,z) = d(w,z) = 1$ and $2 = d(v,w) \leq d(u,v) = d(u,w) = d(u,z) - 1$, there exists a common neighbor x of v and w such that $d(u,x) = d(u,v) - 1$.

A *weakly median* graph is a weakly modular graph in which the vertex x defined in the triangle and quadrangle conditions is always unique. Equivalently, weakly median graphs can be defined as the weakly modular graphs in which each triplet of vertices has a unique quasi-median. Median graphs are the bipartite weakly median graphs and, equivalently, can be defined as the graphs in which each triplet of vertices u,v,w has a unique median vertex, i.e., $|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$. Bridged graphs constitute another important subclass of weakly modular graphs. Recall that a graph is called *bridged* [17, 29] if it does not contain any isometric cycle of length greater than 3, or alternatively, if the *closed neighborhood* $N[A] = A \cup \{y \in V : \ y \text{ is adjacent to some } x \in A\}$ of every convex set A of G is convex. Chordal graphs constitute the most famous subclass of bridged graphs. A graph is said to be *chordal* if it does not contain induced cycles of length greater than 3. In this paper we will investigate the finite graphs G which are obtained from Cartesian products of chordal graphs

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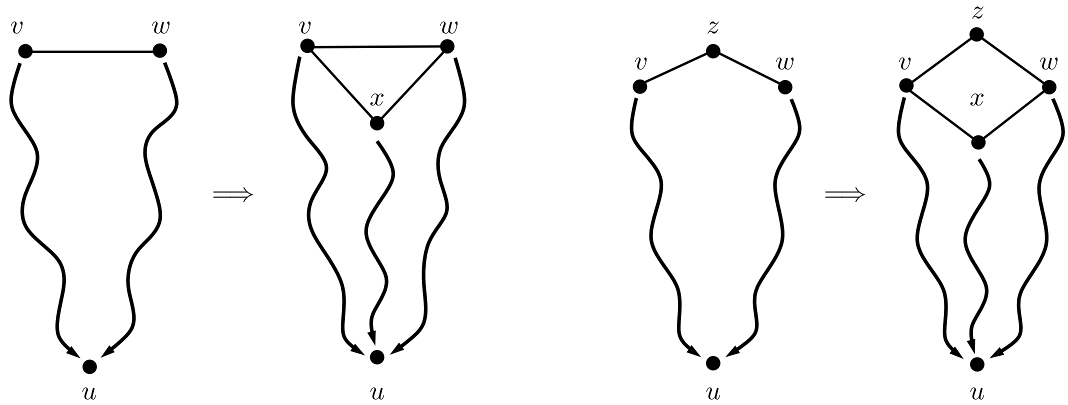

Figure 1: Triangle and quadrangle conditions

via gated amalgamations. These graphs have been introduced by Brešar and Tepeh Horvat [11] and called cage-amalgamation graphs. More precisely, the Cartesian products of chordal graphs were called in [11] *cages*, and the graphs that can be obtained by a sequence of gated amalgamations from cages were called *cage-amalgamation graphs*. It can be easily shown that cage-amalgamation graphs are weakly modular graphs and that they do not contain induced $K_{2,3}$, wheels W_k, and almost-wheels W_k^- (the *wheel* W_k is a graph obtained by connecting a single vertex - the *central vertex* - to all vertices of the k-cycle; the *almost wheel* W_k^- is the graph obtained from W_k by deleting a spoke (i.e., an edge between the central vertex and a vertex of the k-cycle), see Figure 2 for examples). It was conjectured in [11] that in fact this list of forbidden subgraphs completely characterizes the cage-amalgamation graphs. The main result of our paper proves this conjecture:

Theorem 1. *For a finite graph $G = (V, E)$, the following conditions are equivalent:*

(i) G is a retract of the Cartesian product of chordal graphs;

(ii) G is a weakly modular graph not containing induced $K_{2,3}$, wheels W_k, and almost wheels W_k^- for $k \geq 4$;

(iii) G is a cage-amalgamation graph, i.e., it can be obtained by successive applications of gated amalgamations from Cartesian products of 2-connected chordal graphs and K_2's.

The proof of this theorem is provided in the following section. The most difficult part of the proof is the implication (ii)⇒(iii), which we establish in two steps. First, we show that if G is a weakly modular graph not containing induced $K_{2,3}$, wheels W_k, and almost wheels W_k^- for $k \geq 3$, then all its primes are 2-connected chordal graphs or a K_2. In the second part, using the techniques developed in [3], we show that G can be obtained via gated amalgamations from Cartesian products of prime graphs.

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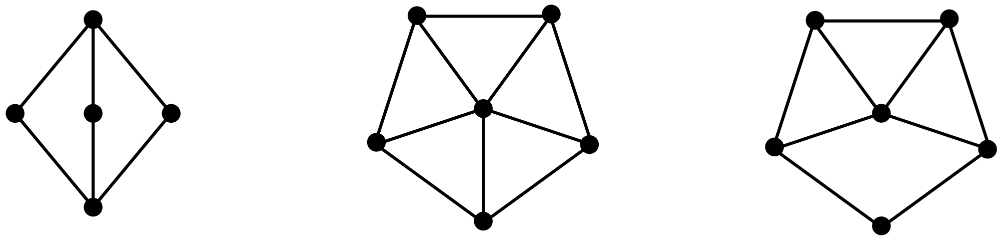

$K_{2,3}$ W_5 W_5^-

Figure 2: The complete bipartite graph $K_{2,3}$, the wheel W_5, and the almost-wheel W_5^-.

Our second result, concerns the geometry of polyhedral complexes derived from cage-amalgamation graphs.

An abstract *simplicial complex* $\mathcal{X}$ is a collection of sets (called *simplices*) such that $\sigma \in \mathcal{X}$ and $\sigma' \subseteq \sigma$ implies $\sigma' \in \mathcal{X}$. A *cubical complex* $\mathcal{C}$ is a set of (graph) cubes of any dimensions which is closed under taking subcubes and nonempty intersections. Simplices or cubes of the respective complex are called *faces*. For a complex $\mathcal{K}$ denote by $V(\mathcal{K})$ and $E(\mathcal{K})$ the *vertex set* and the *edge set* of $\mathcal{K}$, namely, the set of all 0-dimensional and 1-dimensional faces of $\mathcal{K}$. The pair $(V(\mathcal{K}), E(\mathcal{K}))$ is called the *(underlying) graph* or the *1-skeleton* of $\mathcal{K}$ and is denoted by $G(\mathcal{K})$. The *link* of a vertex v in a simplicial complex $\mathcal{X}$, denoted $\text{link}(v)$, is the graph consisting of all edges $e = xy$ such that $v \neq x, y$ and $\{x, y, v\}$ is a simplex of $\mathcal{X}$.

A simplicial complex $\mathcal{X}$ is a *flag complex* (or a *clique complex*) if any set of vertices is included in a face of $\mathcal{X}$ whenever each pair of its vertices is contained in a face of $\mathcal{X}$. (In the theory of hypergraphs this condition is called conformality.) A flag complex can therefore be recovered by its underlying graph $G(\mathcal{X})$: the complete subgraphs of $G(\mathcal{X})$ are exactly the simplices of $\mathcal{X}$. Conversely, for a graph G one can derive a simplicial complex $\mathcal{X}(G)$ by taking all complete subgraphs (simplices) as faces of the complex. Analogously, for a graph G one can also derive a cubical complex $\mathcal{C}(G)$ by taking all induced subhypercubes as faces. If G is a median graph, then $\mathcal{C}(G)$ consists of all hypercubes which are obtained as Cartesian products of the primes graphs (as we noticed above, they are all two-vertex complete graphs K_2). The simplicial complexes arising as clique complexes of bridged graphs were characterized in [16] as simply connected simplicial complexes in which the links of vertices do not contain induced 4- and 5-cycles (these complexes have been rediscovered and investigated by Januszkiewicz and Swiatkowski [24], who called them "systolic complexes" and consider them as simplicial complexes satisfying combinatorial nonpositive curvature property, see the definition below).

In the context of graphs G obtained via Cartesian products and gated amalgamations from prime graphs containing cliques of arbitrary size, it is natural to define on each prime graph G_i its clique complex $\mathcal{X}(G_i)$ and to derive a complex $\mathcal{H}(G)$ by taking all Hamming subgraphs of G (Cartesian products of complete subgraphs of prime graphs) as faces. We call $\mathcal{H}(G)$ the *Hamming complex* of G. If G is a median graph (or more generally, a triangle-free graph), then the Hamming complex of G coincides with the cubical complex defined before. A *geometric realization* $|\mathcal{K}|$ of a simplicial or cubical complex $\mathcal{K}$ is the polyhedral complex obtained by replacing every face σ by a "solid" simplex or "solid" axis-parallel box $|\sigma|$ of the same dimension such that realization commutes with intersection, that is, $|\sigma'|\cap|\sigma''| = |\sigma'\cap\sigma''|$ for any two faces σ' and σ''. Then $|\mathcal{K}| = \bigcup\{|\sigma| : \sigma \in \mathcal{K}\}$. If $\mathcal{H}$ is a Hamming complex, then each face τ of $\mathcal{H}$ is the Cartesian product of simplices $\tau = \sigma_1 \times \cdots \times \sigma_k$. This is consistent with the standard definition of the product of two (or more) polytopes given on pp. 9-10 of the book of Ziegler [34]: given two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, the *product* of P and Q is the set $P \times Q = \{(x, y) : x \in P, y \in Q\}$. $P \times Q$ is a polytope of dimension $\dim(P)+\dim(Q)$, whose nonempty faces are the products of nonempty faces of P and nonempty faces of Q. It is well known (see, for example p.110 of [34]) that the product $|\sigma_1| \times \cdots \times |\sigma_k|$ of solid simplices $|\sigma_1|, \ldots, |\sigma_k|$ is a convex polyhedron $|\tau|$. With some abuse of language, we will call $|\tau|$ a *Hamming prism* or simply a *prism*. The *geometric realization* $|\mathcal{H}|$ of a Hamming complex $\mathcal{H}$ is obtained by replacing each cell $\tau = \sigma_1 \times \cdots \times \sigma_k$ of $\mathcal{H}$ by a "solid" Hamming prism $|\tau| = |\sigma_1| \times \cdots \times |\sigma_k|$.

Any geometric realization $|\mathcal{H}(G)|$ of $\mathcal{H}(G)$ can be endowed with an intrinsic l_2-metric [12], transforming $|\mathcal{H}(G)|$ into a complete geodesic space. Suppose that inside every Hamming prism of $|\mathcal{H}(G)|$ the distance is measured according to the Euclidean l_2-metric. The *intrinsic l_2-metric* d_2 of $|\mathcal{H}(G)|$ is defined by assuming that the distance between two points $x, y \in |\mathcal{H}(G)|$ lying in different Hamming prisms equals to the infimum of the lengths of the paths joining them. Here a *path* in $|\mathcal{H}(G)|$ from x to y is a sequence P of points $x = x_0, x_1 \ldots x_{m-1}, x_m = y$ such that for each $i = 0, \ldots, m-1$ there exists a Hamming cell $|\tau_i|$ containing x_i and x_{i+1}; the *length* of P is $l(P) = \sum_{i=0}^{m-1} d(x_i, x_{i+1})$, where $d(x_i, x_{i+1})$ is computed inside $|\tau_i|$ according to the Euclidean l_2-metric.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three distinct points in X (the vertices of Δ) and a geodesic between each pair of vertices (the sides of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is defined to be a *CAT(0) space* [20] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov:

If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2,$ then $d(x_3, y) \leq d_{\mathbb{E}^2}(x'_3, y')$.

CAT(0) spaces can be characterized in several natural ways (for a full account of this theory consult the book [12]). A geodesic metric space (X, d) is CAT(0) if and only if any two points of this space can be joined by a unique geodesic. CAT(0) is also equivalent to convexity of the function $f : [0, 1] \to X$ given by $f(t) = d(\alpha(t), \beta(t))$, for any geodesics α and β (which is further equivalent to convexity of the neighborhoods of convex sets). This implies that CAT(0) spaces are contractible.

Now, we formulate the second result of this paper:

Theorem 2. *If G is a cage-amalgamation graph, then any geometric realization $|\mathcal{H}(G)|$ of its Hamming complex $\mathcal{H}(G)$ equipped with the intrinsic l_2-metric d_2 is a CAT(0) metric space.*

That cliques complexes of chordal graphs lead to CAT(0) polyhedral complexes was already noticed in [16] and [21]; Gromov called them *tree-like polyhedra*.

3 Proof of Theorem 1

The implication (i)⇒(ii) is obvious: chordal graphs are weakly modular and do not contain induced $K_{2,3}$, wheels W_k, and almost wheels W_k^- ($k \geq 4$). Weakly modular graphs are closed by taking Cartesian products. If a Cartesian product of k graphs $H_1, \ldots, H_k$ contains an induced $K_{2,3}, W_k$, or W_k^-, then necessarily this graph occurs in one of the factors H_i because these graphs cannot be obtained by proper Cartesian products. As a consequence, Cartesian products $H = H_1 \square \cdots \square H_k$ of chordal graphs do not contain induced $K_{2,3}, W_k$, and W_k^-. If G is a retract of $H = H_1 \square \cdots \square H_k$, then G is an isometric subgraph of H and therefore G does not contain induced $K_{2,3}, W_k$, and W_k^- as well. It remains to notice that triangle and quadrangle conditions are preserved by Cartesian products and retractions, thus G is a weakly modular graph, establishing that (i)⇒(ii). The implication (iii)⇒(i) is a particular case of Theorem 1 and Corollary 4 of [4] (the proof of Corollary 4 also follows from a more general result of Chastand [14]). By Theorem 1 of [4] any cage-amalgamation graph G embeds isometrically into the Cartesian product $H = H_1 \square \cdots \square H_k$ of its primes. Corollary 4 of [4] then shows that there exists a retraction from H to G, establishing (iii)⇒(i).

The proof of the implication (ii)⇒(iii) is the main contribution of this section. It employs the fact that each finite chordal graph admits a perfect elimination scheme which can be computed by Maximum Cardinality Search algorithm [19, 28, 30]. Running a modification of MCS on the gated hull of a triangle in a graph G satisfying the condition (ii) of Theorem 1, we show that the level subgraphs returned by MCS are all convex subgraphs of G. This allows us to show that the gated hull of each triangle of G is a 2-connected chordal graph, thus identifying the prime graphs of G. To show that G can be obtained from Cartesian products of 2-connected chordal graphs and edges using successive amalgamations, we adapt the second part of the proof of Theorem 1 of [3].

A *simplicial vertex* of a graph G is a vertex v such that its neighborhood $N(v) = \{u \in V(G) : u \text{ is adjacent to } v\}$ induces a complete subgraph of G. A *Perfect Elimination Ordering* (PEO) of a graph $G = (V, E)$ with n vertices is a total ordering $v_1, \ldots, v_n$ of its vertices such that each v_i is a simplicial vertex in the subgraph G_i induced by the level set $L_i = \{v_1, \ldots, v_i\}$. It is well known (see [19]) that a finite graph G admits a perfect elimination ordering if and only if G is chordal. A PEO of a chordal graph G can be found (in linear time) either using *Lexicographic Breadth-First-Search* (LexBFS) [28] or *Maximum Cardinality Search* (MCS) introduced by Tarjan and Yannakakis [30]. MCS algorithm works as follows: the first vertex is chosen arbitrarily, and the $(i + 1)$-th vertex is the unlabeled vertex that has the largest number of already numbered neighbors, breaking ties arbitrarily. We will denote by $\alpha(v)$ the

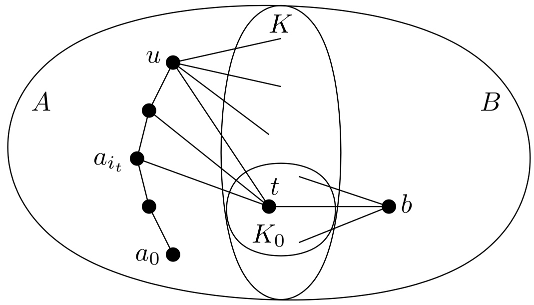

Figure 3: Illustration of the proof of Lemma 2

number of v in a total ordering $v_1, \ldots, v_n$, i.e., if $\alpha(v) = i$, then $v = v_i$. We start with two properties of MCS in chordal graphs.

Lemma 1. *Let G be a chordal graph and α an ordering of vertices produced by MCS. If a vertex z belongs to an induced path between two vertices x, y, then $\alpha(z) < \max\{\alpha(x), \alpha(y)\}$.*

Proof. Assume without loss of generality that $\alpha(x) < \alpha(y)$ and let P be an induced path between x and y. Suppose by way of contradiction that P contains a vertex z such that $\alpha(z) > \alpha(y)$ and suppose without loss of generality that z is the vertex of P with the largest index $i = \alpha(z)$. Then among all vertices of P the vertex z was labeled last. Hence z and its neighbors z', z'' in P all belong to the subgraph G_i. Since z' and z'' are not adjacent, z is not a simplicial vertex of G_i, contradicting the fact that on chordal graphs MCS returns a perfect elimination ordering. $\square$

A *minimal (vertex) separator* of a graph $G = (V, E)$ is a subset of vertices K of G such that the subgraph of G induced by $V - K$ contains at least two connected components A and B, and that K is minimal by inclusion with respect to this separating property. Then K necessarily *separates* any two vertices $x \in A$ and $y \in B$ in the sense that all (x, y)-paths share a vertex with K. It is well-known [19] that any minimal separator K of a chordal graph G induces a complete subgraph of G and, moreover, K separates two vertices x and y such that both x and y are adjacent to all vertices of K.

Lemma 2. *Let K be a minimal separator of a chordal graph G, let A and B be two connected components of $G - K$, and let $u \in A, v \in V$ be two vertices that are adjacent to all vertices of K. Let α be an ordering of vertices produced by MCS. If α labels some vertex of A before any vertex of B is labeled, then α labels u before any vertex of B.*

Proof. Let $a_0 \in A$ be the vertex with the smallest index $\alpha(a_0)$ among all vertices of $A \cup B$. Since A is connected, we can choose $P := (a_0, a_1, \ldots, a_k = u)$ to be a shortest (and therefore induced) path connecting the vertices a_0 and u in A. Suppose by the way of contradiction that there exists $b \in B$ that was labeled before u, i.e. $\alpha(b) < \alpha(u)$, and let b be the first such vertex with respect to α. Denote by $L(x)$ the set of labeled neighbors of a vertex x at the moment of time when b was labeled. Let $K_0 := L(b)$. Since K separates A from $b \in B$, from the choice of b and our assumption we conclude that $K_0 \subseteq K$ (see also Figure 3).

We assert that for each vertex a_i of P, the inequality $\alpha(a_i) < \alpha(b)$ holds. Indeed, let t be an arbitrary vertex in K_0 and let $P_t := (a_0, \ldots, a_k, t)$ be the path from a_0 to t, obtained from P by adding t at the end. Since $\alpha(u) > \alpha(b) > \alpha(t)$ by the assumption and $\alpha(u) > \alpha(a_0)$ from the choice of a_0, by Lemma 1 the path P_t is not induced. Since P is induced, the only possible chords on this path are the chords of the form ta_i, where $0 \leq i < k$. Let i_t be the smallest index such that t and a_{i_t} are adjacent. To avoid induced cycles of length greater than 3 in G, for all j comprised between i_t+1 and k, the vertices t and a_j must be adjacent as well. Since the subpath $(a_0, \ldots, a_{i_t}, t)$ of P_t is induced, by Lemma 1 we infer that all vertices of this path must be labeled either before t or before a_0, but in either case we have $\alpha(a_j) < \alpha(b)$ for all $0 \leq j \leq i_t$ because $\alpha(b) > \max\{\alpha(t), \alpha(a_0)\}$. Set $q = \max\{i_t : t \in K_0\}$. As a result, we obtain the following property for the vertices of P: every vertex $a_j \in \{a_0, \ldots, a_q\}$ was labeled before b, i.e., $\alpha(a_j) < \alpha(b)$. On the other hand, all vertices $a_{q+1}, \ldots, a_k = u$ are adjacent to all vertices of K_0, i.e., $K_0 \subseteq \cap_{j=q+1}^{k} L(a_j)$. We assert that the inclusions $K_0 \subseteq L(a_j)$, $j = q+1, \ldots, k$, are strict. Since $a_q \in L(a_{q+1})$, this inclusion is indeed strict for a_{q+1}. Let $\ell > q+1$ be the smallest index for which $L(a_\ell) = K_0$. Then, as $L(b) = K_0$ is a proper subset of $L(a_{\ell-1})$, MCS must label $a_{\ell-1}$ before b, i.e., $\alpha(a_{\ell-1}) < \alpha(b)$. Hence $a_{\ell-1} \in L(a_\ell)$, a contradiction. This implies, in particular, that the vertices $a_{q+1}, \ldots, a_k$ have been all labeled by MCS before b, i.e., $\alpha(a_j) < \alpha(b)$ for $q < j \leq k$. The claimed assertion is thus proven. Now, since $a_k = u$, this assertion implies that $\alpha(u) < \alpha(b)$, as desired. □

For the remainder of this section, let G be a weakly modular graph that does not contain any of $K_{2,3}$, W_k, and W_k^-, $k \geq 4$, as an induced subgraph. We will show that G can be obtained by a sequence of gated amalgamations from Cartesian products of chordal graphs. We commence by establishing a number of auxiliary results. A subgraph H of G is said to be Δ–*closed* if, for every triangle having two vertices in H, the third vertex belongs to H as well; then the smallest Δ–closed subgraph containing S is the Δ–*closure* of S. In order to check whether a given subgraph of G is convex or gated the following lemma is useful. This essentially coincides with Theorem 7 of [15] and can be proved quite easily by induction.

Lemma 3. *A connected subgraph H of a weakly modular graph G is convex if and only if H is locally convex, i.e., for every pair of nonadjacent vertices u, v of H all common neighbors of u and v belong to H whenever at least one common neighbor does. Moreover, a convex subgraph is gated if and only if it is Δ-closed.*

Now we will prove that the gated hull H of each triangle $T = \{a, b, c\}$ of G is a convex chordal subgraph of G. For this, we perform a (partial) Maximum Cardinality Search α in G starting with $\alpha(a) = 1, \alpha(b) = 2, \alpha(c) = 3$ until the moment when all yet unlabeled vertices have at most one previously labeled neighbor. Denote by H the subgraph of G induced by

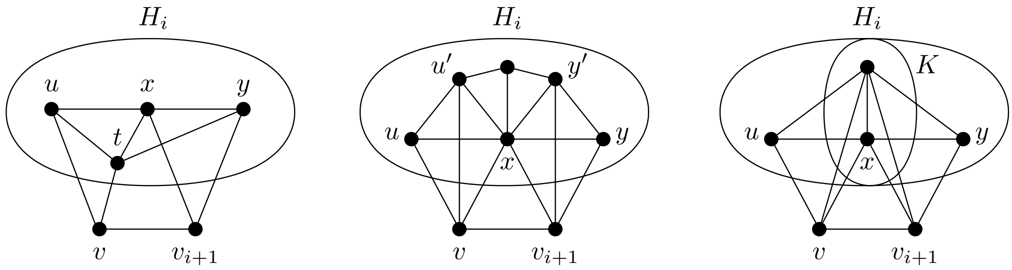

Figure 4: Different cases in the proof of Claim 1

all labeled vertices at the end of the procedure, and let H_i be the subgraph of H induced by the first i labeled vertices.

Proposition 1. *For any i, H_i is a chordal and convex subgraph of G.*

Proof. We proceed by induction on i. Clearly, H_1, H_2, and H_3 are all chordal and convex subgraphs of G. By way of contradiction, assume that for some $i \geq 3$, H_i is convex and chordal but $H_{i+1} = H_i \cup \{v_{i+1}\}$ is not convex. By Lemma 3, H_{i+1} is not locally convex. Then there exists $u \in V(H_i)$ such that $d_{H_{i+1}}(u, v_{i+1}) = d_G(u, v_{i+1}) = 2$ and two vertices $x \in V(H_i), v \notin V(H_i)$ which are both adjacent to u and v_{i+1}. Now, we will prove that any vertex in H_i, adjacent to v_{i+1} is also adjacent to v.

Claim 1. $N(v_{i+1}) \cap H_i \subseteq N(v)$.

Proof of Claim 1. Let $y \in H_i$ be any neighbor of v_{i+1} in H_i different from x. From the definition of the labeling α, we know that such a vertex exists. By induction assumption, H_i is convex, hence x and y are adjacent because they have a common neighbor v_{i+1} not in H_i. First suppose that the vertices u and y are adjacent. To avoid forbidden W_4^- and W_4, the vertex v must be adjacent to x and to y, and we are done. Thus we may assume that u and y are not adjacent. We distinguish two cases.

Case 1: v and x are not adjacent.

If v and y are adjacent, then we obtain a forbidden induced W_4^-. Thus we may further assume that the vertices v and y are not adjacent (see Figure 4, left). By the triangle condition, there exists a common neighbor t of u, v, and y. Since H_i is convex and $t \in I(u, y)$, necessarily $t \in V(H_i)$. To avoid an induced C_4 in H_i (which is chordal by the induction hypothesis) formed by vertices u, t, y, x, the vertex t must be adjacent to x since u is not adjacent to y. But this leads to a contradiction, since, as v is not adjacent to x and v_{i+1} is not adjacent to u, the vertices u, v, v_{i+1}, x, t induce a W_4^- or a W_4.

Case 2: v and x are adjacent.

By construction, the graph H_i is 2-connected, thus the vertices u and y can be connected in H_i by an induced path P that avoids x. Since H_i is chordal and the path P is induced, to avoid an induced cycle of length ≥ 4 formed by some vertices of $P \cup \{x\}$, the vertex x must be adjacent to all vertices of P. To avoid a forbidden wheel W_k induced by v, v_{i+1}, x and the vertices of P, necessarily v or v_{i+1} is adjacent to some vertex of P. Since P is induced and H_i is convex, v can be adjacent only with the neighbor u' of u in P and v_{i+1} can be adjacent only with the neighbor y' of y in P. If $u' \neq y'$ (see Figure 4, center) or only one of the edges vu' or $v_{i+1}y'$ exists, then still we can find there an induced wheel W_k, $k \geq 4$. Hence $u' = y'$ and v, v_{i+1} are both adjacent to $u' = y'$ (see Figure 4, right). Since the induced path P is arbitrary, we infer that each induced path in H_i between u and y is of length 2, and all common neighbors of u and y are adjacent to both v and v_{i+1} . As a conclusion, the set $K = \{z \in H_i \,:\, u, y \in N(z)\}$ is a minimal (by inclusion) (u, y)-separator of the chordal graph H_i, and thus is a clique. Both vertices v and v_{i+1} are adjacent to all vertices of K. Let A be the connected component of $H_i - K$ containing u, and let B be the connected component of $H_i - K$ containing y. Suppose that the first vertex of $A \cup B$ labeled by α belongs to A. By Lemma 2, u was labeled before any vertex of B. Let b be the first vertex labeled by α in B. Let $L(x)$ denote the set of labeled vertices at the moment of time when b is labeled. Then $L(b) \subseteq K$. Since, $K \cup \{u\} \subseteq L(v)$, we obtain a contradiction with the choice of MCS to label b before v. By symmetry of v and v_{i+1}, a similar contradiction is obtained when the first vertex of $A \cup B$ labeled by α belongs to B. This concludes the proof of the claim.

Now, Claim 1 yields $N(v_{i+1}) \cap H_i \subseteq N(v) \cap H_i$. Since $u \in H_i$ is adjacent to v but not to v_{i+1}, we obtain a contradiction with the fact that MCS labels v_{i+1} before v. Hence H_{i+1} is locally convex and, therefore, a convex subgraph of G. It is easy to see that H_{i+1} is also chordal. Indeed, since H_i is convex, the neighborhood of v_{i+1} in H_i induces a complete subgraph, thus v_{i+1} is a simplicial vertex of H_{i+1}. On the other hand, by the induction assumption H_i is chordal and therefore the ordering $v_1, \dots, v_i$ returned by MCS is a perfect elimination ordering of H_i. As a consequence, $v_1, \dots, v_i, v_{i+1}$ is a perfect elimination ordering of H_{i+1}, whence H_{i+1} is chordal. □

Proposition 2. *The gated hull of* $T = \{a, b, c\}$ *in* G *is the chordal subgraph* H.

Proof. From Proposition 1 and the definition of H (H is the last of the subgraphs H_i) we infer that H is a chordal convex subgraph of G. H is Δ-closed because every vertex in $G - H$ has at most one neighbor in H, and, since H is convex, by Lemma 3, H is a gated subgraph of G. On the other hand, if $H = H_k$, then for any index $i \leq k$, the vertex v_i has at least two neighbors in H_{i-1}, thus v_i belongs to the gated hull of H_{i-1}. Now, if by induction assumption H_{i-1} is included in the gated hull of the triangle $T = \{a, b, c\}$, then v_i belongs to this gated hull as well, whence H_i is contained in the gated hull of T, establishing the induction assertion. This shows that H is contained in the gated hull of T. Hence, H is indeed the gated hull of T. □

Let uv be an edge in G and, from now on, let H be the gated hull of the graph induced by $\{u, v\}$ in G. If uv does not belong to a triangle of G, then $\{u, v\}$ is convex and Δ-closed, thus

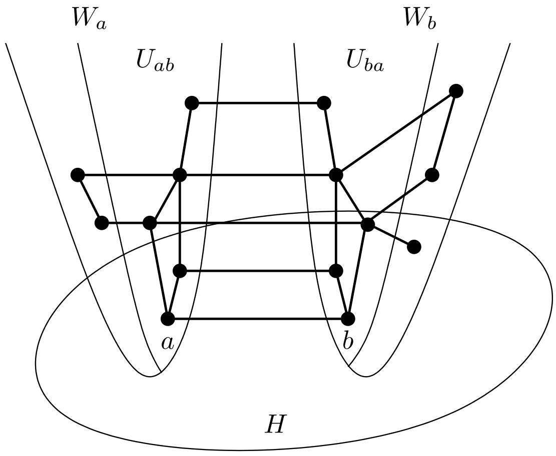

Figure 5: The fibers W_a, W_b of the vertices $a, b \in V(H)$.

$\{u, v\}$ itself is a gated set of G. In this case, H is isomorphic to K_2 and is clearly chordal. If u, v lie in a triangle T, then H coincides with the gated hull of T and can obtained by the (partial) MCS procedure as described above. By Proposition 2, H is chordal as well.

Any gated subset S of G gives rise to a partition W_a ($a \in S$) of the vertex-set of G; viz., the *fiber* W_a of a relative to S consists of all vertices x (including a itself) having a as their gate in S. For adjacent vertices a, b of S, let U_{ab} be the set of vertices from W_a which are adjacent to vertices from W_b. Let also $U_a = \{x \in W_a : \exists y \notin W_a, xy \in E(G)\}$. By some abuse of notation, W_a, U_a, and U_{ab} will denote both the sets and the subgraphs induced by these sets. An example is given in Figure 5.

Lemma 4. *Each fiber W_a relative to H is gated. There exists an edge between two distinct fibers W_a and W_b if and only if a and b are adjacent.*

Proof. To show that W_a is gated, since W_a is connected because $I(u, a) \subset W_a$ for any $u \in W_a$, by Lemma 3 it suffices to prove that W_a is locally convex and Δ-closed. Let $x, y \in W_a$ have a common neighbor z, and, for the purposes of contradiction, suppose that $z \notin W_a$. Hence $z \in W_b$ for some $b \in V(H)$ different from a. Since a (resp. b) is the unique vertex that minimizes the distance from x (resp. z) to H, we infer that $d(x, a) = d(z, b) = k$ and analogously that $d(y, a) = d(z, b) = k$. We claim that a and b are adjacent. Indeed, since $z \in W_b$, there must be a shortest path from z to a, going through b. Since $d(z, b) = k$ and $d(z, a) = d(x, a) + 1 = k + 1$, we infer $d(a, b) \leq 1$ which implies that a and b are adjacent.

By using the quadrangle condition for a, x, y, and z (or, if x and y are adjacent, using the triangle condition for a, x, and y) we conclude that x and y have a common neighbor t such that $d(a, t) = k - 1$. Since $t \in I(x, a)$, clearly $t \in W_a$ and thus $d(b, t) = k$. Applying the quadrangle condition for b, t, z, and x, we infer that t and z have a common neighbor s such that $d(b, s) = k - 1$. It is easy to see that t is not adjacent to z and that s is not adjacent to

x and y. Consequently, the vertices x, y, z, s, and t induce a $K_{2,3}$ if x and y are not adjacent, or a W_4^- otherwise. This leads to a contradiction. Hence W_a is locally convex and Δ-closed, whence each fiber W_a is gated.

Now suppose that there exists an edge uv with $u \in W_a$ and $v \in W_b$. Since a is the gate of u in H and b is the gate of v in H, we conclude that $d(u,a)+d(a,b) = d(u,b) \leq 1+d(v,b)$ and $d(v,b)+d(b,a) = d(v,a) \leq 1+d(u,a)$. From these two inequalities we deduce that $d(a,b) = 1$. □

Lemma 5. *Let $a, b \in V(H)$ be two adjacent vertices. Then $U_{ab} = U_a$ and $U_{ba} = U_b$.*

Proof. If H has only two vertices, the assertion is trivial. Otherwise, since H is a 2-connected chordal subgraph, there exists a vertex $c \in V(H)$ such that a, b, c form a triangle. We first claim that $U_{ab} = U_{ac}$. Let $x \in U_{ab}$. Then there exists $y \in U_b$ that is adjacent to x and clearly $d(a,x) = d(b,y)$. Since $c \in W_c$, we have $d(c,x) = d(c,y) = k \geq 2$, and by the triangle condition there exists a common neighbor z of x and y such that $d(c,z) = k-1$. It is easy to see that $z \in W_c$, which implies that $x \in U_{ac}$. By symmetry, we infer that $U_{ab} = U_{ac}$. Now, let $x \in U_a$. Then $x \in U_{ad}$ for some $d \in N(a) \cap H$. Since H is 2-connected and chordal, there exists a sequence of vertices $b = c_0, c_1, \ldots, c_m = d$ of H such that a, c_i, and c_{i+1} form a triangle for all $i = 0, \ldots m-1$. By the previous reasoning, this implies that $U_{ab} = U_{ac_i} = U_{ad}$. In particular, $x \in U_{ab}$, showing that $U_{ab} = U_a$. □

By Lemma 4, we infer that any vertex $x \in U_{ab} = U_a$ has exactly one neighbor in $U_{ba} = U_b$. Indeed, since each fiber W_b is gated there cannot be a vertex not in W_b adjacent to two vertices of W_b. This fact combined with Lemma 5 gives rise to the following natural mapping $f_{ab} : U_a \longrightarrow U_b$ that maps $x \in U_a$ to the neighbor of x in U_b.

Lemma 6. *Let a, b be two adjacent vertices of H. Then U_a and U_b are isomorphic subgraphs of G and f_{ab} is an isomorphism between the graphs U_a and U_b.*

Proof. Let x, y be two adjacent vertices of U_a, and suppose that their neighbors x', y' in U_b are not adjacent. Since W_b is convex, we infer that $d_{W_b}(x', y') = 2$. Let $z' \in W_b$ be a common neighbor of x' and y'. Since $d(y, z') = d(y, x') = 2$, by the triangle condition we infer that there exists a common neighbor u of y, x', and z'. Since W_b is Δ-closed, we conclude that $u \in W_b$. But then $y \in U_a$ has two neighbors u and y' in U_b, which is impossible. □

Lemma 7. *The subgraphs U_a are gated for all $a \in V(H)$ and are mutually isomorphic. Their union is isomorphic to $H \Box U$, where U is any of U_a.*

Proof. Since H is connected, from Lemma 6, we immediately infer that the subgraphs U_a are all mutually isomorphic. Since each fiber W_a is gated, to prove that U_a is gated it suffices to show that U_a is locally convex and Δ-closed in the subgraph W_a.

Let $x, y \in U_a$ be two vertices having a common neighbor $z \in U_a$ and suppose that there is a vertex $s \in W_a \setminus U_a$ that is adjacent to both x and y but not to z (the case when s is adjacent to z is covered by Δ-closedness of U_a established below). Let b be a neighbor of a in H and let $x', z', y' \in U_b$ be the neighbors of x, z, y, respectively. By Lemma 6 we conclude that z' is adjacent to x' and y' but x' and y' are not adjacent. Then $d(s, x') = d(s, y') = d(s, z') - 1 = 2$ and by the quadrangle condition we find that x', y' and s have a common neighbor s'. Since W_b is convex, $s' \in U_b$ which in turn implies that $s \in U_a$, a contradiction. This shows that U_a is locally convex.

Let $x, y \in U_a$ be two adjacent vertices and suppose that there is a vertex $s \in W_a \setminus U_a$ adjacent to both x and y. Let b be a neighbor of a in H and let $x', y' \in U_b$ be the neighbors of x, y respectively. By Lemma 6, we know that x' is adjacent to y'. Then $d(s, x') = d(s, y') = 2$ and by the triangle condition we find that x', y' and s have a common neighbor s'. Since $N(s) \subseteq U_a$, it implies that either x' or y' has two neighbors in U_a, a contradiction. This shows that U_a is Δ-closed. Thus U_a is indeed gated.

The structure of the union of all U_a, $a \in V(H)$, is now completely described. Its vertex set is isomorphic to $V(H) \times V(U)$, where U is isomorphic to U_a for any $a \in V(H)$. For any vertices $a, c \in V(H)$ and any $x \in U_a$, $y \in U_c$, x is adjacent to y if and only if either $a = c$ and $xy \in E(U_a)$, or a and c are adjacent and y is the unique neighbor of x in U_c. Hence the union of U_a over all $a \in V(H)$ is isomorphic to $H \square U$. □

We collected all results to conclude the proof of the implication (ii)⇒(iii) of Theorem 1. We proceed by induction on the cardinality of G. First, if H (the gated hull of $\{u, v\}$ in G) is equal to whole graph G, then G is chordal, hence G is a cage-amalgamation graph. Therefore, we can suppose that H is a proper subgraph of G. Now, suppose that for any $a \in V(H)$, the set W_a coincides with U_a. By Lemma 7, G is isomorphic to $H \square W_a = H \square U_a$, where H is a chordal graph. Since W_a has smaller cardinality than G and since W_a is a weakly modular graph without $K_{2,3}$, W_k, and W_k^-, $k \geq 4$ (as a gated subgraph of G), by induction hypothesis W_a is a cage-amalgamation graph. Since Cartesian products and gated amalgams commute (see also Lemma 3.1 of [11]), $G = H \square W_a$ is a cage-amalgamation graph as well. Finally, suppose that for some $a \in V(H)$ the set $W_a - U_a$ is nonempty. Since U_a is gated and is a separator of G, we conclude that G is the gated amalgam of W_a and $G - (W_a - U_a)$ along the common gated subgraph U_a. Since both those graphs W_a and $G - (W_a - U_a)$ have smaller cardinality that G, they are cage-amalgamation graphs, and thus so is G. This concludes the proofs of the implication (ii)⇒(iii) and of Theorem 1.

4 Proof of Theorem 2

The proof uses the decomposition scheme from Theorem 1 and runs in three steps: first we show that a geometric realization of the clique complex of a chordal graph is CAT(0), then we establish that a geometric realization of the Hamming complex of a Cartesian product of chordal graphs is CAT(0) as well, and finally we show that gated amalgams of cage-amalgamation graphs preserve the CAT(0) property of their complexes. The proof employs the following known property of CAT(0) spaces due to Reshetnyak and which is a particular case of Basic Gluing Theorem 11.1 of [12]:

Preprint series, IMFM, ISSN 2232-2094, no. 1134, December 14, 2010

Gluing theorem: *If (X_1, d_1) and (X_2, d_2) are two CAT(0) spaces, A_i is a convex non-empty subset of $(X_i, d_i), i = 1, 2,$ and there exists an isometry φ between A_1 and A_2, then the metric space $(X_1 \cup X_2, d)$ obtained by gluing X_1 and X_2 along the sets A_1 and A_2 is CAT(0).*

The metric space $(X_1 \cup X_2, d)$ is obtained by identifying A_1 and A_2 according to φ and d is defined to be d_1 on X_1, d_2 on X_2, and $d(x, y) = \inf\{d_1(x, a) + d_2(a, y) : a \in A_2 = \varphi(A_1)\}$ if $x \in A_1$ and $y \in A_2$.

It was noticed in Corollary 8.4 of [16] that a geometric realization $|\mathcal{X}(G)|$ of the clique complex $\mathcal{X}(G)$ of a finite chordal graph G is CAT(0). We recall this short proof here because the proof of Theorem 2 is based on the same principle. We proceed by induction on the number of vertices of G. Let x be a simplicial vertex of G. Then x belongs to the unique maximal by inclusion simplex σ of $\mathcal{X}(G)$ induced by x and all its neighbors in G. Consequently, $|\mathcal{X}(G)|$ can be obtained by gluing $|\sigma|$ and $|\mathcal{X}'|$, where $\mathcal{X}'$ is the subcomplex of $\mathcal{X}$ spanned by $\sigma' := \sigma - \{x\}$ and the maximal simplexes of $\mathcal{X}(G)$ distinct from σ (in fact, $\mathcal{X}'$ is the clique complex of the chordal graph $G' := G - \{x\}$). Since the gluing is performed along a convex set $|\sigma'|$ of both complexes $|\sigma|$ and $|\mathcal{X}'|$, from the result of Reshetnyak mentioned above we obtain that $|\mathcal{X}(G)|$ is CAT(0) if and only if $|\sigma|$ and $|\mathcal{X}'|$ are CAT(0). Since $\mathcal{X}' = \mathcal{X}(G')$ and the graph G' is chordal, by the induction assumption $|\mathcal{X}'|$ is CAT(0), and we are done. In view of perfect elimination schemes of chordal graphs G, $|\mathcal{X}(G)|$ can be written as a directed union $\bigcup_{i=1}^{n} |\mathcal{X}_i|$ where $|\mathcal{X}_i| = |\mathcal{X}_{i-1}| \cup |\sigma_i|$ and the simplex $|\sigma_i|$ meets $|\mathcal{X}_{i-1}|$ over a single face $|\sigma'_i|$. Such simplicial polyhedral complexes have been called by Gromov (p.121 of [21]) *tree-like polyhedra* and also noticed to be CAT(0).

Now suppose that G is a cage-amalgamation graph whose prime graphs are the chordal graphs $G_1, \ldots, G_m$. Each of these graphs occurs as a gated subgraph of G. Let x be a simplicial vertex of G_1. Denote by σ_x the unique maximal by inclusion simplex of $\mathcal{X}(G_1)$ induced by x and all its neighbors in G_1 and let $\sigma'_x := \sigma_x - \{x\}$. For each vertex a of G_1, denote by W_a its fiber in G relative to some copy of the gated subgraph G_1. From Lemma 4, each such fiber W_a is gated. From Lemmas 6 and 7 we conclude that the boundaries U_a of these fibers W_a are isomorphic gated subgraphs of G. Denote by $\mathcal{H}_{\sigma_x}$ and $\mathcal{H}_{\sigma'_x}$ the Hamming complexes of the subgraphs of G induced by the unions $\bigcup_{a\in\sigma_x} U_a$ and $\bigcup_{a\in\sigma'_x} U_a$. Notice that $\mathcal{H}_{\sigma'_x}$ is a subcomplex of $\mathcal{H}_{\sigma_x}$ and that both $\mathcal{H}_{\sigma'_x}$ and $\mathcal{H}_{\sigma_x}$ are subcomplexes of $\mathcal{H}(G)$.

Lemma 8. *If p, q are two points of $|\mathcal{H}_{\sigma'_x}|$, then any geodesic connecting p and q in $|\mathcal{H}_{\sigma_x}|$ is contained in $|\mathcal{H}_{\sigma'_x}|$.*

Proof. Suppose by way of contradiction that such a geodesic $\gamma(p, q)$ contains a point in the set $|\mathcal{H}_{\sigma_x}| - |\mathcal{H}_{\sigma'_x}|$ (see Figure 6 for an illustration). Let $\pi_1, \ldots, \pi_k$ be the maximal by inclusion Hamming prisms of $|\mathcal{H}_{\sigma_x}|$ intersected by $\gamma(p, q)$ labeled in order in which they are traversed by $\gamma(p, q)$. Let π'_i be the facet of π_i in $|\mathcal{H}_{\sigma'_x}|$, i.e., $\pi'_i = \pi_i \cap |\mathcal{H}_{\sigma'_x}|$. The intersection of any two consecutive prisms π_i and π_{i+1} is a face τ_i of each of them. Let τ'_i denote the facet of τ_i in π'_i (and π'_{i+1}). Let $r_i \in \gamma(p, q) \cap \tau_i$. The orthogonal projection of each prism π_i on its facet π'_i is a non-expansive map f_i. Moreover, each point r_i is mapped by f_i and f_{i+1} to the same point r'_i belonging to τ'_i. As a result, the length of the path $\gamma'(p, q)$ between $p = r'_0$ and $q = r'_k$

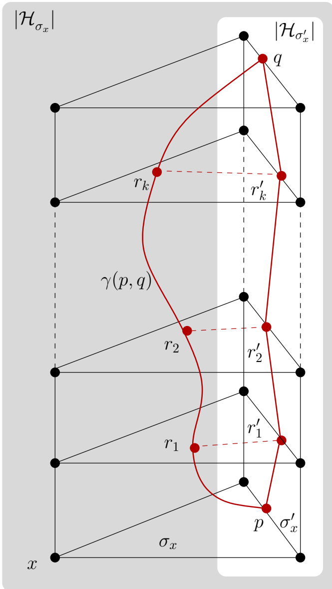

Figure 6: To the proof of Lemma 8

consisting of line segments connecting the consecutive points $p, r'_1, r'_2, \ldots, r'_{k-1}, q$ is at most the length of $\gamma(p,q)$. Since $p, q \in |\mathcal{H}_{\sigma'_x}|$ and $\gamma(p,q)$ passes via a point of $|\mathcal{H}_{\sigma_x}| - |\mathcal{H}_{\sigma'_x}|$, at least one of the orthogonal projections $r'_i r'_{i+1}$ must be strictly smaller than the length of $\gamma(r_i, r_{i+1})$ (the portion of $\gamma(p,q)$ comprised between r_i and r_{i+1}), thus $\gamma'(p,q)$ is strictly shorter than $\gamma(p,q)$, completing the proof of the lemma. $\square$

Now, by induction on the number of vertices of G, we will establish that if G is a Cartesian product of chordal graphs $G_1, \ldots, G_m$, then $|\mathcal{H}(G)|$ is CAT(0). This is obviously true if each G_i is a clique. So, suppose without loss of generality that G_1 is not a clique. Let x be a simplicial vertex of G_1. From Lemma 8 we know that the subcomplex $|\mathcal{H}_{\sigma'_x}|$ is convex (with respect to the d_2-metric) in $|\mathcal{H}_{\sigma_x}|$. Let $G'_1 := G_1 - \{x\}$ and $G' := G'_1 \Box G_2 \Box \ldots \Box G_m$. By induction assumption, $|\mathcal{H}_{\sigma_x}|$ and $|\mathcal{H}(G')|$ are CAT(0) spaces. Since $|\mathcal{H}(G)|$ is obtained by gluing $|\mathcal{H}_{\sigma_x}|$ and $|\mathcal{H}(G')|$ along $|\mathcal{H}_{\sigma'_x}|$ and $|\mathcal{H}_{\sigma'_x}|$ is convex in $|\mathcal{H}_{\sigma_x}|$, to apply the basic gluing theorem it suffices to show that $|\mathcal{H}_{\sigma'_x}|$ is convex in $|\mathcal{H}(G')|$. This is obviously true when $G'_1 = \sigma'_x$. Otherwise, G'_1 contains a simplicial vertex $y \notin \sigma'_x$. Let $G'' = G'_1 - \{y\}$ and assume by induction assumption that $|\mathcal{H}_{\sigma'_x}|$ is convex in $|\mathcal{H}(G'')|$. Therefore, if $|\mathcal{H}_{\sigma'_x}|$ is not convex in $|\mathcal{H}(G'_1)|$, then we can find two points $p, q \in |\mathcal{H}_{\sigma'_x}|$ and a geodesic $\gamma(p,q)$ between p and q in $|\mathcal{H}(G'_1)|$ containing at least one point $z \in |\mathcal{H}(G'_1)| - |\mathcal{H}(G'')| = |\mathcal{H}_{\sigma_y}| - |\mathcal{H}_{\sigma'_y}|$. Then $\gamma(p,q)$ contains two points $p', q' \in |\mathcal{H}_{\sigma'_y}|$ such that z belongs to the portion $\gamma(p',q')$ of $\gamma(p,q)$ comprised between p' and q'. Since $\gamma(p,q)$ is a geodesic, necessarily $\gamma(p',q')$ is a geodesic between p' and q'. This however contradicts the convexity of $|\mathcal{H}_{\sigma'_y}|$ in $|\mathcal{H}_{\sigma_y}|$ established in

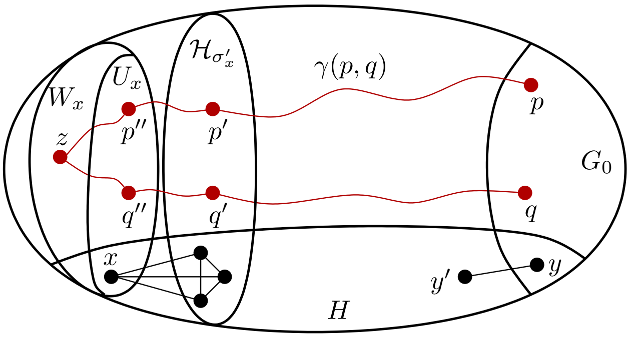

Figure 7: To the proof of Lemma 9

Lemma 8. This shows that $|\mathcal{H}_{\sigma'_x}|$ is convex in $|\mathcal{H}(G')|$ as well, and therefore we can apply the gluing theorem.

Finally, suppose that a graph G is a gated amalgam of two cage-amalgamation graphs G' and G'' along a gated subgraph G_0. Suppose by induction assumption that $|\mathcal{H}(G')|$ and $|\mathcal{H}(G'')|$ are CAT(0) spaces. To use the gluing theorem again, it suffices to show that $|\mathcal{H}(G_0)|$ is convex (with respect to the intrinsic d_2-metric) in both $|\mathcal{H}(G')|$ and $|\mathcal{H}(G'')|$, say in $|\mathcal{H}(G')|$.

Lemma 9. *If G_0 is a gated subgraph of a cage-amalgamation graph G', then $|\mathcal{H}(G_0)|$ is convex in $|\mathcal{H}(G')|$.*

Proof. We proceed by induction on the number of vertices of G'. Since G_0 is different from G', there exists a vertex y of G_0 that has a neighbor $y' \in V(G') \setminus V(G_0)$. Let H be the gated hull of the edge yy'. Consider the partition of G' into fibers W_a with respect to the vertices a of H. Clearly, the gated subgraph G_0 is completely contained in the fiber W_y of y. By Proposition 2, H is either a 2-connected chordal graph or an edge. In both cases, H contains a simplicial vertex x different from y. Denote by σ_x the unique maximal complete subgraph of H containing x and let $\sigma'_x = \sigma_x - \{x\}$. Let D be the subgraph of G' induced by all vertices not belonging to the fiber W_x. Since x is a simplicial vertex of H, it can be easily seen that D is an isometric (in fact a convex) subgraph of G'. D is a cage-amalgamation graph: its primes are the same as those of G' with the single exception that H is replaced by $H - \{x\}$. Moreover, G_0 is a gated subgraph of D. Thus, by induction assumption, we can suppose that $|\mathcal{H}(G_0)|$ is a convex subcomplex of $|\mathcal{H}(D)|$. Now, suppose by way of contradiction that $|\mathcal{H}(G_0)|$ is not convex in $|\mathcal{H}(G')|$. Then there exist two points $p, q \in |\mathcal{H}(G_0)|$ such that the geodesic $\gamma(p,q)$ connecting p and q in $|\mathcal{H}(G')|$ does not belong to $|\mathcal{H}(G_0)|$. Since $|\mathcal{H}(G_0)|$ is convex in $|\mathcal{H}(D)|$, $\gamma(p,q)$ contains at least one point z not belonging to $|\mathcal{H}(D)|$. Then $\gamma(p,q)$ necessarily contains two points $p', q' \in |\mathcal{H}_{\sigma'_x}|$ (where $|\mathcal{H}_{\sigma'_x}|$ is defined as before) such that z belongs to the part $\gamma(p',q')$ of $\gamma(p,q)$ comprised between the points p' and q'. Since $\gamma(p',q')$ is a part of a geodesic, $\gamma(p',q')$ is a geodesic itself. If $\gamma(p',q')$ (and therefore z) is contained in the subcomplex $|\mathcal{H}_{\sigma_x}|$ of $|\mathcal{H}(W_x)|$, then we obtain a contradiction with Lemma 8 asserting the convexity of $|\mathcal{H}_{\sigma'_x}|$ in $|\mathcal{H}_{\sigma_x}|$. Thus we can suppose that $\gamma(p', q')$ contains some points (say z itself) in $|\mathcal{H}(W_x)| - |\mathcal{H}_{\sigma_x}|$. Then necessarily $\gamma(p', q')$ contains two points $p'', q'' \in |\mathcal{H}(U_x)|$ such that z belongs to the portion $\gamma(p'', q'')$ between p'' and q''. Again, $\gamma(p'', q'')$ is a geodesic as a part of a larger geodesic. But this means that $|\mathcal{H}(U_x)|$ is not a convex subcomplex of $|\mathcal{H}(W_x)|$, contrary to the fact that U_x is a gated subgraph of a cage-amalgamation graph W_x having less vertices than the graph G'. This contradiction establishes Lemma 9. □

From Lemma 9 we conclude that $|\mathcal{H}(G_0)|$ is convex in $|\mathcal{H}(G')|$ and $|\mathcal{H}(G'')|$, therefore the gated amalgamation of G' and G'' along G_0 translates into gluing two CAT(0) spaces $|\mathcal{H}(G')|$ and $|\mathcal{H}(G'')|$ along a convex subspace $|\mathcal{H}(G_0)|$, thus $|\mathcal{H}(G)|$ is CAT(0) by the gluing theorem. This concludes the proof of Theorem 2.

We conclude the paper with two open questions:

Question 1. Is it true that the graphs G which can be obtained by successive gated amalgams from Cartesian products of *bridged graphs* are exactly the *weakly modular graphs not containing* $K_{2,3}$*, the wheels* W_4 *and* W_5*, and the almost wheels* W_k^- *for* $k \geq 4$?

Question 2. Characterize the triangle-square complexes (i.e., the 2-dimensional complexes obtained by taking all graph triangles C_3 and squares C_4 as faces) of cage-amalgamation graphs. In particular, is it true that those complexes are exactly *the simply connected triangle-square complexes whose underlying graphs do not contain* $K_{2,3}$*, the wheels* W_k*, and the almost wheels* W_k^- *for* $k \geq 4$*?* In other words, is it possible to replace the global metric condition of "weak modularity" by a topological condition of "simple connectivity"?

Acknowledgement

This research was supported in part by the French-Slovenian Egide PROTEUS project "Distances, structure and products of graphs". The first and the fourth author are also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, and were also supported by the Ministry of Science and Technology of Slovenia under the grants J1-2043 and P1-0297. The third author was also supported by the ANR Projet Blanc THEOMATRO.

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