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Accepted Manuscripts – Yet Another Step to Reduce the Backlog

It takes a long time for a paper accepted into this journal to appear in final form, and certainly much longer than we would like. At the time of writing this, AMC has 42 articles in the processing queue – accepted but not yet scheduled for final publication – yet with 35 of them available for online display. As part of our effort to reduce the backlog of papers in this queue, we have introduced a category of *Accepted Manuscripts*. When in this state, a paper is waiting for final proof-reading and scheduling for full publication, but will be available online to our readers. In doing this, we have joined many other scientific journals that follow a similar practice.

Klavdija Kutnar, Dragan Marušič and Tomaž Pisanski
Editors in Chief

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On a certain class of 1-thin distance-regular graphs

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Abstract

Let Γ denote a non-bipartite distance-regular graph with vertex set X, diameter $D \geq 3$, and valency $k \geq 3$. Fix $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. For any $z \in X$ and for $0 \leq i \leq D$, let $\Gamma_i(z) = \{w \in X : \partial(z, w) = i\}$. For $y \in \Gamma_1(x)$, abbreviate $D^i_j = D^i_j(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$ $(0 \leq i, j \leq D)$. For $1 \leq i \leq D$ and for a given y, we define maps $H_i \colon D^i_i \to \mathbb{Z}$ and $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|, \qquad V_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|.$$

We assume that for every $y \in \Gamma_1(x)$ and for $2 \leq i \leq D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. We further assume that the constant value of H_i is nonzero for $2 \leq i \leq D$. We show that every irreducible T-module of endpoint 1 is thin. Furthermore, we show Γ has exactly three irreducible T-modules of endpoint 1, up to isomorphism, if and only if three certain combinatorial conditions hold. As examples, we show that the Johnson graphs $J(n, m)$ where $n \geq 7$, $3 \leq m < n/2$ satisfy all of these conditions.

Keywords: Distance-regular graph, Terwilliger algebra, subconstituent algebra.

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1 Introduction

This paper is motivated by a desire to find a combinatorial characterization of the distance-regular graphs with exactly three irreducible modules (up to isomorphism) of the Terwilliger algebra with endpoint 1, all of which are thin (see Sections 2, 3 for formal definitions). This is a difficult problem which we will not complete in this paper. To begin, we find combinatorial conditions under which a distance-regular graph is 1-thin. When these combinatorial conditions hold, we identify additional combinatorial conditions that hold if and only if the distance-regular graph has exactly three irreducible T-modules of endpoint 1, up to isomorphism.

Let Γ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let X denote the vertex set of Γ. For $x \in X$, let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. It is known that there exists a unique irreducible T-module with endpoint 0, and this module is thin [5, Proposition 8.4]. It is also known that Γ is bipartite or almost-bipartite if and only if Γ has exactly one irreducible T-module of endpoint 1, up to isomorphism, and this module is thin [4, Theorem 1.3]. Furthermore, Curtin and Nomura have shown that Γ is pseudo-1-homogeneous with respect to x with $a_1 \neq 0$ if and only if Γ has exactly two irreducible T-modules of endpoint 1, up to isomorphism, both of which are thin [4, Theorem 1.6].

For any $z \in X$ and any integer $i \geq 0$, let $\Gamma_i(z) = \{w \in X : \partial(z,w) = i\}$. For $y \in \Gamma_1(x)$ and integers $i, j \geq 0$, abbreviate $D^i_j = D^i_j(x,y) = \Gamma_i(x) \cap \Gamma_j(y)$. For $1 \leq i \leq D$ and for a given y, we define maps $H_i \colon D^i_i \to \mathbb{Z}$, $K_i \colon D^i_i \to \mathbb{Z}$ and $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|, \qquad K_i(z) = |\Gamma_1(z) \cap D^{i+1}_{i+1}|, \quad V_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|.$$

Our main result is the following.

Theorem 1.1. *Let $\Gamma = (X, \mathcal{R})$ denote a non-bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$, and fix vertex $x \in X$. Assume that for every $y \in \Gamma_1(x)$ and for $2 \leq i \leq D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. Also assume that the constant value of H_i is nonzero for $2 \leq i \leq D$. Then Γ is 1-thin with respect to x.*

We need the following definition.

Definition 1.2. With the assumptions of Theorem 1.1, for $y \in \Gamma_1(x)$ let $D^i_j = D^i_j(x,y)$ $(0 \leq i,j \leq D)$ and let K_1 denote the corresponding map. Let $B = B(y)$ denote the adjacency matrix of the subgraph of Γ induced on D^1_1. Observe that $B \in \mathrm{Mat}_{D^1_1}(\mathbb{C})$, and so the rows and the columns of B are indexed by the elements of D^1_1. Let $\boldsymbol{j} \in \mathbb{C}^{D^1_1}$ denote the all-ones column vector with rows indexed by the elements of D^1_1.

With reference to Definition 1.2, we denote by P1, P2 and P3 the following properties of Γ:

- P1: There exists $y \in \Gamma_1(x)$ such that K_1 is not a constant.
- P2: For every $y, z \in \Gamma_1(x)$ with $\partial(y,z) \in \{0,2\}$, the number of walks of length 3 inside $\Gamma_1(x)$ from y to z is a constant number, which depends only on $\partial(y,z)$ (and not on the choice of y, z).

P3: There exist scalars α, β such that for every $y \in \Gamma_1(x)$ we have

$$B^2 \boldsymbol{j} = \alpha B \boldsymbol{j} + \beta \boldsymbol{j}.$$

We prove the following.

Theorem 1.3. *With reference to Definition 1.2,* Γ *has exactly three irreducible* T*-modules of endpoint* 1*, up to isomorphism, if and only if properties P1, P2, and P3 hold. We note these three* T*-modules are all thin by Theorem 1.1.*

Finally, we show that the Johnson graphs $J(n, m)$ where $n \geq 7,\ 3 \leq m < n/2$ satisfy the assumptions in Theorem 1.1 and the equivalent conditions in Theorem 1.3.

2 Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitian inner product $\langle\, ,\, \rangle$ that satisfies $\langle u, v\rangle = u^t \overline{v}$ for $u, v \in V$, where t denotes transpose and $^-$ denotes complex conjugation. For $y \in X$ let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following will be useful: for each $B \in \mathrm{Mat}_X(\mathbb{C})$ we have

$$\langle u, Bv\rangle = \langle \overline{B}^t u, v\rangle \qquad (u, v \in V). \tag{2.1}$$

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{R}$. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For a vertex $x \in X$ and an integer $i \geq 0$ let $\Gamma_i(x)$ denote the set of vertices at distance i from x. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$ we say Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers h, i, j ($0 \leq h, i, j \leq D$) and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ.

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 3$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \leq h, i, j \leq D$. For convenience set $c_i := p^i_{1,i-1}$ ($1 \leq i \leq D$), $a_i := p^i_{1i}$ ($0 \leq i \leq D$), $b_i := p^i_{1,i+1}$ ($0 \leq i \leq D-1$), $k_i := p^0_{ii}$ ($0 \leq i \leq D$), and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \leq h, i, j \leq D$: (i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $p^h_{ij} \neq 0$ if one of h, i, j equals the sum of the other two. In particular $c_i \neq 0$ for $1 \leq i \leq D$ and $b_i \neq 0$ for $0 \leq i \leq D-1$.

We observe that Γ is regular with valency $k = k_1 = b_0$ and that $c_i + a_i + b_i = k$ for $0 \le i \le D$. Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \le i \le D$.

We recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X). \tag{2.2}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{D} A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (aiv) $A_i^t = A_i$ $(0 \le i \le D)$; (av) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [2, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $\sum_{i=0}^{D} E_i = I$; (eiii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (eiv) $E_i^t = E_i$ $(0 \le i \le D)$; (ev) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

3 The Terwilliger algebra

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \ge 3$ and valency $k \ge 3$. In this section we recall the dual Bose-Mesner algebra and the Terwilliger algebra of Γ. Fix a vertex $x \in X$. We view x as a "base vertex." For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [11, p. 378]. We observe (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le D)$; (iii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [11, p. 378]. For $0 \le i \le D$ we have

$$E_i^* V = \mathrm{span}\{\hat{y} \mid y \in \Gamma_i(x)\}$$

so $\dim E_i^* V = k_i$. We call $E_i^* V$ the ith *subconstituent of* Γ with respect to x. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \qquad \text{(orthogonal direct sum)}.$$

Moreover E_i^* is the projection from V onto $E_i^* V$ for $0 \le i \le D$.

We recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the *Terwilliger algebra of* Γ with respect to x [11, Definition 3.3]. Recall M (resp. M^*) is generated by A (resp. $E_0^*, E_1^*, \ldots, E_D^*$) so T is generated by $A, E_0^*, E_1^*, \ldots, E_D^*$. We observe T has finite dimension. By construction T is closed under the conjugate-transpose map so T is semi-simple [11, Lemma 3.4(i)].

By a *T-module* we mean a subspace W of V such that $SW \subseteq W$ for all $S \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W.

By [6, Corollary 6.2] any T-module is an orthogonal direct sum of irreducible T-modules. In particular the standard module V is an orthogonal direct sum of irreducible T-modules. Let W, W' denote T-modules. By an *isomorphism of* T*-modules* from W to W' we mean an isomorphism of vector spaces $\sigma\colon W \to W'$ such that $(\sigma S - S\sigma)W = 0$ for all $S \in T$. The T-modules W, W' are said to be *isomorphic* whenever there exists an isomorphism of T-modules from W to W'. By [3, Lemma 3.3] any two non-isomorphic irreducible T-modules are orthogonal. Let W denote an irreducible T-module. By [11, Lemma 3.4(iii)] W is an orthogonal direct sum of the nonvanishing spaces among $E_0^*W, E_1^*W, \ldots, E_D^*W$. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}| - 1$. We say W is *thin* if $\dim(E_i^*W) \le 1$ for $0 \le i \le D$. We say Γ is 1-*thin* with respect to x if every T-module with endpoint 1 is thin.

By [5, Proposition 8.3, Proposition 8.4] $M\hat{x}$ is the unique irreducible T-module with endpoint 0 and the unique irreducible T-module with diameter D. Moreover $M\hat{x}$ is the unique irreducible T-module on which E_0 does not vanish. We call $M\hat{x}$ the *primary module*. We observe that vectors s_i $(0 \le i \le D)$ form a basis for $M\hat{x}$, where

$$s_i = \sum_{y \in \Gamma_i(x)} \hat{y}. \tag{3.1}$$

Lemma 3.1. *Let* $\Gamma = (X, \mathcal{R})$ *denote a distance-regular graph with diameter* $D \ge 3$ *and distance matrices* A_i $(0 \le i \le D)$*. Fix a vertex* $x \in X$ *and let* $E_i^* = E_i^*(x)$ $(0 \le i \le D)$ *denote the dual idempotents with respect to* x*. For* $0 \le h, i, j \le D$*, the matrix* $E_h^* A_i E_j^* = 0$ *whenever any one of* h, i, j *is bigger than the sum of the other two.*

Proof. Routine using elementary matrix multiplication. □

The following result will be crucial later in the paper.

Lemma 3.2. *Let* $\Gamma = (X, \mathcal{R})$ *denote a distance-regular graph with diameter* $D \ge 3$*. Fix a vertex* $x \in X$ *and let* $E_i^* = E_i^*(x)$ $(0 \le i \le D)$ *denote the dual idempotents with respect to* x*. Let* $T = T(x)$ *denote the Terwilliger algebra of* Γ *with respect to* x*. Assume that (up to isomorphism)* Γ *has exactly three irreducible* T*-modules with endpoint* 1*, and that these modules are all thin. Let* F_1*,* F_2*,* F_3*,* F_4*,* $F_5 \in T$ *and pick an integer* i*,* $1 \le i \le D$*. Then the matrices*

$$E_i^*F_1E_1^*,\ E_i^*F_2E_1^*,\ E_i^*F_3E_1^*,\ E_i^*F_4E_1^*, E_i^*F_5E_1^*$$

are linearly dependent.

Proof. Let V_0 denote the primary module of Γ, and let V_ℓ $(1 \le \ell \le 3)$ denote pairwise non-isomorphic irreducible T-modules with endpoint 1. Define vectors v_ℓ $(0 \le \ell \le 3)$ as follows. If $E_i^*V_\ell = 0$, then let $v_\ell = 0$. Otherwise, let v_ℓ be an arbitrary nonzero vector of $E_i^*V_\ell$. Furthermore, for $0 \le \ell \le 3$ fix a nonzero $u_\ell \in E_1^*V_\ell$. As modules V_ℓ $(0 \le \ell \le 3)$ are thin, there exist scalars λ_j^ℓ $(1 \le j \le 5,\ 0 \le \ell \le 3)$ such that

$$E_i^*F_jE_1^*u_\ell = \lambda_j^\ell v_\ell.$$

Consider now the following homogeneous system of linear equations:

$$\begin{pmatrix} \lambda_1^0 & \lambda_2^0 & \lambda_3^0 & \lambda_4^0 & \lambda_5^0 \\ \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & \lambda_5^3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

Observe that the above system has a nontrivial solution, and let $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)^t$ denote one of its nontrivial solutions. We will now show that $\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* = 0$. First, pick an arbitrary $u \in E_1^* V_\ell$, for some ℓ $(0 \leq \ell \leq 3)$. As module V_ℓ is thin, there exists a scalar λ, such that $u = \lambda u_\ell$. Now we have

$$\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* u = \lambda \sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* u_\ell = \lambda \sum_{j=1}^{5} \mu_j \lambda_j^\ell v_\ell = \lambda v_\ell \sum_{j=1}^{5} \mu_j \lambda_j^\ell = 0. \tag{3.3}$$

Assume now that W is an irreducible T-module with endpoint 1 and note that W is isomorphic to V_ℓ for some $1 \leq \ell \leq 3$. Pick arbitrary $w \in E_1^* W$. Let $\sigma \colon V_\ell \mapsto W$ be a T-module isomorphism and let $u \in E_1^* V_\ell$ be such that $w = \sigma(u)$. Now by (3.3) we have that

$$\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* w = \sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* \sigma(u) = \sigma\Bigg(\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* u \Bigg) = 0. \tag{3.4}$$

For $1 \leq \ell \leq 3$ let $\mathcal{V}_\ell$ denote the sum of all irreducible T-modules with endpoint 1, which are isomorphic to V_ℓ. Observe that

$$E_1^* V = E_1^* V_0 + E_1^* \mathcal{V}_1 + E_1^* \mathcal{V}_2 + E_1^* \mathcal{V}_3 \qquad \text{(orthogonal sum).} \tag{3.5}$$

Pick now an arbitrary $v \in E_1^* V$. Note that by (3.5) v is a sum of vectors v_ξ, where ξ belongs to some index set Ξ, and each v_ξ is contained in $E_1^* W_\xi$, where W_ξ is either V_0, or isomorphic to V_ℓ for some $1 \leq \ell \leq 3$. By (3.4) we have that $\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* v_\xi = 0$ for each $\xi \in \Xi$, and consequently $\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* v = 0$. This shows that $\sum_{j=1}^{5} \mu_j E_i^* F_j E_1^* = 0$. As at least one of μ_j $(1 \leq j \leq 5)$ is nonzero (recall that $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)^t$ is a nontrivial solution of (3.2)), the result follows. □

4 The local eigenvalues

In order to discuss the thin irreducible T-modules with endpoint 1, we first recall some parameters called the local eigenvalues. We will use the notation from [7].

Definition 4.1. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$ and adjacency matrix A. Fix a vertex $x \in X$. We let $\Delta = \Delta(x)$ denote the graph $(\breve{X}, \breve{\mathcal{R}})$, where

$$\breve{X} = \{y \in X \mid \partial(x, y) = 1\},$$
$$\breve{\mathcal{R}} = \{yz \mid y, z \in \breve{X},\ \partial(y, z) = 1\}.$$

The graph Δ has exactly k vertices and is regular with valency a_1. We let $\breve{A}$ denote the adjacency matrix of Δ. The matrix $\breve{A}$ is symmetric with real entries, and thus $\breve{A}$ is diagonalizable with real eigenvalues. We let $\eta_1, \eta_2, \dots, \eta_k$ denote the eigenvalues of $\breve{A}$. We call $\eta_1, \eta_2, \dots, \eta_k$ the *local eigenvalues of* Γ *with respect to* x.

We now consider the first subconstituent E_1^*V. We recall the dimension of E_1^*V is k. Observe E_1^*V is invariant under the action of $E_1^*AE_1^*$. We note that for an appropriate ordering of the vertices of Γ, we have

$$E_1^*AE_1^* = \begin{pmatrix} \breve{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\breve{A}$ is from Definition 4.1. Hence the action of $E_1^*AE_1^*$ on E_1^*V is essentially the adjacency map for Δ. In particular the action of $E_1^*AE_1^*$ on E_1^*V is diagonalizable with eigenvalues $\eta_1, \eta_2, \dots, \eta_k$. We observe the vector s_1 from (3.1) is contained in E_1^*V. One may easily show that s_1 is an eigenvector for $E_1^*AE_1^*$ with eigenvalue a_1. Reordering the eigenvalues if necessary, we have $\eta_1 = a_1$. For the rest of this paper, we assume the local eigenvalues are ordered in this way. Now consider the the orthogonal complement of s_1 in E_1^*V. By (2.1), this space is invariant under multiplication by $E_1^*AE_1^*$. Thus the restriction of the matrix $E_1^*AE_1^*$ to this space is diagonalizable with eigenvalues $\eta_2, \eta_3, \dots, \eta_k$.

Definition 4.2. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$ and adjacency matrix A. Fix a vertex $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let W denote a thin irreducible T-module with endpoint 1. Observe E_1^*W is a 1-dimensional eigenspace for $E_1^*AE_1^*$; let η denote the corresponding eigenvalue. We observe E_1^*W is contained in E_1^*V so η is one of $\eta_2, \eta_3, \dots, \eta_k$. We refer to η as the *local eigenvalue* of W.

Theorem 4.3 ([14, Theorem 12.1])**.** *Let* $\Gamma = (X, \mathcal{R})$ *denote a distance-regular graph with diameter* $D \geq 3$ *and valency* $k \geq 3$*. Fix a vertex* $x \in X$*, and let* $T = T(x)$ *denote the Terwilliger algebra of* Γ *with respect to* x*. Let* W *denote a thin irreducible* T*-module with endpoint* 1 *and local eigenvalue* η*. Let* W' *denote an irreducible* T*-module. Then the following (i), (ii) are equivalent.*

(i) W *and* W' *are isomorphic as* T*-modules.*

(ii) W' *is thin with endpoint* 1 *and local eigenvalue* η*.*

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix a vertex $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Recall that in Section 3, we said that the standard module V is an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity* of W, we mean the number of irreducible T-modules in the above decomposition which are isomorphic to W. It is well-known that this number is independent of the decomposition of V.

Theorem 4.4 ([14, Theorem 12.9])**.** *Let* $\Gamma = (X, \mathcal{R})$ *denote a distance-regular graph with diameter* $D \geq 3$ *and valency* $k \geq 3$*. Fix a vertex* $x \in X$*, and let* $T = T(x)$ *denote the Terwilliger algebra of* Γ *with respect to* x*. With reference to Definition 4.1, the following are equivalent.*

(i) *For every i $(2 \leq i \leq k)$, there exists a thin irreducible T-module W of endpoint 1 with local eigenvalue η_i. Moreover, the multiplicity with which η_i appears in the list $\eta_2, \eta_3, \dots, \eta_k$ is equal to the multiplicity with which W appears in the standard decomposition of V.*

(ii) *Γ is 1-thin with respect to x.*

With reference to Theorem 4.4, we note that if Γ is 1-thin with respect to x, then the number of non-isomorphic irreducible T-modules of endpoint 1 is equal to the number of distinct local eigenvalues in the list $\eta_2, \eta_3, \dots, \eta_k$. We will need this fact later in the paper.

5 The matrices L, F, R

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$. Fix a vertex $x \in X$. In this section we recall certain matrices L, F, R of the Terwilliger algebra $T = T(x)$.

Definition 5.1. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and adjacency matrix A. Fix a vertex $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$ denote the dual idempotents with respect to x. We define matrices $L = L(x)$, $F = F(x)$, $R = R(x)$ by

$$L = \sum_{h=1}^{D} E_{h-1}^* A E_h^*, \qquad F = \sum_{h=0}^{D} E_h^* A E_h^*, \qquad R = \sum_{h=0}^{D-1} E_{h+1}^* A E_h^*.$$

Note that $A = L + F + R$ [3, Lemma 4.4]. We call L, F, and R the *lowering matrix*, the *flat matrix*, and the *raising matrix of* Γ with respect to x, respectively.

Lemma 5.2. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. We fix $x \in X$ and let $L = L(x)$, $F = F(x)$ and $R = R(x)$ be as in Definition 5.1. For $y, z \in X$ the following (i)–(iii) hold.*

(i) *$L_{zy} = 1$ if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) - 1$, and 0 otherwise.*

(ii) *$F_{zy} = 1$ if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y)$, and 0 otherwise.*

(iii) *$R_{zy} = 1$ if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) + 1$, and 0 otherwise.*

Proof. Immediate from Definition 5.1 and elementary matrix multiplication. □

With the notation of Lemma 5.2, we display the (z, y)-entry of certain products of the matrices L, F and R. To do this we need another definition.

A sequence of vertices $[y_0, y_1, \dots, y_t]$ of Γ is a *walk* in Γ if $y_{i-1}y_i$ is an edge for $1 \leq i \leq t$.

Lemma 5.3. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. We fix $x \in X$ and let $L = L(x)$, $F = F(x)$ and $R = R(x)$ be as in Definition 5.1. Choose $y, z \in X$ and let m denote a positive integer. Assume that $y \in \Gamma_i(x)$. Then the following (i)–(vi) hold.*

(i) *The (z, y)-entry of R^m is equal to the number of walks $[y = y_0, y_1, \dots, y_m = z]$, such that $y_j \in \Gamma_{i+j}(x)$ for $0 \leq j \leq m$.*

(ii) *The (z, y)-entry of $R^m L$ is equal to the number of walks $[y = y_0, y_1, \dots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i-2+j}(x)$ for $1 \leq j \leq m + 1$.*

(iii) *The (z,y)-entry of LR^m is equal to the number of walks $[y = y_0, y_1, \dots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i+j}(x)$ for $0 \leq j \leq m$ and $y_{m+1} \in \Gamma_{i+m-1}(x)$.*

(iv) *The (z,y)-entry of R^mF is equal to the number of walks $[y = y_0, y_1, \dots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i-1+j}(x)$ for $1 \leq j \leq m+1$.*

(v) *The (z,y)-entry of FR^m is equal to the number of walks $[y = y_0, y_1, \dots, y_{m+1} = z]$, such that $y_j \in \Gamma_{i+j}(x)$ for $0 \leq j \leq m$ and $y_{m+1} \in \Gamma_{i+m}(x)$.*

(vi) *The (z,y)-entry of F^m is equal to the number of walks $[y = y_0, y_1, \dots, y_m = z]$, such that $y_j \in \Gamma_i(x)$ for $0 \leq j \leq m$.*

Proof. Immediate from Lemma 5.2 and elementary matrix multiplication. □

6 The sets D^i_j

Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$. In this section we display a certain partition of X that we find useful.

Definition 6.1. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Pick $x \in X$ and $y \in \Gamma(x)$. For $0 \leq i, j \leq D$ we define $D^i_j = D^i_j(x,y)$ by

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(y).$$

For notational convenience we set $D^i_j = \emptyset$ if i or j is contained in $\{-1, D+1\}$. Please refer to Figure 1 for a diagram of this partition.

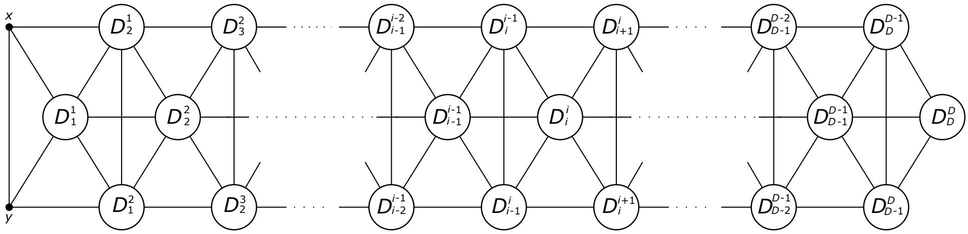

Figure 1: The partition with reference to Definition 6.1.

We now recall some properties of sets D^i_j.

Lemma 6.2 ([10, Lemma 4.2])**.** *With reference to Definition 6.1 the following (i), (ii) hold for $0 \leq i, j \leq D$.*

(i) $|D^i_j| = p^1_{ij}$.

(ii) $D^i_j = \emptyset$ *if and only if* $p^1_{ij} = 0$.

Observe that for $1 \leq i \leq D$ we have $p^1_{i,i-1} = c_i k_i / k \neq 0$ by [2, p. 134]. Therefore, D^i_{i-1} and D^{i-1}_i are nonempty for $1 \leq i \leq D$.

Lemma 6.3 ([9, Lemma 2.11])**.** *With reference to Definition 6.1 pick an integer i ($1 \leq i \leq D$). Then the following (i), (ii) hold.*

(i) Each $z \in D^i_{i-1}$ (resp. D^{i-1}_i) is adjacent to

(a) *precisely*	c_{i-1}	*vertices in D^{i-1}_{i-2} (resp. D^{i-2}_{i-1}),*
(b) *precisely*	$c_i - c_{i-1} - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^{i-1}_i (resp. D^i_{i-1}),*
(c) *precisely*	$a_{i-1} - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^i_{i-1} (resp. D^{i-1}_i),*
(d) *precisely*	b_i	*vertices in D^{i+1}_i (resp. D^i_{i+1}),*
(e) *precisely*	$a_i - a_{i-1} + \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^i_i.*

(ii) Each $z \in D^i_i$ is adjacent to

(a) *precisely*	$c_i - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^i_{i-1},*
(b) *precisely*	$c_i - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^{i-1}_i,*
(c) *precisely*	$b_i - \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in D^{i+1}_i,*
(d) *precisely*	$b_i - \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in D^i_{i+1},*
(e) *precisely*	$a_i - b_i - c_i + \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert + \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in D^i_i.*

In view of the above lemma we have the following definition.

Definition 6.4. With reference to Definition 6.1, for $1 \le i \le D$ we define maps $H_i \colon D^i_i \to \mathbb{Z}$, $K_i \colon D^i_i \to \mathbb{Z}$ and $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma(z) \cap D^{i-1}_{i-1}|, \qquad K_i(z) = |\Gamma(z) \cap D^{i+1}_{i+1}|, \quad V_i(z) = |\Gamma(z) \cap D^{i-1}_{i-1}|.$$

We have the following observation.

Lemma 6.5. *With reference to Definition 6.4, fix an integer i ($2 \le i \le D$) and assume that there exist integers m_1, m_2, such that $V_i(z) = m_1$ for every $z \in D^{i-1}_i$ and $V_i(z) = m_2$ for every $z \in D^i_{i-1}$. Then $m_1 = m_2$.*

Proof. By Lemma 6.3(i) and using a simple double-counting argument we find that

$$|D^{i-1}_i|(c_i - c_{i-1} - m_1) = |D^i_{i-1}|(c_i - c_{i-1} - m_2).$$

As $|D^{i-1}_i| = |D^i_{i-1}| \neq 0$ by the comment below Lemma 6.2, the result follows. □

For the rest of the paper we assume the following situation.

Definition 6.6. Let $\Gamma = (X, \mathcal{R})$ denote a non-bipartite distance-regular graph with diameter $D \ge 3$, valency $k \ge 3$, and distance matrices A_i ($0 \le i \le D$). We abbreviate $A := A_1$. Fix $x \in X$ and let $E^*_i = E^*_i(x)$ ($0 \le i \le D$) denote the dual idempotents with respect to x. Let $T = T(x)$ denote the Terwilliger algebra with respect to x. Let $\Delta = \Delta(x)$ be as in Definition 4.1. Let matrices $L = L(x), F = F(x), R = R(x)$ be as defined in Definition 5.1. For $y \in \Gamma(x)$, let sets $D^i_j(x,y)$ ($0 \le i,j \le D$) and the corresponding maps H_i, K_i, V_i ($1 \le i \le D$) be as defined in Definition 6.1 and Definition 6.4. We assume that for every $y \in \Gamma(x)$ and for every $2 \le i \le D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. We denote the constant value of H_i (V_i, respectively) by h_i (v_i, respectively). We further assume that $h_i \neq 0$ for $2 \le i \le D$.

Remark 6.7. With reference to Definition 6.6, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$ ($0 \le i,j \le D$). Since Γ is assumed to be non-bipartite, $a_j \neq 0$ for some integer j ($1 \le j \le D$). It follows that $D^j_j \neq \emptyset$ by Lemma 6.2(ii) and [2, p. 127]. But since each

$h_i \neq 0$ ($2 \leq i \leq D$), we conclude each of sets $D^{j-1}_{j-1}, D^{j-2}_{j-2}, \ldots, D^1_1$ is nonempty. Since $D^1_1 \neq \emptyset$, we have $a_1 \neq 0$. Now by [2, Proposition 5.5.1], we find $a_i \neq 0$ for $1 \leq i \leq D-1$. Thus $D^i_i \neq \emptyset$ for $1 \leq i \leq D-1$. However, with our assumptions of Definition 6.6, it is possible that $a_D = 0$ and $D^D_D = \emptyset$. In this case, we make the convention that $h_D := 1$. Finally, we wish to make clear that while we are assuming the maps H_i and V_i are constant for $2 \leq i \leq D$, we are not making any such global assumptions about the maps K_i.

7 Some products in T

With reference to Definition 6.6, in this section we display the values of the entries of certain products in T.

Lemma 7.1. *With reference to Definition 6.6, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$ ($0 \leq i,j \leq D$). Pick an integer i ($1 \leq i \leq D$), and let $z \in \Gamma_i(x)$. Then the following (i)–(iii) hold.*

(i) $$(R^{i-1})_{zy} = \begin{cases} c_{i-1}c_{i-2}\cdots c_1 & \textit{if } z \in D^i_{i-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) $(R^iL)_{zy} = c_ic_{i-1}\cdots c_1$.

(iii) $$(LR^i)_{zy} = \begin{cases} b_ic_ic_{i-1}\cdots c_1 & \textit{if } z \in D^i_{i-1}, \\ (b_i - K_i(z))c_ic_{i-1}\cdots c_1 & \textit{if } z \in D^i_i, \\ (c_{i+1} - c_i - v_{i+1})c_ic_{i-1}\cdots c_1 & \textit{if } z \in D^i_{i+1}. \end{cases}$$

Proof. First we observe that, by the triangle inequality, we have $\partial(y,z) \in \{i-1, i, i+1\}$.

(i): By Lemma 5.3(i), the (z,y)-entry of R^{i-1} is equal to the number of walks $[y = y_0, y_1, \ldots, y_{i-1} = z]$, such that $y_j \in \Gamma_{1+j}(x)$ for $0 \leq j \leq i-1$. Observe that there are no such walks if $\partial(y,z) \geq i$. If $\partial(y,z) = i-1$, then it is easy to see that $y_j \in \Gamma_{j+1}(x) \cap \Gamma_j(y) = D^{j+1}_j$ for $0 \leq j \leq i-1$. Lemma 6.3(i) now implies that the number of such walks is equal to $c_{i-1}c_{i-2}\cdots c_1$.

(ii): By Lemma 5.3(ii), the (z,y)-entry of R^iL is equal to the number of walks $[y = y_0, y_1, \ldots, y_{i+1} = z]$, such that $y_j \in \Gamma_{j-1}(x)$ for $1 \leq j \leq i+1$. Observe that this implies that $y_1 = x$. On the other hand, since $z \in \Gamma_i(x)$, there are $c_ic_{i-1}\cdots c_1$ walks $[x = y_1, y_2, \ldots, y_{i+1} = z]$, such that $y_j \in \Gamma_{j-1}(x)$ for $1 \leq j \leq i+1$. The result follows.

(iii): By Lemma 5.3(iii), the (z,y)-entry of LR^i is equal to the number of walks $[y = y_0, y_1, \ldots, y_{i+1} = z]$, such that $y_j \in \Gamma_{j+1}(x)$ for $0 \leq j \leq i$. It follows that $y_j \in D^{j+1}_j$ for $0 \leq j \leq i$. Furthermore, observe that by Lemma 6.3, z has exactly $c_{i+1} - c_i - v_{i+1}$ neighbours in D^{i+1}_i if $\partial(y,z) = i+1$ (that is, if $z \in D^i_{i+1}$), exactly $b_i - K_i(z)$ neighbours in D^{i+1}_i if $\partial(y,z) = i$ (that is, if $z \in D^i_i$), and exactly b_i neighbours in D^{i+1}_i if $\partial(y,z) = i-1$ (that is, if $z \in D^i_{i-1}$). Moreover, by Lemma 6.3(i), for any vertex $y_i \in D^{i+1}_i$, the number of walks $[y = y_0, y_1, \ldots, y_i]$, such that $y_j \in D^{j+1}_j$ for $0 \leq j \leq i$, is equal to $c_ic_{i-1}\cdots c_1$. The result follows. □

Lemma 7.2. *With reference to Definition 6.6, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$ ($0 \leq i,j \leq D$). Pick an integer i ($1 \leq i \leq D$), and let $z \in \Gamma_i(x)$. Then the following (i), (ii) hold.*

$$\textit{(i)}\ \ (R^{i-1}F)_{zy} = \begin{cases} \sum_{j=1}^{i-1} c_{i-1}c_{i-2}\cdots c_{j+1}v_{j+1}h_jh_{j-1}\cdots h_2 & \textit{if } z \in D^i_{i-1}, \\ h_ih_{i-1}\cdots h_2 & \textit{if } z \in D^i_i, \\ 0 & \textit{if } z \in D^i_{i+1}. \end{cases}$$

$$\textit{(ii)}\ \ (FR^{i-1})_{zy} = \begin{cases} (a_{i-1}-v_i)c_{i-1}c_{i-2}\cdots c_1 & \textit{if } z \in D^i_{i-1}, \\ (c_i-h_i)c_{i-1}c_{i-2}\cdots c_1 & \textit{if } z \in D^i_i, \\ 0 & \textit{if } z \in D^i_{i+1}. \end{cases}$$

Proof. The proof is very similar to the proof of Lemma 7.1, so we omit the details. We only provide a sketch of the proof.

(i): We would like to count the number of walks of length $i-1$ from z to D^1_1. First, this number is 0 if $z \in D^i_{i+1}$. If $z \in D^i_i$, then this walk must pass through sets $D^{i-1}_{i-1}, D^{i-2}_{i-2}, \ldots, D^2_2$. Observe the number of such walks is equal to $h_ih_{i-1}\cdots h_2$. Finally, suppose $z \in D^i_{i-1}$. For any walk of length $i-1$ from z to D^1_1, there must exist some integer $1 \le j \le i-1$ such that this walk passes through sets D^{i-1}_{i-2}, $D^{i-2}_{i-3}, \ldots, D^{j+1}_j$, $D^j_j, D^{j-1}_{j-1}, \ldots, D^2_2, D^1_1$. By Lemma 6.3, the number of such walks (for a fixed j) is $c_{i-1}c_{i-2}\cdots c_{j+1}v_{j+1}h_jh_{j-1}\cdots h_2$. The result follows.

(ii): Here we note that z has 0 neighbours in D^i_{i-1} if $z \in D^i_{i+1}$, $c_i - h_i$ neighbours in D^i_{i-1} if $z \in D^i_i$, and $a_{i-1} - v_i$ neighbours in D^i_{i-1} if $z \in D^i_{i-1}$. Moreover, there are $c_{i-1}c_{i-2}\cdots c_1$ walks of length $i-1$ from each vertex of D^i_{i-1} to y. □

8 Proof of the main result

In this section we will prove our main result. With reference to Definition 6.6, we will show that Γ is 1-thin with respect to x.

Lemma 8.1. *With reference to Definition 6.6, fix an integer i ($1 \le i \le D$). Then there exist scalars λ_1, λ_2 such that*

$$E^*_iFR^{i-1}E^*_1 = \lambda_1E^*_iR^{i-1}E^*_1 + \lambda_2E^*_iR^{i-1}FE^*_1. \tag{8.1}$$

Proof. Let $z, y \in X$. We shall show the (z, y)-entry of both sides of (8.1) agree. Note that we may assume $z \in \Gamma_i(x), y \in \Gamma(x)$; otherwise the (z, y)-entry of both sides of (8.1) is zero. Let $D^\ell_j = D^\ell_j(x, y)$ ($0 \le \ell, j \le D$) and define scalars λ_1, λ_2 as follows:

$$\lambda_1 = a_{i-1} - v_i - \frac{(c_i-h_i)\sum_{j=1}^{i-1} c_{i-1}c_{i-2}\cdots c_{j+1}v_{j+1}h_jh_{j-1}\cdots h_2}{h_ih_{i-1}\cdots h_2},$$
$$\lambda_2 = \frac{(c_i-h_i)c_{i-1}c_{i-2}\cdots c_1}{h_ih_{i-1}\cdots h_2}.$$

Treating separately the cases where $z \in D^i_{i-1}, D^i_i, D^i_{i+1}$, it's now routine using Lemma 7.1(i) and Lemma 7.2 to check that the (z, y)-entry of both sides of (8.1) agree. □

Lemma 8.2. *With reference to Definition 6.6,*

$$E^*_iA_{i-1}E^*_1 = \frac{1}{c_1c_2\cdots c_{i-1}}E^*_iR^{i-1}E^*_1 \qquad (1 \le i \le D). \tag{8.2}$$

Proof. Let $z, y \in X$. Observe the (z, y)-entries of both sides of (8.2) are zero unless $z \in \Gamma_i(x), y \in \Gamma(x)$. When $z \in \Gamma_i(x), y \in \Gamma(x)$, the (z, y)-entries of both sides of (8.2) are equal by (2.2) and Lemma 7.1(i). The result follows. □

Lemma 8.3. *With reference to Definition 6.6, assume $v \in E_1^*V$ is an eigenvector for F. Then*

$$E_i^* A_i E_1^* v \in \operatorname{span}\{R^{i-1}v\} \qquad (1 \le i \le D). \tag{8.3}$$

Proof. We proceed by induction on i. For $i = 1$, the result is immediate since v is an eigenvector for F. Now assume the result is true for a fixed i, $1 \le i \le D-1$. By [2, p. 127],

$$c_{i+1}A_{i+1} = AA_i - a_iA_i - b_{i-1}A_{i-1}.$$

Using this equation, Lemma 3.1, Definition 5.1, and Lemma 8.2, we find

$$\begin{aligned} c_{i+1}E_{i+1}^*A_{i+1}E_1^*v &= E_{i+1}^*AA_iE_1^*v - a_iE_{i+1}^*A_iE_1^*v \\ &= E_{i+1}^*(R+F+L)A_iE_1^*v - \frac{a_i}{c_1c_2\cdots c_i}E_{i+1}^*R^iE_1^*v \\ &= RE_i^*A_iE_1^*v + FE_{i+1}^*A_iE_1^*v - \frac{a_i}{c_1c_2\cdots c_i}E_{i+1}^*R^iE_1^*v. \end{aligned} \tag{8.4}$$

Observe $FE_{i+1}^*A_iE_1^*v = (c_1c_2\cdots c_i)^{-1}E_{i+1}^*FR^iE_1^*v$ by (8.2), and $E_{i+1}^*FR^iE_1^*v \in \operatorname{span}\{R^iv\}$ by Lemma 8.1 and the fact that v is an eigenvector for F. Using this information along with (8.4) and the inductive hypothesis, we find $E_{i+1}^*A_{i+1}E_1^*v \in \operatorname{span}\{R^iv\}$, as desired. □

Lemma 8.4. *With reference to Definition 6.6, let U denote the sum of all T-modules of endpoint* 1*. Assume $v \in E_1^*U$ is an eigenvector for F. Then $Lv = 0$ and $LR^iv \in \operatorname{span}\{R^{i-1}v\}$ for $1 \le i \le D-1$.*

Proof. Since v is contained in a sum of irreducible T-modules of endpoint 1, we find $Lv = 0$. By [5, Propositions 8.3(ii), 8.4], the primary module is the unique irreducible T-module upon which J does not vanish. Thus $JE_1^*v = 0$, and for $1 \le j \le D-1$,

$$\begin{aligned} 0 = E_j^*JE_1^*v &= E_j^*(\sum_{t=0}^{D} A_t)E_1^*v \\ &= E_j^*A_{j-1}E_1^*v + E_j^*A_jE_1^*v + E_j^*A_{j+1}E_1^*v. \end{aligned}$$

Thus $E_j^*A_{j+1}E_1^*v = -E_j^*A_{j-1}E_1^*v - E_j^*A_jE_1^*v$, and so by Lemma 8.2 and Lemma 8.3,

$$E_j^*A_{j+1}E_1^*v \in \operatorname{span}\{R^{j-1}v\} \qquad (1 \le j \le D-1). \tag{8.5}$$

Now fix an integer i ($1 \le i \le D-1$). By [2, p. 127],

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}.$$

Thus

$$E_i^*AA_iE_1^*v = c_{i+1}E_i^*A_{i+1}E_1^*v + a_iE_i^*A_iE_1^*v + b_{i-1}E_i^*A_{i-1}E_1^*v. \tag{8.6}$$

In view of (8.6), and using (8.5), (8.3), (8.2), we find

$$E_i^*AA_iE_1^*v \in \operatorname{span}\{R^{i-1}v\}. \tag{8.7}$$

Now using Definition 5.1 and (8.2),

$$\begin{aligned} E_i^* A A_i E_1^* v &= E_i^*(R+F+L) A_i E_1^* v \\ &= R E_{i-1}^* A_i E_1^* v + F E_i^* A_i E_1^* v + L E_{i+1}^* A_i E_1^* v \\ &= R E_{i-1}^* A_i E_1^* v + F E_i^* A_i E_1^* v + \frac{1}{c_1 c_2 \cdots c_i} L R^i v. \end{aligned}$$

Thus

$$L R^i v = c_1 c_2 \cdots c_i (E_i^* A A_i E_1^* v - R E_{i-1}^* A_i E_1^* v - F E_i^* A_i E_1^* v).$$

Recalling that v is an eigenvector for F, the result now follows from (8.7), (8.5), (8.3), (8.1). □

We now present our main result. With reference to Definition 6.6, let W denote an irreducible T-module of endpoint 1, and observe by Definition 5.1 that $F E_1^* W \subseteq E_1^* W$. Thus, there is a nonzero vector $v \in E_1^* W$ such that v is an eigenvector for F. We shall show W is thin.

Theorem 8.5. *With reference to Definition 6.6, let W denote an irreducible T-module with endpoint 1. Choose nonzero $v \in E_1^* W$ which is an eigenvector for F. Then the following set spans W:*

$$\{v, Rv, R^2 v, \ldots, R^{D-1} v\}. \tag{8.8}$$

In particular, W is thin.

Proof. We first show that W is spanned by the vectors in (8.8). Let W' denote the subspace of V spanned by the vectors in (8.8) and note that $W' \subseteq W$. We claim that W' is T-invariant. Observe that since $R E_j^* V \subseteq E_{j+1}^* V$ for $0 \le j \le D-1$, W' is invariant under the action of E_j^* for $0 \le j \le D$, and so W' is M^*-invariant. By definition and since $R E_D^* V = 0$, W' is invariant under R. From Lemma 8.1, Lemma 8.4, and the fact that v is an eigenvector for F, it follows that W' is also invariant under F and L. Since $A = R + F + L$ and since A generates M, W' is M-invariant. The claim follows. Hence W' is a T-module, and it is nonzero since $v \in W'$. By the irreducibility of W we have that $W' = W$. Since for $0 \le j \le D-1$ we have $R^j v \in E_{j+1}^* W$, it follows that W is thin. □

9 Special case – two modules with endpoint 1

With reference to Definition 6.6, in this section we consider the case where Γ has (up to isomorphism) exactly two irreducible T-modules with endpoint 1. Note that these modules are thin by Theorem 8.5. Observe that in this case it follows from the comments of Section 4 that the local graph $\Delta = \Delta(x)$ has either two or three distinct eigenvalues. In the former case Δ is a disjoint union of complete graphs (with order $a_1 + 1$), while in the latter case Δ is a strongly regular graph (see [8, Chapter 10, Lemma 1.5]). We observe that Δ has one of these two forms if and only if the map K_1 is constant for every $y \in \Gamma(x)$, and this constant does not depend on y.

Proposition 9.1. *With reference to Definition 6.6, assume that Δ is a disjoint union of $k/(a_1+1)$ cliques of order $a_1 + 1$. Let W denote an irreducible T-module with endpoint 1. Then W is thin with local eigenvalue a_1 or -1.*

Proof. Recall that W is thin by Theorem 8.5. Let η denote the local eigenvalue of W, and note that η is an eigenvalue of Δ by the comments of Section 4. But the eigenvalues of Δ are a_1 (with multiplicity $k/(a_1+1) > 1$) and -1 (with multiplicity $k - k/(a_1+1) = ka_1/(a_1+1)$). The result follows. □

Proposition 9.2. *With reference to Definition 6.6, assume that Δ is a connected strongly regular graph with parameters (k, a_1, λ, v_2). Let W denote an irreducible T-module with endpoint* 1. *Then W is thin with local eigenvalue η_2 or η_3, where*

$$\eta_2, \eta_3 = \frac{\lambda - v_2 \pm \sqrt{(\lambda - v_2)^2 + 4(a_1 - v_2)}}{2}. \tag{9.1}$$

Proof. Recall that W is thin by Theorem 8.5. Let η denote the local eigenvalue of W, and recall that η is an eigenvalue of Δ. Therefore, by the well-known formula for the eigenvalues of a connected strongly regular graph, the eigenvalues of $\Gamma(x)$ are $\eta_1 = a_1$ (with multiplicity 1), and scalars η_2, η_3 from (9.1). The result follows. □

Theorem 9.3. *With reference to Definition 6.6, assume that for every $y \in \Gamma(x)$ the map K_1 is constant, and that this constant does not depend on y. Then Γ has (up to isomorphism) exactly two irreducible T-modules with endpoint* 1*, both of which are thin. In particular, for every $1 \leq i \leq D-1$, the map K_i is constant, and this constant does not depend on y (in other words, Γ is pseudo-*1*-homogeneous with respect to x in the sense of Curtin and Nomura [4]).*

Proof. Recall that every irreducible T-module of Γ is thin by Theorem 8.5. Therefore, by Theorem 4.3, two irreducible T-modules with endpoint 1 are isomorphic if and only if they have the same local eigenvalue. As K_1 is constant and this constant does not depend on y, the local graph Δ is either a disjoint union of cliques of order a_1+1, or connected strongly regular graph. The first part of the above theorem now follows from Propositions 9.1 and 9.2. The second part follows from [4, Theorem 1.6]. □

10 Special case – three modules with endpoint 1

With reference to Definition 6.6, in this section we consider the case where Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Note that these modules are thin by Theorem 8.5. It follows from the comments in Section 4 that this situation occurs if and only if the local graph Δ is either disconnected with exactly three distinct eigenvalues, or connected with exactly four distinct eigenvalues. Moreover, Δ is not connected if and only if $v_2 = 0$. But if $v_2 = 0$, then it is easy to see that Δ is a disjoint union of complete graphs (with order a_1+1), and has therefore 2 distinct eigenvalues. This shows that $v_2 \neq 0$, and so Δ is connected with exactly four distinct eigenvalues. To describe this case we need the following definition.

Definition 10.1. With reference to Definition 6.6, for $y \in \Gamma(x)$ let $B = B(y)$ denote the adjacency matrix of the subgraph of Γ induced on D^1_1. Observe that $B \in \mathrm{Mat}_{D^1_1}(\mathbb{C})$, and so the rows and the columns of B are indexed by the elements of D^1_1. Let $\boldsymbol{j} \in \mathbb{C}^{D^1_1}$ denote the all-ones column vector with rows indexed by the elements of D^1_1.

Lemma 10.2. *With reference to Definition 10.1, pick $y \in \Gamma(x)$. Then for every $z \in D^1_1$ we have*

$$K_1(z) = b_1 - a_1 + (B\boldsymbol{j})_z + 1.$$

Proof. Observe that $(B\boldsymbol{j})_z$ is equal to the number of neighbours that z has in D_1^1. Therefore, z has $a_1 - 1 - (B\boldsymbol{j})_z$ neighbours in D_2^1. But as z also has $K_1(z)$ neighbours in D_2^2 and no neighbours in D_2^3, it must have $b_1 - K_1(z)$ neighbours in D_2^1. The result follows. □

With reference to Definition 10.1, we now describe three properties that Γ could have.

Definition 10.3. With reference to Definition 10.1, we denote by P1, P2 and P3 the following properties of Γ:

P1: There exists $y \in \Gamma(x)$ such that K_1 is not a constant.

P2: For every $y, z \in \Gamma(x)$ with $\partial(y, z) \in \{0, 2\}$, the number of walks of length 3 from y to z in graph Δ is a constant number, which depends only on $\partial(y, z)$ (and not on the choice of y, z).

P3: There exist scalars α, β such that for every $y \in \Gamma(x)$ we have

$$B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}.$$

With reference to Definition 10.3, in the rest of this section we prove that Γ has properties P1, P2, P3 if and only if Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1.

Proposition 10.4. *With reference to Definition 10.3, assume that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Then Γ has property P1.*

Proof. Assume on the contrary that K_1 is a constant for every $y \in \Gamma(x)$. We claim that this constant is independent of the choice of $y \in \Gamma(x)$. Pick $y \in \Gamma(x)$ and let $D_j^i = D_j^i(x, y)$. Denote the constant value of $K_1 = K_1(y)$ by $\kappa = \kappa(y)$. Observe that every vertex in D_2^1 has v_2 neighbours in D_1^1, and that every vertex in D_1^1 has $b_1 - \kappa$ neighbours in D_2^1. As $|D_2^1| = b_1$ and $|D_1^1| = a_1$, this gives us $a_1(b_1 - \kappa) = b_1 v_2$. This shows that κ is independent of the choice of $y \in \Gamma(x)$. By Theorem 9.3, Γ has up to isomorphism at most two irreducible modules with endpoint 1, a contradiction. This shows that Γ has property P1. □

Lemma 10.5. *With reference to Definition 10.3, assume that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. Then*

$$E_1^* F^3 E_1^* = E_1^*\big(\mu_1 LR + \mu_2 RL + \mu_3 F + \mu_4 F^2\big)E_1^* \tag{10.1}$$

for some scalars μ_i $(1 \le i \le 4)$.

Proof. By Lemma 3.2, there exist scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, not all zero, such that

$$E_1^*\big(\lambda_1 LR + \lambda_2 RL + \lambda_3 F + \lambda_4 F^2 + \lambda_5 F^3\big)E_1^* = 0. \tag{10.2}$$

We claim that $\lambda_5 \neq 0$. Assume on the contrary that $\lambda_5 = 0$. By Proposition 10.4, there exists $y \in \Gamma(x)$ such that $K_1 = K_1(y)$ is not a constant. Pick such y and let $D_j^i = D_j^i(x, y)$. Let $z \in D_1^1$. We now compute the (z, y)-entry of (10.2). By Lemma 7.1(ii),(iii), the (z, y) entry of $E_1^* LR E_1^*$ ($E_1^* RL E_1^*$, respectively) is $b_1 - K_1(z)$ (1, respectively). By Lemma 5.3(vi), the (z, y)-entry of $E_1^* F E_1^*$ is 1, and the (z, y)-entry of $E_1^* F^2 E_1^*$ is equal to the number of neighbours of z in D^1_1. But by Lemma 10.2, the number of neighbours of z in D^1_1 is equal to $a_1 - 1 - b_1 + K_1(z)$. It follows from the above comments that

$$\lambda_1(b_1 - K_1(z)) + \lambda_2 + \lambda_3 + \lambda_4(a_1 - 1 - b_1 + K_1(z)) = 0.$$

Note that by the assumption the map K_1 is not constant, and so the above equality implies $\lambda_4 = \lambda_1$. Therefore $\lambda_1(a_1 - 1) + \lambda_2 + \lambda_3 = 0$.

We now compute the (y, y)-entry of (10.2). Similarly as above we get

$$\lambda_1(k - 1) + \lambda_2 = 0.$$

Finally, pick $z \in D^1_2$. By computing the (y, z)-entry of (10.2) we get

$$\lambda_1(c_2 - 1) + \lambda_2 = 0.$$

It follows easily from the above equations that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, a contradiction. This shows that $\lambda_5 \neq 0$ and so

$$E^*_1 F^3 E^*_1 = E^*_1 \big(\mu_1 LR + \mu_2 RL + \mu_3 F + \mu_4 F^2\big) E^*_1,$$

where $\mu_i = -\lambda_i/\lambda_5$ for $1 \le i \le 4$. □

Theorem 10.6. *With reference to Definition 10.3, assume that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint* 1*. Then Γ has properties P2 and P3.*

Proof. Note that for every $y, z \in \Gamma(x)$, the (z, y)-entry of $E^*_1 F^3 E^*_1$ is equal to the number of walks of length 3 from y to z in graph Δ. Pick $y, z \in \Gamma(x)$ such that $\partial(y, z) \in \{0, 2\}$. We compute the (z, y)-entry of (10.1). Using Lemma 5.3(vi) and Lemma 7.1(ii),(iii) we find that

$$(E^*_i F^3 E^*_1)_{zy} = \begin{cases} \mu_1 b_1 + \mu_2 + \mu_4 a_1 & \text{if } z = y, \\ \mu_1(c_2 - v_2 - 1) + \mu_2 + \mu_4 v_2 & \text{if } z \neq y. \end{cases}$$

This shows that Γ has property P2.

Pick now $y, z \in \Gamma(x)$ such that $\partial(y, z) = 1$ and let $D^i_j = D^i_j(x, y)$. Let K_1 denote the corresponding map, and let $B = B(y)$. Let $[y = y_0, y_1, y_2, y_3 = z]$ be a walk of length 3 from y to z in Δ. We will say that this walk is of type 0 if $y_2 = y$, of type 1 if $y_2 \in D^1_1$, and of type 2 if $y_2 \in D^1_2$. It is clear that we have a_1 walks of type 0 and $(a_1 - 1 - (B\boldsymbol{j})_z)v_2$ walks of type 2. Similarly, there are $(B^2\boldsymbol{j})_z$ walks of type 1. So there are in total

$$a_1 + (a_1 - 1 - (B\boldsymbol{j})_z)v_2 + (B^2\boldsymbol{j})_z$$

walks of length 3 from y to z in Δ.

We now compute the (z, y)-entry of the right side of (10.1). Using Lemma 7.1(iii) and Lemma 10.2, we find that the (z, y)-entry of $E^*_1 LR E^*_1$ is equal to

$$b_1 - K_1(z) = a_1 - (B\boldsymbol{j})_z - 1.$$

It is easy to see that the (z, y)-entries of $E^*_1 RL E^*_1$ and $E^*_1 F E^*_1$ are both equal to 1. Finally, the (z, y)-entry of $E^*_1 F^2 E^*_1$ is equal to the number of neighbours of z in D^1_1, that is to $(B\boldsymbol{j})_z$. It now follows from the above comments that

$$a_1 + (a_1 - 1 - (B\boldsymbol{j})_z)v_2 + (B^2\boldsymbol{j})_z = \mu_1(a_1 - (B\boldsymbol{j})_z - 1) + \mu_2 + \mu_3 + \mu_4(B\boldsymbol{j})_z.$$

This shows that

$$(B^2\boldsymbol{j})_z = \alpha(B\boldsymbol{j})_z + \beta$$

for some scalars α, β, which are independent of the choice of vertices y, z. This proves that Γ has property P3. □

We now assume that Γ has properties P1, P2 and P3. We will show that this implies that Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1.

Definition 10.7. With reference to Definition 10.3, assume that Γ has properties P1, P2 and P3, and recall that $\breve{X} = \Gamma(x)$. Recall also that for any $y, z \in \breve{X}$ with $\partial(y, z) \in \{0, 2\}$, the number of walks of length 3 from y to z in Δ is a constant number, which depends just on the distance between y and z. We denote this number by w_0 if $y = z$ and by w_2 if $\partial(y, z) = 2$. Recall that $\breve{A} = \breve{A}(x) \in \mathrm{Mat}_{\breve{X}}(\mathbb{C})$ denotes the adjacency matrix of Δ. Furthermore, let $\breve{I}$ denote the identity matrix of $\mathrm{Mat}_{\breve{X}}(\mathbb{C})$ and let $\breve{J}$ denote the all-ones matrix of $\mathrm{Mat}_{\breve{X}}(\mathbb{C})$.

We now display the entries of $\breve{A}$, $\breve{A}^2$ and $\breve{A}^3$.

Proposition 10.8. *With reference to Definition 10.7, the following (i)–(iii) hold for all* $z, y \in \breve{X}$.

(i)

$$(\breve{A})_{zy} = \begin{cases} 1 & \text{if } \partial(y, z) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$(\breve{A}^2)_{zy} = \begin{cases} a_1 & \text{if } y = z, \\ (B\boldsymbol{j})_z & \text{if } \partial(y, z) = 1, \\ v_2 & \text{if } \partial(y, z) = 2, \end{cases}$$

where $B = B(y)$.

(iii)

$$(\breve{A}^3)_{zy} = \begin{cases} w_0 & \text{if } y = z, \\ a_1 + v_2(a_1 - 1) + (B\boldsymbol{j})_z(\alpha - v_2) + \beta & \text{if } \partial(y, z) = 1, \\ w_2 & \text{if } \partial(y, z) = 2, \end{cases}$$

where $B = B(y)$ *and* α, β *are from Definition 10.3.*

Proof. Recall that for $i \geq 0$, the (z, y)-entry of $\breve{A}^i$ is equal to the number of walks of length i from y to z in Δ. Parts (i), (ii) follow. We now prove part (iii).

Note that the result is clear if $y = z$ or if $\partial(y, z) = 2$. Therefore, assume $\partial(y, z) = 1$. Similarly as in the proof of Theorem 10.6, we split the walks of length 3 between y and z into three types, depending on whether the third vertex of the walk is equal to y, or is a neighbour of y, or is at distance 2 from y. There are a_1 walks of the first type, $(B^2\boldsymbol{j})_z$ walks of the second type, and $(a_1 - 1 - (B\boldsymbol{j})_z)v_2$ walks of the third type. Recall that by property P3 we have $B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}$, and so the result follows. □

Proposition 10.9. *With reference to Definition 10.7, we have*

$$\begin{aligned}\breve{A}^3 = (\alpha - v_2)\breve{A}^2 &+ (a_1 + \beta + v_2(a_1 - 1 + \alpha - v_2) - w_2)\breve{A} \\ &+ (w_0 - w_2 + (\alpha - v_2)(v_2 - a_1))\breve{I} + (w_2 - (\alpha - v_2)v_2)\breve{J},\end{aligned} \tag{10.3}$$

where α, β *are from Definition 10.3.*

Proof. Pick $y, z \in \breve{X}$. It follows from Proposition 10.8 that the (z, y)-entry of the left side and the right side of (10.3) agree. This proves the proposition. □

Theorem 10.10. *With reference to Definition 10.7,* Δ *has exactly four distinct eigenvalues.*

Proof. Observe that Δ is connected and regular with valency a_1, so a_1 is an eigenvalue of Δ with multiplicity 1. The corresponding eigenvector is the all-ones vector in $\mathbb{C}^{\breve{X}}$, which we denote by $\breve{\boldsymbol{j}}$. Let θ denote an eigenvalue of Δ which is different from a_1, and let $\boldsymbol{w}$ denote a corresponding eigenvector. Note that $\boldsymbol{w}$ and $\breve{\boldsymbol{j}}$ are orthogonal, and so applying (10.3) to $\boldsymbol{w}$ we get

$$\begin{aligned}\theta^3\boldsymbol{w} = (\alpha - v_2)\theta^2\boldsymbol{w} &+ (a_1 + \beta + v_2(a_1 - 1 + \alpha - v_2) - w_2)\theta\boldsymbol{w} \\ &+ (w_0 - w_2 + (\alpha - v_2)(v_2 - a_1))\boldsymbol{w}.\end{aligned}$$

As $\boldsymbol{w}$ is nonzero, we have

$$\theta^3 = (\alpha - v_2)\theta^2 + (a_1 + \beta + v_2(a_1 - 1 + \alpha - v_2) - w_2)\theta + w_0 - w_2 + (\alpha - v_2)(v_2 - a_1).$$

This shows that Δ could have at most four different eigenvalues. Now if Δ has fewer than four different eigenvalues, then Δ is strongly regular [8, Chapter 10, Lemma 1.5], and so $(B\boldsymbol{j})_z$ is constant for every $y, z \in \breve{X}$ with $z \in \Gamma(y)$, where $B = B(y)$ and $\boldsymbol{j}$ is from Definition 10.1. By Lemma 10.2, K_1 is constant for every $y \in \breve{X}$, contradicting property P1. □

Theorem 10.11. *With reference to Definition 10.7,* Γ *has (up to isomorphism) exactly three irreducible* T*-modules with endpoint* 1*.*

Proof. Recall that Γ is 1-thin with respect to x by Theorem 8.5. The result now follows from Theorems 4.3, 4.4, and 10.10. □

11 Example: Johnson graphs

Pick a positive integer $n \geq 2$ and let m denote an integer ($0 \leq m \leq n$). The vertices of the Johnson graph $J(n, m)$ are the m-element subsets of $\{1, 2, \ldots, n\}$. Vertices x, y are adjacent if and only if the cardinality of $x \cap y$ is equal to $m - 1$. It follows that if x, y are arbitrary vertices of $J(n, m)$, then $\partial(x, y) = m - |x \cap y|$. Therefore, the diameter D of $J(n, m)$ is equal to $\min\{m, n - m\}$. Recall that $J(n, m)$ is distance-transitive (see [2, Theorem 9.1.2]), and so it is also distance-regular. It is well known that $J(n, m)$ is isomorphic to $J(n, n - m)$, so we will assume that $m \leq n/2$, which implies $D = m$. In fact, if n is even and $m = n/2$, then $J(2m, m)$ is 1-homogeneous (see [9]), and so we assume from here on that $m < n/2$. As we are also assuming that $D \geq 3$, we therefore have $m \geq 3$, $n \geq 7$. For more details on Johnson graphs, see [2, Section 9.1].

Pick adjacent vertices x, y of $J(n,m)$, and let $D_j^i = D_j^i(x,y)$ be as defined in Definition 6.1. For $1 \leq i \leq D$ let maps H_i, K_i and V_i be as defined in Definition 6.4. The main purposes of this section are to describe maps H_i, K_i and V_i in detail and to show $J(n,m)$ satisfies the assumptions of Definitions 6.6 and 10.7. As $J(n,m)$ is distance-transitive, it is also arc-transitive, and so we can assume that $x = \{1,2,\ldots,m\}$, $y = \{2,3,\ldots,m+1\}$. We start with a description of the sets D_j^i.

Proposition 11.1. *Pick positive integers n and m with $n \geq 7$, $3 \leq m < n/2$, and let $x = \{1,2,\ldots,m\}$, $y = \{2,3,\ldots,m+1\}$ be adjacent vertices of $J(n,m)$. Let $D_j^i = D_j^i(x,y)$ be as defined in Definition 6.1. Then for $1 \leq i \leq D$, the set D_i^{i-1} (D_{i-1}^i, respectively) consists of vertices of the form $\{1\} \cup A \cup B$ ($\{m+1\} \cup A \cup B$, respectively), where $A \subseteq \{2,3,\ldots,m\}$ with $|A| = m-i$ and $B \subseteq \{m+2,m+3,\ldots,n\}$ with $|B| = i-1$.*

Proof. Routine. □

To describe sets D_i^i, we need the following definition.

Definition 11.2. Pick positive integers n and m with $n \geq 7$, $3 \leq m < n/2$, and let $x = \{1,2,\ldots,m\}$, $y = \{2,3,\ldots,m+1\}$ be adjacent vertices of $J(n,m)$.

(i) For $1 \leq i \leq D-1$, define set $D_i^i(0)$ to be the set of vertices of the form $\{1,m+1\} \cup A \cup B$, where $A \subseteq \{2,3,\ldots,m\}$ with $|A| = m-i-1$ and $B \subseteq \{m+2,m+3,\ldots,n\}$ with $|B| = i-1$. We define $D_0^0(0) = D_D^D(0) = \emptyset$.

(ii) For $1 \leq i \leq D$, define set $D_i^i(1)$ to be the set of vertices of the form $A \cup B$, where $A \subseteq \{2,3,\ldots,m\}$ with $|A| = m-i$, and $B \subseteq \{m+2,m+3,\ldots,n\}$ with $|B| = i$. We define $D_0^0(1) = \emptyset$.

Please refer to Figure 2 for a diagram of this partition.

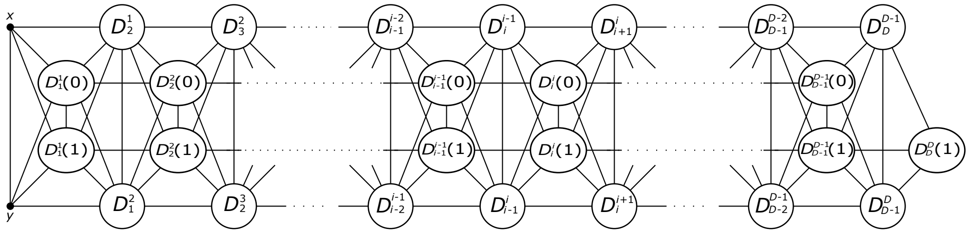

Figure 2: The partition with reference to Definition 11.2. For further information about which sets in the diagram are connected by edges, please refer to the propositions and corollaries later in this section.

Proposition 11.3. *Pick positive integers n and m with $n \geq 7$, $3 \leq m < n/2$, and let $x = \{1,2,\ldots,m\}$, $y = \{2,3,\ldots,m+1\}$ be adjacent vertices of $J(n,m)$. Let $D_j^i = D_j^i(x,y)$ be as defined in Definition 6.1 and let $D_i^i(0)$, $D_i^i(1)$ be as in Definition 11.2. Then for $1 \leq i \leq D-1$ we have that D_i^i is a disjoint union of $D_i^i(0)$ and $D_i^i(1)$. Moreover, $D_D^D = D_D^D(1)$.*

Proof. Routine. □

We now first describe the maps V_i.

Proposition 11.4. *With the notation of Proposition 11.3, let the maps V_i be as defined in Definition 6.4. Then for $1 \leq i \leq D$ and any $z \in D^i_{i-1} \cup D^{i-1}_i$ we have*

$$V_i(z) = 2(i-1).$$

In particular, the maps V_i are constant.

Proof. Note that the result is clear for $i = 1$, so pick $2 \leq i \leq D$ and assume $z \in D^{i-1}_i$ (case $z \in D^i_{i-1}$ is treated similarly and we omit the details). First recall that by the definition of map V_i and by Proposition 11.3 we have

$$V_i(z) = |\Gamma(z) \cap D^{i-1}_{i-1}| = |\Gamma(z) \cap D^{i-1}_{i-1}(0)| + |\Gamma(z) \cap D^{i-1}_{i-1}(1)|.$$

Recall also that by Proposition 11.1 there exist subsets $A \subseteq \{2, 3, \ldots, m\}$ with $|A| = m-i$ and $B \subseteq \{m+2, m+3, \ldots, n\}$ with $|B| = i-1$, such that $z = \{1\} \cup A \cup B$. We first count the number of neighbours of z in $D^{i-1}_{i-1}(1)$. As vertices contained in $D^{i-1}_{i-1}(1)$ do not contain the number 1 as an element, vertex $w \in D^{i-1}_{i-1}(1)$ will be adjacent with z if and only if

$$w = A \cup B \cup \{\ell\}$$

for some $\ell \in \{2, 3, \ldots, m\} \setminus A$. Therefore, there are exactly $m - 1 - (m - i) = i - 1$ neighbours of z in $D^{i-1}_{i-1}(1)$. We now count the number of neighbours of z in $D^{i-1}_{i-1}(0)$. As vertices contained in $D^{i-1}_{i-1}(0)$ contain numbers 1 and $m+1$ as elements, vertex $w \in D^{i-1}_{i-1}(0)$ will be adjacent with z if and only if

$$w = (\{1, m+1\} \cup A \cup B) \setminus \{\ell\}$$

for some $\ell \in B$. Therefore, there are exactly $i - 1$ neighbours of z in $D^{i-1}_{i-1}(0)$. The result follows. $\square$

Proposition 11.5. *With the notation of Proposition 11.3, for $1 \leq i \leq D-1$ and for any $z \in D^i_i(0)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D^{i-1}_{i-1}(0)| = i(i-1)$.

(ii) $|\Gamma(z) \cap D^{i-1}_{i-1}(1)| = 0$.

Proof. Note that the result is clear for $i = 1$, so pick $2 \leq i \leq D-1$ and $z \in D^i_i(0)$. Recall that $z = \{1, m+1\} \cup A \cup B$ for some subsets $A \subseteq \{2, 3, \ldots, m\}$ with $|A| = m-i-1$ and $B \subseteq \{m+2, m+3, \ldots, n\}$ with $|B| = i-1$.

(i): Note that $w \in D^{i-1}_{i-1}(0)$ is adjacent with z if and only if $w = \{1, m+1\} \cup A' \cup B'$, where $A' = A \cup \{\ell_1\}$ for some $\ell_1 \in \{2, 3, \ldots, m\} \setminus A$ and $B' = B \setminus \{\ell_2\}$ for some $\ell_2 \in B$. We have $m - 1 - (m - i - 1) = i$ choices for ℓ_1 and $i - 1$ choices for ℓ_2. It follows that z has $i(i-1)$ neighbours in $D^{i-1}_{i-1}(0)$.

(ii): Recall that if w is an element of $D^{i-1}_{i-1}(1)$, then 1 and $m+1$ are not elements of w. On the other hand, 1 and $m+1$ are elements of z, and so z and w are not adjacent. $\square$

Proposition 11.6. *With the notation of Proposition 11.3, for $1 \leq i \leq D$ and for any $z \in D^i_i(1)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i-1}^{i-1}(1)| = i(i-1)$.

(ii) $|\Gamma(z) \cap D_{i-1}^{i-1}(0)| = 0$.

Proof. Similar to the proof of Proposition 11.5. □

Corollary 11.7. *With the notation of Proposition 11.3, let the maps H_i be as defined in Definition 6.4. Then for $1 \leq i \leq D$ and any $z \in D_i^i$ we have*

$$H_i(z) = i(i-1).$$

In particular, the maps H_i are constant.

Proof. Immediate from Propositions 11.5 and 11.6 and since D_i^i is a disjoint union of $D_i^i(0)$ and $D_i^i(1)$. □

Proposition 11.8. *With the notation of Proposition 11.3, for $1 \leq i \leq D-1$ and for any $z \in D_i^i(0)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i+1}^{i+1}(0)| = (m-i-1)(n-m-i)$.

(ii) $|\Gamma(z) \cap D_{i+1}^{i+1}(1)| = 0$.

Proof. Pick $1 \leq i \leq D-1$ and $z \in D_i^i(0)$. Recall that $z = \{1, m+1\} \cup A \cup B$ for some subsets $A \subseteq \{2, 3, \ldots, m\}$ with $|A| = m-i-1$ and $B \subseteq \{m+2, m+3, \ldots, n\}$ with $|B| = i-1$.

(i): Note that $w \in D_{i+1}^{i+1}(0)$ is adjacent with z if and only if $w = \{1, m+1\} \cup A' \cup B'$, where $A' = A \setminus \{\ell_1\}$ for some $\ell_1 \in A$ and $B' = B \cup \{\ell_2\}$ for some $\ell_2 \in \{m+2, m+3, \ldots, n\} \backslash B$. We therefore have $m-i-1$ choices for ℓ_1 and $(n-m-1)-(i-1) = n-m-i$ choices for ℓ_2. It follows that z has $(m-i-1)(n-m-i)$ neighbours in $D_{i+1}^{i+1}(0)$.

(ii): Immediate from Proposition 11.6(ii). □

Proposition 11.9. *With the notation of Proposition 11.3, for $1 \leq i \leq D-1$ and for any $z \in D_i^i(1)$ the following (i), (ii) hold.*

(i) $|\Gamma(z) \cap D_{i+1}^{i+1}(1)| = (m-i)(n-m-i-1)$.

(ii) $|\Gamma(z) \cap D_{i+1}^{i+1}(0)| = 0$.

Proof. Similar to the proof of Proposition 11.8. □

Corollary 11.10. *With the notation of Proposition 11.3, let the maps K_i be as defined in Definition 6.4. Then for $1 \leq i \leq D-1$ and any $z \in D_i^i$ we have*

$$K_i(z) = \begin{cases} (m-i-1)(n-m-i) & \text{if } z \in D_i^i(0), \\ (m-i)(n-m-i-1) & \text{if } z \in D_i^i(1). \end{cases}$$

In particular, maps K_i are not constant.

Proof. The first part of the corollary follows immediately from Propositions 11.8 and 11.9 and since D_i^i is a disjoint union of $D_i^i(0)$ and $D_i^i(1)$. For the second part, observe that if K_i is a constant, then we have $n = 2m$, contradicting our assumption $m < n/2$. □

Proposition 11.11. *With the notation of Proposition 11.3, the following (i)–(iii) hold.*

(i) Every $z \in D^1_2$ has 1 neighbour in $D^1_1(0)$, 1 neighbour in $D^1_1(1)$, and $n-4$ neighbours in D^1_2.

(ii) Every $z \in D^1_1(0)$ has $n-m-1$ neighbours in D^1_2, $m-2$ neighbours in $D^1_1(0)$, and no neighbours in $D^1_1(1)$.

(iii) Every $z \in D^1_1(1)$ has $m-1$ neighbours in D^1_2, $n-m-2$ neighbours in $D^1_1(1)$, and no neighbours in $D^1_1(0)$.

Consequently, the partition $\{\{y\}, D^1_1(0), D^1_1(1), D^1_2\}$ of $\Gamma(x)$ is equitable.

Proof. First observe that it follows from the proof of Proposition 11.4 that each $z \in D^1_2$ has 1 neighbour in $D^1_1(0)$ and 1 neighbour in $D^1_1(1)$. Consequently, z has $a_1 - 2 = n - 4$ neighbours in D^1_2. Next observe that each vertex from $D^1_1(0)$ contains 1 and $m+1$ as elements, while 1 and $m+1$ are not elements of any vertex from $D^1_1(1)$. Consequently, there are no edges between vertices of $D^1_1(0)$ and $D^1_1(1)$. Furthermore, by Corollary 11.10, each vertex in $D^1_1(0)$ has $(m-2)(n-m-1)$ neighbours in D^2_2, while each vertex in $D^1_1(1)$ has $(m-1)(n-m-2)$ neighbours in D^2_2. The other claims of the above proposition now follow from the fact that intersection numbers a_1 and b_1 of $J(n,m)$ are equal to $n-2$ and $(m-1)(n-m-1)$, respectively. □

Theorem 11.12. *Pick positive integers n and m with $n \geq 7$, $3 \leq m < n/2$, and let $\Gamma = J(n,m)$. Pick $x \in V(\Gamma)$ and let $T = T(x)$. Then Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1, and these modules are all thin.*

Proof. As Γ is arc transitive, it follows from Proposition 11.4 and Corollary 11.7 that maps V_i and H_i $(2 \leq i \leq D)$ are constant for every $y \in \Gamma(x)$, and that these constants are nonzero and independent of the choice of y. By Theorem 8.5, Γ is 1-thin. By Corollary 11.10, the map K_1 is not constant for any $y \in \Gamma(x)$. Pick $y, z \in \Gamma(x)$ and let $B = B(y)$ be as defined in Definition 10.1. It follows from Proposition 11.11 that the number of walks of length 3 from y to z in $\Delta = \Delta(x)$ depends only on the distance between y and z when $\partial(y,z) \in \{0,2\}$. Finally, by Proposition 11.11 we also have that $B^2\boldsymbol{j} = \alpha B\boldsymbol{j} + \beta\boldsymbol{j}$, where $\alpha = n-4$, $\beta = -(n-m-2)(m-2)$, and $\boldsymbol{j}$ is from Definition 10.1. Therefore Γ has properties P1, P2 and P3, and so, by Theorem 10.11, Γ has (up to isomorphism) exactly three irreducible T-modules with endpoint 1. □

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Sums of r-Lah numbers and r-Lah polynomials

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Abstract

The total number of partitions of a finite set into nonempty ordered subsets such that r distinguished elements belong to distinct ordered blocks can be described as sums of r-Lah numbers. In this paper we study this possible variant of Bell-like numbers, as well as the related r-Lah polynomials.

Keywords: Summed r-Lah numbers, r-Lah polynomials.

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1 Introduction

Bell numbers play a crucial role in enumerative combinatorics. The nth Bell number B_n counts the number of partitions of an n-element set, or in other words, it is the sum of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ $(k = 0, \ldots, n)$. In connection with these numbers, it is possible to introduce the nth Bell polynomial

$$B_n(x) = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} x^j,$$

whose value at 1 is simply $B_n(1) = B_n$. (These polynomials should not be confused with partial Bell polynomials which are multivariate polynomials.)

Using r-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r$ defined by L. Carlitz [5], A. Z. Broder [4], and later rediscovered by R. Merris [12], I. Mező [13, 14] introduced and investigated the corresponding r-Bell numbers $B_{n,r}$ as the number of partitions of a set with $n + r$

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elements such that r distinguished elements belong to distinct blocks, and the r-Bell polynomials as

$$B_{n,r}(x) = \sum_{j=0}^{n} \left\{ {n \atop j} \right\}_r x^j.$$

(We have to mention that there is some confusion in notation of r-Stirling numbers in the literature, therefore we need to underline that for various reasons, we prefer to denote by $\left\{ {n \atop k} \right\}_r$ the number of partitions of an $(n+r)$-element set into $k+r$ nonempty subsets such that r distinguished elements belong to distinct blocks.) The r-Bell numbers were studied from a graph theoretical point of view by Zs. Kereskényi-Balogh and G. Nyul [9]. We shall discuss these numbers and polynomials in detail in Section 2.

Lah numbers $\left\lfloor {n \atop k} \right\rfloor$, named after I. Lah [10, 11], are close relatives of Stirling numbers. Sometimes they are called Stirling numbers of the third kind. G. Nyul and G. Rácz [19] defined and extensively studied the r-generalization of Lah numbers. The r-Lah number $\left\lfloor {n \atop k} \right\rfloor_r$ is the number of partitions of a set with $n+r$ elements into $k+r$ nonempty ordered subsets such that r distinguished elements have to be in distinct ordered blocks. We notice that some identities for r-Lah numbers were derived by H. Belbachir, A. Belkhir [1] and H. Belbachir, E. Bousbaa [2], and they appear as the results of substitutions into partial r-Bell polynomials by M. Mihoubi and M. Rahmani [17]. The r-Lah numbers are special cases of r-Whitney-Lah numbers defined by G.-S. Cheon and J.-H. Jung [6] (see also [8]), and recently M. Shattuck [21] introduced a further generalization of these numbers.

Similarly to Bell numbers, one could be interested in summation

$$L_n = \sum_{j=0}^{n} \left\lfloor {n \atop j} \right\rfloor$$

of Lah numbers. Although these numbers slightly appear in the literature [7, 18, 20, 22], they have not been studied systematically yet. This will be done in our paper at a more general level, namely we shall prove several properties of sums $L_{n,r}$ of r-Lah numbers and r-Lah polynomials $L_{n,r}(x)$, for instance, we express summed r-Lah numbers by sums of $(r-s)$-Lah numbers, we derive Spivey and Dobiński type identities, second-order linear recurrence relations, exponential generating functions. Finally, we show that r-Lah polynomials have only real roots. We prefer purely combinatorial arguments in the proofs where it is possible. As we shall see, some of these results could be viewed as the summed or polynomial counterparts of certain theorems from [19]. They are also included in this paper, because we aim to give a self-contained presentation of these numbers and polynomials.

2 r-Bell numbers and r-Bell polynomials

Above, we have defined r-Bell numbers and r-Bell polynomials. In the following table we collect their properties, especially those ones which correspond to our theorems about summed r-Lah numbers and r-Lah polynomials. We indicate the references for the known identities (star symbol means that a certain paper contains the formula only for r-Bell numbers, not for polynomials), but it also contains some new results. For example, to the best of our knowledge, the Spivey type identity never appeared previously in this full generality. All of these properties can be proved along the lines of our proofs in the next section. We notice that these proofs are based on a completely new idea even for several known identities of the table. We should draw attention to that our purely combinatorial argument will fail to work in the most general case (Theorem 3.3) for r-Lah polynomials, but even so, it works for r-Bell numbers and polynomials.

Table 1: Properties of r-Bell numbers and r-Bell polynomials.

$B_{n,0}(x) = B_n(x)$ [14], $xB_{n,1}(x) = B_{n+1}(x)$
$B_{n,r}(x) = \sum\limits_{j=0}^{n} \binom{n}{j} B_{j,r-s}(x) s^{n-j}$ [16] $B_{n,r}(x) = \sum\limits_{j=0}^{n} \binom{n}{j} B_{j,r-1}(x)$ [14] $B_{n,r}(x) = \sum\limits_{j=0}^{n} \binom{n}{j} B_j(x) r^{n-j}$ [5]*, [14]
$B_{m+n,r}(x) = \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} \left\{ {m \atop i} \right\}_r \binom{n}{j} B_{j,r-s}(x)(i+s)^{n-j} x^i$ $B_{m+n,r}(x) = \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} \left\{ {m \atop i} \right\}_r \binom{n}{j} B_{j,r}(x) i^{n-j} x^i$ [16] $B_{m+n,r}(x) = \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} \left\{ {m \atop i} \right\}_r \binom{n}{j} B_{j,r-1}(x)(i+1)^{n-j} x^i$ $B_{m+n,r}(x) = \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} \left\{ {m \atop i} \right\}_r \binom{n}{j} B_j(x)(i+r)^{n-j} x^i$ [15]*, [16]
$B_{n,r}(x) = \frac{1}{\exp(x)} \sum\limits_{j=0}^{\infty} \frac{(j+r)^n}{j!} x^j$ [14]
$\sum\limits_{n=0}^{\infty} \frac{B_{n,r}(x)}{n!} y^n = \exp\left(x\left(\exp(y)-1\right)+ry\right)$ [5]*, [14]
The roots of $B_{n,r}(x)$ are simple, real and negative ($r \geq 1$). [13]

3 Summed r-Lah numbers and r-Lah polynomials

We begin this section with the exact definitions of summed r-Lah numbers and r-Lah polynomials, which can be viewed as relatives of r-Bell numbers and polynomials (in the sense that r-Lah numbers are relatives of r-Stirling numbers of the second kind).

For non-negative integers n, r, not both 0, denote by $L_{n,r}$ the number of partitions of a set with $n + r$ elements into nonempty ordered subsets such that r distinguished elements belong to distinct ordered blocks. Moreover, let $L_{0,0} = 1$. We can call $L_{n,r}$ the nth *summed r-Lah number*, because the formula

$$L_{n,r} = \sum_{j=0}^{n} \left\lfloor {n \atop j} \right\rfloor_r$$

immediately follows from the definitions. This suggests us to define the polynomial analogues of these numbers. If $n, r \geq 0$, then the nth *r-Lah polynomial* is

$$L_{n,r}(x) = \sum_{j=0}^{n} \left\lfloor {n \atop j} \right\rfloor_r x^j.$$

If we have no distinguished elements, then the summands in the first formula and the coefficients of the polynomial are the ordinary Lah numbers. In this case, we simply call them the nth summed Lah number and Lah polynomial, and denote them by L_n and $L_n(x)$.

Obviously, $L_{n,r}(x)$ is a monic polynomial of degree n with non-negative integer coefficients. Since $L_{n,r}(1) = L_{n,r}$, it is enough to state our theorems for r-Lah polynomials throughout this paper, the corresponding properties for summed r-Lah numbers follows simply by the substitution $x = 1$.

It will be useful to associate a combinatorial interpretation to r-Lah polynomials, as well. If $n, r \geq 0$, not both 0, and $c \geq 1$, then $L_{n,r}(c)$ counts the number of partitions of a set with $n + r$ elements into nonempty ordered subsets and colourings of the blocks with c colours such that r distinguished elements belong to distinct uncoloured ordered blocks. For brevity, in the rest of the paper we shall call these objects c-coloured r-Lah partitions of an $(n + r)$-element set into ordered blocks.

If $r = 0$ or $r = 1$, then we have no restriction for the partition into ordered blocks, hence $L_{n,0}(x) = L_n(x)$ and $xL_{n,1}(x) = L_{n+1}(x)$ $(n \geq 0)$.

In our first theorem, we express r-Lah polynomials in terms of $(r-s)$-Lah polynomials. It is the polynomial counterpart and could be derived directly from [19, Theorem 3.4], but we carry out the necessary modification of the combinatorial proof.

Theorem 3.1. *If $n, r, s \geq 0$ and $s \leq r$, then*

$$L_{n,r}(x) = \sum_{j=0}^{n} \binom{n}{j} L_{j,r-s}(x)(2s)^{\overline{n-j}}.$$

Proof. We may assume that n, r are not both 0, and let c be a positive integer. Then, $L_{n,r}(c)$ is the number of c-coloured r-Lah partitions of an $(n+r)$-element set into ordered blocks. These can be enumerated in another way:

Let j be the number of those non-distinguished elements which belong to other ordered blocks than the first s distinguished elements $(j = 0, \ldots, n)$. We can choose them in $\binom{n}{j}$ ways, thereafter we have $L_{j,r-s}(c)$ possibilities for their c-coloured $(r - s)$-Lah partitions into ordered blocks together with the last $r - s$ distinguished elements. Finally, we can put the remaining $n - j$ non-distinguished elements into the ordered blocks of the first s distinguished elements in $(2s)^{\overline{n-j}}$ ways. It means that, for a fixed j, the number of possibilities is $\binom{n}{j} L_{j,r-s}(c)(2s)^{\overline{n-j}}$. □

Remark 3.2. For the most important choices $s = 1$ and $s = r$, the identity becomes

$$L_{n,r}(x) = \sum_{j=0}^{n} \binom{n}{j} L_{j,r-1}(x)(n-j+1)!,$$

$$L_{n,r}(x) = \sum_{j=0}^{n} \binom{n}{j} L_j(x)(2r)^{\overline{n-j}}.$$

Now, we prove a general Spivey type formula for r-Lah polynomials. It is named after M. Z. Spivey [23], who discovered his remarkable formula for Bell numbers just over a decade ago.

Theorem 3.3. *If $m, n, r, s \geq 0$ and $s \leq r$, then*

$$L_{m+n,r}(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} L_{j,r-s}(x)(m+i+2s)^{\overline{n-j}} x^i.$$

Proof. By [19, Theorem 3.2], we get

$$(x+2r)^{\overline{m+n}} = \sum_{k=0}^{m+n} \left\lfloor {m+n \atop k} \right\rfloor_r x^{\underline{k}}.$$

On the other hand, using again [19, Theorem 3.2] and the binomial theorem for rising factorials, we also have

$$\begin{aligned}
(x+2r)^{\overline{m+n}} &= (x+2r)^{\overline{m}}(x+2r+m)^{\overline{n}} \\
&= \sum_{i=0}^{m} \left\lfloor {m \atop i} \right\rfloor_r x^{\underline{i}}(x-i+2r-2s+m+i+2s)^{\overline{n}} \\
&= \sum_{i=0}^{m} \left\lfloor {m \atop i} \right\rfloor_r x^{\underline{i}} \sum_{j=0}^{n} \binom{n}{j} (x-i+2r-2s)^{\overline{j}}(m+i+2s)^{\overline{n-j}} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \left\lfloor {m \atop i} \right\rfloor_r x^{\underline{i}} \binom{n}{j} (m+i+2s)^{\overline{n-j}} \sum_{k=0}^{j} \left\lfloor {j \atop k} \right\rfloor_{r-s} (x-i)^{\underline{k}} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{j} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k} \right\rfloor_{r-s} x^{\underline{i+k}} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=i}^{i+j} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k-i} \right\rfloor_{r-s} x^{\underline{k}} \\
&= \sum_{k=0}^{m+n} \sum_{i=0}^{\min\{m,k\}} \sum_{j=\max\{0,k-i\}}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k-i} \right\rfloor_{r-s} x^{\underline{k}}.
\end{aligned}$$

Comparing the coefficients of $x^{\underline{k}}$ in the above two expressions gives

$$\left\lfloor {m+n \atop k} \right\rfloor_r = \sum_{i=0}^{\min\{m,k\}} \sum_{j=\max\{0,k-i\}}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k-i} \right\rfloor_{r-s},$$

which identity is interesting on its own.

If we multiply both sides by x^k and sum for k ($k = 0, \ldots, m+n$), we obtain

$$\begin{aligned}
L_{m+n,r}(x) &= \sum_{k=0}^{m+n} \left\lfloor {m+n \atop k} \right\rfloor_r x^k \\
&= \sum_{k=0}^{m+n} \sum_{i=0}^{\min\{m,k\}} \sum_{j=\max\{0,k-i\}}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k-i} \right\rfloor_{r-s} x^k
\end{aligned}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=i}^{i+j} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k-i} \right\rfloor_{r-s} x^k$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{j} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} \left\lfloor {j \atop k} \right\rfloor_{r-s} x^{i+k}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} (m+i+2s)^{\overline{n-j}} x^i L_{j,r-s}(x). \qquad \square$$

Remark 3.4. First, we note that this formula gives back Theorem 3.1 and the definition of r-Lah polynomials for $m = 0$ and $n = 0$, respectively.

While, in the special cases of $s = 0$, $s = 1$ and $s = r$, we have

$$L_{m+n,r}(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} L_{j,r}(x)(m+i)^{\overline{n-j}} x^i,$$

$$L_{m+n,r}(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} L_{j,r-1}(x)(m+i+2)^{\overline{n-j}} x^i,$$

$$L_{m+n,r}(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} \left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} L_j(x)(m+i+2r)^{\overline{n-j}} x^i.$$

For the last identity, we give a combinatorial proof, as well. The reason is that the extension of Spivey's idea works for r-Lah polynomials only if $s = r$. However, as we mentioned previously, a similar argument proves the Spivey type formula listed in the table of Section 2 for r-Bell polynomials in full generality. It would be interesting to find a purely combinatorial proof of the general identity as stated in Theorem 3.3.

Proof. We may assume that m, n, r are not all 0, and let c be a positive integer. Then, $L_{m+n,r}(c)$ gives the number of c-coloured r-Lah partitions of an $(m+n+r)$-element set into ordered blocks. We find an alternative way to count them:

First, we consider a c-coloured r-Lah partition of the distinguished elements and the first m non-distinguished elements into $i + r$ ordered blocks ($i = 0, \dots, m$). We have $\left\lfloor {m \atop i} \right\rfloor_r c^i$ such partitions. Denote by j the number of those non-distinguished elements among the last n ones which do not belong to these $i + r$ ordered blocks ($j = 0, \dots, n$). They can be chosen in $\binom{n}{j}$ ways, and there are $L_j(c)$ possibilities to partition them into coloured ordered blocks with c colours. As our last step, we place the remaining $n - j$ non-distinguished elements into the $i + r$ original ordered blocks, which can be done in $(m+i+2r)^{\overline{n-j}}$ ways. Summarizing, the number of possibilities is

$$\left\lfloor {m \atop i} \right\rfloor_r \binom{n}{j} L_j(c)(m+i+2r)^{\overline{n-j}} c^i$$

for a fixed pair of i, j. $\square$

The r-Lah polynomials satisfy the following second-order linear recurrence relation. In the special case of sums of ordinary Lah numbers (i.e., for $r = 0$), it appears in [18, 20, 22] in different contexts.

Theorem 3.5. *If $n \geq 1$ and $r \geq 0$, then*

$$L_{n+1,r}(x) = (x + 2n + 2r)L_{n,r}(x) - n(n + 2r - 1)L_{n-1,r}(x).$$

Proof. Let c be a positive integer. Then, $L_{n+1,r}(c)$ counts the number of c-coloured r-Lah partitions of an $(n + r + 1)$-element set into ordered blocks. The rest of the proof gives another enumeration of them:

We have $L_{n,r}(c)$ c-coloured r-Lah partitions of our set excluding the last non-distinguished element into ordered blocks. If this last element constitutes a singleton, then we only need to colour its one-element ordered block with c colours. Otherwise, we can place the excluded element before or after any other elements, i.e., to $2n + 2r$ places. It means that there would be $(c + 2n + 2r)L_{n,r}(c)$ possibilities.

But, of course, we counted twice those cases when our last element is put between two elements. This could happen in two different ways. If the jth non-distinguished element stands directly before the originally excluded element ($j = 1, \dots, n$), then there are $L_{n-1,r}(c)$ c-coloured r-Lah partitions of our set without these two elements into ordered blocks, and this pair of elements can be put back to $n + r - 1$ places (they cannot be at the end of an ordered block). If a distinguished element stands directly before and the jth non-distinguished element stands directly after the originally excluded element ($j = 1, \dots, n$), then we have $L_{n-1,r}(c)$ c-coloured r-Lah partitions of our set without the latter two elements into ordered blocks, and they can be put back to r places (directly after one of the distinguished elements). Therefore, the number of the possibilities to be subtracted is $(n(n + r - 1) + nr)L_{n-1,r}(c)$, altogether. $\square$

We can derive a Dobiński type formula for r-Lah polynomials, named after the well-known Dobiński formula for Bell numbers.

Theorem 3.6. *If $n, r \geq 0$, then*

$$L_{n,r}(x) = \frac{1}{\exp(x)} \sum_{j=0}^{\infty} \frac{(j + 2r)^{\overline{n}}}{j!} x^j.$$

Proof. I. First, we prove it for polynomials. Through this proof, let $\left\lfloor {n \atop i} \right\rfloor_r = 0$ if $i > n$. Applying [19, Theorem 3.2], we have

$$(j + 2r)^{\overline{n}} = \sum_{i=0}^{n} \left\lfloor {n \atop i} \right\rfloor_r j^{\underline{i}} = \sum_{i=0}^{\infty} \left\lfloor {n \atop i} \right\rfloor_r j^{\underline{i}} = \sum_{i=0}^{j} \left\lfloor {n \atop i} \right\rfloor_r \frac{j!}{(j - i)!}.$$

Dividing both sides by $j!$ gives

$$\frac{(j + 2r)^{\overline{n}}}{j!} = \sum_{i=0}^{j} \left\lfloor {n \atop i} \right\rfloor_r \frac{1}{(j - i)!},$$

which means that $\left(\frac{(j+2r)^{\overline{n}}}{j!} \right)_{j=0}^{\infty}$ is the convolution of the sequences $\left(\left\lfloor {n \atop j} \right\rfloor_r \right)_{j=0}^{\infty}$ and $\left(\frac{1}{j!} \right)_{j=0}^{\infty}$. Therefore, its generating function is

$$\sum_{j=0}^{\infty} \frac{(j + 2r)^{\overline{n}}}{j!} x^j = L_{n,r}(x) \exp(x).$$

II. Now, we can give another proof for summed r-Lah numbers using probability theory. Let λ be a positive real number and ξ a Poisson random variable with parameter λ. Then, again by [19, Theorem 3.2], we get

$$\begin{aligned}
\mathsf{E}\left(\xi+2r\right)^{\overline{n}} &= \sum_{j=0}^{\infty}(j+2r)^{\overline{n}}\frac{\lambda^j}{j!}e^{-\lambda} = e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\sum_{i=0}^{n}\left\lfloor {n \atop i} \right\rfloor_r j^{\underline{i}} \\
&= e^{-\lambda}\sum_{i=0}^{n}\left\lfloor {n \atop i} \right\rfloor_r \sum_{j=0}^{\infty}\frac{j^{\underline{i}}}{j!}\lambda^j = e^{-\lambda}\sum_{i=0}^{n}\left\lfloor {n \atop i} \right\rfloor_r \sum_{j=i}^{\infty}\frac{\lambda^j}{(j-i)!} \\
&= e^{-\lambda}\sum_{i=0}^{n}\left\lfloor {n \atop i} \right\rfloor_r \lambda^i\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \sum_{i=0}^{n}\left\lfloor {n \atop i} \right\rfloor_r \lambda^i = L_{n,r}\left(\lambda\right).
\end{aligned}$$

Especially, for $\lambda=1$, we have

$$L_{n,r} = L_{n,r}\left(1\right) = \mathsf{E}\left(\xi+2r\right)^{\overline{n}} = \sum_{j=0}^{\infty}(j+2r)^{\overline{n}}\frac{1}{j!}e^{-1}. \qquad \square$$

The next theorem gives the exponential generating function of the sequence of r-Lah polynomials. We note that a special case, the exponential generating function of $(L_n)_{n=0}^{\infty}$ can be found in [7, 18, 22].

Theorem 3.7. *For $r \geq 0$, the exponential generating function of $(L_{n,r}(x))_{n=0}^{\infty}$ is*

$$\sum_{n=0}^{\infty}\frac{L_{n,r}(x)}{n!}y^n = \exp\left(\frac{xy}{1-y}\right)\frac{1}{(1-y)^{2r}}.$$

Proof. I. We use [19, Theorem 3.10] to get

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{L_{n,r}(x)}{n!}y^n &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\left\lfloor {n \atop j} \right\rfloor_r x^j\frac{1}{n!}y^n = \sum_{j=0}^{\infty}x^j\sum_{n=j}^{\infty}\left\lfloor {n \atop j} \right\rfloor_r \frac{1}{n!}y^n \\
&= \sum_{j=0}^{\infty}x^j\frac{1}{j!}\left(\frac{y}{1-y}\right)^j\frac{1}{(1-y)^{2r}} = \frac{1}{(1-y)^{2r}}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{xy}{1-y}\right)^j \\
&= \exp\left(\frac{xy}{1-y}\right)\frac{1}{(1-y)^{2r}}.
\end{aligned}$$

II. We can prove the theorem in another way for summed r-Lah numbers. Denote by $\ell_r(y)$ the exponential generating function to be find.

From the first special case of Theorem 3.1, it follows that $(L_{n+1})_{n=0}^{\infty} = (L_{n,1})_{n=0}^{\infty}$ is the binomial convolution of the sequences $(L_n)_{n=0}^{\infty}$ and $((n+1)!)_{n=0}^{\infty}$, hence their exponential generating functions give the differential equation

$$\ell_0'(y) = \ell_0(y)\frac{1}{(1-y)^2}.$$

For $n \geq 0$, it shows that $[y^j]\ell_0(y)$ $(j=0,\ldots,n)$ uniquely determine $[y^{n+1}]\ell_0(y)$, whence our differential equation with the initial condition $[y^0]\ell_0(y) = \frac{L_0}{0!} = 1$ is uniquely solvable among formal power series, and this solution is $\ell_0(y) = \exp\left(\frac{y}{1-y}\right)$.

The second special case of Theorem 3.1 says that $(L_{n,r})_{n=0}^{\infty}$ is the binomial convolution of the sequences $(L_n)_{n=0}^{\infty}$ and $\left((2r)^{\overline{n}}\right)_{n=0}^{\infty}$, therefore its exponential generating function is

$$\ell_r(y) = \ell_0(y)\sum_{n=0}^{\infty}\frac{(2r)^{\overline{n}}}{n!}y^n = \exp\left(\frac{y}{1-y}\right)\frac{1}{(1-y)^{2r}}. \qquad \square$$

In the following theorem, we show the real-rootedness of r-Lah polynomials, where the proof will contain a further recurrence for them.

Theorem 3.8. *If $n \geq 1$, then the roots of $L_n(x)$ are simple, real, one of them is 0 and the others are negative. If $n, r \geq 1$, then the roots of $L_{n,r}(x)$ are simple, real and negative. Furthermore, for any $r \geq 0$, $(L_{n,r}(x))_{n=0}^{\infty}$ is an interlacing sequence of polynomials.*

Proof. We perform the proof by induction on n only for $r \geq 1$. We can easily check the assertion for $n = 1, 2$, and assume that it holds for some n.

Using [19, Theorem 3.1] and the special values of r-Lah numbers, we get

$$\begin{aligned}
L_{n+1,r}(x) &= \sum_{k=0}^{n+1}\left\lfloor {n+1 \atop k} \right\rfloor_r x^k = \left\lfloor {n+1 \atop 0} \right\rfloor_r + \sum_{k=1}^{n}\left\lfloor {n+1 \atop k} \right\rfloor_r x^k + \left\lfloor {n+1 \atop n+1} \right\rfloor_r x^{n+1} \\
&= (2r)^{\overline{n+1}} + \sum_{k=1}^{n}\left(\left\lfloor {n \atop k-1} \right\rfloor_r + (n+k+2r)\left\lfloor {n \atop k} \right\rfloor_r\right)x^k + x^{n+1} \\
&= \sum_{k=0}^{n-1}\left\lfloor {n \atop k} \right\rfloor_r x^{k+1} + x^{n+1} + (n+2r)\sum_{k=1}^{n}\left\lfloor {n \atop k} \right\rfloor_r x^k + (2r)^{\overline{n+1}} + \sum_{k=1}^{n} k\left\lfloor {n \atop k} \right\rfloor_r x^k \\
&= x\sum_{k=0}^{n}\left\lfloor {n \atop k} \right\rfloor_r x^k + (n+2r)\sum_{k=0}^{n}\left\lfloor {n \atop k} \right\rfloor_r x^k + x\sum_{k=1}^{n} k\left\lfloor {n \atop k} \right\rfloor_r x^{k-1} \\
&= xL_{n,r}(x) + (n+2r)L_{n,r}(x) + xL'_{n,r}(x).
\end{aligned}$$

Then, multiplying this equation by $e^x x^{n+2r-1}$ gives

$$e^x x^{n+2r-1}L_{n+1,r}(x) = \left(e^x x^{n+2r}L_{n,r}(x)\right)'.$$

The induction hypothesis tells us that $L_{n,r}(x)$ has n simple real roots which are negative, hence $e^x x^{n+2r}L_{n,r}(x)$ has exactly $n+1$ zeros, one of them is 0, and the others are negative. Moreover, $\lim_{x\to-\infty} e^x x^{n+2r}L_{n,r}(x) = 0$. Then it follows from Rolle's mean value theorem that $\left(e^x x^{n+2r}L_{n,r}(x)\right)' = e^x x^{n+2r-1}L_{n+1,r}(x)$ has at least $n+1$ negative zeros, therefore $L_{n+1,r}(x)$ has $n+1$ distinct negative roots.

The proof also shows the interlacing property. $\square$

This result together with a theorem of Newton (see, e.g., [24]) immediately implies the following consequence, which was proved in [19, Theorem 3.8] by different means.

Corollary 3.9. *If $n \geq 1$ and $r \geq 0$, then the sequence $\left(\left\lfloor {n \atop j} \right\rfloor_r\right)_{j=0}^{n}$ is strictly log-concave and unimodal.*

The theorem also allows us to give a good approximation of the quotient of two consecutive summed r-Lah numbers.

Corollary 3.10. *If $n \geq 1$ and $r \geq 0$, then*

$$\left| \frac{L_{n+1,r}}{L_{n,r}} - (n+r+1) - \left\lfloor \sqrt{n+r^2+1} \right\rfloor \right| < 1.$$

Proof. From the recurrence derived in the proof of Theorem 3.8, we get

$$L'_{n,r}(1) = L_{n+1,r} - (n+2r+1)L_{n,r}.$$

Then the assertion follows from Theorem 3.8, a theorem of Darroch (see, e.g., [3]) and [19, Theorem 3.9]. □

Finally, we prove that the r-Stirling transform of the first kind of the sequence of s-Bell polynomials is the sequence of $\frac{r+s}{2}$-Lah polynomials if r and s have the same parity.

Theorem 3.11. *If $n, r, s \geq 0$ and $r + s$ is even, then*

$$L_{n,\frac{r+s}{2}}(x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,s}(x).$$

Proof. By [19, Theorem 3.11], we have

$$\begin{aligned} L_{n,\frac{r+s}{2}}(x) &= \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{\frac{r+s}{2}} x^k = \sum_{k=0}^{n} \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r \begin{Bmatrix} j \\ k \end{Bmatrix}_s x^k \\ &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r \sum_{k=0}^{j} \begin{Bmatrix} j \\ k \end{Bmatrix}_s x^k = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,s}(x). \end{aligned}$$

□

Remark 3.12. If $r = s$, then the identity simply becomes

$$L_{n,r}(x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r B_{j,r}(x).$$

In this case, we can provide a combinatorial proof.

Proof. We may again assume that n, r are not both 0, and let c be a positive integer. A c-coloured r-Lah partition of an $(n+r)$-element set into ordered blocks can be constructed as follows: First, we decompose the elements into $j + r$ disjoint cycles such that the r distinguished elements belong to distinct cycles ($j = 0, \dots, n$). These latter cycles will be referred to as distinguished cycles. After that, we partition all the cycles such that distinguished cycles are in distinct blocks, and we colour the blocks containing no distinguished cycle with c colours. Finally, we multiply the cycles in each block to obtain the ordered blocks of the original $(n+r)$-element set. Therefore, for a fixed j, the number of c-coloured r-Lah partitions is $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_r B_{j,r}(c)$. □

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Generation of local symmetry-preserving operations on polyhedra

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Abstract

We introduce a new practical and more general definition of local symmetry-preserving operations on polyhedra. These can be applied to arbitrary embedded graphs and result in embedded graphs with the same or higher symmetry. With some additional properties we can restrict the connectivity, e.g. when we only want to consider polyhedra. Using some base structures and a list of 10 extensions, we can generate all possible local symmetry-preserving operations isomorph-free.

Keywords: Graph theory, polyhedra, symmetry, chamber systems.

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1 Introduction

Symmetry-preserving operations on polyhedra have a long history – from Plato and Archimedes to Kepler [11], Goldberg [9], Caspar and Klug [4], Coxeter [6], Conway [5], and many others. Notwithstanding their utility, until recently we had no unified way of defining or describing these operations without resorting to ad-hoc descriptions and drawings. In [2] the concept of local symmetry-preserving operations on polyhedra (*lsp operations* for short) was introduced. These are operations that are locally defined – on the *chamber* level, as explained in the next section – and therefore preserve the symmetries of the polyhedron to which they are applied. This established a general framework in which the class of all lsp operations can be studied, without having to consider individual operations separately. It was shown that many of the most frequently used operations on polyhedra (e.g. dual, ambo, truncate, . . .) fit into this framework.

But of course we sometimes do want to examine the operations individually, e.g. to check conjectures on as many examples as possible before we try to prove them, or to

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find operations with certain properties. We can do this for a few operations by hand, but a computer can do this a lot faster, and in a systematic way such that no operations are missed.

In this paper we shall slightly extend the definition of lsp operation so it can be applied to any graph embedded on a compact closed surface[1], and at the same time provide a reformulation of these operations as decorations, which will turn out to be easier to use in practice.

2 Decorations and lsp operations

Every embedded graph G has an associated chamber system C_G [7]. This chamber system is obtained by constructing a barycentric subdivision of G by adding one vertex in the center of each edge and face of G, and edges from each center of a face to its vertices and centers of edges. These vertices can be chosen invariant under the symmetries of G. In C_G, each vertex has a type that is 0, 1, or 2, indicating the dimension of its corresponding structure in G. Each edge has the type of the opposite vertex in the adjacent triangles. In Figure 1, the chamber system of the plane graph of a cube is given. The original graph consists of the edges of type 2 in the chamber system.

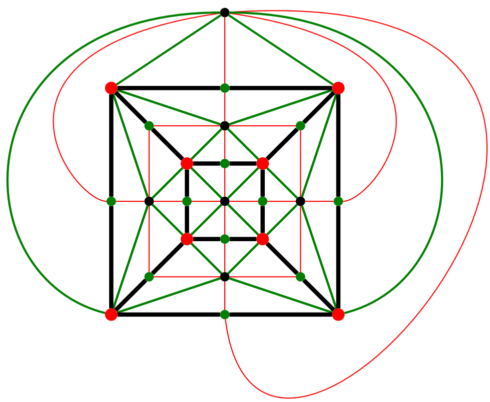


Figure 1: The barycentric subdivision of the plane graph of a cube. Edges of type 0 are red, edges of type 1 are green and edges of type 2 are black.

We use the drawing conventions from Figure 1 for the types of the edges in all figures. Since the vertex types can be deduced from the edge types, we do not display them in the figures.

Definition 2.1. A *decoration* D is a 2-connected plane graph with vertex set V and edge set E, together with a labeling function $t\colon V \cup E \to \{0, 1, 2\}$, and an outer face which contains vertices v_0, v_1, v_2, such that

1. all inner faces are triangles;

2. for each edge $e = (v, w)$, $\{t(e), t(v), t(w)\} = \{0, 1, 2\}$;

[1]All graphs in this paper are embedded graphs, and a subgraph has the induced embedding.

3. for each vertex v with $t(v) = i$, the types of incident edges are j and k with $\{i, j, k\} = \{0, 1, 2\}$. Two consecutive edges with an inner face in between can not have the same type;

4. for each inner vertex v

$$\begin{aligned} t(v) = 1 \quad &\Rightarrow \quad \deg(v) = 4 \\ t(v) \neq 1 \quad &\Rightarrow \quad \deg(v) > 4 \end{aligned}$$

for each vertex v in the outer face and different from v_0, v_1, v_2

$$\begin{aligned} t(v) = 1 \quad &\Rightarrow \quad \deg(v) = 3 \\ t(v) \neq 1 \quad &\Rightarrow \quad \deg(v) > 3 \end{aligned}$$

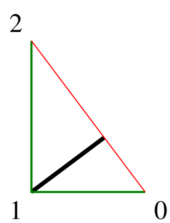
and

$$\begin{aligned} &t(v_0), t(v_2) \neq 1 \\ &t(v_1) = 1 \quad \Rightarrow \quad \deg(v_1) = 2 \\ &t(v_1) \neq 1 \quad \Rightarrow \quad \deg(v_1) > 2. \end{aligned}$$

Note that condition 3 implies that all inner vertices have an even degree.

For all $\{i, j, k\} = \{0, 1, 2\}$, the k-side of a decoration D is the path on the border of the outer face between v_i and v_j that does not pass through v_k.

We can fill each triangular face of a chamber system C_G with a decoration, by identifying the vertex of type i with v_i for $i \in \{0, 1, 2\}$ and identifying corresponding vertices on the boundary. This results in a new chamber system $C_{G'}$ of a new graph G', as can be seen in Figure 2.



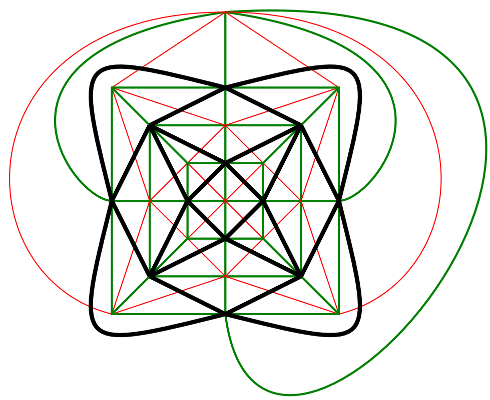


Figure 2: The decoration *ambo* applied to the cube of Figure 1. The resulting graph G' is the one in black.

This is very similar to the lsp operations of [2]. We are constructing graphs by subdividing the chambers of the chamber system. One key difference is that we impose no restrictions on the connectivity. This means that we can apply decorations to arbitrary embedded graphs, but when applied to a polyhedron – i.e. a 3-connected plane graph – it is possible that the result has a lower connectivity. We will address this problem later with additional restrictions on decorations.

For now, we will repeat Definition 5.1 of [2] without the restrictions on the connectivity.

Definition 2.2. Let T be a connected periodic tiling of the Euclidean plane with chamber system C_T, that is given by a barycentric subdivision that is invariant under the symmetries of T. Let v_0, v_1, v_2 be points in the Euclidean plane so that for $0 \leq i < j \leq 2$ the line $L_{i,j}$ through v_i and v_j is a mirror axis of the tiling.

If the angle between $L_{0,1}$ and $L_{2,1}$ is 90 degrees, the angle between $L_{2,1}$ and $L_{2,0}$ is 30 degrees and consequently the angle between $L_{0,1}$ and $L_{0,2}$ is 60 degrees, then the triangle v_0, v_1, v_2 subdivided into chambers as given by C_T and the corners v_0, v_1, v_2 labelled with their names v_0, v_1, v_2 is called a *local symmetry-preserving operation*, *lsp operation* for short.

The result $O(G)$ of applying an lsp operation O to a connected graph G is given by subdividing each chamber C of the chamber system C_G with O by identifying for $0 \leq i \leq 2$ the vertices of O labelled v_i with the vertices labelled i in C.

An lsp operation is called k*-connected* for $k \in \{1, 2, 3\}$ if it is derived from a k-connected tiling T. So the original definition was for 3-connected lsp operations only. In order to correctly determine the connectivity, we first need to identify which chamber systems correspond to k-connected graphs. To decide whether a graph G is k-connected based on its chamber system C_G, we can look at the *type*-1 *cycles* in C_G. A type-1 cycle is a cycle in the subgraph of C_G that consists of the type-1 edges only. A type-1 cycle is empty if there are no vertices on the inside or on the outside of the cycle in this type-1 subgraph. Note that in the graph C_G these cycles are not necessarily empty.

Lemma 2.3. *A plane graph G is*

1. 2*-connected if and only if C_G contains no type-*1 *cycles of length* 2*;*
2. 3*-connected if and only if G is* 2*-connected and C_G contains no non-empty type-*1 *cycles of length* 4*.*

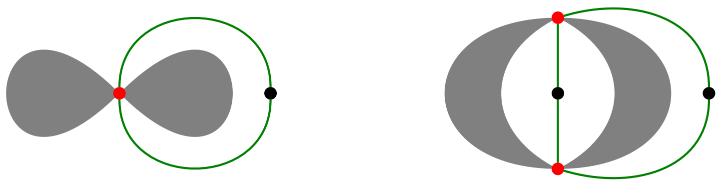
Proof.

1. Suppose C_G contains a type-1 cycle of length 2. This cycle contains one type-0 vertex v, incident to at least one type-2 edge inside the cycle and at least one type-2 edge outside the cycle (see Figure 3a), because C_G is a barycentric subdivision. It is clear that v has to be a cut-vertex of G.

   Conversely, if G has a cut-vertex v, there is a face of G for which v occurs at least two times in its border. In C_G this face corresponds with a type-2 vertex, incident with at least two type-1 edges to v. These edges form a type-1 cycle in C_G.

2. Suppose C_G contains a non-empty type-1 cycle of length 4, as can be seen in Figure 3b. This cycle contains two type-0 vertices v and w, with incident type-2 edges at both sides of the cycle. Removing v and w from G results in a disconnected graph.

   If G is 2-connected but not 3-connected, there are two vertices v and w that disconnect G when removed. So there are two non-empty subgraphs of G that are only connected by v and w, as in Figure 3b. This means that there is a non-empty type-1 cycle in C_G. □



(a) not 2-connected (b) 2-connected but not 3-connected

Figure 3: Two graphs with type-1 cycles. The gray area contains the graph. Only the type-1 edges of the chamber system are shown. The type-0 vertices are red and the type-2 vertices are black.

Note that this theorem only holds for plane graphs, since the proof relies on the Jordan curve theorem. A counterexample to an equivalent theorem for embedded graphs of higher genus is the dual of a 3-connected graph on the torus, which can have a 2-cut (see [1]).

Since we introduced a more general definition of lsp operations, we can also formulate a more general version of Theorem 5.2 in [2].

Theorem 2.4. *If G is a k-connected plane graph with $k \in \{1, 2, 3\}$, and O is a k-connected lsp operation, then $O(G)$ is a k-connected plane graph.*

Proof. It is clear that $O(G)$ is a plane graph. For $k = 1$, we know that T and G are connected, and it follows easily that $O(G)$ is connected. For $k = 3$, the proof is given in [2]. For $k = 2$, we will prove that there is no cut-vertex in $O(G)$.

A type-1 cycle of length 2 in $C_{O(G)}$ is either completely contained in one chamber of C_G[2], or it is split between two chambers of C_G (see Figure 4). Both cases cannot appear, as for any chamber (resp. any pair of adjacent chambers) there is an isomorphism between this chamber (resp. these two chambers) and the corresponding area in T, and according to Lemma 2.3 T has no type-1 cycles of length 2.

This implies that $C_{O(G)}$ contains no type-1 cycles of length 2, and thus, invoking once again Lemma 2.3, $O(G)$ contains no cut-vertices. □

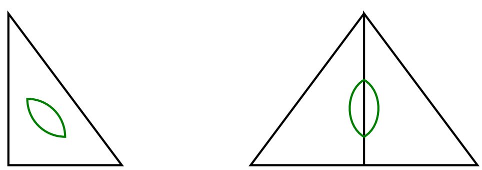


Figure 4: The different situations where type-1 cycles of length 2 can occur.

[2]With a chamber of C_G in $C_{O(G)}$, we mean the area that was a chamber of C_G before it was subdivided by O.

We can prove similar properties for decorations, but it is easier to use the correspondence between lsp operations and decorations. Although the way they are defined is rather different, in reality they are the same thing. The triangle v_0, v_1, v_2 of an lsp operation that is derived from a tiling has exactly the properties of a decoration, and each decoration can be derived as an lsp operation from a tiling.

Theorem 2.5. *Each decoration defines an lsp operation and vice versa.*

Proof. It is straightforward that the graph defined by an lsp operation is unique and satisfies the conditions of Definition 2.1. We still have to prove that each decoration defines an lsp operation.

Given a decoration D, we can take the hexagonal lattice H and use D to decorate each chamber of the chamber system C_H. The result will be a chamber system C_T of a tiling T.

We will first prove that the type-2 subgraph of D is connected, by induction on the number of triangles. There is always at least one triangle in D that shares one or two edges with the outer face. We remove these edges, and call the result D'. It is clear that D' still satisfies properties 1–3 of Definition 2.1, and by induction its type-2 subgraph is connected. If one of the removed edges has type 2, it is connected to D' by a vertex of type 0 or 1 with degree at least 3, and therefore it is connected to the type-2 subgraph of D'.

Given vertices u and v in the type-2 subgraph of C_T, there exists a sequence of chambers $C_0, \dots, C_n$ of H such that two consecutive chambers C_i and C_{i+1} share one side, and u is contained in C_0 and v in C_n. Since there are at least two vertices on each side of D, and they are not both of type 2, at least one of them is in the type-2 subgraph of C_T. Thus, there is a type-2 path between u and v that passes through all chambers in the sequence $C_0, \dots, C_n$, and the type-2 subgraph of C_T is connected. It follows immediately that T is connected too.

We can choose the vertices of one chamber of C_H in T as v_0, v_1 and v_2. This satisfies the properties of Definition 2.2, and it is clear that the decoration defined by the triangle v_0, v_1, v_2 is equal to D. □

This correspondence can be further extended to 2-connected and 3-connected operations.

Definition 2.6. A 2*-connected decoration* is a decoration with

1. no type-1 cycles of length 2;
2. no internal type-1 edges between two vertices on a single side.

Definition 2.7. A 3*-connected decoration* is a 2-connected decoration with

1. no type-1 edge between sides 0 and 2;
2. no non-empty type-1 cycles of length 4.

Note that, when seen as a graph, a decoration is always at least 2-connected.

Theorem 2.8. *Each* 2*-connected decoration* D *defines a* 2*-connected lsp operation and vice versa.*

Proof. A 2-connected decoration is a decoration, so it follows from Theorem 2.5 that D defines an lsp operation. We still have to prove that the corresponding tiling T is 2-connected. If T is not 2-connected, there is a type-1 cycle of length 2 in C_T. If this cycle is completely contained in the triangle v_0, v_1, v_2, there is a cycle of length 2 in D too, which is impossible. The only other possibility is that the cycle of length 2 is cut in half by L_{ij}, but then there would be an internal type-1 edge between 2 vertices on L_{ij}, which is a side of D.

A 2-connected lsp operation with corresponding tiling T defines a decoration D according to Theorem 2.5. We still have to prove that the extra conditions of Definition 2.6 are satisfied. If there is a type-1 cycle of length 2 in D, this cycle occurs in C_T too, and T would not be 2-connected. If there is an internal type-1 edge between 2 vertices on the same side, this will result in a cycle of length 2 in T because this side lies on a mirror axis of T. □

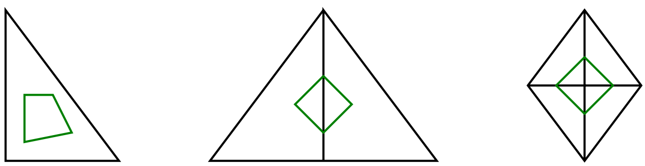


Figure 5: The different situations where non-empty type-1 cycles of length 4 can occur.

Theorem 2.9. *Each* 3*-connected decoration* D *defines a* 3*-connected lsp operation and vice versa.*

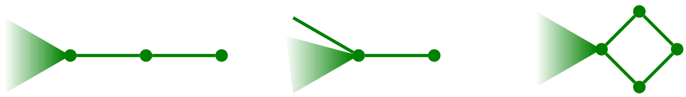
Proof. A 3-connected decoration defines a 2-connected lsp operation. If T is not 3-connected, there is a non-empty type-1 cycle of length 4. If this cycle is completely contained in the triangle v_0, v_1, v_2, there is a type-1 cycle of length 4 in D. If the cycle is cut in half by L_{ij}, there is an internal type-1 path of length 2 between 2 vertices on L_{ij}, which is a side of D. If the cycle is cut in four, as in Figure 5, there is a type-1 edge between sides 0 and 2.

A 3-connected lsp operation with corresponding tiling T defines a 2-connected decoration D. If there is a type-1 cycle of length 4 in D, this cycle occurs in C_T too, and T would not be 3-connected. If there is an internal type-1 path of length 2 between 2 vertices on the same side, or a type-1 edge between sides 0 and 2, this will result in a cycle of length 4 in T. □

3 Predecorations

The generation of all decorations will be split into two phases. In the first phase, we will construct the type-1 subgraph, consisting of all edges of type 1.

Let n_A be the number of vertices in the type-1 subgraph of degree 1 with a neighbouring vertex of degree 2, n_B the number of remaining vertices of degree 1, and n_C the number of quadrangles with three vertices of degree 2.



(a) n_A (b) n_B (c) n_C

Figure 6: The subgraphs counted as n_A, n_B and n_C.

Lemma 3.1. *Let D be a decoration. The type-1 subgraph D_1 of D has the following properties:*

1. *all inner faces are quadrangles;*
2. *each inner vertex has degree at least 3;*
3. *$n_A \leq 2$ and $n_A + n_B + n_C \leq 3$.*

Proof. It follows immediately from the properties of a decoration (Definition 2.1) that the inner faces of D_1 are quadrangles and the inner vertices have degree at least 3.

Each area bounded by a quadrangle in D_1 contains one vertex of type 1 in D. The only other difference between D and D_1 is in the outer face of D_1, where type-1 vertices of degree 3 in D (a 3-completion), and at most one of degree 2 in D (a 2-completion), can be present in D. If there is a type-1 vertex of degree 2, then that vertex is v_1. An example can be seen in Figure 7.

The subgraph in Figure 6c can only occur if the rightmost vertex v of degree two is v_0, v_1 or v_2, or if v_1 is a type-1 vertex of degree 2 connected to this vertex. Each of the three vertices of degree 2 in this subgraph of D_1 corresponds to v_0, v_1, v_2 or a vertex of degree at least 4 in D. The inner edges of the quadrangle in D contribute exactly one to the degree of these vertices. This implies that either there is a 2-completion here (in which case v_1 is connected to v), or there are two 3-completions which do not involve v (in which case v is v_0, v_1 or v_2).

The subgraph in Figure 6b can only occur if the rightmost vertex v is v_0, v_1 or v_2. This vertex of degree 1 in D_1 corresponds to a vertex of degree at most 3 in D, which is only possible in v_0, v_1 or v_2.

The subgraph in Figure 6a can only occur if the rightmost vertex v is v_0 or v_2. There are two neighbouring cut-vertices of D_1 in this subgraph, which do not correspond to cut-vertices in D. This is only possible if both of these vertices are the middle vertex of a 3-completion. This increases the degree of v in D to 2, which is only possible in v_0 or v_2. The degree of v can be 3 if there is a 2-completion too, but then v_1 is contained in this 2-completion and v still has to be v_0 or v_2.

We find that $n_A \leq |\{v_0, v_2\}| = 2$ and $n_A + n_B + n_C \leq |\{v_0, v_1, v_2\}| = 3$. □

Definition 3.2. A *predecoration* is a connected plane graph with an outer face that satisfies the properties of Lemma 3.1.

Given a predecoration P, we can try to add edges, vertices and labels to get a decoration with P as its type-1 subgraph. We will have to add one type-1 vertex in each inner face of P, as in Figure 7. Then we can add type-1 vertices in the outer face, and connect them to three consecutive vertices of P. Finally, we can add a type-1 vertex in the outer face and connect it to two consecutive vertices of P. This vertex has to be v_1.

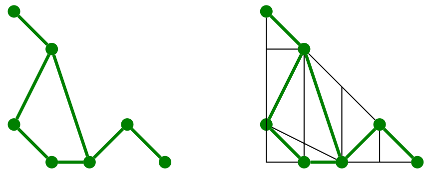


Figure 7: A predecoration with a possible completion. The edges of type 0 and 2 are both shown in black.

By definition, the type-1 subgraph of a decoration D is a predecoration. Unfortunately, not each predecoration corresponds to a type-1 subgraph of some decoration. This is e.g. the case if there are too many cut-vertices, as in Figure 8.

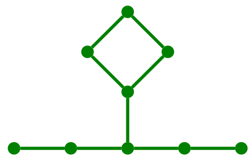


Figure 8: A predecoration that cannot be completed.

4 Construction of predecorations

All predecorations can be constructed from the base decorations K_2 and C_4 (see Figure 9) using the 10 extension operations shown in Figure 10. We will prove this by showing that each predecoration, with the exception of K_2 and C_4, can be reduced by the inverse of one of the extension operations. We will then use the *canonical construction path* method [12] to generate all predecorations without isomorphic copies.

Figure 9: The base predecorations.

Given a predecoration P, we will choose a *canonical parent* of P. This is a predecoration obtained by applying one of the reductions to P. We will always use the reduction with the smallest number among all possible reductions. It is possible that there is more than one way to apply this reduction to P, and if P has non-trivial symmetry, some of these can result in the same parent. If we choose one special edge in the subgraph that is affected by the reduction operation, each way to apply this reduction corresponds to an edge of P. We can choose an orbit of edges under the symmetry group of P by constructing a canonical labeling of the vertices – similar to [3] – and choosing the orbit of the edge with the lowest numbered vertices. The canonical parent of P is then obtained by applying the corresponding reduction.

During the construction, we will try each possible extension in all possible ways, and then check if it is the inverse of the reduction used to get the canonical parent of the result-

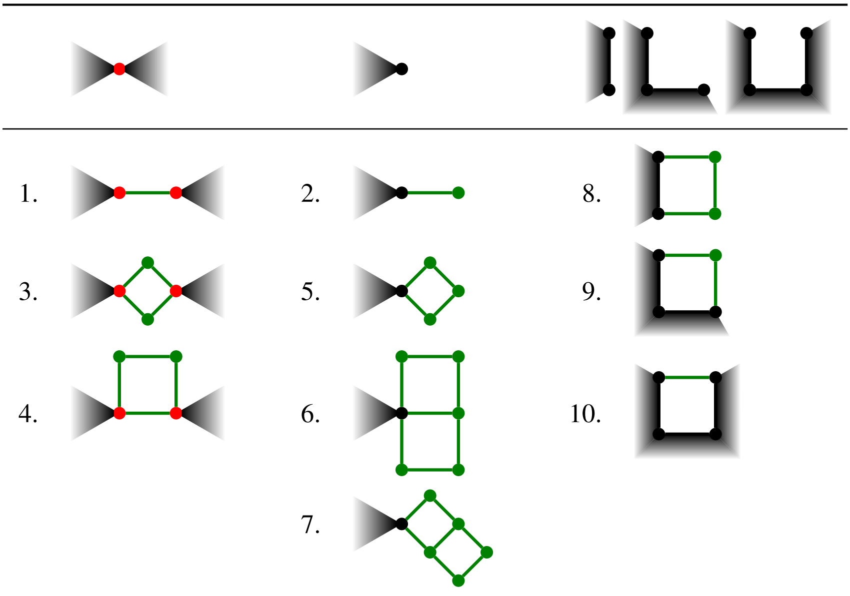


Figure 10: The extensions. In the first row, the subgraphs before the extension is applied are given. New edges and vertices are green, and vertices that are broken apart in two new vertices are red. The outer face is always on the outside, and shadowed parts contain at least one vertex.

ing predecoration. If that is the case, we can continue to extend this predecoration.

It is possible to construct all predecorations with fewer extensions, but it is important that a canonical reduction always results in a valid predecoration. The order of extensions 1–4 ensures that a canonical reduction never increases n_A, and extensions 5–7 ensure that a canonical reduction never increases $n_A + n_B + n_C$. Extensions 8–10 are necessary when none of the other reductions are possible, so that each predecoration different from the base decorations has a possible reduction. We will prove this in Lemma 4.2 and Theorem 4.3.

Lemma 4.1. *An extension applied to a predecoration results in another predecoration if it keeps* $n_A \leq 2$ *and* $n_A + n_B + n_C \leq 3$*. Only extensions 1, 2 and 5 possibly violate this condition.*

Proof. It is easy to see that each extension can only create new inner faces that are quadrangles, and inner vertices with degree at least 3.

The only extensions that can increase n_A are extensions 1 and 2. The only extension that can increase n_B is extension 2. The only extension that can increase n_C is extension 5. □

This makes it easier to keep count of n_A, n_B and n_C during the construction.

Lemma 4.2. *Let* P *be a predecoration different from the base predecorations. By applying one of the reductions from Figure 10,* P *can be reduced to a graph containing fewer vertices or a graph containing the same number of vertices but fewer edges.*

Furthermore, if we apply the reduction with the smallest number among all possible reductions, the resulting graph is again a predecoration.

Proof. For the first part, it is clear that each reduction results in a 'smaller' graph, so we only need to verify that at least one reduction can be applied. If P contains at least one quadrangle, there is at least one quadrangle Q with an edge in the outer face. Since P is not C_4, there is at least one other vertex not contained in Q in the graph, and reduction 10 is possible. If there is no quadrangle in P, reduction 1 is possible.

For the second part, it is immediately clear that all reductions preserve the properties that all inner faces are quadrangles and that all inner vertices have degree at least 3. It remains to be proven that for the new graph $n_A \leq 2$ and $n_A + n_B + n_C \leq 3$.

Some reductions can increase n_A, n_B or n_C, but only if another reduction with a smaller number can also be applied. This is the reason that we need so many extension operations in that particular order. In Table 1, all these situations are given.

Table 1: Table with possible reductions. Read this table as:
Reduction i can increase n_X, but only if n_Y is decreased by the same amount.
Reduction i can increase n_X, but only if reduction j/k can be applied too.

reduction	n_A	n_B	n_C
1		n_A	
2	1	1	n_B
3, 4	1		
5, 6, 7	1	1	3/4
8	2/5	5	6/7
9	2/8	8	8
10	2/9	9	9

It is impossible to increase n_A with a reduction that has the smallest possible number. Therefore, we still have $n_A \leq 2$ in the new graph.

Reduction 1 can increase n_B, but only by removing a vertex of degree 2 neighbouring a vertex of degree 1, i.e. by decreasing n_A by the same amount. Therefore, we still have $n_A + n_B + n_C \leq 3$ in the new graph.

Reduction 2 can increase n_C, but only by decreasing n_B by the same amount. Therefore, we still have $n_A + n_B + n_C \leq 3$ in the new graph. □

Theorem 4.3. *The algorithm described in Algorithm 1 generates all predecorations.*

Proof. This follows immediately from [12] and Lemma 4.2. □

5 Construction of decorations

Now that we can construct all predecorations, we can use the homomorphism principle [10] and complete each predecoration in all possible ways to get all k-decorations with Algorithm 2. We first have to compute the symmetry group of the predecoration, in order to avoid completions that result in the same decoration. After the first 4 steps, all symmetry is broken by choosing v_0, v_1 and v_2.

Algorithm 1 Construction of predecorations

```
function EXTEND(P)
    output P
    for i = 1, ..., 10 do
        for O an orbit of edges in the outer face of P do
            e ← edge in O
            P' ← apply extension i to edge e of P
            if P canonical parent of P' then
                EXTEND(P')
for G a base predecoration do
    EXTEND(P)
```

Algorithm 2 Complete a predecoration in all possible ways

1. If $n_A > 0$, label the corresponding vertices of degree 1 with v_0 or v_2 in all non-isomorphic ways.
2. If $n_B + n_C > 0$, label the corresponding vertices with v_0, v_1 or v_2 in all non-isomorphic ways.
3. If v_1 is not yet chosen, label an outer vertex with v_1 or add a new type-1 vertex v_1 of degree 2 in the outer face in all non-isomorphic ways.
4. If v_0 or v_2 is not yet chosen, label two outer vertices with v_0 and v_2 in all non-isomorphic ways.
5. Fill all inner quadrangles with a type-1 vertex.
6. Add type-1 vertices of degree 3 in the outer face in all possible ways, such that there are no cut-vertices or vertices of degree 2 left.
7. Check whether the result is a k-decoration.

We do not have to take isomorphisms into account, since two isomorphic decorations will have isomorphic predecorations.

Note that it might not be possible to complete a predecoration in Step 6 such that there are no cut-vertices left.

5.1 Connectivity

In Step 7, we will always obtain a decoration. The additional properties for 2-connected decorations and 3-connected decorations have to be checked. The properties in the outer face cannot be checked earlier in the construction process, because they depend on the chosen completion. But we can prevent type-1 cycles of length 2 and cycles of length 4 during the construction. It is clear that once a type-1 cycle is created during the construction, it cannot be destroyed later. So we only have to avoid the creation of the first type-1 cycle of length 2 or 4.

The only way to create a first type-1 cycle of length 2 is by applying extension 10 to a predecoration with an outer face of size 4. This can easily be avoided. The only way to create a non-empty type-1 cycle of length 4 is by applying extension 10 to a predecoration with an outer face of size 6. We can avoid this too.

To check the other properties after the completion, we can loop over the outer face of the decoration, and mark all vertices one inner edge away from side i with i. If we encounter a vertex on side i that is marked with i, the decoration is not 2-connected. If a vertex is marked two times with the same number, or a vertex on side 1 is marked with 0 or vice versa, the decoration is not 3-connected.

5.2 Inflation rate

As mentioned in [2], the impact of an operation on the size of a polyhedron can be measured by the *inflation rate*. This is the ratio of the number of edges before and after the operation, and is equal to the number of chambers in the decoration.

Although it is interesting to construct all possible decorations, we are more interested in the decorations with a given inflation rate. Unfortunately, we cannot determine the inflation rate before the predecoration is completed as decorations with different inflation rates might have the same predecoration, but we can compute lower and upper bounds.

Given a predecoration P, for each decoration that has P as its underlying predecoration, each quadrangle of P corresponds to 4 chambers and each cut-vertex of which the removal leaves $k \geq 2$ components requires $2(k-1)$ extra chambers. So

$$4 \cdot (\text{number of quadrangles}) + 2 \cdot \sum_{\text{cut-vertices}} (\text{occurences in outer face} - 1)$$

is a lower bound for the inflation rate. The maximal inflation rate of a predecoration is reached by adding as much type-1 vertices as possible in the outer face. This will result in exactly one chamber for each edge in the outer face. In combination with the 4 chambers in each quadrangle, this results in 2 chambers (one at each side) for each edge of the predecoration. So the maximal inflation rate is

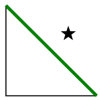
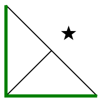
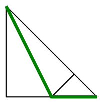
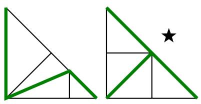
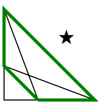
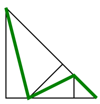
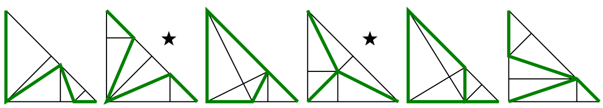
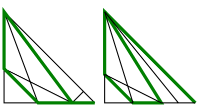
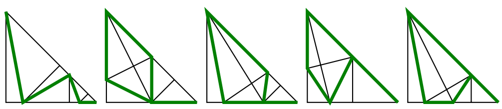
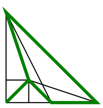
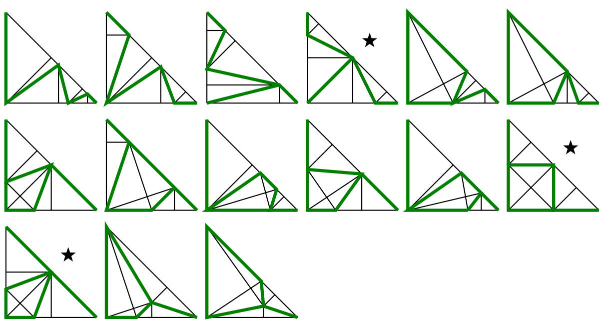
$$2 \cdot (\text{number of edges}).$$

If the lower bound for the inflation rate of a predecoration is already higher than the desired inflation rate, we do not have to extend it further as it can only increase. If the upper bound is lower than the desired inflation rate, we have to extend it, but we do not have to try to complete it.

Table 2: The number of k-connected decorations up to inflation rate 40. The number of predecorations that can be completed to a decoration with given inflation rate are given too. Not all of these predecorations are constructed for 2-connected or 3-connected decorations.

	k-connected decorations			
inflation rate	$k=1$	$k=2$	$k=3$	predecorations
1	2	2	2	1
2	2	2	2	1
3	4	4	4	1
4	6	6	6	2
5	6	6	4	2
6	20	20	20	4
7	28	28	20	7
8	58	58	54	8
9	82	82	64	7
10	170	168	144	19
11	204	200	132	16
12	496	492	404	50
13	650	640	396	42
14	1432	1400	1112	118
15	1824	1786	1100	109
16	4114	3952	2958	298
17	5078	4900	2769	300
18	11874	11150	7972	749
19	14808	14058	7560	782
20	33978	30998	21300	1902
21	41794	38964	20076	2056
22	97096	85976	56296	4893
23	118572	107784	52380	5419
24	277208	237482	148956	12615
25	337216	298546	138384	14153
26	788342	652236	392096	32665
27	953060	820960	362499	36953
28	2239396	1786222	1027488	84853
29	2697088	2250816	945612	96491
30	6350014	4875076	2687408	220646
31	7618068	6153604	2466156	251104
32	17972390	13262574	7007118	573547
33	21487746	16773086	6409664	654663
34	50805716	35985748	18222032	1491540
35	60573248	45592594	16623268	1706755
36	143425040	97394726	47287986	3878836
37	170530518	123628298	43038260	4446426
38	404413576	262983002	122451618	10085305
39	479711448	334473144	111200316	11582891
40	1139138344	708583784	316474370	26222191

Table 3: All decorations with inflation rate r up to 8. The green lines are edges of type 1. The black lines are edges of type 0 and 2. For each of the given decorations, the edges of type 0 and 2 can be chosen in two different ways. All decorations except the symmetric ones (marked with a star) can be mirrored. So each starred decoration represents two related lsp operations, and the unstarred ones represent four related lsp operations.

r	$k = 2$	$k = 3$
1		
2		
3		
4		
5		
6		
7		
8		

6 Results

Using Algortihms 1 and 2, we implemented a computer program [8] to generate all k-decorations with a given inflation rate. The results of this program are given in Table 2. The decorations for inflation rates $r \leq 8$ are given in Table 3.

The two lsp operations with inflation rate 1 are obviously identity and dual. The lsp operations with inflation rate 2 are ambo and join, and the ones with inflation rate 3 are truncate, zip, needle and kiss. Up to here, all lsp operations were already described by Conway [5] or others. For the left decoration with inflation rate 4, only two of the 4 related lsp operations (chamfer and subdivide) are already named. The first decoration for which none of the related lsp operations (including dual and mirrored ones) are already named, is the 2-connected lsp operation with inflation rate 5. The first unnamed 3-connected lsp operations are the three leftmost decorations with inflation rate 6.

These results are verified for inflation rate up to 23 by an independent implementation that constructs all triangulations, filters the decorations out, applies them to a polyhedron, checks the connectivity and filters the isomorphic ones out.

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Relative Heffter arrays and biembeddings

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Abstract

Relative Heffter arrays, denoted by $\mathrm{H}_t(m, n; s, k)$, have been introduced as a generalization of the classical concept of Heffter array. A $\mathrm{H}_t(m, n; s, k)$ is an $m \times n$ partially filled array with elements in $\mathbb{Z}_v$, where $v = 2nk + t$, whose rows contain s filled cells and whose columns contain k filled cells, such that the elements in every row and column sum to zero and, for every $x \in \mathbb{Z}_v$ not belonging to the subgroup of order t, either x or $-x$ appears in the array. In this paper we show how relative Heffter arrays can be used to construct biembeddings of cyclic cycle decompositions of the complete multipartite graph $K_{\frac{2nk+t}{t}\times t}$ into an orientable surface. In particular, we construct such biembeddings providing integer globally simple square relative Heffter arrays for $t = k = 3, 5, 7, 9$ and $n \equiv 3 \pmod 4$ and for $k = 3$ with $t = n, 2n$, any odd n.

Keywords: Heffter array, biembedding, complete multipartite graph.

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1 Introduction

An $m \times n$ partially filled (p.f., for short) array on a set Ω is an $m \times n$ matrix whose elements belong to Ω and where we also allow some cells to be empty. The following class of p.f. arrays was introduced in [15], generalizing the ideas of [2]:

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Definition 1.1. Let $v = 2nk + t$ be a positive integer and let J be the subgroup of $\mathbb{Z}_v$ of order t. A $\mathrm{H}_t(m, n; s, k)$ *Heffter array over* $\mathbb{Z}_v$ *relative to* J is an $m \times n$ p.f. array with elements in $\mathbb{Z}_v$ such that:

(a) each row contains s filled cells and each column contains k filled cells;

(b) for every $x \in \mathbb{Z}_{2nk+t} \setminus J$, either x or $-x$ appears in the array;

(c) the elements in every row and column sum to zero.

Trivial necessary conditions for the existence of a $\mathrm{H}_t(m, n; s, k)$ are that t divides $2nk$, $nk = ms$, $3 \le s \le n$ and $3 \le k \le m$. If $\mathrm{H}_t(m, n; s, k)$ is a square array, it will be denoted by $\mathrm{H}_t(n; k)$. A relative Heffter array is called *integer* if Condition (c) in Definition 1.1 is strengthened so that the elements in every row and in every column, viewed as integers in $\pm\{1, \dots, \lfloor \frac{2nk+t}{2} \rfloor\}$, sum to zero in $\mathbb{Z}$. We remark that, if $t = 1$, namely if J is the trivial subgroup of $\mathbb{Z}_{2nk+1}$, we find again the classical concept of a (integer) Heffter array, see [2, 3, 4, 9, 10, 13, 16, 17]. In particular, in [10] it was proved that Heffter arrays $\mathrm{H}_1(n; k)$ exist for all $n \ge k \ge 3$, while by [4, 17] integer Heffter arrays $\mathrm{H}_1(n; k)$ exist if and only if the additional condition $nk \equiv 0, 3 \pmod 4$ holds. At the moment, the only known results concerning relative Heffter arrays are described in [15, 22]. Some necessary conditions for the existence of an integer $\mathrm{H}_t(n; k)$ are given by the following.

Proposition 1.2 ([15])**.** *Suppose that there exists an integer* $\mathrm{H}_t(n; k)$ *for some* $n \ge k \ge 3$ *and some divisor* t *of* $2nk$*.*

(1) If t *divides* nk*, then* $nk \equiv 0 \pmod 4$ *or* $nk \equiv -t \equiv \pm 1 \pmod 4$*.*

(2) If $t = 2nk$*, then* k *must be even.*

(3) If $t \neq 2nk$ *does not divide* nk*, then* $t + 2nk \equiv 0 \pmod 8$*.*

We point out that these conditions are not sufficient, in fact in the same paper the authors show that there is no integer $\mathrm{H}_{3n}(n; 3)$ and no integer $\mathrm{H}_8(4; 3)$.

The *support* of an integer Heffter array A, denoted by $\mathrm{supp}(A)$, is defined to be the set of the absolute values of the elements contained in A. It is immediate to see that an integer $\mathrm{H}_2(n; k)$ is nothing but an integer $\mathrm{H}_1(n; k)$, since in both cases the support is $\{1, 2, \dots, nk\}$.

In this paper we study the connection between relative Heffter arrays and biembeddings. In particular, in Section 2 we recall well known definitions and results about simple orderings and cycle decompositions. Then, in Section 3 we explain how relative Heffter arrays $\mathrm{H}_t(n; k)$ can be used to construct biembeddings of cyclic k-cycle decompositions of the complete multipartite graph $K_{\frac{2nk+t}{t} \times t}$ into an orientable surface. Direct constructions of globally simple integer $\mathrm{H}_t(n; 3)$ with $t = n, 2n$ for any odd n and of globally simple integer $\mathrm{H}_k(n; k)$ for $k = 7, 9$ and $n \equiv 3 \pmod 4$ are described in Section 4. Combining the results of these sections we prove the following.

Theorem 1.3. *There exists a cellular biembedding of a pair of cyclic* k*-cycle decompositions of* $K_{\frac{2nk+t}{t} \times t}$ *into an orientable surface in each of the following cases:*

(1) $k = 3$*,* $t \in \{n, 2n\}$ *and* n *is odd;*

(2) $k \in \{3, 5, 7, 9\}$*,* $t = k$ *and* $n \equiv 3 \pmod 4$*.*

Finally, in Section 5 we introduce a further generalization, called *Archdeacon array*, of the classical concept of Heffter array. We show some examples and how both cycle decompositions and biembeddings can be obtained also using these arrays.

2 Simple orderings and cycle decompositions

Given two integers $a \leq b$, we denote by $[a, b]$ the interval containing the integers $a, a+1, \ldots, b$. If $a > b$, then $[a, b]$ is empty.

If A is an $m \times n$ p.f. array, the rows and the columns of A will be denoted by $\overline{R}_1, \ldots, \overline{R}_m$ and by $\overline{C}_1, \ldots, \overline{C}_n$, respectively. We will denote by $\mathcal{E}(A)$ the unordered list of the elements of the filled cells of A. Analogously, by $\mathcal{E}(\overline{R}_i)$ and $\mathcal{E}(\overline{C}_j)$ we mean the unordered lists of elements of the i-th row and of the j-th column, respectively, of A. Also, we define the skeleton of A, denoted by $\mathrm{skel}(A)$, to be the set of the filled positions of A.

Given a finite subset T of an abelian group G and an ordering $\omega = (t_1, t_2, \ldots, t_k)$ of the elements in T, let $s_i = \sum_{j=1}^{i} t_j$, for any $i \in [1, k]$, be the i-th partial sum of ω and set $\mathcal{S}(\omega) = (s_1, \ldots, s_k)$. The ordering ω is said to be *simple* if $s_b \neq s_c$ for all $1 \leq b < c \leq k$ or, equivalently, if there is no proper subsequence of ω that sums to 0. Note that if ω is a simple ordering so is $\omega^{-1} = (t_k, t_{k-1}, \ldots, t_1)$. We point out that there are several interesting problems and conjectures about distinct partial sums: see, for instance, [1, 5, 14, 19, 23]. Given an $m \times n$ p.f. array A, by $\omega_{\overline{R}_i}$ and $\omega_{\overline{C}_j}$ we will denote, respectively, an ordering of $\mathcal{E}(\overline{R}_i)$ and of $\mathcal{E}(\overline{C}_j)$. If for any $i \in [1, m]$ and for any $j \in [1, n]$, the orderings $\omega_{\overline{R}_i}$ and $\omega_{\overline{C}_j}$ are simple, we define by $\omega_r = \omega_{\overline{R}_1} \circ \cdots \circ \omega_{\overline{R}_m}$ the simple ordering for the rows and by $\omega_c = \omega_{\overline{C}_1} \circ \cdots \circ \omega_{\overline{C}_n}$ the simple ordering for the columns. Moreover, by *natural ordering* of a row (column) of A we mean the ordering from left to right (from top to bottom). A p.f. array A on an abelian group G is said to be

- *simple* if each row and each column of A admits a simple ordering;
- *globally simple* if the natural ordering of each row and each column of A is simple.

Clearly if $k \leq 5$, then every square relative Heffter array is (globally) simple.

We recall some basic definitions about graphs and graph decompositions. Given a graph Γ, by $V(\Gamma)$ and $E(\Gamma)$ we mean the vertex set and the edge set of Γ, respectively. We will denote by K_v the complete graph of order v and by $K_{q \times r}$ the complete multipartite graph with q parts each of size r. Obviously $K_{q \times 1}$ is nothing but the complete graph K_q. Let G be an additive group (not necessarily abelian) and let $\Lambda \subseteq G \setminus \{0\}$ such that $\Lambda = -\Lambda$, which means that for every $\lambda \in \Lambda$ we have also $-\lambda \in \Lambda$. The Cayley graph on G with connection set Λ, denoted by $\mathsf{Cay}[G : \Lambda]$, is the simple graph having G as vertex set and such that two vertices x and y are adjacent if and only if $x - y \in \Lambda$. Note that, if $\Lambda = G \setminus \{0\}$, the Cayley graph is the complete graph whose vertex set is G and, if $\Lambda = G \setminus J$ for some subgroup J of G, the Cayley graph is the complete multipartite graph $K_{q \times r}$ where $q = |G : J|$ and $r = |J|$.

The following are well known definitions and results which can be found, for instance, in [8]. Let Γ be a subgraph of a graph K. A Γ-*decomposition* of K is a set $\mathcal{D}$ of subgraphs of K isomorphic to Γ whose edges partition $E(K)$. If the vertices of K belong to a group G, given $g \in G$, by $\Gamma + g$ one means the graph whose vertex set is $V(\Gamma) + g$ and whose edge set is $\{\{x + g, y + g\} \mid \{x, y\} \in E(\Gamma)\}$. An *automorphism group* of a Γ-decomposition $\mathcal{D}$ of K is a group of bijections on $V(K)$ leaving $\mathcal{D}$ invariant. A Γ-decomposition of K is said to be *regular under a group* G or G*-regular* if it admits G as an automorphism group acting sharply transitively on $V(K)$. Here we consider cyclic cycle decompositions, namely decompositions which are regular under a cyclic group and with Γ a cycle. Finally, two graph decompositions $\mathcal{D}$ and $\mathcal{D}'$ of a simple graph K are said *orthogonal* if and only if for any B of $\mathcal{D}$ and any B' of $\mathcal{D}'$, B intersects B' in at most one edge.

The relationship between simple relative Heffter arrays and cyclic cycle decompositions of the complete multipartite graph is explained in [15]. Here we briefly recall the following result.

Proposition 2.1 ([15, Proposition 2.9]). *Let A be a $\mathrm{H}_t(m,n;s,k)$ simple with respect to the orderings ω_r and ω_c. Then:*

(1) there exists a cyclic s-cycle decomposition $\mathcal{D}_{\omega_r}$ of $K_{\frac{2ms+t}{t}\times t}$;

(2) there exists a cyclic k-cycle decomposition $\mathcal{D}_{\omega_c}$ of $K_{\frac{2nk+t}{t}\times t}$;

(3) the cycle decompositions $\mathcal{D}_{\omega_r}$ and $\mathcal{D}_{\omega_c}$ are orthogonal.

The arrays we are going to construct are square with a diagonal structure, so it is convenient to introduce the following notation. If A is an $n\times n$ array, for $i\in[1,n]$ we define the i-th diagonal

$$D_i=\{(i,1),(i+1,2),\dots,(i-1,n)\}.$$

Here all the arithmetic on the row and the column indices is performed modulo n, where the set of reduced residues is $\{1,2,\dots,n\}$. We say that the diagonals $D_i,D_{i+1},\dots,D_{i+r}$ are *consecutive diagonals*.

Definition 2.2. Let $k\geq 1$ be an integer. We will say that a square p.f. array A of size $n\geq k$ is

- *k-diagonal* if the non empty cells of A are exactly those of k diagonals;
- *cyclically k-diagonal* if the nonempty cells of A are exactly those of k consecutive diagonals.

Let A be a k-diagonal array of size $n>k$. A set $S=\{D_{r+1},D_{r+2},\dots,D_{r+\ell}\}$ is said to be an *empty strip of width ℓ* if $D_{r+1},D_{r+2},\dots,D_{r+\ell}$ are empty diagonals, while D_r and $D_{r+\ell+1}$ are filled diagonals.

Definition 2.3. Let A be a k-diagonal array of size $n>k$. We will say that A is a *k-diagonal array with width ℓ* if all the empty strips of A have width ℓ.

An array of this kind will be given in Example 4.9.

3 Relation with biembeddings

In [2], Archdeacon introduced Heffter arrays also in view of their applications and, in particular, since they are useful for finding biembeddings of cycle decompositions, as shown, for instance, in [11, 13, 16]. In this section, generalizing some of Archdeacon's results we show how starting from a relative Heffter array it is possible to obtain suitable biembeddings.

We recall the following definition, see [20].

Definition 3.1. An *embedding* of a graph Γ in a surface Σ is a continuous injective mapping $\psi\colon\Gamma\to\Sigma$, where Γ is viewed with the usual topology as 1-dimensional simplicial complex.

The connected components of $\Sigma\setminus\psi(\Gamma)$ are called *ψ-faces*. If each ψ-face is homeomorphic to an open disc, then the embedding ψ is said to be *cellular*.

Definition 3.2. A *biembedding* of two cycle decompositions $\mathcal{D}$ and $\mathcal{D}'$ of a simple graph Γ is a face 2-colorable embedding of Γ in which one color class is comprised of the cycles in $\mathcal{D}$ and the other class contains the cycles in $\mathcal{D}'$.

Following the notation given in [2], for every edge e of a graph Γ, let e^+ and e^- denote its two possible directions and let τ be the involution swapping e^+ and e^- for every e. Let $D(\Gamma)$ be the set of all directed edges of Γ and, for any $v \in V(\Gamma)$, call D_v the set of edges directed out of v. A local rotation ρ_v is a cyclic permutation of D_v. If we select a local rotation for each vertex of Γ, then all together they form a rotation of $D(\Gamma)$. We recall the following result, see [2, 18, 21].

Theorem 3.3. *A rotation ρ on Γ is equivalent to a cellular embedding of Γ in an orientable surface. The face boundaries of the embedding corresponding to ρ are the orbits of $\rho \circ \tau$.*

Given a relative Heffter array $A = \mathrm{H}_t(m, n; s, k)$, the orderings ω_r and ω_c are said to be *compatible* if $\omega_c \circ \omega_r$ is a cycle of length $|\mathcal{E}(A)|$.

Theorem 3.4. *Let A be a relative Heffter array $\mathrm{H}_t(m, n; s, k)$ that is simple with respect to the compatible orderings ω_r and ω_c. Then there exists a cellular biembedding of the cyclic cycle decompositions $\mathcal{D}_{\omega_r^{-1}}$ and $\mathcal{D}_{\omega_c}$ of $K_{\frac{2nk+t}{t}\times t}$ into an orientable surface of genus*

$$g = 1 + \frac{(nk - n - m - 1)(2nk + t)}{2}.$$

Proof. Since the orderings ω_r and ω_c are compatible, we have that $\omega_c \circ \omega_r$ is a cycle of length $|\mathcal{E}(A)|$. Let us consider the permutation $\bar{\rho}_0$ on $\pm\mathcal{E}(A) = \mathbb{Z}_{2nk+t} \setminus \frac{2nk+t}{t}\mathbb{Z}_{2nk+t}$, where $\frac{2nk+t}{t}\mathbb{Z}_{2nk+t}$ denotes the subgroup of $\mathbb{Z}_{2nk+t}$ of order t, defined by:

$$\bar{\rho}_0(a) = \begin{cases} -\omega_r(a) & \text{if } a \in \mathcal{E}(A); \\ \omega_c(-a) & \text{if } a \in -\mathcal{E}(A). \end{cases}$$

Note that, if $a \in \mathcal{E}(A)$, then $\bar{\rho}_0^2(a) = \omega_c \circ \omega_r(a)$ and hence $\bar{\rho}_0^2$ acts cyclically on $\mathcal{E}(A)$. Also $\bar{\rho}_0$ exchanges $\mathcal{E}(A)$ with $-\mathcal{E}(A)$. Thus it acts cyclically on $\pm\mathcal{E}(A)$.

We note that the graph $K_{\frac{2nk+t}{t}\times t}$ is nothing but $\mathsf{Cay}[\mathbb{Z}_{2nk+t} : \mathbb{Z}_{2nk+t} \setminus \frac{2nk+t}{t}\mathbb{Z}_{2nk+t}]$ that is $\mathsf{Cay}[\mathbb{Z}_{2nk+t} : \pm\mathcal{E}(A)]$. Now, we define the map ρ on the set of the oriented edges of the Cayley graph $\mathsf{Cay}[\mathbb{Z}_{2nk+t} : \pm\mathcal{E}(A)]$ so that:

$$\rho((x, x + a)) = (x, x + \bar{\rho}_0(a)).$$

Since $\bar{\rho}_0$ acts cyclically on $\pm\mathcal{E}(A)$ the map ρ is a rotation of $\mathsf{Cay}[\mathbb{Z}_{2nk+t} : \pm\mathcal{E}(A)]$. Hence, by Theorem 3.3, there exists a cellular embedding σ of $\mathsf{Cay}[\mathbb{Z}_{2nk+t} : \pm\mathcal{E}(A)]$ in an orientable surface so that the face boundaries correspond to the orbits of $\rho \circ \tau$ where $\tau((x, x + a)) = (x + a, x)$. Let us consider the oriented edge $(x, x + a)$ with $a \in \mathcal{E}(A)$, and let $\overline{C}$ be the column containing a. Since $a \in \mathcal{E}(A)$, $-a \in -\mathcal{E}(A)$ and we have that:

$$\rho \circ \tau((x, x + a)) = \rho((x + a, (x + a) - a)) = (x + a, x + a + \omega_c(a)).$$

Thus $(x, x + a)$ belongs to the boundary of the face F_1 delimited by the oriented edges:

$$(x, x + a), (x + a, x + a + \omega_c(a)),$$
$$(x + a + \omega_c(a), x + a + \omega_c(a) + \omega_c^2(a)), \ldots, \left(x + \sum_{i=0}^{|\mathcal{E}(\overline{C})|-2} \omega_c^i(a), x\right).$$

We note that the cycle associated to the face F_1 is:

$$\left(x, x+a, x+a+\omega_c(a), \ldots, x+\sum_{i=0}^{|\mathcal{E}(\overline{C})|-2} \omega_c^i(a)\right).$$

Let us now consider the oriented edge $(x, x+a)$ with $a \notin \mathcal{E}(A)$. Hence $-a \in \mathcal{E}(A)$, and we name by $\overline{R}$ the row containing the element $-a$. Since $-a \in \mathcal{E}(A)$ we have that:

$$\rho \circ \tau((x, x+a)) = \rho((x+a, (x+a)-a)) = (x+a, x+a-\omega_r(-a)).$$

Thus $(x, x+a)$ belongs to the boundary of the face F_2 delimited by the oriented edges:

$$(x, x+a), (x-(-a), x-(-a)-\omega_r(-a)),$$
$$(x-(-a)-\omega_r(-a), x-(-a)-\omega_r(-a)-\omega_r^2(-a)), \ldots, \left(x-\sum_{i=0}^{|\mathcal{E}(\overline{R})|-2} \omega_r^i(-a), x\right).$$

Since A is a Heffter array and ω_r acts cyclically on $\mathcal{E}(\overline{R})$, for any $j \in [1, |\mathcal{E}(\overline{R})|]$ we have that:

$$-\sum_{i=0}^{j-1} \omega_r^i(-a) = \sum_{i=j}^{|\mathcal{E}(\overline{R})|-1} \omega_r^i(-a) = \sum_{i=1}^{|\mathcal{E}(\overline{R})|-j} \omega_r^{|\mathcal{E}(\overline{R})|-i}(-a) = \sum_{i=1}^{|\mathcal{E}(\overline{R})|-j} \omega_r^{-i}(-a).$$

It follows that the cycle associated to the face F_2 can be written also as:

$$\left(x, x+\sum_{i=1}^{|\mathcal{E}(\overline{R})|-1} \omega_r^{-i}(-a), x+\sum_{i=1}^{|\mathcal{E}(\overline{R})|-2} \omega_r^{-i}(-a), \ldots, x+\omega_r^{-1}(-a)\right).$$

Therefore any nonoriented edge $\{x, x+a\}$ belongs to the boundaries of exactly two faces: one of type F_1 and one of type F_2. Hence the embedding is 2-colorable.

Moreover, it is easy to see that those face boundaries are the cycles obtained from the relative Heffter array A following the orderings ω_c and ω_r^{-1}.

To calculate the genus g it suffices to recall that $V-S+F = 2-2g$, where V, S and F denote the number of vertices, edges and faces determined by the embedding on the surface, respectively. We have $V = 2nk+t$, $S = nk(2nk+t)$ and $F = (2nk+t)(n+m)$. □

Looking for compatible orderings in the case of a globally simple Heffter array led us to investigate the following problem introduced in [12]. Let A be an $m \times n$ *toroidal* p.f. array. By r_i we denote the orientation of the i-th row, precisely $r_i = 1$ if it is from left to right and $r_i = -1$ if it is from right to left. Analogously, for the j-th column, if its orientation c_j is from top to bottom then $c_j = 1$ otherwise $c_j = -1$. Assume that an orientation $\mathcal{R} = (r_1, \ldots, r_m)$ and $\mathcal{C} = (c_1, \ldots, c_n)$ is fixed. Given an initial filled cell (i_1, j_1) consider the sequence $L_{\mathcal{R},\mathcal{C}}(i_1, j_1) = ((i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell), (i_{\ell+1}, j_{\ell+1}), \ldots)$ where $j_{\ell+1}$ is the column index of the filled cell $(i_\ell, j_{\ell+1})$ of the row $\overline{R}_{i_\ell}$ next to (i_ℓ, j_ℓ) in the orientation r_{i_ℓ}, and where $i_{\ell+1}$ is the row index of the filled cell of the column $\overline{C}_{j_{\ell+1}}$ next to $(i_\ell, j_{\ell+1})$ in the orientation $c_{j_{\ell+1}}$. The problem is the following:

Crazy Knight's Tour Problem. Given a toroidal p.f. array A, do there exist $\mathcal{R}$ and $\mathcal{C}$ such that the list $L_{\mathcal{R},\mathcal{C}}$ covers all the filled cells of A?

By $P(A)$ we will denote the Crazy Knight's Tour Problem for a given array A. Also, given a filled cell (i,j), if $L_{\mathcal{R},\mathcal{C}}(i,j)$ covers all the filled positions of A we will say that $(\mathcal{R},\mathcal{C})$ is a solution of $P(A)$. For known results about this problem see [12]. The relationship between the Crazy Knight's Tour Problem and globally simple relative Heffter arrays is explained in the following result which is an easy consequence of Theorem 3.4.

Corollary 3.5. *Let A be a globally simple relative Heffter array $\mathrm{H}_t(m,n;s,k)$ such that $P(A)$ admits a solution $(\mathcal{R},\mathcal{C})$. Then there exists a biembedding of the cyclic cycle decompositions $\mathcal{D}_{\omega_r^{-1}}$ and $\mathcal{D}_{\omega_c}$ of $K_{\frac{2nk+t}{t}\times t}$ into an orientable surface.*

Extending [11, Theorem 1.1] to the relative case, we have the following result (see also [12, Theorem 2.7]).

Proposition 3.6. *If there exist compatible simple orderings ω_r and ω_c for a $\mathrm{H}_t(m,n;s,k)$, then one of the following cases occurs:*

(1) m,n,s,k are all odd;

(2) m is odd and n,k are even;

(3) n is odd and m,t are even.

Given a positive integer n, let $0<\ell_1<\ell_2<\cdots<\ell_k<n$ be integers. We denote by $A_n=A_n(\ell_1,\ell_2,\ldots,\ell_k)$ a k-diagonal p.f. array of size n whose filled diagonals are $D_{\ell_1},D_{\ell_2},\ldots,D_{\ell_k}$. Let $M=\mathrm{lcm}(\ell_2-\ell_1,\ell_3-\ell_2,\ldots,\ell_k-\ell_{k-1},\ell_k-\ell_1)$ and set $A_{n+M}=A_{n+M}(\ell_1,\ell_2,\ldots,\ell_k)$. We now study the Crazy Knight's Tour Problem for such arrays A_n. As a consequence, we will obtain new biembeddings of cycle decompositions of complete graphs on orientable surfaces.

Theorem 3.7. *Suppose that the problem $P(A_n)$ admits a solution $(\mathcal{R},\mathcal{C})$ where $\mathcal{R}=(1,1,\ldots,1)$ and $\mathcal{C}=(c_1,c_2,\ldots,c_{n-\ell_k+1},1,1,\ldots,1)$. Then $P(A_{n+M})$ admits the solution $(\mathcal{R}',\mathcal{C}')$ where $\mathcal{R}'=(1,1,\ldots,1)$ and $\mathcal{C}'=(c_1,c_2,\ldots,c_{n-\ell_k+1},1,1,\ldots,1)$.*

Proof. We denote by E the set of indices i such that $c_i=-1$ and by B_n the p.f. array of size n obtained from A_n by replacing each column $\overline{C}_j$, when $j\notin E$, with an empty column. Also, we denote by B_{n+M} the p.f. array of size $n+M$ obtained from A_{n+M} in the same way using the same set E. As $E\subseteq[1,n-\ell_k+1]$, the nonempty cells of B_n are of the form $((e-1)+\ell_i,e)$ for $e\in E$ and $i\in[1,k]$. Since $(e-1)+\ell_i\leq n$, we have $\mathrm{skel}(B_n)=\mathrm{skel}(B_{n+M})$. So we can set $B=\mathrm{skel}(B_n)=\mathrm{skel}(B_{n+M})$.

For any $x=(i_1,j_1)\in B$, consider the sequence $X=L_{\mathcal{R},\mathcal{C}}(i_1,j_1)$ defined on $\mathrm{skel}(A_n)$ and let y be the second element of X that belongs to B if $|X\cap B|\geq 2$, $y=x$ otherwise. Define $\vartheta_n\colon B\to B$ by setting $\vartheta_n(x)=y$. Take $(\mathcal{R}',\mathcal{C}')$ as in the statement and define the map $\vartheta_{n+M}\colon B\to B$ as before considering the sequence $L_{\mathcal{R}',\mathcal{C}'}(x)$ defined on $\mathrm{skel}(A_{n+M})$.

In order to prove that $\vartheta_n(x)=\vartheta_{n+M}(x)$, for any $h\in[1,k]$, we set:

$$\sigma(h)=\begin{cases}\ell_1-\ell_{k-1} & \text{if } h=1;\\ \ell_2-\ell_k & \text{if } h=2;\\ \ell_h-\ell_{h-2} & \text{otherwise}\end{cases}\quad\text{and}\quad \delta(h)=\begin{cases}\ell_1-\ell_k & \text{if } h=1;\\ \ell_h-\ell_{h-1} & \text{otherwise.}\end{cases}$$

Set $x = (i_1, j_1) \in B$, hence $x \in D_{\ell_h}$ for some $h \in [1, k]$. We have that

$$\vartheta_n(x) = (i_1 + \delta(h)\lambda - \sigma(h), j_1 + \delta(h)\lambda) \pmod{n}$$

where λ is the minimum positive integer such that $(j_1 + \delta(h)\lambda) \pmod{n} \in E$. Similarly

$$\vartheta_{n+M}(x) = (i_1 + \delta(h)\lambda' - \sigma(h), j_1 + \delta(h)\lambda') \pmod{n+M}$$

where λ' is the minimum positive integer such that $(j_1 + \delta(h)\lambda') \pmod{n+M} \in E$. Write $j_1 + \delta(h)\lambda = qn + r$ where $1 \le r \le n$, which means $r \in E$.

If $q = 0$, we clearly have $\lambda' = \lambda$ and hence $\vartheta_{n+M}(x) = \vartheta_n(x)$. Otherwise, since the last M elements of $\mathcal{C}'$ are equal to 1, we have that $\lambda' = \lambda + \frac{qM}{\delta(h)}$. Hence:

$$\begin{aligned}\vartheta_{n+M}(x) &= \left(i_1 + \delta(h)\left(\lambda + \frac{qM}{\delta(h)}\right) - \sigma(h), j_1 + \delta(h)\left(\lambda + \frac{qM}{\delta(h)}\right)\right) \pmod{n+M}\\ &= (i_1 + \delta(h)\lambda + qM - \sigma(h), j_1 + \delta(h)\lambda + qM) \pmod{n+M}\\ &= ((i_1 - j_1) + q(n+M) + r - \sigma(h), q(n+M) + r) \pmod{n+M}\\ &= ((i_1 - j_1) + r - \sigma(h), r) \pmod{n+M}.\end{aligned}$$

It is not hard to see that $1 \le (i_1 - j_1) + r - \sigma(h) \le n$; also recall that $1 \le r \le n$. Hence

$$\vartheta_{n+M}(x) = ((i_1 - j_1) + r - \sigma(h), r).$$

On the other hand, by $j_1 + \delta(h)\lambda = qn + r$, we obtain:

$$((i_1 - j_1) + r - \sigma(h), r) = (i_1 + \delta(h)\lambda - \sigma(h), j_1 + \delta(h)\lambda) \pmod{n} = \vartheta_n(x).$$

So we have proved that $\vartheta_{n+M}(x) = \vartheta_n(x)$ for any $x \in B$.

For any $(i, j) \in \operatorname{skel}(A_n)$, since $(\mathcal{R}, \mathcal{C})$ is a solution of $P(A_n)$, we have $L_{\mathcal{R},\mathcal{C}}(i, j) \cap B = B$. Moreover, since $\vartheta_n(x) = \vartheta_{n+M}(x)$ for any $x \in B$, it follows that for any $(i', j') \in \operatorname{skel}(A_{n+M})$ we have $L_{\mathcal{R}',\mathcal{C}'}(i', j') \cap B$ is either B or $\emptyset$. If there exists $(\bar{\imath}, \bar{\jmath}) \in \operatorname{skel}(A_{n+M})$ such that $L_{\mathcal{R}',\mathcal{C}'}(\bar{\imath}, \bar{\jmath}) \cap B = \emptyset$ then for any $\lambda' \in \mathbb{N}$, the cell $(\bar{\imath} + \delta(\bar{h})\lambda', \bar{\jmath} + \delta(\bar{h})\lambda') \pmod{n+M}$ is not in B. On the other hand there exists $\lambda \in \mathbb{N}$, such that $(\bar{\imath} + \delta(\bar{h})\lambda, \bar{\jmath} + \delta(\bar{h})\lambda) \pmod{n} \in B$, since $(\mathcal{R}, \mathcal{C})$ is a solution of $P(A_n)$. Also, since $\delta(\bar{h})$ divides M there exists $\bar{q} \in \mathbb{N}$ such that $(\bar{\imath} + \delta(\bar{h})\bar{\lambda}, \bar{\jmath} + \delta(\bar{h})\bar{\lambda}) \pmod{n+M} \in B$, where $\bar{\lambda} = \lambda + \bar{q}M/\delta(\bar{h})$. Hence $L_{\mathcal{R}',\mathcal{C}'}(\bar{\imath}, \bar{\jmath}) \cap B \neq \emptyset$, which is a contradiction. Thus it follows that $(\mathcal{R}', \mathcal{C}')$ is a solution of $P(A_{n+M})$. □

Corollary 3.8. *Let $k \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$ be such that $n \ge k$ and $3 \le k \le 119$. Let A_n be a k-diagonal array whose filled diagonals are $D_1, D_2, \ldots, D_{k-3}$, D_{k-1}, D_k and D_{k+1}. Then $P(A_n)$ admits a solution.*

Proof. Let $k = 4h + 3$ and $M = \operatorname{lcm}(2, 4h + 3)$, that is $M = 2(4h + 3)$. For any $1 \le h \le 29$, with the help of a computer, we have checked the existence of a solution of $P(A_n)$ for any $n \in [4h+5, 4h+5+M] = [4h+5, 12h+11]$, that satisfies the hypothesis of Theorem 3.7. Hence the claim follows by this theorem. □

Corollary 3.9. *Let $k \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$ such that $n \ge k$ and $3 \le k \le 119$. Then there exists a globally simple $\mathrm{H}_1(n; k)$ with orderings ω_r and ω_c which are both simple and compatible. As a consequence, there exists a biembedding of cyclic k-cycle decompositions of the complete graph K_{2nk+1} into an orientable surface.*

Proof. The existence of a globally simple $\mathrm{H}_1(n;k)$, whose filled diagonals are $D_1, D_2, \ldots, D_{k-3}, D_{k-1}, D_k, D_{k+1}$, was proven in [9]. The result follows from Corollaries 3.5 and 3.8. □

4 Direct constructions of globally simple $\mathrm{H}_t(n;k)$

Many of the constructions we will present are based on filling in the cells of a set of diagonals. In order to describe these constructions we use the same procedure introduced in [17]. In an $n \times n$ array A the procedure $diag(r, c, s, \Delta_1, \Delta_2, \ell)$ installs the entries

$$A[r + i\Delta_1, c + i\Delta_1] = s + i\Delta_2 \text{ for } i \in [0, \ell - 1],$$

where by $A[i, j]$ we mean the element of A in position (i, j). The parameters used in the $diag$ procedure have the following meaning:

- r denotes the starting row,
- c denotes the starting column,
- s denotes the entry $A[r, c]$,
- Δ_1 denotes the increasing value of the row and column at each step,
- Δ_2 denotes how much the entry is changed at each step,
- ℓ is the length of the chain.

We will write $[a, b]_{(W)}$ to mean $\mathrm{supp}(W) = [a, b]$.

Proposition 4.1. *For every odd $n \geq 3$ there exists an integer cyclically 3-diagonal Heffter array $\mathrm{H}_n(n;3)$.*

Proof. We construct an $n \times n$ array A using the following procedures labeled `A` to `E`:

`A`: $diag\left(1, 1, -\frac{7n-9}{2}, 1, 7, n\right)$; `B`: $diag\left(1, 2, \frac{7n-3}{2}, 2, -7, \frac{n+1}{2}\right)$;
`C`: $diag\left(2, 3, -5, 2, -7, \frac{n-1}{2}\right)$; `D`: $diag\left(2, 1, \frac{7n-13}{2}, 2, -7, \frac{n+1}{2}\right)$;
`E`: $diag\left(3, 2, -10, 2, -7, \frac{n-1}{2}\right)$.

We prove that the array constructed above is an integer cyclically 3-diagonal $\mathrm{H}_n(n;3)$. To aid in the proof we give a schematic picture of where each of the diagonal procedures fills cells (see Figure 1). Note that each row and each column contain exactly 3 elements. We now check that the elements in every row sum to zero (in $\mathbb{Z}$).

Row 1. There is the first value of the `A` diagonal and of the `B` diagonal and the last of the `D` diagonal. The sum is

$$-\frac{7n-9}{2} + \frac{7n-3}{2} - 3 = 0.$$

Row 2 to n. There are two cases depending on whether the row r is even or odd. If r is even, then write $r = 2i + 2$ where $i \in \left[0, \frac{n-3}{2}\right]$. Notice that from the `D`, `A` and `C` diagonal cells we get the following sum:

$$\left(\frac{7n-13}{2} - 7i\right) + \left(-\frac{7n-23}{2} + 14i\right) + (-5 - 7i) = 0.$$

A	B							D
D	A	C						
	E	A	B					
		D	A	C				
			E	A	B			
				D	A	C		
					E	A	B	
						D	A	C
B							E	A

Figure 1: Scheme of construction with $n = 9$.

If r is odd, then write $r = 2i + 3$ where $i \in \left[0, \frac{n-3}{2}\right]$. From the $\mathtt{E}$, $\mathtt{A}$ and $\mathtt{B}$ diagonal cells we get the following sum:

$$(-10 - 7i) + \left(-\frac{7n - 37}{2} + 14i\right) + \left(\frac{7n - 17}{2} - 7i\right) = 0.$$

So we have shown that all row sums are zero. Next we check that the columns all add to zero.

Column 1. There is the first value of the $\mathtt{A}$ diagonal and of the $\mathtt{D}$ diagonal and the last of the $\mathtt{B}$ diagonal. The sum is

$$-\frac{7n - 9}{2} + \frac{7n - 13}{2} + 2 = 0.$$

Column 2 to n. There are two cases depending on whether the column c is even or odd. If c is even, then write $c = 2i + 2$ where $i \in \left[0, \frac{n-3}{2}\right]$. Notice that from the $\mathtt{B}$, $\mathtt{A}$ and $\mathtt{E}$ diagonal cells we get the following sum:

$$\left(\frac{7n - 3}{2} - 7i\right) + \left(-\frac{7n - 23}{2} + 14i\right) + (-10 - 7i) = 0.$$

If c is odd, then write $c = 2i + 3$ where $i \in \left[0, \frac{n-3}{2}\right]$. From the $\mathtt{C}$, $\mathtt{A}$ and $\mathtt{D}$ diagonal cells we get the following sum:

$$(-5 - 7i) + \left(-\frac{7n - 37}{2} + 14i\right) + \left(\frac{7n - 27}{2} - 7i\right) = 0.$$

So we have shown that each column sums to zero. Also, it is not hard to see that:

$$\begin{aligned}
\operatorname{supp}(\mathtt{A}) &= \left\{1, 8, 15, \ldots, \tfrac{7n-5}{2}\right\} \cup \left\{6, 13, 20, \ldots, \tfrac{7n-9}{2}\right\}, \\
\operatorname{supp}(\mathtt{B}) &= \left\{2, 9, 16, \ldots, \tfrac{7n-3}{2}\right\}, \\
\operatorname{supp}(\mathtt{C}) &= \left\{5, 12, 19, \ldots, \tfrac{7n-11}{2}\right\}, \\
\operatorname{supp}(\mathtt{D}) &= \{3\} \cup \left\{4, 11, 18, \ldots, \tfrac{7n-13}{2}\right\}, \\
\operatorname{supp}(\mathtt{E}) &= \left\{10, 17, 24, \ldots, \tfrac{7n-1}{2}\right\},
\end{aligned}$$

hence $\operatorname{supp}(A) = \left[1, \frac{7n-1}{2}\right] \setminus \left\{7, 14, 21, \ldots, \frac{7n-7}{2}\right\}$. This concludes the proof. $\square$

Example 4.2. Following the proof of Proposition 4.1 we obtain the integer $\mathrm{H}_9(9;3)$ below.

-27	30							-3
25	-20	-5						
	-10	-13	23					
		18	-6	-12				
			-17	1	16			
				11	8	-19		
					-24	15	9	
						4	22	-26
2							-31	29

We can use this example to briefly explain how the construction has been obtained (a similar idea will be used also in Proposition 4.3 below). First of all, we have to avoid the multiples of $\frac{2nk}{t}+1=7$, so we work modulo 7. The diagonal D_1 consists of elements, all congruent to 1 modulo 7, arranged in arithmetic progression where, for instance, the central cell is filled with 1. The other two filled diagonals are obtained in such a way that the elements of D_9 are all congruent to 2 modulo 7 and the elements of D_2 are all congruent to -3 modulo 7. This can be achieved filling the cell $(9,1)$ with the integer 2: it is now easy to obtain the elements in the remaining cells, remembering that the row/column sums are 0.

Proposition 4.3. *For every odd* $n \geq 3$ *there exists an integer cyclically* 3*-diagonal Heffter array* $\mathrm{H}_{2n}(n;3)$.

Proof. We construct an $n \times n$ array A using the following procedures labeled `A` to `E`:

`A`: $diag(1,1,-(4n-5),1,8,n)$; `B`: $diag\big(1,2,4n-2,2,-8,\frac{n+1}{2}\big)$;
`C`: $diag\big(2,3,-6,2,-8,\frac{n-1}{2}\big)$; `D`: $diag\big(2,1,4n-7,2,-8,\frac{n+1}{2}\big)$;
`E`: $diag\big(3,2,-11,2,-8,\frac{n-1}{2}\big)$.

We prove that the array constructed above is an integer cyclically 3-diagonal $\mathrm{H}_{2n}(n;3)$. To aid in the proof we give a schematic picture of where each of the diagonal procedures fills cells (see Figure 1). Note that each row and each column contain exactly 3 elements. We now check that the elements in every row sum to zero (in $\mathbb{Z}$).

Row 1. There is the first value of the `A` diagonal and of the `B` diagonal and the last of the `D` diagonal. The sum is

$$-(4n-5)+(4n-2)-3=0.$$

Row 2 to n. There are two cases depending on whether the row r is even or odd. If r is even, then write $r=2i+2$ where $i \in \big[0,\frac{n-3}{2}\big]$. Notice that from the `D`, `A` and `C` diagonal cells we get the following sum:

$$(4n-7-8i)+(-4n+13+16i)+(-6-8i)=0.$$

If r is odd, then write $r=2i+3$ where $i \in \big[0,\frac{n-3}{2}\big]$. From the `E`, `A` and `B` diagonal cells we get the following sum:

$$(-11-8i)+(-4n+21+16i)+(4n-10-8i)=0.$$

So we have shown that all row sums are zero. Next we check that the columns all add to zero.

Column 1. There is the first value of the $\mathtt{A}$ diagonal and of the $\mathtt{D}$ diagonal and the last of the $\mathtt{B}$ diagonal. The sum is

$$-(4n-5)+(4n-7)+2=0.$$

Column 2 to n. There are two cases depending on whether the column c is even or odd. If c is even, then write $c=2i+2$ where $i\in\left[0,\frac{n-3}{2}\right]$. Notice that from the $\mathtt{B}$, $\mathtt{A}$ and $\mathtt{E}$ diagonal cells we get the following sum:

$$(4n-2-8i)+(-4n+13+16i)+(-11-8i)=0.$$

If c is odd, then write $c=2i+3$ where $i\in\left[0,\frac{n-3}{2}\right]$. From the $\mathtt{C}$, $\mathtt{A}$ and $\mathtt{D}$ diagonal cells we get the following sum:

$$(-6-8i)+(-4n+21+16i)+(4n-15-8i)=0.$$

So we have shown that each column sums to zero. Also, it is not hard to see that:

$$\begin{aligned}
\operatorname{supp}(\mathtt{A}) &= \{1,9,17,\dots,4n-3\}\cup\{7,15,23,\dots,4n-5\},\\
\operatorname{supp}(\mathtt{B}) &= \{2,10,18,\dots,4n-2\},\\
\operatorname{supp}(\mathtt{C}) &= \{6,14,22,\dots,4n-6\},\\
\operatorname{supp}(\mathtt{D}) &= \{3\}\cup\{5,13,21,\dots,4n-7\},\\
\operatorname{supp}(\mathtt{E}) &= \{11,19,27,\dots,4n-1\},
\end{aligned}$$

hence $\operatorname{supp}(A)=[1,4n-1]\setminus\{4,8,12,\dots,4n-4\}$. This concludes the proof. □

Example 4.4. Following the proof of Proposition 4.3 we obtain the integer $\mathrm{H}_{18}(9;3)$ below.

-31	34							-3
29	-23	-6						
	-11	-15	26					
		21	-7	-14				
			-19	1	18			
				13	9	-22		
					-27	17	10	
						5	25	-30
2							-35	33

In the following propositions, since $k>5$, in order to prove that the relative Heffter array $\mathrm{H}_k(n;k)$ constructed is globally simple we have to show that the partial sums of each row and of each column are distinct modulo $2nk+k$. From now on, the sets $\mathcal{E}(\overline{R}_i)$ and $\mathcal{E}(\overline{C}_i)$ are considered ordered with respect to the natural ordering. Also, by $\mathcal{S}(\overline{R}_i)$ and $\mathcal{S}(\overline{C}_i)$ we will denote the sequence of the partial sums of $\mathcal{E}(\overline{R}_i)$ and $\mathcal{E}(\overline{C}_i)$, respectively. In order to check that the partial sums are distinct the following remark allows to reduce the computations.

Remark 4.5. Let A be a $\mathrm{H}_t(n;k)$. By the definition of a (relative) Heffter array it easily follows that the i-th partial sum s_i of a row (or a column) is different from the partial sums $s_{i-2}, s_{i-1}, s_{i+1}$ and s_{i+2} of the same row (column).

Proposition 4.6. *For every $n \geq 7$ with $n \equiv 3 \pmod 4$ there exists an integer cyclically 7-diagonal globally simple* $\mathrm{H}_7(n;7)$.

Proof. We construct an $n \times n$ array A using the following procedures labeled `A` to `N`:

`A`: $diag\left(3,3,-\frac{n+1}{2},2,-1,\frac{n-1}{2}\right)$; `B`: $diag\left(4,4,1,2,1,\frac{n-3}{2}\right)$;
`C`: $diag(n-2,n-1,-(5n+3),2,-1,n)$; `D`: $diag(2,1,-(4n+3),2,-1,n)$;
`E`: $diag\left(1,3,\frac{7n+3}{4},4,1,\frac{n+1}{4}\right)$; `F`: $diag\left(2,4,\frac{3n+1}{2},4,-1,\frac{n+1}{4}\right)$;
`G`: $diag\left(3,5,\frac{11n+7}{4},4,1,\frac{n+1}{4}\right)$; `H`: $diag\left(4,6,\frac{5n+1}{2},4,-1,\frac{n-3}{4}\right)$;
`I`: $diag\left(3,1,-\frac{9n+5}{4},4,1,\frac{n+1}{4}\right)$; `J`: $diag\left(4,2,-\frac{5n+3}{2},4,-1,\frac{n+1}{4}\right)$;
`K`: $diag\left(5,3,-\frac{5n+1}{4},4,1,\frac{n+1}{4}\right)$; `L`: $diag\left(6,4,-\frac{3n+3}{2},4,-1,\frac{n-3}{4}\right)$;
`M`: $diag(n-2,1,6n+4,2,1,n)$; `N`: $diag(2,n-1,3n+2,2,1,n)$.

We also fill the following cells in an *ad hoc* manner:

$$A[1,1]=n, \qquad A[2,2]=-\tfrac{n-1}{2}.$$

We prove that the array constructed above is an integer cyclically 7-diagonal globally simple $\mathrm{H}_7(n;7)$. To aid in the proof we give a schematic picture of where each of the diagonal procedures fills cells (see Figure 2). We have placed an `X` in the *ad hoc* cells. Note that each row and each column contains exactly 7 elements. We now list the elements and the partial sums of each row. We leave to the reader the direct check that the partial sums are distinct modulo $14n+7$; for a quicker check keep in mind Remark 4.5.

X	C	E	M					N	J	D
D	X	C	F	M					N	K
I	D	A	C	G	M					N
N	J	D	B	C	H	M				
	N	K	D	A	C	E	M			
		N	L	D	B	C	F	M		
			N	I	D	A	C	G	M	
				N	J	D	B	C	H	M
M					N	K	D	A	C	E
F	M					N	L	D	B	C
C	G	M					N	I	D	A

Figure 2: Scheme of construction with $n = 11$.

Row 1. There is an ad hoc element, the $(\frac{n+5}{2})^{\text{th}}$ value of the `C` diagonal, the first one of the `E` diagonal, the $(\frac{n+5}{2})^{\text{th}}$ value of the `M` diagonal, the $(\frac{n+1}{2})^{\text{th}}$ value of the `N` diagonal, the last value of the `J` diagonal and the $(\frac{n+1}{2})^{\text{th}}$ value of the `D` diagonal. Namely,

$$\mathcal{E}(\overline{R}_1)=\left(n,-\frac{11n+9}{2},\frac{7n+3}{4},\frac{13n+11}{2},\frac{7n+3}{2},-\frac{11n+3}{4},-\frac{9n+5}{2}\right)$$

and

$$\mathcal{S}(\overline{R}_1) = \left(n, -\frac{9n+9}{2}, -\frac{11n+15}{4}, \frac{15n+7}{4}, \frac{29n+13}{4}, \frac{9n+5}{2}, 0\right).$$

Row 2. There is the first value of the D diagonal, an ad hoc element, the third value of the C diagonal, the first value of the F diagonal, the third value of the M diagonal, the first value of the N diagonal and the last value of the K diagonal. Hence

$$\mathcal{E}(\overline{R}_2) = \left(-(4n+3), -\frac{n-1}{2}, -(5n+5), \frac{3n+1}{2}, 6n+6, 3n+2, -(n+1)\right)$$

and

$$\mathcal{S}(\overline{R}_2) = \left(-(4n+3), -\frac{9n+5}{2}, -\frac{19n+15}{2}, -(8n+7), -(2n+1), n+1, 0\right).$$

Row 3 to n. There are four cases depending on the congruence class of r modulo 4. If $r \equiv 3 \pmod 4$, then write $r = 4i+3$ where $i \in \left[0, \frac{n-3}{4}\right]$. It is not hard to see that from the N, I, D, A, C, G and M diagonal cells we get:

$$\begin{aligned}\mathcal{E}(\overline{R}_{4i+3}) = \Bigg(&\frac{7n+5}{2} + 2i, -\frac{9n+5}{4} + i, -\frac{9n+7}{2} - 2i, -\frac{n+1}{2} - 2i,\\ &-\frac{11n+11}{2} - 2i + \varepsilon, \frac{11n+7}{4} + i, \frac{13n+13}{2} + 2i - \varepsilon\Bigg),\end{aligned}$$

where $\varepsilon = 0$ for $i \in \left[0, \frac{n-7}{4}\right]$ while $\varepsilon = n$ for $i = \frac{n-3}{4}$, and

$$\begin{aligned}\mathcal{S}(\overline{R}_{4i+3}) = \Bigg(&\frac{7n+5}{2} + 2i, \frac{5n+5}{4} + 3i, -\frac{13n+9}{4} + i,\\ &-\frac{15n+11}{4} - i, -\frac{37n+33}{4} - 3i + \varepsilon, -\frac{13n+13}{2} - 2i + \varepsilon, 0\Bigg).\end{aligned}$$

If $r \equiv 0 \pmod 4$, then write $r = 4i+4$ where $i \in \left[0, \frac{n-7}{4}\right]$. It is not hard to see that from the N, J, D, B, C, H and M diagonal cells we get:

$$\begin{aligned}\mathcal{E}(\overline{R}_{4i+4}) = \Bigg(&3n+3+2i, -\frac{5n+3}{2} - i, -(4n+4+2i),\\ &1+2i, -(5n+6+2i), \frac{5n+1}{2} - i, 6n+7+2i\Bigg)\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(\overline{R}_{4i+4}) = \Bigg(&3n+3+2i, \frac{n+3}{2} + i, -\frac{7n+5}{2} - i,\\ &-\frac{7n+3}{2} + i, -\frac{17n+15}{2} - i, -(6n+7+2i), 0\Bigg).\end{aligned}$$

If $r \equiv 1 \pmod 4$, then write $r = 4i + 5$ where $i \in \left[0, \frac{n-7}{4}\right]$. It is not hard to see that from the N, K, D, A, C, E and M diagonal cells we get:

$$\mathcal{E}(\overline{R}_{4i+5}) = \left(\frac{7n+7}{2} + 2i, -\frac{5n+1}{4} + i, -\frac{9n+9}{2} - 2i, -\frac{n+3}{2} - 2i, \right.$$
$$\left. -\frac{11n+13}{2} - 2i + \varepsilon, \frac{7n+7}{4} + i, \frac{13n+15}{2} + 2i - \varepsilon\right),$$

where $\varepsilon = 0$ for $i \in \left[0, \frac{n-11}{4}\right]$ while $\varepsilon = n$ for $i = \frac{n-7}{4}$, and

$$\mathcal{S}(\overline{R}_{4i+5}) = \left(\frac{7n+7}{2} + 2i, \frac{9n+13}{4} + 3i, -\frac{9n+5}{4} + i, \right.$$
$$\left. -\frac{11n+11}{4} - i, -\frac{33n+37}{4} - 3i + \varepsilon, -\frac{13n+15}{2} - 2i + \varepsilon, 0\right).$$

If $r \equiv 2 \pmod 4$, then write $r = 4i + 6$ where $i \in \left[0, \frac{n-7}{4}\right]$. It is not hard to see that from the N, L, D, B, C, F and M diagonal cells we get:

$$\mathcal{E}(\overline{R}_{4i+6}) = \left(3n + 4 + 2i, -\frac{3n+3}{2} - i, -(4n+5+2i), \right.$$
$$\left. 2 + 2i, -(5n+7+2i), \frac{3n-1}{2} - i, 6n + 8 + 2i\right)$$

and

$$\mathcal{S}(\overline{R}_{4i+6}) = \left(3n + 4 + 2i, \frac{3n+5}{2} + i, -\frac{5n+5}{2} - i, \right.$$
$$\left. -\frac{5n+1}{2} + i, -\frac{15n+15}{2} - i, -(6n+8+2i), 0\right).$$

Now we list the elements and the partial sums of the columns.

Column 1. There is an ad hoc element, the first value of the D diagonal and of the I diagonal, the second value of the N diagonal, the first value of the M diagonal, the last value of the F diagonal and the second value of the C diagonal. Namely,

$$\mathcal{E}(\overline{C}_1) = \left(n, -(4n+3), -\frac{9n+5}{4}, 3n+3, 6n+4, \frac{5n+5}{4}, -(5n+4)\right)$$

and

$$\mathcal{S}(\overline{C}_1) = \left(n, -(3n+3), -\frac{21n+17}{4}, -\frac{9n+5}{4}, \frac{15n+11}{4}, 5n+4, 0\right).$$

Column 2. There is the $(\frac{n+5}{2})^{\text{th}}$ value of the C diagonal, an ad hoc element, the $(\frac{n+3}{2})^{\text{th}}$ value of the D diagonal, the first value of the J diagonal, the $(\frac{n+5}{2})^{\text{th}}$ value of the N diagonal and of the M diagonal and the last value of the G diagonal. Namely,

$$\mathcal{E}(\overline{C}_2) = \left(-\frac{11n+9}{2}, -\frac{n-1}{2}, -\frac{9n+7}{2}, -\frac{5n+3}{2}, \frac{7n+7}{2}, \frac{13n+9}{2}, 3n+1\right)$$

and

$$\mathcal{S}(\overline{C}_2) = \Big(-\frac{11n+9}{2}, -(6n+4), -\frac{21n+15}{2}, -(13n+9), -\frac{19n+11}{2}, -(3n+1), 0\Big).$$

Column 3 to n. There are four cases depending on the congruence class of c modulo 4. If $c \equiv 3 \pmod 4$, then write $c = 4i+3$ where $i \in \big[0, \frac{n-3}{4}\big]$. It is not hard to see that from the M, E, C, A, D, K and N diagonal cells we get:

$$\mathcal{E}(\overline{C}_{4i+3}) = \Big(6n+5+2i, \frac{7n+3}{4}+i, -(5n+5+2i), -\frac{n+1}{2}-2i, -(4n+4+2i), -\frac{5n+1}{4}+i, 3n+4+2i\Big)$$

and

$$\mathcal{S}(\overline{C}_{4i+3}) = \Big(6n+5+2i, \frac{31n+23}{4}+3i, \frac{11n+3}{4}+i, \frac{9n+1}{4}-i, -\frac{7n+15}{4}-3i, -(3n+4+2i), 0\Big).$$

If $c \equiv 0 \pmod 4$, then write $c = 4i+4$ where $i \in \big[0, \frac{n-7}{4}\big]$. It is not hard to see that from the M, F, C, B, D, L and N diagonal cells we get:

$$\mathcal{E}(\overline{C}_{4i+4}) = \Big(\frac{13n+11}{2}+2i, \frac{3n+1}{2}-i, -\frac{11n+11}{2}-2i, 1+2i, -\frac{9n+9}{2}-2i, -\frac{3n+3}{2}-i, \frac{7n+9}{2}+2i\Big)$$

and

$$\mathcal{S}(\overline{C}_{4i+4}) = \Big(\frac{13n+11}{2}+2i, 8n+6+i, \frac{5n+1}{2}-i, \frac{5n+3}{2}+i, -(2n+3+i), -\frac{7n+9}{2}-2i, 0\Big).$$

If $c \equiv 1 \pmod 4$, then write $c = 4i+5$ where $i \in \big[0, \frac{n-7}{4}\big]$. It is not hard to see that from the M, G, C, A, D, I and N diagonal cells we get:

$$\mathcal{E}(\overline{C}_{4i+5}) = \Big(6n+6+2i, \frac{11n+7}{4}+i, -(5n+6+2i), -\frac{n+3}{2}-2i, -(4n+5+2i), -\frac{9n+1}{4}+i, 3n+5+2i\Big)$$

and

$$\mathcal{S}(\overline{C}_{4i+5}) = \Big(6n+6+2i, \frac{35n+31}{4}+3i, \frac{15n+7}{4}+i, \frac{13n+1}{4}-i, -\frac{3n+19}{4}-3i, -(3n+5+2i), 0\Big).$$

If $c \equiv 2 \pmod 4$, then write $c = 4i+6$ where $i \in \left[0, \frac{n-7}{4}\right]$. It is not hard to see that from the $\mathtt{M}$, $\mathtt{H}$, $\mathtt{C}$, $\mathtt{B}$, $\mathtt{D}$, $\mathtt{J}$ and $\mathtt{N}$ diagonal cells we get:

$$\mathcal{E}(\overline{C}_{4i+6}) = \left(\frac{13n+13}{2}+2i, \frac{5n+1}{2}-i, -\frac{11n+13}{2}-2i+\varepsilon,\right.$$
$$\left. 2+2i, -\frac{9n+11}{2}-2i, -\frac{5n+5}{2}-i, \frac{7n+11}{2}+2i-\varepsilon\right),$$

where $\varepsilon = 0$ for $i \in \left[0, \frac{n-11}{4}\right]$ while $\varepsilon = n$ for $i = \frac{n-7}{4}$, and

$$\mathcal{S}(\overline{C}_{4i+6}) = \left(\frac{13n+13}{2}+2i, 9n+7+i, \frac{7n+1}{2}-i+\varepsilon,\right.$$
$$\left.\frac{7n+5}{2}+i+\varepsilon, -(n+3+i)+\varepsilon, -\frac{7n+11}{2}-2i+\varepsilon, 0\right).$$

Finally we consider the support of A:

$$\begin{aligned}\mathrm{supp}(A) &= \left[1, \tfrac{n-3}{2}\right]_{(\mathtt{B})} \cup \left\{\tfrac{n-1}{2}\right\} \cup \left[\tfrac{n+1}{2}, n-1\right]_{(\mathtt{A})} \cup \{n\}\\ &\cup \left[n+1, \tfrac{5n+1}{4}\right]_{(\mathtt{K})} \cup \left[\tfrac{5n+5}{4}, \tfrac{3n+1}{2}\right]_{(\mathtt{F})} \cup \left[\tfrac{3n+3}{2}, \tfrac{7n-1}{4}\right]_{(\mathtt{L})}\\ &\cup \left[\tfrac{7n+3}{4}, 2n\right]_{(\mathtt{E})} \cup \left[2n+2, \tfrac{9n+5}{4}\right]_{(\mathtt{I})} \cup \left[\tfrac{9n+9}{4}, \tfrac{5n+1}{2}\right]_{(\mathtt{H})}\\ &\cup \left[\tfrac{5n+3}{2}, \tfrac{11n+3}{4}\right]_{(\mathtt{J})} \cup \left[\tfrac{11n+7}{4}, 3n+1\right]_{(\mathtt{G})} \cup [3n+2, 4n+1]_{(\mathtt{N})}\\ &\cup [4n+3, 5n+2]_{(\mathtt{D})} \cup [5n+3, 6n+2]_{(\mathtt{C})} \cup [6n+4, 7n+3]_{(\mathtt{M})}\\ &= [1, 7n+3] \setminus \{2n+1, 4n+2, 6n+3\}.\end{aligned}$$

This concludes the proof. □

Example 4.7. Following the proof of Proposition 4.6 we obtain the integer globally simple $\mathrm{H}_7(11;7)$ below.

11	−65	20	77					40	−31	−52
−47	−5	−60	17	72					35	−12
−26	−53	−6	−66	32	78					41
36	−29	−48	1	−61	28	73				
	42	−14	−54	−7	−67	21	79			
		37	−18	−49	2	−62	16	74		
			43	−25	−55	−8	−68	33	80	
				38	−30	−50	3	−63	27	75
70					44	−13	−56	−9	−58	22
15	76					39	−19	−51	4	−64
−59	34	71					45	−24	−57	−10

Proposition 4.8. *For every $n \geq 11$ with $n \equiv 3 \pmod 4$ there exists an integer 9-diagonal globally simple $\mathrm{H}_9(n;9)$ with width $\frac{n-9}{2}$.*

Proof. We construct an $n \times n$ array A using the following procedures labeled `A` to `R`:

`A`: $diag(3, 1, 5n + 3, 1, 1, n)$;
`B`: $diag(4, 1, -(6n + 4), 1, -1, n)$;
`C`: $diag(3, 6, -(7n + 4), 1, -1, n)$;
`D`: $diag(4, 6, 8n + 5, 1, 1, n)$;
`E`: $diag\big(1, \frac{n+3}{2}, -(2n), 1, 2, \frac{n-1}{2}\big)$;
`F`: $diag\big(\frac{n+3}{2}, 1, 2n + 2, 1, 2, \frac{n-1}{2}\big)$;
`G`: $diag\big(2, 2, -(n - 2), 1, 1, \frac{n-3}{2}\big)$;
`H`: $diag\big(\frac{n+3}{2}, 2, -(2n + 3), 1, -2, \frac{n-3}{2}\big)$;
`I`: $diag\big(2, \frac{n+3}{2}, 2n - 1, 1, -2, \frac{n-3}{2}\big)$;
`J`: $diag\big(\frac{n+3}{2}, \frac{n+3}{2}, \frac{n-3}{2}, 1, -1, \frac{n-5}{2}\big)$;
`K`: $diag\big(2, 1, -(3n + 4), 2, -1, \frac{n+1}{4}\big)$;
`L`: $diag\big(1, 2, 5n, 2, -1, \frac{n+1}{4}\big)$;
`M`: $diag\big(3, 2, -(4n + 3), 2, -1, \frac{n-3}{4}\big)$;
`N`: $diag\big(2, 3, 4n + 1, 2, -1, \frac{n-3}{4}\big)$;
`O`: $diag\big(\frac{n+1}{2}, \frac{n+3}{2}, \frac{17n+9}{4}, 2, 1, \frac{n-3}{4}\big)$;
`P`: $diag\big(\frac{n+3}{2}, \frac{n+1}{2}, -\frac{15n+7}{4}, 2, 1, \frac{n-3}{4}\big)$;
`Q`: $diag\big(\frac{n+3}{2}, \frac{n+5}{2}, \frac{13n+17}{4}, 2, 1, \frac{n-3}{4}\big)$;
`R`: $diag\big(\frac{n+5}{2}, \frac{n+3}{2}, -\frac{19n-1}{4}, 2, 1, \frac{n-3}{4}\big)$.

We also fill the following cells in an *ad hoc* manner:

$$A[1, 1] = n - 1, \quad A[1, \tfrac{n+1}{2}] = n + 2, \quad A[1, n] = -(5n + 1),$$
$$A[\tfrac{n+1}{2}, 1] = -(3n), \quad A[\tfrac{n+1}{2}, \tfrac{n+1}{2}] = n, \quad A[\tfrac{n+1}{2}, n] = n + 1,$$
$$A[n - 1, n - 1] = -\tfrac{n-1}{2}, \quad A[n - 1, n] = 5n + 2, \quad A[n, 1] = 3n + 3,$$
$$A[n, \tfrac{n+1}{2}] = -(3n + 1), \quad A[n, n - 1] = -(3n + 2), \quad A[n, n] = 1.$$

We prove that the array constructed above is an integer 9-diagonal globally simple $\mathrm{H}_9(n; 9)$ with width $\frac{n-9}{2}$. To aid in the proof we give a schematic picture of where each of the diagonal procedures fills cells (see Figure 3). We have placed an `X` in the *ad hoc* cells. Note that each row and each column contains exactly 9 elements. Since the filled diagonals are $D_1, D_2, D_3, D_4, D_{\frac{n+1}{2}}, D_{\frac{n+3}{2}}, D_{n-2}, D_{n-1}$ and D_n, A has two empty strips of size $\frac{n-9}{2}$. We now list the elements and the partial sums of every row. We leave to the reader the direct check that the partial sums are distinct modulo $18n + 9$; for a quicker check keep in mind Remark 4.5.

X	L	D	C				X	E				B	A	X
K	G	N	D	C				I	E				B	A
A	M	G	L	D	C				I	E				B
B	A	K	G	N	D	C				I	E			
	B	A	M	G	L	D	C				I	E		
		B	A	K	G	N	D	C				I	E	
			B	A	M	G	L	D	C				I	E
X				B	A	K	X	O	D	C				X
F	H				B	A	P	J	Q	D	C			
	F	H				B	A	R	J	O	D	C		
		F	H				B	A	P	J	Q	D	C	
			F	H				B	A	R	J	O	D	C
C				F	H				B	A	P	J	Q	D
D	C				F	H				B	A	R	X	X
X	D	C				F	X				B	A	X	X

Figure 3: Scheme of construction with $n = 15$.

Row 1. There are three ad hoc values plus the elements of the L, D, C, E, B and A diagonals. Namely:

$$\mathcal{E}(\overline{R}_1) = (n-1, 5n, 9n+2, -(8n+2), n+2, -2n, -(7n+1), 6n+1, -(5n+1))$$

and

$$\mathcal{S}(\overline{R}_1) = (n-1, 6n-1, 15n+1, 7n-1, 8n+1, 6n+1, -n, 5n+1, 0).$$

Row 2. It is not hard to see that from the K, G, N, D, C, I, E, B and A diagonal cells we get:

$$\begin{aligned}\mathcal{E}(\overline{R}_2) = (&-(3n+4), -(n-2), 4n+1, 9n+3,\\ &-(8n+3), 2n-1, -(2n-2), -(7n+2), 6n+2)\end{aligned}$$

and

$$\mathcal{S}(\overline{R}_2) = (-(3n+4), -(4n+2), -1, 9n+2, n-1, 3n-2, n, -(6n+2), 0).$$

Row 3. It is not hard to see that from the A, M, G, L, D, C, I, E and B diagonal cells we get:

$$\begin{aligned}\mathcal{E}(\overline{R}_3) = (&5n+3, -(4n+3), -(n-3), 5n-1,\\ &9n+4, -(7n+4), 2n-3, -(2n-4), -(7n+3))\end{aligned}$$

and

$$\mathcal{S}(\overline{R}_3) = (5n+3, n, 3, 5n+2, 14n+6, 7n+2, 9n-1, 7n+3, 0).$$

Row 4 to $\frac{n-1}{2}$. We have to distinguish two cases, depending on the parity of the row r. If r is even, then write $r = 4+2i$ where $i \in \left[0, \frac{n-11}{4}\right]$. It is not hard to see that from the B, A, K, G, N, D, C, I and E diagonal cells we get:

$$\begin{aligned}\mathcal{E}(\overline{R}_{4+2i}) = (&-(6n+4+2i), 5n+4+2i, -(3n+5+i), -(n-4-2i),\\ &4n-i, 8n+5+2i, -(7n+5+2i), 2n-5-4i, -(2n-6-4i))\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(\overline{R}_{4+2i}) = (&-(6n+4+2i), -n, -(4n+5+i),\\ &-(5n+1-i), -(n+1), 7n+4+2i, -1, 2n-6-4i, 0).\end{aligned}$$

If r is odd, then write $r = 5+2i$, where $i \in \left[0, \frac{n-11}{4}\right]$. It is not hard to see that from the B, A, M, G, L, D, C, I and E diagonal cells we get:

$$\begin{aligned}\mathcal{E}(\overline{R}_{5+2i}) = (&-(6n+5+2i), 5n+5+2i, -(4n+4+i), -(n-5-2i),\\ &5n-2-i, 8n+6+2i, -(7n+6+2i), 2n-7-4i, -(2n-8-4i))\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(\overline{R}_{5+2i}) = (&-(6n+5+2i), -n, -(5n+4+i),\\ &-(6n-1-i), -(n+1), 7n+5+2i, -1, 2n-8-4i, 0).\end{aligned}$$

Row $\frac{n+1}{2}$. There are three ad hoc values plus the elements of the B, A, K, O, D and C diagonals. Namely:

$$\mathcal{E}\Big(\overline{R}_{\frac{n+1}{2}}\Big) = \Big(-3n, -\frac{13n+1}{2}, \frac{11n+1}{2}, \\ -\frac{13n+13}{4}, n, \frac{17n+9}{4}, \frac{17n+3}{2}, -\frac{15n+3}{2}, n+1\Big)$$

and

$$\mathcal{S}\Big(\overline{R}_{\frac{n+1}{2}}\Big) = \Big(-3n, -\frac{19n+1}{2}, -4n, -\frac{29n+13}{4}, \\ -\frac{25n+13}{4}, -(2n+1), \frac{13n+1}{2}, -(n+1), 0\Big).$$

Row $\frac{n+3}{2}$ to $n-2$. We have to distinguish two cases, depending on the parity of the row r. If r is odd, then write $r = \frac{n+3}{2} + 2i$ where $i \in \big[0, \frac{n-7}{4}\big]$. It is not hard to see that from the F, H, B, A, P, J, Q, D and C diagonal cells we get:

$$\mathcal{E}\Big(\overline{R}_{\frac{n+3}{2}+2i}\Big) = \Big(2n+2+4i, -(2n+3+4i), -\frac{13n+3}{2} - 2i, \frac{11n+3}{2} + 2i, \\ -\frac{15n+7}{4} + i, \frac{n-3}{2} - 2i, \frac{13n+17}{4} + i, \frac{17n+5}{2} + 2i, -\frac{15n+5}{2} - 2i\Big)$$

and

$$\mathcal{S}\Big(\overline{R}_{\frac{n+3}{2}+2i}\Big) = \Big(2n+2+4i, -1, -\frac{13n+5}{2} - 2i, -(n+1), \\ -\frac{19n+11}{4} + i, -\frac{17n+17}{4} - i, -n, \frac{15n+5}{2} + 2i, 0\Big).$$

If r is even, then write $r = \frac{n+5}{2} + 2i$ where $i \in \big[0, \frac{n-11}{4}\big]$. It is not hard to see that from the F, H, B, A, R, J, O, D and C diagonal cells we get:

$$\mathcal{E}\Big(\overline{R}_{\frac{n+5}{2}+2i}\Big) = \Big(2n+4+4i, -(2n+5+4i), -\frac{13n+5}{2} - 2i, \frac{11n+5}{2} + 2i, \\ -\frac{19n-1}{4} + i, \frac{n-5}{2} - 2i, \frac{17n+13}{4} + i, \frac{17n+7}{2} + 2i, -\frac{15n+7}{2} - 2i\Big)$$

and

$$\mathcal{S}\Big(\overline{R}_{\frac{n+5}{2}+2i}\Big) = \Big(2n+4+4i, -1, -\frac{13n+7}{2} - 2i, -(n+1), \\ -\frac{23n+3}{4} + i, -\frac{21n+13}{4} - i, -n, \frac{15n+7}{2} + 2i, 0\Big).$$

Row $n-1$. There are two ad hoc values plus the elements of the `D`, `C`, `F`, `H`, `B`, `A` and `R` diagonals. Namely:

$$\mathcal{E}(\overline{R}_{n-1}) = \Big(9n, -8n, 3n-3, -(3n-2), \\ -(7n-1), 6n-1, -\frac{9n+3}{2}, -\frac{n-1}{2}, 5n+2\Big)$$

and

$$\mathcal{S}(\overline{R}_{n-1}) = \Big(9n, n, 4n-3, n-1, -6n, -1, -\frac{9n+5}{2}, -(5n+2), 0\Big).$$

Row n. There are four ad hoc values plus the elements of the `D`, `C`, `F`, `B` and `A` diagonals. Namely:

$$\mathcal{E}(\overline{R}_{n}) = (3n+3, 9n+1, -(8n+1), 3n-1, -(3n+1), -7n, 6n, -(3n+2), 1)$$

and

$$\mathcal{S}(\overline{R}_{n}) = (3n+3, 12n+4, 4n+3, 7n+2, 4n+1, -(3n-1), 3n+1, -1, 0).$$

Now we list the elements and the partial sums of the columns.

Column 1. There are three ad hoc values plus the elements of the `K`, `A`, `B`, `F`, `C` and `D` diagonals. Namely:

$$\mathcal{E}(\overline{C}_1) = (n-1, -(3n+4), 5n+3, -(6n+4), -3n, 2n+2, -(8n-1), 9n, 3n+3)$$

and

$$\mathcal{S}(\overline{C}_1) = (n-1, -(2n+5), 3n-2, -(3n+6), \\ -(6n+6), -(4n+4), -(12n+3), -(3n+3), 0).$$

Column 2. It is not hard to see that from the `L`, `G`, `M`, `A`, `B`, `H`, `F`, `C` and `D` diagonal cells we get:

$$\mathcal{E}(\overline{C}_2) = (5n, -(n-2), -(4n+3), 5n+4, -(6n+5), -(2n+3), 2n+4, -8n, 9n+1)$$

and

$$\mathcal{S}(\overline{C}_2) = (5n, 4n+2, -1, 5n+3, -(n+2), -(3n+5), -(n+1), -(9n+1), 0).$$

Column 3. It is not hard to see that from the `D`, `N`, `G`, `K`, `A`, `B`, `H`, `F` and `C` diagonal cells we get:

$$\mathcal{E}(\overline{C}_3) = (9n+2, 4n+1, -(n-3), -(3n+5), \\ 5n+5, -(6n+6), -(2n+5), 2n+6, -(8n+1))$$

and

$$\mathcal{S}(\overline{C}_3) = (9n+2, 13n+3, 12n+6, 9n+1, 14n+6, 8n, 6n-5, 8n+1, 0).$$

Column 4. It is not hard to see that from the C, D, L, G, M, A, B, H and F diagonal cells we get:

$$\mathcal{E}(\overline{C}_4) = \big(-(8n+2), 9n+3, 5n-1, -(n-4),\\ -(4n+4), 5n+6, -(6n+7), -(2n+7), 2n+8\big)$$

and

$$\mathcal{S}(\overline{C}_4) = (-(8n+2), n+1, 6n, 5n+4, n, 6n+6, -1, -(2n+8), 0).$$

Column 5. It is not hard to see that from the C, D, N, G, K, A, B, H and F diagonal cells we get:

$$\mathcal{E}(\overline{C}_5) = \big(-(8n+3), 9n+4, 4n, -(n-5),\\ -(3n+6), 5n+7, -(6n+8), -(2n+9), 2n+10\big)$$

and

$$\mathcal{S}(\overline{C}_5) = (-(8n+3), n+1, 5n+1, 4n+6, n, 6n+7, -1, -(2n+10), 0).$$

Column 6 to $\frac{n-1}{2}$. We have to distinguish two cases, depending on the parity of the column c. If c is even, then write $c = 6+2i$ where $i \in [0, \frac{n-15}{4}]$. It is not hard to see that from the C, D, L, G, M, A, B, H and F diagonal cells we get:

$$\mathcal{E}(\overline{C}_{6+2i}) = (-(7n+4+2i), 8n+5+2i, 5n-2-i, -(n-6-2i),\\ -(4n+5+i), 5n+8+2i, -(6n+9+2i), -(2n+11+4i), 2n+12+4i)$$

and

$$\mathcal{S}(\overline{C}_{6+2i}) = \big(-(7n+4+2i), n+1, 6n-1-i,\\ 5n+5+i, n, 6n+8+2i, -1, -(2n+12+4i), 0\big).$$

If c is odd, then write $c = 7+2i$ where $i \in [0, \frac{n-15}{4}]$. It is not hard to see that from the C, D, N, G, K, A, B, H and F diagonal cells we get:

$$\mathcal{E}(\overline{C}_{7+2i}) = (-(7n+5+2i), 8n+6+2i, 4n-1-i, -(n-7-2i),\\ -(3n+7+i), 5n+9+2i, -(6n+10+2i), -(2n+13+4i), 2n+14+4i)$$

and

$$\mathcal{S}(\overline{C}_{7+2i}) = \big(-(7n+5+2i), n+1, 5n-i,\\ 4n+7+i, n, 6n+9+2i, -1, -(2n+14+4i), 0\big).$$

Column $\frac{n+1}{2}$. The are three ad hoc values plus the elements of the C, D, L, P, A and B diagonals. Namely:

$$\mathcal{E}\Big(\overline{C}_{\frac{n+1}{2}}\Big) = \Big(n+2, -\frac{15n-3}{2}, \frac{17n-1}{2}, \frac{19n+3}{4},\\ n, -\frac{15n+7}{4}, \frac{11n+5}{2}, -\frac{13n+7}{4}, -(3n+1)\Big)$$

and

$$\mathcal{S}\Big(\overline{C}_{\frac{n+1}{2}}\Big) = \bigg(n+2, -\frac{13n-7}{2}, 2n+3, \frac{27n+15}{4}, \frac{31n+1}{4}, 4n+2, \frac{19n+9}{2}, 3n+1, 0\bigg).$$

Column $\frac{n+3}{2}$ to $n-2$. We have to distinguish two cases, depending on the parity of the column c. If c is odd, then write $c = \frac{n+3}{2} + 2i$ where $i \in [0, \frac{n-7}{4}]$. It is not hard to see that from the E, I, C, D, O, J, R, A and B diagonal cells we get:

$$\mathcal{E}\bigg(\overline{C}_{\frac{n+3}{2}+2i}\bigg) = \bigg(-(2n-4i), 2n-1-4i, -\frac{15n-1}{2} - 2i, \frac{17n+1}{2} + 2i, \frac{17n+9}{4} + i, \frac{n-3}{2} - 2i, -\frac{19n-1}{4} + i, \frac{11n+7}{2} + 2i, -\frac{13n+9}{2} - 2i,\bigg)$$

and

$$\mathcal{S}\Big(\overline{C}_{\frac{n+3}{2}+2i}\Big) = \bigg(-(2n-4i), -1, -\frac{15n+1}{2} - 2i, n, \frac{21n+9}{4} + i, \frac{23n+3}{4} - i, n+1, \frac{13n+9}{2} + 2i, 0\bigg).$$

If c is even, then write $c = \frac{n+5}{2} + 2i$ where $i \in [0, \frac{n-11}{4}]$. It is not hard to see that from the E, I, C, D, Q, J, P, A and B diagonal cells we get:

$$\mathcal{E}\Big(\overline{C}_{\frac{n+5}{2}+2i}\Big) = \bigg(-(2n-2-4i), 2n-3-4i, -\frac{15n+1}{2} - 2i, \frac{17n+3}{2} + 2i, \frac{13n+17}{4} + i, \frac{n-5}{2} - 2i, -\frac{15n+3}{4} + i, \frac{11n+9}{2} + 2i, -\frac{13n+11}{2} - 2i\bigg)$$

and

$$\mathcal{S}\Big(\overline{C}_{\frac{n+5}{2}+2i}\Big) = \bigg(-(2n-2-4i), -1, -\frac{15n+3}{2}, n, \frac{17n+17}{4} + i, \frac{19n+7}{4} - i, n+1, \frac{13n+11}{2} + 2i, 0\bigg).$$

Column $n-1$. There are two ad hoc values plus the elements of the A, B, E, I, C, D and Q diagonals. Namely:

$$\mathcal{E}(\overline{C}_{n-1}) = \bigg(6n+1, -(7n+2), -(n+5), n+4, -(8n-3), 9n-2, \frac{7n+5}{2}, -\frac{n-1}{2}, -(3n+2)\bigg)$$

and

$$\mathcal{S}(\overline{C}_{n-1}) = \bigg(6n+1, -(n+1), -(2n+6), -(n+2), -(9n-1), -1, \frac{7n+3}{2}, 3n+2, 0\bigg).$$

Column n. There are four ad hoc values plus the elements of the A, B, E, C and D diagonals. Namely:

$$\mathcal{E}(\overline{C}_n) = (-(5n+1), 6n+2, -(7n+3), -(n+3), n+1, -(8n-2), 9n-1, 5n+2, 1)$$

and

$$\begin{aligned} S(\overline{C}_n) = (&- (5n+1), n+1, -(6n+2), -(7n+5), \\ &- (6n+4), -(14n+2), -(5n+3), -1, 0). \end{aligned}$$

Finally, we consider the support of A:

$$\begin{aligned} \mathrm{supp}(A) = &\{1\} \cup \left[2, \tfrac{n-3}{2}\right]_{\texttt{(J)}} \cup \left\{\tfrac{n-1}{2}\right\} \cup \left[\tfrac{n+1}{2}, n-2\right]_{\texttt{(G)}} \\ &\cup \{n-1, n, n+1, n+2\} \cup [n+3, 2n]_{\texttt{(E}\cup\texttt{I)}} \cup [2n+2, 3n-1]_{\texttt{(F}\cup\texttt{H)}} \\ &\cup \{3n, 3n+1, 3n+2, 3n+3\} \cup \left[3n+4, \tfrac{13n+13}{4}\right]_{\texttt{(K)}} \\ &\cup \left[\tfrac{13n+17}{4}, \tfrac{7n+5}{2}\right]_{\texttt{(Q)}} \cup \left[\tfrac{7n+7}{2}, \tfrac{15n+7}{4}\right]_{\texttt{(P)}} \cup \left[\tfrac{15n+11}{4}, 4n+1\right]_{\texttt{(N)}} \\ &\cup \left[4n+3, \tfrac{17n+5}{4}\right]_{\texttt{(M)}} \cup \left[\tfrac{17n+9}{4}, \tfrac{9n+1}{2}\right]_{\texttt{(O)}} \cup \left[\tfrac{9n+3}{2}, \tfrac{19n-1}{4}\right]_{\texttt{(R)}} \\ &\cup \left[\tfrac{19n+3}{4}, 5n\right]_{\texttt{(L)}} \cup \{5n+1, 5n+2\} \cup [5n+3, 6n+2]_{\texttt{(A)}} \\ &\cup [6n+4, 7n+3]_{\texttt{(B)}} \cup [7n+4, 8n+3]_{\texttt{(C)}} \cup [8n+5, 9n+4]_{\texttt{(D)}} \\ = &[1, 9n+4] \setminus \{2n+1, 4n+2, 6n+3, 8n+4\}. \end{aligned}$$

This concludes the proof. □

Example 4.9. Following the proof of Proposition 4.8 we obtain the integer globally simple $\mathrm{H}_9(15; 9)$ given in Figure 4.

Lemma 4.10. *For any $n \equiv 7 \pmod{14}$ such that $n \geq 21$, write $r = \frac{n-7}{2}$. Let A_n be a 9-diagonal array whose filled diagonals are $D_1, D_2, \ldots, D_7$, D_{r+7} and D_{r+8}. Then $(\mathcal{R}, \mathcal{C})$, where $\mathcal{R} = (1, 1, \ldots, 1)$ and $\mathcal{C} = (\underbrace{-1, \ldots, -1}_{8}, 1, 1, \ldots, 1)$, is a solution of $P(A_n)$.*

Proof. For any $i \in [1, 7] \cup \{r+7, r+8\}$ set $D_i = (d_{i,1}, d_{i,2}, d_{i,3}, \ldots, d_{i,n})$, where $d_{i,1}$ is the position $[i, 1]$ of A_n. Also, we set

$$\begin{aligned} \mathbf{A}_i &= d_{i,8}, d_{i,9}, d_{i,10}, \ldots, d_{i,n}; \\ \mathbf{B}_i &= d_{1,i}, d_{1,i+r}, d_{1,i+2r}, \ldots, d_{1,i+\frac{2r}{7}r}; \\ \mathbf{C}_i &= d_{r+7,i}, d_{r+7,i+r}, d_{r+7,i+2r}, \ldots, d_{r+7,i+\frac{2r}{7}r}; \\ \mathbf{D}_1 &= d_{1,1}, d_{1,1+r}, d_{1,1+2r}, \ldots, d_{1,1+(\frac{2r}{7}-2)r}; \\ \mathbf{D}_2 &= d_{1,8}, d_{1,8+r}; \\ \mathbf{E}_1 &= d_{r+7,1}, d_{r+7,1+r}, d_{r+7,1+2r}, \ldots, d_{r+7,1+(\frac{2r}{7}-2)r}; \\ \mathbf{E}_2 &= d_{r+7,8}, d_{r+7,8+r}. \end{aligned}$$

14	75	137	−122				17	−30				−106	91	−76
−49	−13	61	138	−123				29	−28				−107	92
78	−63	−12	74	139	−109				27	−26				−108
−94	79	−50	−11	60	125	−110				25	−24			
	−95	80	−64	−10	73	126	−111				23	−22		
		−96	81	−51	−9	59	127	−112				21	−20	
			−97	82	−65	−8	72	128	−113				19	−18
−45				−98	83	−52	15	66	129	−114				16
32	−33				−99	84	−58	6	53	130	−115			
	34	−35				−100	85	−71	5	67	131	−116		
		36	−37				−101	86	−57	4	54	132	−117	
			38	−39				−102	87	−70	3	68	133	−118
−119				40	−41				−103	88	−56	2	55	134
135	−120				42	−43				−104	89	−69	−7	77
48	136	−121				44	−46				−105	90	−47	1

Figure 4: An integer globally simple $\mathrm{H}_9(15; 9)$.

To aid in the proof, at the webpage

`http://anita-pasotti.unibs.it/Publications.html`,

we give a schematic picture of where each of these sequences fills cells. By a direct check, one can verify that

$$\begin{aligned} L_{\mathcal{R},\mathcal{C}}(d_{6,8}) = (&\mathbf{A}_6, d_{4,1}, d_{2,2}, d_{r+8,3}, d_{7,4}, d_{5,5}, d_{3,6}, \mathbf{B}_7, \mathbf{C}_7, d_{6,7}, \\ &\mathbf{A}_4, d_{2,1}, d_{r+8,2}, d_{7,3}, d_{5,4}, d_{3,5}, \mathbf{B}_6, \mathbf{C}_6, d_{6,6}, d_{4,7}, \\ &\mathbf{A}_2, d_{r+8,1}, d_{7,2}, d_{5,3}, d_{3,4}, \mathbf{B}_5, \mathbf{C}_5, d_{6,5}, d_{4,6}, d_{2,7}, \\ &\mathbf{A}_{r+8}, d_{7,1}, d_{5,2}, d_{3,3}, \mathbf{B}_4, \mathbf{C}_4, d_{6,4}, d_{4,5}, d_{2,6}, d_{r+8,7}, \\ &\mathbf{A}_7, d_{5,1}, d_{3,2}, \mathbf{B}_3, \mathbf{C}_3, d_{6,3}, d_{4,4}, d_{2,5}, d_{r+8,6}, d_{7,7}, \\ &\mathbf{A}_5, d_{3,1}, \mathbf{B}_2, \mathbf{C}_2, d_{6,2}, d_{4,3}, d_{2,4}, d_{r+8,5}, d_{7,6}, d_{5,7}, \\ &\mathbf{A}_3, \mathbf{D}_1, \mathbf{E}_2, d_{6,1}, d_{4,2}, d_{2,3}, d_{r+8,4}, d_{7,5}, d_{5,6}, d_{3,7}, \mathbf{D}_2, \mathbf{E}_1). \end{aligned}$$

Hence, it is easy to see that $L_{\mathcal{R},\mathcal{C}}(d_{6,8})$ covers all the filled cells of A_n. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The result follows from Theorem 3.4, once we have proved the existence of a relative Heffter array with compatible simple orderings ω_r and ω_c.

(1): For any n odd, a $\mathrm{H}_n(n;3)$ and a $\mathrm{H}_{2n}(n;3)$ are constructed in Propositions 4.1 and 4.3, respectively. Clearly these are globally simple Heffter arrays. Since they are cyclically 3-diagonal their compatibility follows from [13, Proposition 3.4].

(2): Let $n \equiv 3 \pmod 4$. A $\mathrm{H}_3(n;3)$ and a $\mathrm{H}_5(n;5)$ are constructed in [15, Propositions 5.1 and 5.5], respectively. As before these are globally simple Heffter arrays and since they are cyclically 3-diagonal and 5-diagonal, respectively, their compatibility follows from [13, Proposition 3.4]. A globally simple $\mathrm{H}_7(n;7)$ is given in Proposition 4.6. Since this is cyclically 7-diagonal its compatibility follows from [13, Propositions 3.4 and 3.6]. Finally, a globally simple $\mathrm{H}_9(n;9)$ is given in Proposition 4.8. Since this is 9-diagonal with width $\frac{n-9}{2}$, if $\gcd\big(n, \frac{n-7}{2}\big) = \gcd(n,7) = 1$ its compatibility follows from [12, Proposition 4.19]. If $\gcd(n,7) \neq 1$ the result follows from Lemma 4.10. □

5 Archdeacon arrays

In this section we introduce a further generalization of the concept of Heffter array. In particular we will consider p.f. arrays where the number of filled cells in each row and in each column is not fixed.

Definition 5.1. An *Archdeacon array* A over an abelian group $(G,+)$ is an $m \times n$ p.f. array with elements in G, such that:

(a) $\mathcal{E}(A)$ is a set;

(b) for every $g \in G$, $g \in \mathcal{E}(A)$ implies $-g \notin \mathcal{E}(A)$;

(c) the elements in every row and column sum to 0.

An example of this kind of arrays will be given in Figure 5. We note that, in the special case $G = \mathbb{Z}_v$, $\pm\mathcal{E}(A) = \mathbb{Z}_v \setminus J$ where J is a subgroup of $\mathbb{Z}_v$ and all the rows (resp. columns) have the same number of filled cells, we meet again the definition of a relative

Heffter array. The purpose of this section is to show how Archdeacon arrays can be used in order to obtain biembeddings and orthogonal cycle decompositions. First of all we need a generalization of [7, Proposition 2.6], stated by Buratti in [6, Theorem 3.3]. All the well known concepts about the differences method can be found in [6, 15].

Theorem 5.2. *Let G be an additive group and $\mathcal{B}$ be a set of cycles with vertices in G. If the list of differences of $\mathcal{B}$ is a set, say Λ, then $\mathcal{B}$ is a set of base cycles of a G-regular cycle decomposition of* $\mathsf{Cay}[G:\Lambda]$.

Generalizing Proposition 2.1, an Archdeacon array can be used to obtain regular cycle decompositions of Cayley graphs as follows.

Proposition 5.3. *Let A be an $m \times n$ Archdeacon array on an abelian group G with simple orderings $\omega_r = \omega_{\overline{R}_1} \circ \cdots \circ \omega_{\overline{R}_m}$ for the rows and $\omega_c = \omega_{\overline{C}_1} \circ \cdots \circ \omega_{\overline{C}_n}$ for the columns. Then:*

(1) $\mathcal{B}_{\omega_r} = \{\mathcal{S}(\omega_{\overline{R}_i}) \mid i \in [1,m]\}$ is a set of base cycles of a G-regular cycle decomposition $\mathcal{D}_{\omega_r}$ of $\mathsf{Cay}[G:\pm\mathcal{E}(A)]$*;*

(2) $\mathcal{B}_{\omega_c} = \{\mathcal{S}(\omega_{\overline{C}_j}) \mid j \in [1,n]\}$ is a set of base cycles of a G-regular cycle decomposition $\mathcal{D}_{\omega_c}$ of $\mathsf{Cay}[G:\pm\mathcal{E}(A)]$*;*

(3) the cycle decompositions $\mathcal{D}_{\omega_r}$ and $\mathcal{D}_{\omega_c}$ are orthogonal.

Proof. (1): Since the ordering ω_r is simple the elements of $\mathcal{B}_{\omega_r}$ are cycles of lengths $|\mathcal{E}(\overline{R}_1)|, \ldots, |\mathcal{E}(\overline{R}_m)|$ and by definition of partial sums the list of differences of $\mathcal{S}(\omega_{\overline{R}_i})$ is $\pm\mathcal{E}(\overline{R}_i)$, for any $i \in [1,m]$. Hence, the list of differences of $\mathcal{B}_{\omega_r}$ is $\pm\mathcal{E}(A)$ and so the thesis follows from Theorem 5.2. Obviously, (2) can be proved in the same way. Note that, in general, the cycles of $\mathcal{B}_{\omega_r}$ and those of $\mathcal{B}_{\omega_c}$ have different lengths. (3) follows from the requirement that the elements of $\pm\mathcal{E}(A)$ are pairwise distinct. □

Moreover the pair of cycles decompositions obtained from an Archdeacon array can be biembedded under the same hypothesis of Theorem 3.4. In fact, within the same proof, we have that:

Theorem 5.4. *Let A be an Archdeacon array on an abelian group G that is simple with respect to two compatible orderings ω_r and ω_c. Then there exists a biembedding of the G-regular cycle decompositions $\mathcal{D}_{\omega_r^{-1}}$ and $\mathcal{D}_{\omega_c}$ of* $\mathsf{Cay}[G:\pm\mathcal{E}(A)]$ *into an orientable surface.*

We observe that if an Archdeacon array has no empty rows/columns, then a necessary condition for the existence of compatible orderings is $|\operatorname{skel}(A)| \equiv m+n-1 \pmod 2$. This can be proved with the same proof of [11, Theorem 1.1] and of [12, Theorem 2.7].

Finally, as an easy consequence of Theorem 5.4, we obtain the relationship between the Crazy Knight's Tour Problem and globally simple Archdeacon arrays.

Corollary 5.5. *Let A be a globally simple Archdeacon array on an abelian group G such that $P(A)$ admits a solution $(\mathcal{R},\mathcal{C})$. Then there exists a biembedding of the G-regular cycle decompositions $\mathcal{D}_{\omega_r^{-1}}$ and $\mathcal{D}_{\omega_c}$ of* $\mathsf{Cay}[G:\pm\mathcal{E}(A)]$ *into an orientable surface.*

Given two $m\times n$ p.f. arrays A and B defined on abelian groups G_1 and G_2, respectively, we define their direct sum $A \oplus B$ as the $m \times n$ p.f. array E whose skeleton is $\mathrm{skel}(A) \cup \mathrm{skel}(B)$ and whose entries in $G_1 \oplus G_2$ are so defined:

$$E[i,j] = \begin{cases} (A[i,j], B[i,j]) & \text{if } (i,j) \in \mathrm{skel}(A) \cap skel(B), \\ (A[i,j], 0_{G_2}) & \text{if } (i,j) \in \mathrm{skel}(A) \setminus skel(B), \\ (0_{G_1}, B[i,j]) & \text{if } (i,j) \in \mathrm{skel}(B) \setminus skel(A). \end{cases}$$

In the following we will denote by $\overline{R}_i(A)$ and $\overline{C}_j(A)$ the i-th row and the j-th column of A, respectively.

Lemma 5.6. *Let A and B be $m \times n$ globally simple p.f. arrays over abelian groups G_1 and G_2, respectively, such that:*

(1) for any $i \in [1,m]$ for which the i-th rows of A and B are both nonempty, we have $\mathrm{skel}(\overline{R}_i(A)) \cap \mathrm{skel}(\overline{R}_i(B)) \neq \emptyset$;

(2) for any $j \in [1,n]$ for which the j-th columns of A and B are both nonempty, we have $\mathrm{skel}(\overline{C}_j(A)) \cap \mathrm{skel}(\overline{C}_j(B)) \neq \emptyset$;

(3) the elements in every nonempty row/column of both A and B sum to zero.

Then $A \oplus B$ is a globally simple p.f. array, whose nonempty rows and columns sum to zero.

Proof. Since the elements in every nonempty row and column of both A and B sum to zero, the same holds for $A \oplus B$.

Let us suppose, by contradiction, that there exists a row (resp. a column) $\overline{R}_i$ of $A \oplus B$ that is not simple with respect to the natural ordering. Then there would be a subsequence L of consecutive elements of $\overline{R}_i$ that sum to zero. Denoted by L_1 the subsequence of the first coordinates of L (ignoring the zeros) and by L_2 the one of the second coordinates, we have that both L_1 and L_2 sums to zero. Since both $\overline{R}_i(A)$ and $\overline{R}_i(B)$ are simple with respect to the natural ordering, it follows that either $L_1 = \emptyset$ (we are ignoring zeros) or $L_1 = \mathcal{E}(A)$. Similarly, for $\overline{R}_i(B)$. If $L_1 = \emptyset$, then $L_2 = \mathcal{E}(\overline{R}_i(B))$ and hence L is $\mathcal{E}(\overline{R}_i)$. Similarly, if $L_2 = \emptyset$. Finally, if L_1 and L_2 are both nonempty, the only possibility is that $L = \mathcal{E}(\overline{R}_i)$ since $\mathrm{skel}(\overline{R}_i(A)) \cap \mathrm{skel}(\overline{R}_i(B)) \neq \emptyset$. □

Proposition 5.7. *Let A be an Archdeacon array over an abelian group G_1 and let B be a p.f. array of the same size defined over an abelian group G_2. Suppose that the hypotheses of* Lemma 5.6 *are satisfied, that $\mathcal{E}(A \oplus B)$ is a set and that if $(0_{G_1}, x) \in \mathcal{E}(A \oplus B)$, then $(0_{G_1}, -x) \notin \mathcal{E}(A \oplus B)$. Then $A \oplus B$ is a globally simple Archdeacon array over $G_1 \oplus G_2$.*

Proof. By Lemma 5.6, $E = A \oplus B$ is a globally simple p.f. array whose rows and columns sum to zero. We now show that condition (b) of Definition 5.1 holds. Suppose that $g = (g_1, g_2) \in G_1 \oplus G_2$ belongs to $\mathcal{E}(E)$. Then, either $g_1 \in \mathcal{E}(A)$ or $g_1 = 0_{G_1}$. In the first case, $-g_1 \notin \mathcal{E}(A)$ and so $-g = (-g_1, -g_2) \notin \mathcal{E}(E)$. If $g_1 = 0_{G_1}$, then $(0_{G_1}, -g_2) \notin \mathcal{E}(E)$ by hypothesis, proving the statement. □

Now we consider the $m \times n$ p.f. array $B_{m,n,d}(i_1, i_2; j_1, j_2)$ over $\mathbb{Z}_d$ which has only four nonempty cells: those in positions $(i_1, j_1), (i_2, j_2)$ that we fill with $+1$ and those in positions $(i_2, j_1), (i_1, j_2)$ that we fill with -1. The following result is a consequence of Proposition 5.7.

Corollary 5.8. *Let $k < n$ and let us suppose there exists a globally simple cyclically k-diagonal $\mathrm{H}_t(n;k)$, say A, whose filled diagonals are $D_1, \dots, D_k$. Then considering the array $B = B_{n,n,d}(1,2;1,2)$, where $d > 2$, we have that $E = A \oplus B$ is a globally simple Archdeacon array over the group $\mathbb{Z}_{2nk+t} \oplus \mathbb{Z}_d$.*

We know that there exists a (globally simple) cyclically 3-diagonal $\mathrm{H}_t(n;3)$ in each of the following cases:

(1) $t \in \{1,2\}$ and $n \equiv 0,1 \pmod 4$, see [4, Theorems 3.4 and 3.9];

(2) $t = 3$ and $n \equiv 0,3 \pmod 4$, see [15, Propositions 5.1 and 5.3];

(3) $t = n$ and n is odd, see Proposition 4.1;

(4) $t = 2n$ and n is odd, see Proposition 4.3.

Therefore in these cases, we can apply Corollary 5.8: for any $d \geq 3$ there exists a globally simple Archdeacon array E of size $n \geq 4$ defined over $\mathbb{Z}_{6n+t} \oplus \mathbb{Z}_d$ whose skeleton is $D_1 \cup D_2 \cup D_3 \cup \{(1,2)\}$.

Moreover, because of [12, Proposition 5.9], there exists a solution of $P(E)$ whenever n is also even. In those cases we have a biembedding of $\mathsf{Cay}[\mathbb{Z}_{6n+t} \oplus \mathbb{Z}_d : \pm\mathcal{E}(E)]$ in an orientable surface whose faces classes contain triangles and exactly one quadrangle.

As example of such construction, in Figure 5 we give a globally simple Archdeacon array over $\mathbb{Z}_{51} \oplus \mathbb{Z}_d$, where $d \geq 3$.

$(-9,1)$	$(0,-1)$					$(16,0)$	$(-7,0)$
$(-3,-1)$	$(-22,1)$						$(25,0)$
$(12,0)$	$(1,0)$	$(-13,0)$					
	$(21,0)$	$(2,0)$	$(-23,0)$				
		$(11,0)$	$(8,0)$	$(-19,0)$			
			$(15,0)$	$(5,0)$	$(-20,0)$		
				$(14,0)$	$(-4,0)$	$(-10,0)$	
					$(24,0)$	$(-6,0)$	$(-18,0)$

Figure 5: An Archdeacon array over $\mathbb{Z}_{51} \oplus \mathbb{Z}_d$.

We recall that the existence of a (globally simple) cyclically 4-diagonal $\mathrm{H}_t(n;4)$ for any n and $t \in \{1,2,4\}$ has been proved in [17, Theorem 2.2] and [15, Proposition 4.9]. Therefore, for any $d \geq 3$, because of Corollary 5.8 there exists a globally simple Archdeacon array E of size $n \geq 4$ over $\mathbb{Z}_{8n+t} \oplus \mathbb{Z}_d$ whose skeleton is $D_1 \cup D_2 \cup D_3 \cup D_4 \cup \{(1,2)\}$.

Moreover, because of [12, Proposition 5.13], there exists a solution of $P(E)$ whenever $n \not\equiv 0 \pmod 3$. In these cases we have a biembedding of $\mathsf{Cay}[\mathbb{Z}_{8n+t} \oplus \mathbb{Z}_d : \pm\mathcal{E}(E)]$ in an orientable surface whose faces classes contain quadrangles and exactly one pentagon.

An example of such construction is given in Figure 6 where we provide a globally simple Archdeacon array over $\mathbb{Z}_{60} \oplus \mathbb{Z}_d$, where $d \geq 3$.

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$(25,1)$	$(0,-1)$			$(1,0)$	$(-8,0)$	$(-18,0)$
$(-19,-1)$	$(26,1)$				$(2,0)$	$(-9,0)$
$(-10,0)$	$(-20,0)$	$(27,0)$				$(3,0)$
$(4,0)$	$(-11,0)$	$(-21,0)$	$(28,0)$			
	$(5,0)$	$(-12,0)$	$(-22,0)$	$(29,0)$		
		$(6,0)$	$(-13,0)$	$(-16,0)$	$(23,0)$	
			$(7,0)$	$(-14,0)$	$(-17,0)$	$(24,0)$

Figure 6: An Archdeacon array over $\mathbb{Z}_{60} \oplus \mathbb{Z}_d$.

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On the general position problem on Kneser graphs*

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Abstract

In a graph G, a *geodesic* between two vertices x and y is a shortest path connecting x to y. A subset S of the vertices of G is *in general position* if no vertex of S lies on any geodesic between two other vertices of S. The size of a largest set of vertices in general position is the *general position number* that we denote by $gp(G)$. Recently, Ghorbani et al. proved that for any k if $n \geq k^3 - k^2 + 2k - 2$, then $gp(Kn_{n,k}) = \binom{n-1}{k-1}$, where $Kn_{n,k}$ denotes the Kneser graph. We improve on their result and show that the same conclusion holds for $n \geq 2.5k - 0.5$ and this bound is best possible. Our main tools are a result on cross-intersecting families and a slight generalization of Bollobás's inequality on intersecting set pair systems.

Keywords: General position problem, Kneser graphs, intersection theorems.

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1 Introduction

A recently studied extremal problem [4, 6, 12] in graph theory is the following. In a graph G, a *geodesic* between two vertices x and y is a shortest path connecting x to y. We say that a subset S of the vertices of G is *in general position* if no vertex of S lies on any geodesic between two other vertices of S. The size of a largest set of vertices in general position

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is the *general position number* which we denote by $gp(G)$. Our graph of interest in this paper is the *Kneser graph* $Kn_{n,k}$ whose vertex is $\binom{[n]}{k}$, the set of all k-element subsets of the set $[n] = \{1, 2, \ldots, n\}$ and two k-subsets S and T are joined by an edge if and only if $S \cap T = \emptyset$. Ghorbani et al. [10] determined $gp(Kn_{n,2})$ and $gp(Kn_{n,3})$ for all n and showed that for any fixed k if n is large enough, then $gp(Kn_{n,k}) = \binom{n-1}{k-1}$ holds.

Theorem 1.1 ([10])**.** *Let $n, k \geq 2$ be integers with $n \geq 3k - 1$. If for all t, where $2 \leq t \leq k$, the inequality $k^t\binom{n-t}{k-t} + t \leq \binom{n-1}{k-1}$ holds, then $gp(Kn_{n,k}) = \binom{n-1}{k-1}$.*

For fixed k and $t = 2$ the above inequality is satisfied when $n \geq k^3 - k^2 + 2k - 1$ holds. We improve on this and the main result of this note is the following.

Theorem 1.2. *If $n, k \geq 4$ are integers with $n \geq 2k + 1$, then $gp(Kn_{n,k}) \leq \binom{n-1}{k-1}$ holds. Moreover, if $n \geq 2.5k - 0.5$, then we have $gp(Kn_{n,k}) = \binom{n-1}{k-1}$, while if $2k + 1 \leq n < 2.5k - 0.5$, then $gp(Kn_{n,k}) < \binom{n-1}{k-1}$ holds.*

The threshold $n \geq 2.5k - 0.5$ comes from the fact that $\operatorname{diam}(Kn_{n,k}) \leq 3$ holds if and only if this inequality is satisfied. The proof of Theorem 1.1 uses the following general result of Anand et al. [2] that characterizes vertex subsets in general position.

Theorem 1.3 ([2])**.** *If G is a connected graph, then a subset S of the vertices of G is in general position if and only if all the components $S_1, S_2, \ldots, S_h$ of $G[S]$ are cliques in G and*

- *for any $1 \leq i < j \leq h$ and $s_i, s_i' \in S_i$, $s_j, s_j' \in S_j$ we have $d(s_i, s_j) = d(s_i', s_j') =: d(S_i, S_j)$ (where $d(x, y)$ denotes the distance of x and y in G),*
- *$d(S_i, S_j) \neq d(S_i, S_l) + d(S_l, S_j)$ for any $1 \leq i, j, l \leq h$.*

In Kneser graphs a clique corresponds to a family $\mathcal{F} \subseteq \binom{[n]}{k}$ of pairwise disjoint sets. There is no edge between different components of any general position set S. It follows that if $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_h$ correspond to the components of $G[S]$, then for any $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$ with $i \neq j$ we have $F_i \cap F_j \neq \emptyset$. Families with this property are called *cross-intersecting*. So the upper bound in Theorem 1.2 will follow from the next result unless $n = 2k + 1$ in which case we will need some further reasonings.

Theorem 1.4. *Let $n \geq 2k + 2$, $k \geq 4$ and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_h \subseteq \binom{[n]}{k}$ such that*

- *$\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ for all $1 \leq i < j \leq h$,*
- *$F_i \cap F_i' = \emptyset$ for all pairs of distinct sets $F_i, F_i' \in \mathcal{F}_i$ for any $i = 1, 2, \ldots, h$,*
- *$F_i \cap F_j \neq \emptyset$ for any $1 \leq i < j \leq j$ and any $F_i \in \mathcal{F}_i$, $F_j \in \mathcal{F}_j$*

hold. Then we have $\sum_{i=1}^{h} |\mathcal{F}_i| \leq \binom{n-1}{k-1}$.

Note that the first condition cannot be omitted as otherwise we could repeat some families that consist of a single set.

The remainder of the paper is organized as follows: Section 2 contains the proof of Theorem 1.4 and in Section 3 we list some open problems along with some remarks.

2 Proofs

Proof of Theorem 1.4. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_h \subseteq \binom{[n]}{k}$ satisfy the conditions of the theorem. As the $\mathcal{F}_i$'s are families of pairwise disjoint sets, each of them are of size at most n/k and we may assume that $|\mathcal{F}_1| \leq |\mathcal{F}_2| \leq \cdots \leq |\mathcal{F}_h| =: t \leq n/k$. If $t = 1$, then $\mathcal{F} = \cup_{i=1}^h \mathcal{F}_i$ form an intersecting family and therefore by the celebrated theorem of Erdős, Ko and Rado [5] we have $\sum_{i=1}^h |\mathcal{F}_i| = h \leq \binom{n-1}{k-1}$.

Suppose next that $t \geq 2$ holds. Then we claim $h \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Indeed, let us fix one set F_i from each $\mathcal{F}_i$ for $i = 1, 2, \dots, h-1$ and two sets $F_h, F'_h \in \mathcal{F}_h$. Hence if

- $|\cap_{i=1}^{h-1} F_i| \geq 2$, then $h - 1 \leq \binom{n-2}{k-2} < \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$,
- $\cap_{i=1}^{h-1} F_i$ consists of a single element x, then either F_h or F'_h cannot contain x and as all F_i's meet both F_h and F'_h we must have $h - 1 \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$,
- $\cap_{i=1}^{h-1} F_i = \emptyset$, then $\{F_1, F_2, \dots, F_{h-1}, F_h\}$ is intersecting with no common elements, and a result of Hilton and Milner [11] states that families with this property can have size at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, so we obtain $h \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

Let m_i denote the number of j's such that $|\mathcal{F}_j| \geq i$ holds. Then clearly we have

$$\sum_{i=1}^{h} |\mathcal{F}_i| = h + \sum_{j=2}^{t} m_j \leq h + \left(\frac{n}{k} - 1\right) m_2. \tag{2.1}$$

To bound m_2 we apply Bollobás's famous inequality [3] that states that if $\{(A_1, B_1)\}_{i=1}^{l}$ are pairs of disjoint sets such that for any $1 \leq i \neq j \leq l$ we have $A_i \cap B_j \neq \emptyset$, then $\sum_{i=1}^{l} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$ holds. For any $1 \leq i \leq m_2$ we can pick two sets $F_i, G_i \in \mathcal{F}_{h-m_2+i}$. Then we can define $2m_2$ pairs $\{(A_j, B_j)\}_{j=1}^{2m_2}$ such that for $1 \leq j \leq m_2$ we have $A_j = F_j, B_j = G_j$ and $A_{2m_2-j} = G_j, B_{2m_2-j} = F_j$. As the $\mathcal{F}_i$'s are cross-intersecting families of disjoint sets, therefore the pairs $\{(A_j, B_j)\}_{j=1}^{2m_2}$ satisfy the conditions of Bollobás's inequality and we obtain $\frac{2m_2}{\binom{2k}{k}} \leq 1$ and thus $m_2 \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$. Putting together (2.1) and the bounds on h and m_2 we obtain

$$\sum_{i=1}^{h} |\mathcal{F}_i| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 + \frac{n-k}{k}\binom{2k-1}{k-1}.$$

Therefore it is enough to prove $\binom{n-k-1}{k-1} > \frac{n-k}{k}\binom{2k-1}{k-1}$. Observe that

$$\frac{\binom{n-k}{k-1}}{\binom{n-k-1}{k-1}} = \frac{n-k}{n-2k+1} \geq \frac{n-k+1}{n-k} = \frac{\frac{n-k+1}{k}\binom{2k-1}{k-1}}{\frac{n-k}{k}\binom{2k-1}{k-1}},$$

therefore if $\binom{n_0-k-1}{k-1} > \frac{n_0-k}{k}\binom{2k-1}{k-1}$ holds for some n_0, then $\binom{n-k-1}{k-1} > \frac{n-k}{k}\binom{2k-1}{k-1}$ holds for $n \geq n_0$. Putting $n_0 = 3k+2$ the above inequality is equivalent to

$$k \prod_{i=0}^{k-2} (2k+1-i) > (2k+2) \prod_{i=0}^{k-2} (2k-1-i)$$

which simpifies to

$$k(2k+1)2k > (2k+2)(k+2)(k+1).$$

This holds for $k \geq 5$ and a similar calculation shows that if $k = 4$, then the desired inequality holds if $n \geq 17 = 4k+1$.

In all missing cases, except for $k = 4$, $n = 16$, we have $n < 4k$, therefore we have $m_j = 0$ for all $j \geq 4$. So for the remaining pairs n and k, we need to strengthen our bound on $m_2 + m_3$. We will need the following lemma, a slight generalization of Bollobás's result.

Lemma 2.1. *Let $\{A_i, B_i\}_{i=1}^{\alpha}$ and $\{A_j, B_j, C_j\}_{j=\alpha+1}^{\beta}$ be pairs and triples of pairwise disjoint sets such that for any $1 \leq i < j \leq \alpha+\beta$ we have $X_i \cap Y_j \neq \emptyset$ where X and Y can be any of A, B and C. Then the following inequality holds:*

$$\sum_{i=1}^{\alpha+\beta} \frac{2}{\binom{|A_i|+|B_i|}{|A_i|}} + \sum_{j=1}^{\beta} \left(\frac{2}{\binom{|A_{\alpha+j}|+|C_{\alpha+j}|}{|A_{\alpha+j}|}} + \frac{2}{\binom{|B_{\alpha+j}|+|C_{\alpha+j}|}{|B_{\alpha+j}|}} \right.$$
$$\left. - \frac{2}{\binom{|A_{\alpha+j}|+|B_{\alpha+j}|+|C_{\alpha+j}|}{|A_{\alpha+j}|}} - \frac{2}{\binom{|A_{\alpha+j}|+|B_{\alpha+j}|+|C_{\alpha+j}|}{|B_{\alpha+j}|}} \right) \leq 1.$$

Proof. Let us define M to be $\bigcup_{i=1}^{\alpha}(A_i \cup B_i) \cup \bigcup_{j=1}^{\beta}(A_{\alpha+j} \cup B_{\alpha+j} \cup C_{\alpha+j})$ and let us write $|M| = m$. Just as before, let us introduce a family $\{S_i, T_i\}_{i=1}^{2(\alpha+\beta)}$ of disjoint pairs as $S_i = A_i, T_i = B_i$ and $S_{2(\alpha+\beta)-j} = B_j, T_{2(\alpha+\beta)-j} = A_j$ for all $1 \leq i, j \leq \alpha+\beta$. We count the pairs (π, j) such that π is a permutation of the elements of M and $1 \leq j \leq 2(\alpha+\beta)$ with all elements of S_j preceding all elements of T_j in π that is $\max\{\pi^{i-1}(s) : s \in S_j\} < \min\{\pi^{-1}(t) : t \in T_j\}$. We denote this by $S_j <_\pi T_j$. For every fixed j there exist exactly $|S_j|!|T_j|!(m - |S_j| - |T_j|)!\binom{m}{|S_j|+|T_j|}$ permutations π with $S_j <_\pi T_j$. On the other hand for any fixed π there exists at most one j with $S_j <_\pi T_j$. Indeed, if $i \neq j, 2(\alpha+\beta)-j$, then both S_i and T_i meet both S_j and T_j, while clearly if $S_j <_\pi T_j$, then $S_{2(\alpha+\beta)-j} = T_j \not<_\pi S_j = T_{2(\alpha+\beta)-j}$. These observations would yield Bollobás's original inequality, but we haven't used the existence of the C_j's. Observe that if $A_j <_\pi C_j$, $C_j <_\pi A_j$, $B_j <_\pi C_j$ or $C_j <_\pi B_j$, then again by the cross-intersecting property (π, i) can be a pair counted only if $i = j$ or $i = 2(\alpha+\beta)-j$ and at least one of $A_i <_\pi B_i \cup C_i$, $B_i \cup C_i <_\pi A_i$, $C_i \cup B_i <_\pi A_i$, $C_i \cup A_i <_\pi B_i$ holds. Counting j and $2(\alpha+\beta)-j$ cases together this yields

$$\sum_{j=1}^{\alpha+\beta} 2|A_j|!|B_j|!(m - |A_j| - |B_j|)!\binom{m}{|A_j|+|B_j|}$$
$$\leq m! - \sum_{j=1}^{\alpha+\beta} 2\left[|A_j|!|C_j|!(m - |A_j| - |C_j|)!\binom{m}{|A_j|+|C_j|}\right.$$
$$\left. + |B_j|!|C_j|!(m - |C_j| - |B_j|)!\binom{m}{|C_j|+|B_j|}\right]$$

$$+\sum_{j=1}^{\beta} 2|A_{\alpha+j}|!(|B_{\alpha+j}|+|C_{\alpha+j}|)!(m-|A_{\alpha+j}|-|B_{\alpha+j}|$$
$$-|C_{\alpha+j}|)!\binom{m}{|A_{\alpha+j}|+|B_{\alpha+j}|+|C_{\alpha+j}|}$$
$$+\sum_{j=1}^{\beta} 2|B_{\alpha+j}|!(|A_{\alpha+j}|+|C_{\alpha+j}|)!(m-|A_{\alpha+j}|-|B_{\alpha+j}|$$
$$-|C_{\alpha+j}|)!\binom{m}{|A_{\alpha+j}|+|B_{\alpha+j}|+|C_{\alpha+j}|}$$

Dividing by $m!$ and rearranging yields the statement of the lemma. $\square$

We apply Lemma 2.1 to the families $\mathcal{F}_{h-m_2+1},\dots,\mathcal{F}_h$ with $\beta=m_3$ and $\alpha=m_2-m_3$. As all sets in the $\mathcal{F}_i$'s are of size k we obtain

$$\frac{2(m_2-m_3)}{\binom{2k}{k}}+\frac{6m_3}{\binom{2k}{k}}-\frac{6m_3}{\binom{3k}{k}}\le 1. \tag{2.2}$$

As $\binom{3k}{k}\ge 3\binom{2k}{k}$ for $k\ge 3$, the left hand side of the above equation is greater than $\frac{2(m_2-m_3)}{\binom{2k}{k}}+\frac{4m_3}{\binom{2k}{k}}=\frac{2(m_2+m_3)}{\binom{2k}{k}}$. Therefore we obtain $m_2+m_3\le\frac{1}{2}\binom{2k}{k}=\binom{2k-1}{k-1}$. So for $n<4k$ we have the bound

$$\sum_{i=1}^{h}|\mathcal{F}_i|\le h+m_2+m_3\le\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1+\binom{2k-1}{k-1}. \tag{2.3}$$

Suppose first that $n\ge 3k$ holds. Plugging into (2.3) we obtain the upper bound $\binom{n-1}{k-1}+1$. To get rid of the extra 1, we need to use the uniqueness part of the Hilton-Milner theorem [11] that we used to get our bound on h. It states that if $k\ge 4$ and an intersecting family $\mathcal{F}\subseteq\binom{[n]}{k}$ with $\cap_{F\in\mathcal{F}}F=\emptyset$ has size $\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$, then there exist $x\in[n]$ and $x\notin G\subseteq[n]$ such that $\mathcal{F}=\{G\}\cup\{F: x\in F, F\cap G\neq\emptyset\}$. Observe that for any $H\neq G$ with $x\notin H$ there exist lots of sets $F\in\mathcal{F}$ that are disjoint with H, so only sets H' that contain x can be added to the $\mathcal{F}_j$'s. But as all $\mathcal{F}_j$'s consist of pairwise disjoint sets, such an H' can only be added to the $\mathcal{F}_j$ containing G. Also, at most one such set can be added as again this $\mathcal{F}_j$ consists of pairwise disjoint sets. We obtained that if $t\ge 2$ and $h=\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$, then $\sum_{j=1}^{h}|\mathcal{F}_j|\le\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+2<\binom{n-1}{k-1}$.

Next, we assume that $2k+2\le n<3k$. Then we have $t\le 2$ and therefore the family $\mathcal{F}':=\cup_{i=1}^{h}\mathcal{F}_i$ has the property that for any $F\in\mathcal{F}'$ there exists at most one other $G\in\mathcal{F}'$ that is disjoint with F. Such families are called (≤ 1)-*almost intersecting* and Gerbner et al. [8] proved that whenever $2k+2\le n$ holds, then any (≤ 1)-almost intersecting family $\mathcal{G}\subseteq\binom{[n]}{k}$ has size at most $\binom{n-1}{k-1}$.

Finally, if $n=16$, $k=4$, then we need to bound $h+m_2+m_3+m_4\le h+m_2+2m_3\le h+2m_2+3m_3$. As $\binom{3k}{k}=\binom{12}{4}>6\binom{8}{4}=\binom{2k}{k}$, (2.2) implies $2m_2+3m_3\le\binom{8}{4}$. Using the Hilton-Milner bound $h\le\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$ and plugging in $n=16$, we obtain $\sum_{i=1}^{h}|\mathcal{F}_i|\le h+2m_2+3m_3\le\binom{n-1}{k-1}-\binom{11}{3}+1+\binom{8}{4}<\binom{n-1}{k-1}$. This concludes the proof. $\square$

Proof of Theorem 1.2. Theorem 1.4 shows that $Kn_{n,k} \leq \binom{n-1}{k-1}$ holds if $n \geq 2k+2$. Observe that $\mathrm{diam}(Kn_{n,k}) \leq 3$ if and only if $n \geq 2.5k - 0.5$ (see e.g. [16]). Also, Theorem 1.3 yields that if the diameter of a graph G is at most 3, then any independent set in G is in general position. The largest independent sets in $Kn_{n,k}$ correspond to *stars*, i.e. families $\mathcal{S}_x = \{H \in \binom{[n]}{k} : x \in H\}$ for some $x \in [n]$. Therefore, $gp(Kn_{n,k}) \geq \binom{n-1}{k-1}$ holds provided $n \geq 2.5k - 0.5$.

If $2k+2 \leq n < 2.5k-0.5$, then the upper bound of Theorem 1.4 is based on the result of Gerbner et al. [8] on (≤ 1)-almost intersecting families. Their result also states that the only (≤ 1)-almost intersecting families of size $\binom{n-1}{k-1}$ are stars. But if $n < 2.5k - 0.5$, then $\{H \in \binom{[n]}{k} : 1 \in H\}$ is not in general position as shown by the following example: let $n = 2k + M$ with $1 \leq M < 0.5k - 0.5$ and $F_1 = [k]$, $F_2 = \{1, 2, \dots, k - M - 1\} \cup \{k+1, k+2, \dots, k+M+1\}$. We claim that $d_{Kn_{n,k}}(F_1, F_2) \geq 4$. Indeed, as $C := [n] \setminus (F_1 \cup F_2)$ is of size $k-1$, we have $d_{Kn_{n,k}}(F_1, F_2) \geq 3$. Suppose G_1, G_2 are k-subsets of $[n]$ with $F_1 \cap G_1 = G_1 \cap G_2 = \emptyset$. Let us define $\ell = |G_1 \cap F_2|$. As G_1 is disjoint with F_1, so with $F_1 \cap F_2$, we have $\ell \leq M+1$. Therefore $|C \cap G_1| \geq k - M - 1$ must hold. As G_2 is disjoint with G_1, we obtain $|C \cap G_2| \leq M$, but as $|F_1 \setminus F_2| = M + 1$ and $2M + 1 < k$, G_2 must meet F_2, so indeed $d_{Kn_{n,k}}(F_1, F_2) \geq 4$ holds. On the other hand, for any $x \in F_2 \setminus F_1$ and $y, z \in F_1 \setminus F_2$, the sets $F_1, C \cup \{x\}, F_2 \setminus \{x\} \cup \{y\}, C \cup \{z\}, F_2$ form a path of length 4, therefore a geodesic with $1 \in F_2 \setminus \{x\} \cup \{z\}$. This shows that $\{H \in \binom{[n]}{k} : 1 \in H\}$ is not in general position. Therefore if $2k + 2 \leq n < 2.5k - 0.5$ holds, then we have $gp(Kn_{n,k}) < \binom{n-1}{k-1}$.

Finally, let us consider the case $n = 2k + 1$. Again, vertices corresponding to sets of stars are not in general position and all other independent sets have size smaller than $\binom{n-1}{k-1}$. So suppose F, F' are disjoint sets in a family $\mathcal{F}$ corresponding to vertices in general position. Then by Theorem 1.3, for any set $G \neq F, F'$ in $\mathcal{F}$ we must have $d(G, F) = d(G, F')$. Observe that in $Kn_{2k+1,k}$ we have $d(H, H') = \min\{2(k - |H \cap H'|), 2|H \cap H'| + 1\}$.

Let us first assume that $k = 2l + 1$ is odd. Then by the above, for any $G \in \mathcal{F}$ we must have $|G \cap F| = |G \cap F'| = l$ and the unique element $x \in [2k+1] \setminus (F \cup F')$ must belong to G. Therefore, with the notation of the proof of Theorem 1.4, we have $m_2 = 1$ and $h \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ and thus $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2 < \binom{n-1}{k-1}$.

Let us assume that $k = 2l$ is even. Then by the above, for any $G \neq F, F'$ in $\mathcal{F}$ we must have $|G \cap F| = |G \cap F'| = l$ and thus $G \subseteq F \cup F'$. If we take one set from each disjoint pair, we obtain a family $\mathcal{G} \subseteq \binom{[2k]}{k}$ such that any pairwise intersection is of the same size. By Fisher's inequality, we obtain that the number m_2 of pairs is at most $2k$. Moreover, as all sets of $\mathcal{F}$ are k-subsets of $[2k]$, we must have $h \leq \frac{1}{2}\binom{2k}{k}$. Therefore, we need to show $\frac{1}{2}\binom{2k}{k} + 2k < \binom{2k}{k-1} = \binom{2k}{k}\frac{k}{k+1}$ which is equivalent to $\frac{2k(2k+2)}{k-1} < \binom{2k}{k}$. This holds for $k \geq 4$. □

3 Concluding remarks

First of all, it remains an open problem to determine $gp(Kn_{n,k})$ for $2k + 1 \leq n < 2.5k - 0.5$.

Let us finish this short note with two remarks. First observe that an (≤ 1)-almost intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ corresponds to a subset U of the vertices of $Kn_{n,k}$ such that $Kn_{n,k}[U]$ does not contain a path on three vertices. There have been recent developments

[1, 9, 15] in the general problem of finding the largest possible size of a subset U of the vertices of $Kn_{n,k}$ such that $Kn_{n,k}[U]$ does not contain some fixed forbidden graph F. Note that independently of the host graph G, if a subset S of the vertices of G is in general position, then $G[S]$ cannot contain a path on three vertices as an *induced* subgraph. Returning to the Kneser graph $Kn_{n,k}$ it would be interesting to address the induced version of the vertex Turán problems mentioned above.

There have been lots of applications and generalizations of Bollobás's inequality. Very recently O'Neill and Verstraëte [13] obtained Bollobás type results for k-tuples. Their condition to generalize disjoint pairs is completely different from the condition of Lemma 2.1. More importantly pairwise disjoint, cross-intersecting families were introduced by Rényi [14] as *qualitatively independent partitions* if the extra condition that $\cup_{F\in\mathcal{F}_i}F=[n]$ holds for all $1\le i\le h$ is added, and the uniformity condition $|F|=k$ for all $F\in\cup_{i=1}^{h}\mathcal{F}_i$ is replaced by $|\mathcal{F}_i|=d$ for all $1\le i\le h$. Gargano, Körner and Vaccaro proved [7] that for any fixed $d\ge 2$ as n tends to infinity the maximum number of qualitatively independent d-partitions is $2^{(\frac{2}{d}-o(1))n}$. Based on their construction, for any fixed d one can obtain $2^{(2-o(1))k}$ many pairwise disjoint cross-intersecting d-tuples of k-sets as k tends to infinity.

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Schur numbers involving rainbow colorings

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Abstract

In this paper, we introduce two different generalizations of Schur numbers that involve rainbow colorings. Motivated by well-known generalizations of Ramsey numbers, we first define the rainbow Schur number $RS(n)$ to be the minimum number of colors needed such that every coloring of $\{1, 2, \dots, n\}$, in which all available colors are used, contains a rainbow solution to $a + b = c$. It is shown that

$$RS(n) = \lfloor \log_2(n) \rfloor + 2, \quad \text{for all } n \geq 3.$$

Second, we consider the Gallai-Schur number $GS(n)$, defined to be the least natural number such that every n-coloring of $\{1, 2, \dots, GS(n)\}$ that lacks rainbow solutions to the equation $a + b = c$ necessarily contains a monochromatic solution to this equation. By connecting this number with the n-color Gallai-Ramsey number for triangles, it is shown that for all $n \geq 3$,

$$GS(n) = \begin{cases} 5^k & \text{if } n = 2k \\ 2 \cdot 5^k & \text{if } n = 2k + 1. \end{cases}$$

Keywords: Schur numbers, anti-Ramsey numbers, rainbow triangles, Gallai colorings.

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1 Introduction

One of the earliest results that falls under the blanket of Ramsey theory is a theorem of Issai Schur [11] from 1916. In fact, his work predates Frank Ramsey's foundational paper [10]. Schur proved that for any $n \in \mathbb{N}$, there exists a minimal $S(n) \in \mathbb{N}$ such that every n-coloring of the elements in the set $\{1, 2, \dots, S(n)\}$ contains elements a, b, and c of the same color such that $a + b = c$. Such a triple a, b, and c is called a *monochromatic*

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Schur solution and we note that it is possible that $a = b$. The number $S(n)$ is called a *Schur number* and it is well-known that $S(1) = 2$, $S(2) = 5$, $S(3) = 14$, $S(4) = 45$ (see Golomb and Baumert [6]). Recently, Heule [7] has shown that $S(5) = 161$. We note that some authors define a Schur number to be the largest $f(n) \in \mathbb{N}$ such that some n-coloring of $\{1, 2, \ldots, f(n)\}$ lacks a monochromatic Schur solution. It is easily seen that $S(n) = f(n) + 1$.

A thorough overview of Schur numbers is given in Landman and Robertson's book [9] and in Section 3 of Soifer's article [12]. Schur's theorem is interesting from a combinatorial perspective, but his motivation was a tool for proving that the congruence

$$x^m + y^m \equiv z^m \pmod{p}$$

contains a nontrivial solution when p is a sufficiently large prime (specifically, $p > S(n)$). This result had been originally proved by Dickson [4] in 1908 in his attempt to prove Fermat's Last Theorem.

In this paper, we adapt some common generalizations of Ramsey numbers that involve rainbow colorings to Schur numbers. In Section 2, we consider the minimum number of colors such that every coloring of $\{1, 2, \ldots, n\}$, using all of the colors, contains a rainbow Schur solution. This leads us to the definition of the rainbow Schur number $RS(n)$, which is a Schur number analogue of rainbow numbers (closely related to anti-Ramsey numbers). The number $RS(n)$ is similar in definition to the number $ss(k)$ defined in [5], but does not restrict the number of times each color can be used. In Section 3, we restrict ourselves to colorings of $\{1, 2, \ldots, k\}$ that lack rainbow Schur solutions: a, b, and c with distinct colors such that $a + b = c$. Limiting the colorings in this way leads to the definition of the Gallai-Schur number $GS(n)$. We provide exact evaluations of both $RS(n)$ and $GS(n)$ and offer some related open questions for future inquiry.

2 Rainbow Schur numbers

In this section, we consider Schur number analogues of rainbow numbers and anti-Ramsey numbers (c.f., Chapter 11, Section 4 of [2]). For $n \geq 3$, define the *rainbow Schur number* $RS(n)$ to be the minimum number of colors such that every coloring of $\{1, 2, \ldots, n\}$, using all $RS(n)$ colors, contains a *rainbow Schur solution*: a, b, and c all distinct colors such that $a + b = c$. Observe that $a + b = c$ is never a rainbow Schur solution when $a = b$. As with the case of graphs, the rainbow Schur number is closely related to the *anti-Schur number* $AS(n)$, defined to be the maximum number of colors that can be used to color $\{1, 2, \ldots, n\}$ so that no rainbow Schur solution exists. From these definitions, it follows that

$$RS(n) = AS(n) + 1, \quad \text{for all } n \geq 3.$$

Since determining the values of these two numbers is equivalent, we will focus on $RS(n)$ for the remainder of this section, beginning with a few small values of n.

Observe that at least three colors are needed to have a rainbow triangle. Using all three colors to color $\{1, 2, 3\}$, we find that $1 + 2 = 3$ is rainbow. Thus,

$$RS(3) = 3.$$

Next, consider the following 3-coloring of

$$\{1, 2, 3, 4\}.$$

It is easily checked that no rainbow Schur solutions exist, implying that $RS(4) > 3$. Of course, 4-coloring $\{1,2,3,4\}$ produces a rainbow Schur solution, implying that

$$RS(4) = 4.$$

The following 3-coloring does not contain any rainbow Schur solutions:

$$\{1,2,3,4,5\}.$$

Thus, $RS(5) > 3$. Now consider a 4-coloring of $\{1,2,3,4,5\}$. If 5 is assigned the same color as some $i < 5$, then the coloring induces a 4-coloring of $\{1,2,3,4\}$, which necessarily contains a rainbow Schur solution. Otherwise, the color assigned to 5 is not assigned to any other number. In order to avoid a rainbow Schur solution, 1 and 4 receive the same color, as do 2 and 3. Since all three remaining colors must be used, either $1+4=5$ or $2+3=5$ must be rainbow. Hence,

$$RS(5) = 4.$$

As a crude general bound, note that giving unique colors to the numbers in $\{1,2,\ldots,n\}$ necessarily produces a rainbow Schur solution when $n \geq 3$. Thus,

$$RS(n) \leq n,$$

proving that $RS(n)$ exists for all $n \geq 3$. Suppose that every k-coloring of $\{1,2,\ldots,n\}$ contains a rainbow Schur solution, then every $(k+1)$-coloring of $\{1,2,\ldots,n+1\}$ also contains a rainbow Schur solution. It follows that

$$RS(n+1) \leq RS(n) + 1.$$

If there exists a k-coloring of $\{1,2,\ldots,n+1\}$ that lacks a rainbow Schur solution, then it induces such a coloring on $\{1,2,\ldots,n\}$. Hence,

$$RS(n) \leq RS(n+1), \quad \text{for all } n \geq 3.$$

The following lemma will allow us to show that equality holds for most values of n.

Lemma 2.1. *Let $n \geq 6$ and suppose that $RS(n-1) = k$ and $RS\left(\lfloor \frac{n}{2} \rfloor\right) \leq k-1$. Then $RS(n) = k$.*

Proof. Suppose that $RS(n-1) = k$ and $RS\left(\lfloor \frac{n}{2} \rfloor\right) \leq k-1$ and consider a k-coloring of $\{1,2,\ldots,n\}$. If the color assigned to n is shared with some $i < n$, then this coloring induces a k-coloring of $\{1,2,\ldots,n-1\}$, which necessarily contains a rainbow Schur solution. So, assume that n is assigned a unique color. If n is even, and a rainbow Schur solution is avoided, then numbers in each of the sets

$$\{1,n-1\},\{2,n-2\},\ldots,\left\{\frac{n}{2}-1,\frac{n}{2}+1\right\},\left\{\frac{n}{2}\right\}$$

are colored according to the set they are in. That is, 1 and $n-1$ receive the same color, 2 and $n-2$ receive the same color, etc. If n is odd, and a rainbow Schur solution is avoided, then numbers in each of the sets

$$\{1,n-1\},\{2,n-2\},\ldots,\left\{\frac{n-1}{2},\frac{n+1}{2}\right\}$$

are colored according to which set they are in. In both cases, we are reduced to considering a $(k-1)$-coloring of $\left\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\right\}$, which contains a rainbow Schur solution. □

Observe that the colorings that have given us lower bounds for $RS(4)$ and $RS(5)$ have both had the odd numbers grouped into a single color class (red). This leads us to the following lemma.

Lemma 2.2. *For all* $k \geq 2$, $RS(2^k) > k + 1$.

Proof. Define the map $\vartheta_2 \colon \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$ by

$$\vartheta_2(a) = \ell \quad \Longleftrightarrow \quad 2^\ell \mid a \text{ and } 2^{\ell+1} \nmid a.$$

Color the elements of $\{1, 2, \ldots, 2^k\}$ according to their images $\vartheta_2(a) \in \{0, 1, \ldots, k\}$. It can now be confirmed that this $(k + 1)$-coloring does not contain any rainbow Schur solutions. Certainly any Schur solution $a + b = c$ in which $\vartheta_2(a) = \vartheta_2(b)$ is not rainbow colored. Now, consider the case in which $\vartheta_2(a) = \ell < k = \vartheta_2(b)$. Then we can write

$$a + b = 2^\ell(e + f), \quad \text{where } e \text{ is odd and } f \text{ is even.}$$

So, $\vartheta_2(a + b) = \ell$ and we see that such a Schur solution is not rainbow colored. We have produced a $(k + 1)$-coloring of $\{1, 2, \ldots, 2^k\}$ that does not contain any rainbow Schur solutions. It follows that $RS(2^k)$ is greater than $k + 1$. □

Theorem 2.3. *For all* $n \geq 3$, $RS(n) = \lfloor \log_2(n) \rfloor + 2$.

Proof. Proving this theorem is equivalent to proving that if $2^k \leq n \leq 2^{k+1} - 1$, then $RS(n) = k + 2$ for all $n \geq 3$. We have already shown this result to be true for $3 \leq n \leq 5$. We proceed by strong induction on n. Suppose that the theorem is true for all n such that $3 \leq n \leq m$, for some $m \geq 6$ and consider the rainbow Schur number $RS(m + 1)$. There are two cases to consider.

Case 1: If $m + 1$ is not a power of 2, then we can write

$$2^k + 1 \leq m + 1 \leq 2^{k+1} - 1,$$

for some k. It follows that $m \leq 2^{k+1} - 2$ and the inductive hypothesis implies that

$$RS(m) = \lfloor \log_2(m) \rfloor + 2 \quad \text{and} \quad RS\left(\left\lfloor \frac{m+1}{2} \right\rfloor\right) = \left\lfloor \log_2\left(\left\lfloor \frac{m+1}{2} \right\rfloor\right)\right\rfloor + 2$$
$$= \Big\lfloor \log_2(m) \Big\rfloor + 1.$$

Hence, $RS(m + 1) = \lfloor \log_2(m) \rfloor + 2$ by Lemma 2.1.

Case 2: If $m + 1 = 2^k$ for some $k > 2$, then $RS(m) = k + 1$ by the inductive hypothesis. By Lemma 2.2, $RS(m + 1) > k + 1$. Consider a $(k + 2)$-coloring of $\{1, 2, \ldots, m + 1\}$. Regardless of the color assigned to $m+1$, at least $k+1$ colors are assigned to $\{1, 2, \ldots, m\}$, which necessarily contains a rainbow Schur solution. Thus, $RS(m + 1) = k + 2$, when $m + 1 = 2^k$. □

3 Gallai-Schur numbers

A *Gallai n-coloring* of $\{1, 2, \ldots, k\}$ is a coloring that lacks rainbow Schur solutions. For every $n \in \mathbb{N}$, define the *Gallai-Schur number* $GS(n)$ to be the least positive integer such that every Gallai n-coloring of $\{1, 2, \ldots, GS(n)\}$ contains a monochromatic Schur solution. It is easily observed that $GS(1) = S(1) = 2$, $GS(2) = S(2) = 5$, and

$$GS(n) \leq S(n), \quad \text{for all } n \geq 3.$$

The Gallai-Schur number $GS(n)$ is closely related to the *Gallai-Ramsey number* $gr^n(3)$, defined to be the minimum number of vertices p needed to guarantee that every rainbow-triangle-free n-coloring of the edges of the complete graph K_p contains a monochromatic triangle. The following theorem makes this relationship explicit.

Theorem 3.1. *For all* $n \geq 3$, $GS(n) \leq gr^n(3) - 1$.

Proof. Let $p = gr^n(3)$ and identify the vertices in K_p with $\{1, 2, \ldots, p\}$. For every pair of distinct vertices $a, b \in \{1, 2, \ldots, p\}$, color edge ab according to the value of $|b - a| \in \{1, 2, \ldots, p-1\}$. If we consider a Gallai n-coloring of K_p, it necessarily contains a monochromatic triangle. Suppose the vertices of such a triangle are given by $a < b < c$. Then setting $x = b - a$, $y = c - b$, and $z = c - a$, it follows that

$$x + y = (b - a) + (c - b) = c - a = z$$

is monochromatic. Also, note that no rainbow Schur solutions exist because if $x + y = z$ is rainbow, then the triangle with vertices 1, $x + 1$, and $x + y + 1$ would be rainbow as well. Thus, every Gallai n-coloring of $\{1, 2, \ldots, p - 1\}$ produces a monochromatic Schur solution:

$$GS(n) \leq gr^n(3) - 1,$$

completing the proof of the theorem. □

In 1983, Chung and Graham (see Theorem 1 of [3]) proved a result equivalent to

$$gr^n(3) = \begin{cases} 5^k + 1 & \text{if } n = 2k \\ 2 \cdot 5^k + 1 & \text{if } n = 2k + 1. \end{cases}$$

Hence, Theorem 3.1 gives

$$GS(n) \leq \begin{cases} 5^k & \text{if } n = 2k \\ 2 \cdot 5^k & \text{if } n = 2k + 1. \end{cases} \tag{3.1}$$

To find a lower bound for $GS(n)$ when $n \geq 3$, we begin with some preliminary examples. It is straight-foward to check that

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

is a Gallai 3-coloring that lacks a monochromatic Schur solution. It follows that $GS(3) > 9$. Combining this inequality with Theorem 3.1, we find that

$$GS(3) = 10.$$

One can also check that

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$$

is a Gallai 4-coloring that lacks a monochromatic Schur solution, which implies $GS(4) > 24$. Combining this inequality with Theorem 3.1, we find that

$$GS(4) = 25.$$

The following theorem offers a general lower bound for $GS(n)$.

Theorem 3.2. *The set* $\{1, 2, \dots, gr^n(3) - 2\}$ *can be Gallai* n*-colored without producing a monochromatic Schur solution.*

Proof. Similar to the proof of Lemma 2.2, define the map $\vartheta_5 \colon \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$ by

$$\vartheta_5(a) = \ell \quad \Longleftrightarrow \quad 5^\ell \mid a \text{ and } 5^{\ell+1} \nmid a.$$

First, we consider the case in which $n = 2k$, where $n \geq 4$. We will construct a Gallai n-coloring of $S = \{1, 2, \dots, 5^k - 1\}$ that lacks a monochromatic Schur solution. We start by partitioning S according to the images of elements under the map ϑ_5. This gives us the following k sets:

$$S_\ell = \{a \mid \vartheta_5(a) = \ell\}, \quad \text{where } \ell = 0, 1, \dots, k-1.$$

Each S_ℓ is then partitioned into two distinct color classes:

$$\begin{aligned} S_\ell^+ &= \left\{ a \;\middle|\; \vartheta_5(a) = \ell \text{ and } \frac{a}{5^\ell} \equiv \pm 1 \pmod 5 \right\}, \\ S_\ell^- &= \left\{ a \;\middle|\; \vartheta_5(a) = \ell \text{ and } \frac{a}{5^\ell} \equiv \pm 2 \pmod 5 \right\}. \end{aligned}$$

We have now partitioned S into $n = 2k$ color classes. It remains to be shown that such a coloring lacks both rainbow and monochromatic Schur solutions. We consider several cases for adding $a, b \in S$.

Case 1: Suppose that a and b receive different colors. Then there exist two subcases.

Subcase 1.1: Assume that $\vartheta_5(a) = \vartheta_5(b) = \ell$. Since a and b receive different colors, without loss of generality, it follows that

$$\frac{a}{5^\ell} \equiv \pm 1 \pmod 5 \quad \text{and} \quad \frac{b}{5^\ell} \equiv \pm 2 \pmod 5.$$

It follows that $\vartheta_5(a + b) = \ell$, and hence, either a or b receives the same color as $a + b$. So, this subcase does not produce a rainbow or monochromatic Schur solution.

Subcase 1.2: Without loss of generality, assume that $\vartheta_5(a) = \ell_1 < \ell_2 = \vartheta_5(b)$. Then $\vartheta_5(a + b) = \ell_1$ and

$$\frac{a+b}{5^{\ell_1}} \equiv \frac{a}{5^{\ell_1}} + \frac{b}{5^{\ell_2}} \cdot 5^{\ell_2 - \ell_1} \equiv \frac{a}{5^{\ell_1}} \pmod 5.$$

In this subcase, a and $a + b$ receive the same color, avoiding both a rainbow and monochromatic Schur solution.

Case 2: Suppose that a and b receive the same color. Then $\vartheta_5(a) = \vartheta_5(b) = \ell$ and either

$$\frac{a}{5^\ell} \equiv \pm 1 \pmod 5 \qquad \text{or} \qquad \frac{b}{5^\ell} \equiv \pm 2 \pmod 5.$$

Once again, we consider two subcases.

Subcase 2.1: If $\vartheta_5(a+b) > \vartheta_5(a) = \vartheta_5(b)$, then $a+b$ necessarily receives a color different than that of a and b.

Subcase 2.2: Suppose that $\vartheta_5(a+b) = \vartheta_5(a) = \vartheta_5(b) = \ell$. If $\frac{a}{5^\ell} \equiv \frac{b}{5^\ell} \equiv \pm 1 \pmod 5$, then $\frac{a+b}{5^\ell} \equiv \pm 2 \pmod 5$ and if $\frac{a}{5^\ell} \equiv \frac{b}{5^\ell} \equiv \pm 2 \pmod 5$, then $\frac{a+b}{5^\ell} \equiv \pm 1 \pmod 5$.

In all cases, we find that a, b, and $a+b$ never form a rainbow or monochromatic Schur solution. The same construction also provides a Gallai n-coloring of

$$S' = \{1, 2, \dots, 2 \cdot 5^k - 1\}$$

when $n = 2k+1$, and we leave the details to the reader. □

Putting together the results of Theorems 3.1 and 3.2, we find that

$$GS(n) = gr^n(3) - 1 = \begin{cases} 5^k & \text{if } n = 2k \\ 2 \cdot 5^k & \text{if } n = 2k+1. \end{cases}$$

4 Conclusion

We have shown how extremal results from graph theory can be used to prove related number theoretic results. Although we have succeeded in providing exact evaluations of $GS(n)$ and $RS(n)$, the generalizations considered here lead to analogous constructions involving weak Schur numbers (see [8]) and generalized Schur numbers (see [1]). Such work is reserved for future inquiry.

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Complete regular dessins and skew-morphisms of cyclic groups*

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Abstract

A dessin is a 2-cell embedding of a connected 2-coloured bipartite graph into an orientable closed surface. A dessin is regular if its group of orientation- and colour-preserving automorphisms acts regularly on the edges. In this paper we study regular dessins whose underlying graph is a complete bipartite graph $K_{m,n}$, called (m,n)-complete regular dessins. The purpose is to establish a rather surprising correspondence between (m,n)-complete regular dessins and pairs of skew-morphisms of cyclic groups. A skew-morphism

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of a finite group A is a bijection $\varphi\colon A \to A$ that satisfies the identity $\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$ for some function $\pi\colon A \to \mathbb{Z}$ and fixes the neutral element of A. We show that every (m, n)-complete regular dessin $\mathcal{D}$ determines a pair of reciprocal skew-morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Conversely, $\mathcal{D}$ can be reconstructed from such a reciprocal pair. As a consequence, we prove that complete regular dessins, exact bicyclic groups with a distinguished pair of generators, and pairs of reciprocal skew-morphisms of cyclic groups are all in a one-to-one correspondence. Finally, we apply the main result to determining all pairs of integers m and n for which there exists, up to interchange of colours, exactly one isomorphism class of (m, n)-complete regular dessins. We show that the latter occurs precisely when every group expressible as a product of cyclic groups of order m and n is abelian, which eventually comes down to the condition $\gcd(m, \phi(n)) = \gcd(\phi(m), n) = 1$, where ϕ is Euler's totient function.

Keywords: Regular dessin, bicyclic group, skew-morphism, graph embedding.

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1 Introduction

A *dessin* is a cellular embedding $i\colon \Gamma \hookrightarrow \mathcal{C}$ of a connected bipartite graph Γ, endowed with a fixed proper 2-colouring of its vertices, into an orientable closed surface $\mathcal{C}$ such that each component of $\mathcal{C} \setminus i(\Gamma)$ is homeomorphic to the open disc. An automorphism of a dessin is a colour-preserving automorphism of the underlying graph that extends to an orientation-preserving self-homeomorphism of the supporting surface. The action of the automorphism group of a dessin on the edges is well known to be semi-regular; if this action is transitive, and hence regular, the dessin itself is called *regular*.

Dessins – more precisely *dessins d'enfants* – were introduced by Grothendieck in [42] as a combinatorial counterpart of algebraic curves. Grothendieck was inspired by a theorem of Belyĭ [3] which states that a compact Riemann surface $\mathcal{C}$, regarded as a projective algebraic curve, can be defined by an algebraic equation $P(x, y) = 0$ with coefficients from the algebraic number field $\bar{\mathbb{Q}}$ if and only if there exists a non-constant meromorphic function $\beta\colon \mathcal{C} \to \mathbb{P}^1(\mathbb{C})$, branched over at most three points, which can be chosen to be 0, 1, and ∞. It follows that each such curve carries a dessin in which the black and the white vertices are the preimages of 0 and 1, respectively, and the edges are the preimages of the unit interval $I = [0, 1]$. The absolute Galois group $\mathbb{G} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ has a natural action on the curves and thus also on the dessins. As was shown by Grothendieck [42], the action of $\mathbb{G}$ on dessins is faithful. More recently, González-Diez and Jaikin-Zapirain [13] have proved that this action remains faithful even when restricted to regular dessins. It follows that one can study the absolute Galois group through its action on such simple and symmetrical combinatorial objects as regular dessins.

In this paper we study regular dessins whose underlying graph is a complete bipartite graph $K_{m,n}$, which we call *complete regular dessins*, or more specifically (m, n)-*complete regular dessins*. The associated algebraic curves may be viewed as a generalisation of the Fermat curves, defined by the equation $x^n + y^n = 1$ (see Lang [38]). These curves have recently attracted considerable attention, see for example [7, 24, 25, 27, 28]. Classification

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of complete regular dessins is therefore a very natural problem, interesting from algebraic, combinatorial, and geometric points of view.

Jones, Nedela, and Škoviera [23] were first to observe that there is a correspondence between complete regular dessins and exact bicyclic groups. Recall that a finite group G is *bicyclic* if it can be expressed as a product $G = AB$ of two cyclic subgroups A and B; if the two subgroups are *disjoint*, that is, if $A \cap B = \{1\}$, the bicyclic group is called *exact*. Exact bicyclic groups are, in turn, closely related to skew-morphisms of the cyclic groups.

A *skew-morphism* of a finite group A is a bijection $\varphi\colon A \to A$ fixing the identity element of A and obeying the morphism-type rule $\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$ for some integer function $\pi\colon A \to \mathbb{Z}$. In the case where π is the constant function $\pi(x) = 1$, a skew-morphism is just an automorphism. Thus, skew-morphisms may be viewed as a generalisation of group automorphisms. The concept of skew-morphism was introduced by Jajcay and Širáň as an algebraic tool to the investigation of (orientably) regular Cayley maps. In the seminal paper [20] they proved that a Cayley map $\mathrm{CM}(A, X, P)$ of a finite group A is regular if and only if there is a skew-morphism of A such that the restriction of φ to X is equal to P [20, Theorem 1]. Thus the classification problem of regular Cayley maps of a finite group A is reduced to a problem of determining certain skew-morphisms of A. The interested reader is referred to [5, 6, 29, 30, 31, 34, 35, 36, 46, 47] for progress in this direction.

The main purpose of this paper is to establish another surprising connection between skew-morphisms and complete regular dessins. As we have already mentioned above, every (m, n)-complete regular dessin can be represented as an exact bicyclic group factorisation $G = \langle a\rangle\langle b\rangle$ with two distinguished generators a and b of orders m and n, respectively (see [23]). The factorisation gives rise to a pair of closely related skew-morphisms of cyclic groups $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$ which satisfy two simple technical conditions (see Definition 3.2); such a pair of skew-morphisms will be called *reciprocal*. We prove that isomorphic complete regular dessins give rise to the *same* pair of reciprocal skew-morphisms, which is a rather remarkable fact, because every complete regular dessin thus receives a natural algebraic invariant.

Even more surprising is the fact that given a pair of reciprocal skew-morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$, one can reconstruct the original complete regular dessin up to isomorphism. In other words, a pair of reciprocal skew-morphisms of the cyclic groups constitutes a complete set of invariants for a regular dessin whose underlying graph is the complete bipartite graph. One can therefore study and classify complete regular dessins by means of determining pairs of reciprocal skew-morphisms of cyclic groups. Note that the classification of skew-morphisms of the cyclic groups is a prominent open problem, see [1, 2, 5, 6, 32, 33] for partial results.

The relationship between complete regular dessins and exact bicyclic groups has an important implication for the classical classification problem of bicyclic groups in group theory (see [8, 16, 18, 21]). More precisely, suppose that we are given an exact product $G = AB$ of two cyclic groups A and B with distinguished generators $a \in A$ and $b \in B$. The corresponding pair of reciprocal skew-morphisms (φ, φ^*) and associated pair of power functions (π, π^*) can be alternatively derived from the equations

$$ba^x = a^{\varphi(x)}b^{\pi(x)} \quad \text{and} \quad ab^y = b^{\varphi^*(y)}a^{\pi^*(y)},$$

and thus encodes the commuting rules within G. By our main result, determining all exact bicyclic groups with a distinguished generator pair is equivalent to determining all pairs of reciprocal skew-morphisms. Thus to describe all exact bicyclic groups it is sufficient to characterise all pairs of reciprocal skew-morphisms of the cyclic groups.

Our paper is organised as follows. In Section 2 we describe the basic correspondence between complete regular dessins and *exact bicyclic triples* $(G; a, b)$, where G is a group which factorises as $G = \langle a\rangle\langle b\rangle$ with $\langle a\rangle \cap \langle b\rangle = \{1\}$. Given a complete regular dessin $\mathcal{D}$, its automorphism group $G = \mathrm{Aut}(\mathcal{D})$ can be factorised as a product of two disjoint cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$ where $\langle a\rangle$ is the stabiliser of one black vertex and $\langle b\rangle$ is the stabiliser of one white vertex. The triple $(G; a, b)$ is then an exact bicyclic triple. Conversely, each exact bicyclic triple $(G; a, b)$ determines a complete regular dessin where the elements of G are the edges, the cosets of $\langle a\rangle$ are black vertices, the cosets of $\langle b\rangle$ are white vertices, and the local rotations at black and white vertices, respectively, correspond to the multiplication by a and b.

In Section 3 we introduce the concept of a reciprocal skew-morphism and prove the main result, Theorem 3.5, which establishes the aforementioned correspondence between complete regular dessins and pairs of reciprocal skew-morphisms of cyclic groups.

An important part of the classification of complete regular dessins is identifying all pairs of integers m and n for which there exists a unique complete regular dessin up to isomorphism and interchange of colours. This problem will be discussed in Section 4. In view of the correspondence between complete regular dessins and pairs of reciprocal skew-morphisms of cyclic groups, we ask for which integers m and n the only reciprocal pair of skew-morphisms is the trivial pair formed by the two identity automorphisms. In other words, we wish to determine all pairs of positive integers m and n that give rise to only one exact product of cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_m$, which necessarily must be the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$. The answer is given in Theorem 4.4 which states that all this occurs precisely when $\gcd(m, \phi(n)) = \gcd(\phi(m), n) = 1$, where ϕ is the Euler's totient function. This theorem presents six equivalent conditions one of which corresponds to a recent result of Fan and Li [12] about the existence of a unique edge-transitive orientable embedding of a complete bipartite graph. While the proof in [12] is based on the structure of exact bicyclic groups, our proof employs the correspondence theorems established in Section 3.

Theorem 4.4 is a direct generalisation of a result of Jones, Nedela, and Škoviera [23] where it is assumed that the complete dessin in question admits an external symmetry swapping the two partition sets. Theorem 4.4 also strengthens the main result of [12] by extending it to all products of cyclic groups rather than just to those where the intersection of factors is trivial. In particular, we prove that every group that factorises as a product of two cyclic subgroups of orders m and n is abelian if and only if $\gcd(m, \phi(n)) = \gcd(\phi(m), n) = 1$, where ϕ is Euler's totient function. This generalises an old result due to Burnside which states that every group of order n is cyclic if and only if $\gcd(n, \phi(n)) = 1$, see [41, §10.1].

Finally, in Section 5 we deal with the symmetric case, that is, with the case where the reciprocal skew-morphism pairs have the form (φ, φ). In this situation, the corresponding complete regular dessins admit an additional external symmetry transposing the two partition sets, and thus are essentially the same thing as orientably regular embeddings of the complete bipartite graphs $K_{n,n}$ recently classified in a series of papers [9, 10, 11, 23, 25, 26, 40].

2 Complete regular dessins

It is well known that every dessin, as defined in the previous section, can be regarded as a two-generator transitive permutation group acting on a non-empty finite set [24]. Given a dessin $\mathcal{D}$ on an oriented surface $\mathcal{C}$, we can define two permutations ρ and λ on the edge set of $\mathcal{D}$ as follows: For every black vertex v and every white vertex w let ρ_v and λ_w be the cyclic permutations of edges incident with v or w, respectively, induced by the orientation of $\mathcal{C}$. Set $\rho = \prod_v \rho_v$ and $\lambda = \prod_w \lambda_w$, where v and w run through the set of all black and white vertices, respectively. Since the underlying graph of $\mathcal{D}$ is connected, the group $G = \langle \rho, \lambda \rangle$ is transitive. Conversely, given a transitive permutation group $G = \langle \rho, \lambda \rangle$ acting on a finite set Ω, we can reconstruct a dessin $\mathcal{D}$ as follows: Take Ω to be the edge set of $\mathcal{D}$, the orbits of ρ to be the black vertices, and the orbits of λ to be white vertices, with incidence being defined by containment. The vertices and edges of $\mathcal{D}$ clearly form a bipartite graph Γ, the *underlying graph* of $\mathcal{D}$. The underlying graph is connected, because the action of G on Ω is transitive. The cycles of ρ and λ determine the local rotations around black and white vertices, respectively, thereby giving rise to a 2-cell embedding of Γ into an oriented surface. Summing up, we can identify a dessin with a triple $(\Omega; \rho, \lambda)$ where Ω is a nonempty finite set, and ρ and λ are permutations of Ω such that the group $\langle \rho, \lambda \rangle$ is transitive on Ω; this group is called the *monodromy group* of $\mathcal{D}$ and is denoted by $\mathrm{Mon}(\mathcal{D})$.

Two dessins $\mathcal{D}_1 = (\Omega_1; \rho_1, \lambda_1)$ and $\mathcal{D}_2 = (\Omega_2; \rho_2, \lambda_2)$ are *isomorphic* provided that there is a bijection $\alpha\colon \Omega_1 \to \Omega_2$ such that $\alpha\rho_1 = \rho_2\alpha$ and $\alpha\lambda_1 = \lambda_2\alpha$. An isomorphism of a dessin $\mathcal{D}$ to itself is an *automorphism* of $\mathcal{D}$. It follows that the automorphism group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the centraliser of $\mathrm{Mon}(\mathcal{D}) = \langle \rho, \lambda \rangle$ in the symmetric group $\mathrm{Sym}(\Omega)$. As $\mathrm{Mon}(\mathcal{D})$ is transitive, $\mathrm{Aut}(\mathcal{D})$ is semi-regular on Ω. If $\mathrm{Aut}(\mathcal{D})$ is transitive, and hence regular on Ω, the dessin $\mathcal{D}$ itself is called *regular*.

Since every regular action of a group on a set is equivalent to its action on itself by multiplication, every regular dessin can be identified with a triple $\mathcal{D} = (G; a, b)$ where G is a finite group generated by two elements a and b. Given such a triple $\mathcal{D} = (G; a, b)$, we can define the edges of $\mathcal{D}$ to be the elements of G, the black vertices to be the left cosets of the cyclic subgroup $\langle a \rangle$, and the white vertices to be the left cosets of the cyclic subgroup $\langle b \rangle$. An edge $g \in G$ joins the vertices $s\langle a \rangle$ and $t\langle b \rangle$ if and only if $g \in s\langle a \rangle \cap t\langle b \rangle$. In particular, the underlying graph is simple if and only if $\langle a \rangle \cap \langle b \rangle = \{1\}$. The local rotation of edges around a black vertex $s\langle a \rangle$ corresponds to the right translation by the generator a, that is, $sa^i \mapsto sa^{i+1}$ for any integer i. Similarly, the local rotation of edges around a white vertex $t\langle b \rangle$ corresponds to the right translation by the generator b, that is, $tb^i \mapsto tb^{i+1}$ for any integer i. It follows that $\mathrm{Mon}(\mathcal{D})$ can be identified with the group of all right translations of G by the elements of G while $\mathrm{Aut}(\mathcal{D})$ can be identified with the group of all left translations of G by the elements of G. In particular, $\mathrm{Mon}(\mathcal{D}) \cong \mathrm{Aut}(\mathcal{D}) \cong G$ for every regular dessin $\mathcal{D}$.

It is easy to see that two regular dessins $\mathcal{D}_1 = (G_1; a_1, b_1)$ and $\mathcal{D}_2 = (G_2; a_2, b_2)$ are isomorphic if and only if the triples $(G_1; a_1, b_1)$ and $(G_2; a_2, b_2)$ are *equivalent*, that is, whenever there is a group isomorphism $G_1 \to G_2$ such that $a_1 \mapsto a_2$ and $b_1 \mapsto b_2$. Consequently, for a given two-generator group G, the isomorphism classes of regular dessins $\mathcal{D}$ with $\mathrm{Aut}(\mathcal{D}) \cong G$ are in a one-to-one correspondence with the orbits of the action of $\mathrm{Aut}(G)$ on the generating pairs (a, b) of G.

Following Lando and Zvonkin [37], for a regular dessin $\mathcal{D} = (G; a, b)$ we define its *reciprocal dessin* to be the regular dessin $\mathcal{D}^* = (G; b, a)$. Topologically, $\mathcal{D}^*$ arises from $\mathcal{D}$

simply by interchanging the vertex colours of $\mathcal{D}$. Thus the reciprocal dessin has the same underlying graph, the same supporting surface, and the same automorphism group as the original one. Clearly, $\mathcal{D}^*$ is isomorphic to $\mathcal{D}$ if and only if G has an automorphism swapping the generators a and b. If this occurs, the regular dessin $\mathcal{D}$ will be called *symmetric*. A symmetric dessin possesses an external symmetry which transposes the vertex-colours and thus is essentially the same thing as an orientably regular bipartite map.

In this paper we apply the general theory to regular dessins whose underlying graph is a complete bipartite graph. A regular dessin $\mathcal{D}$ will be called an (m, n)*-complete regular dessin*, or simply a *complete regular dessin*, if its underlying graph is the complete bipartite graph $K_{m,n}$ whose m-valent vertices are coloured black and n-valent vertices are coloured white. If $\mathcal{D}$ is an (m, n)-complete regular dessin, then the reciprocal dessin $\mathcal{D}^*$ is an (n, m)-complete regular dessin. Thus all complete regular dessins appear in reciprocal pairs. Note that $m = n$ does not necessarily imply that the dessin is symmetric.

Complete regular dessins can be easily described in group theoretical terms: their automorphism group is just an exact bicyclic group. This fact was first observed by Jones et al. in [23]. A bicyclic group $G = \langle a\rangle\langle b\rangle$ with $|a| = m$ and $|b| = n$ will be called an (m, n)-*bicyclic group* and $(G; a, b)$ an (m, n)*-bicyclic triple*. Note that an exact (m, n)-bicyclic group has precisely mn elements.

The following statement was proved by Jones, Nedela, and Škoviera in [23, Section 2] under the condition that $m = n$. However, the same arguments can be used to prove it for any m and n, so we state it without proof.

Theorem 2.1. *A regular dessin $\mathcal{D} = (G; a, b)$ is complete if and only if $G = \langle a\rangle\langle b\rangle$ is an exact bicyclic group. Furthermore, the isomorphism classes of (m, n)-complete regular dessins are in a one-to-one correspondence with the equivalence classes of exact (m, n)-bicyclic triples.*

Example 2.2. For each pair of positive integers m and n there is an exact bicyclic triple $(G; a, b)$ where

$$G = \langle a, b \mid a^m = b^n = [a, b] = 1\rangle = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n,$$

with $[a, b]$ denoting the commutator $a^{-1}b^{-1}ab$. It is easy to see that this triple is uniquely determined by the group $\mathbb{Z}_m \times \mathbb{Z}_n$ up to order of generators and equivalence, so up to reciprocality this group gives rise to a unique complete regular dessin with underlying graph $K_{m,n}$. We call this dessin the *standard* (m, n)-complete dessin. If $m = n$, the group G has an automorphism transposing a and b, which implies that in this case the dessin is symmetric. The corresponding embedding is the *standard embedding* of $K_{n,n}$ described in [23, Example 1]. The associated algebraic curves coincide with the Fermat curves.

3 Reciprocal skew-morphisms

In this section we establish a correspondence between exact bicyclic triples and certain pairs of skew-morphisms of cyclic groups.

Recall that a skew-morphism φ of a finite group A is a bijection $A \to A$ fixing the identity of A for which there exists an associated power function $\pi\colon A \to \mathbb{Z}$ such that

$$\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$$

for all $x, y \in A$. It may be useful to realise that π is not uniquely determined by φ. However, if φ has order d, then π can be regarded as a function $A \to \mathbb{Z}_d$, which is unique. In the special case where $\pi(x) = 1$ for all $x \in A$, φ is a group automorphism. In general, the set $\{x \in A; \pi(x) = 1\}$ is a subgroup of A, called the *kernel* of φ and denoted by $\ker \varphi$.

Skew-morphisms have a number of important properties, sometimes very different from those of group automorphisms. In our treatment we restrict ourselves to a few basic properties of skew-morphisms needed in this paper. For a more detailed account we refer the reader to [5, 20, 32, 45, 48].

The next three properties of skew-morphisms are well known and were proved in [20, Lemma 2], [19, Lemma 2.1], and [45, Lemma 2.6], respectively.

Lemma 3.1. *Let φ be a skew-morphism of a finite group A with associated power function π. Let d be the order of φ. Then:*

(i) *for any two elements $x, y \in A$ and an arbitrary positive integer k one has*

$$\varphi^k(xy) = \varphi^k(x)\varphi^{\sigma(x,k)}(y) \quad \text{where} \quad \sigma(x, k) = \sum_{i=1}^{k} \pi(\varphi^{i-1}(x));$$

(ii) *for every element $x \in A$ one has $\mathcal{O}_{x^{-1}} = \mathcal{O}_x^{-1}$, where $\mathcal{O}_x$ denotes the orbit of φ containing x;*

(iii) *for every $x \in A$ one has $\sigma(x, d) \equiv 0 \pmod{d}$.*

Let G be a finite group which is expressible as a product AC of two subgroups A and C where C is cyclic and $A \cap C = \{1\}$; in this situation we say that C is a *cyclic complement* of A. Choose a generator c of C. Since $G = AC = CA$, for every element $x \in A$ we can write the product cx in the form yc^k, so

$$cx = yc^k$$

for some $y \in A$ and $k \in \mathbb{Z}_{|c|}$. Note that both $y \in A$ and $k \in \mathbb{Z}_{|c|}$ are uniquely determined by x. Thus we can define functions $\varphi_c\colon A \to A$ and $\pi_c\colon A \to \mathbb{Z}_{|c|}$ by setting

$$\varphi_c(x) = y \quad \text{and} \quad \pi_c(x) = k. \tag{3.1}$$

It is not difficult to verify that φ_c is a skew-morphism of A and π_c is an associated power function (see [4, p. 262] or [5, p. 73]). We call φ_c the skew-morphism *induced* by c. The order $|\varphi_c|$ of this skew-morphism equals the index $|\langle c\rangle : \langle c\rangle_G|$ where $\langle c\rangle_G = \cap_{g\in G}\langle c\rangle^g$; see [5, Lemma 4.1]. It follows that the power function π_c can be further reduced to a function $A \to \mathbb{Z}_{|\varphi_c|}$, still denoted by π_c.

We now focus on the particular case $G = AB$ where both A and B are cyclic and $A \cap B = \{1\}$, which means that G is an exact bicyclic group. The subgroups A and B can now be taken as cyclic complements of each other. Therefore a generator a of A induces a skew-morphism of B and a generator b of B induces a skew-morphism of A. In other words, every exact bicyclic triple $(G; a, b)$ gives rise to a pair of skew-morphisms, one for each of the two cyclic subgroups.

Next we show that this pair of skew-morphisms can be characterised by two simple properties. For this purpose, we need the following definition. We switch to the additive notation.

Definition 3.2. A pair (φ, φ^*) of skew-morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$ with power functions π and π^*, respectively, will be called (m,n)-*reciprocal* if the following two conditions are satisfied:

(i) $|\varphi|$ divides m and $|\varphi^*|$ divides n,

(ii) $\pi(x) = -\varphi^{*-x}(-1)$ and $\pi^*(y) = -\varphi^{-y}(-1)$ are power functions for φ and φ^*, respectively.

If $m = n$ and (φ, φ^*) is an (n,n)-reciprocal pair of skew-morphisms, it may, but need not, happen that $\varphi = \varphi^*$. If it does, then the pair (φ, φ), as well as the skew-morphism φ itself, will be called *symmetric*. Note that a skew-morphism φ of $\mathbb{Z}_n$ is symmetric if and only if $|\varphi|$ divides n and $\pi(x) = -\varphi^{-x}(-1)$ is a power function of φ.

Proposition 3.3. *If $(G; a, b)$ is an exact (m,n)-bicyclic triple with $\langle a\rangle \cong \mathbb{Z}_m$ and $\langle b\rangle \cong \mathbb{Z}_n$, then the pair of induced skew-morphisms (φ_a, φ_b) is an (m,n)-reciprocal pair of skew-morphisms. If, in addition, G has an automorphism transposing a and b, then $\varphi_a = \varphi_b$ and the pair is symmetric.*

Proof. Let $\varphi = \varphi_a$ and $\varphi^* = \varphi_b$ be the skew-morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ determined by the identities

$$ab^x = b^{\varphi(x)}a^{\pi(x)} \quad \text{and} \quad ba^y = a^{\varphi^*(y)}b^{\pi^*(y)} \tag{3.2}$$

where $\pi = \pi_a$ and $\pi^* = \pi_b$ are the power functions associated with φ and φ^*, respectively, and the elements $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$ are arbitrary. As mentioned above, the orders of φ and φ^* coincide with the indices $|\langle a\rangle : \bigcap_{g\in G}\langle a\rangle^g|$ and $|\langle b\rangle : \bigcap_{g\in G}\langle b\rangle^g|$ [5, Lemma 4.1]. Hence $|\varphi|$ divides $|\langle a\rangle| = m$ and $|\varphi^*|$ divides $|\langle b\rangle| = n$.

By applying induction to the equations (3.2) we get

$$a^k b^x = b^{\varphi^k(x)}a^{\sigma(x,k)} \quad \text{and} \quad b^l a^y = a^{\varphi^{*l}(y)}b^{\sigma^*(y,l)},$$

where

$$\sigma(x,k) = \sum_{i=1}^{k} \pi(\varphi^{i-1}(x)) \quad \text{and} \quad \sigma^*(y,l) = \sum_{i=1}^{l} \pi^*(\varphi^{*i-1}(y)).$$

By inverting these identities we obtain

$$b^{-x}a^{-k} = a^{-\sigma(x,k)}b^{-\varphi^k(x)} \quad \text{and} \quad a^{-y}b^{-l} = b^{-\sigma^*(y,l)}a^{-\varphi^{*l}(y)}. \tag{3.3}$$

The first equation of (3.3) with $x = -1$ and $k = -y$ yields $ba^y = a^{-\sigma(-1,-y)}b^{-\varphi^{-y}(-1)}$, which we compare with the rule $ba^y = a^{\varphi^*(y)}b^{\pi^*(y)}$ and get

$$a^{\varphi^*(y)}b^{\pi^*(y)} = a^{-\sigma(-1,-y)}b^{-\varphi^{-y}(-1)}.$$

Consequently $\pi^*(y) = -\varphi^{-y}(-1)$. Similarly, inserting $y = -1$ and $l = -x$ into the second equation of (3.3) we get $ab^x = b^{-\sigma^*(-1,-x)}a^{-\varphi^{*-x}(-1)}$, and combining this with the rule $ab^x = b^{\varphi(x)}a^{\pi(x)}$ we derive $\pi(x) = -\varphi^{*-x}(-1)$. Hence, the pair (φ, φ^*) is (m,n)-reciprocal.

Finally, if G has an automorphism θ transposing a and b, then clearly $m = n$. By applying θ to the identity $ba^x = a^{\varphi^*(x)}b^{\pi^*(x)}$ we obtain $ab^x = \theta(ba^x) = \theta(a^{\varphi^*(x)}b^{\pi^*(x)}) = b^{\varphi^*(x)}a^{\pi^*(x)}$. If we compare the last identity with the rule $ab^x = b^{\varphi(x)}a^{\pi(x)}$ we obtain $\varphi^* = \varphi$, which means that φ is a symmetric skew-morphism of $\mathbb{Z}_n$, as required. □

We have just shown that every exact (m,n)-bicyclic triple determines an (m,n)-reciprocal pair of skew-morphisms. Our next aim is to show that the converse is also true. Let (φ,φ^*) be an (m,n)-reciprocal pair of skew-morphisms of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ with power functions π and π^*, respectively. For the sake of clarity we relabel the elements of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ by setting

$$\mathbb{Z}_n = \{0,1,\ldots,(n-1)\} \quad \text{and} \quad \mathbb{Z}_m = \{0',1',\ldots,(m-1)'\},$$

so that $\mathbb{Z}_n \cap \mathbb{Z}_m = \emptyset$. Let

$$\rho = (0,1,\ldots,(n-1)) \quad \text{and} \quad \rho^* = (0',1',\ldots,(m-1)')$$

denote the *cyclic shifts* in $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. We now extend the permutations φ, ρ, φ^*, and ρ^* to the set $\mathbb{Z}_n \cup \mathbb{Z}_m$ in a natural way, and define a permutation group acting on the set $\mathbb{Z}_m \cup \mathbb{Z}_n$ by

$$G = \langle a,b\rangle, \quad \text{where} \quad a = \varphi\rho^* \quad \text{and} \quad b = \varphi^*\rho.$$

If we regard $\mathbb{Z}_m \cup \mathbb{Z}_n$ as the vertex set of the complete bipartite graph $K_{m,n}$ with natural bipartition, it becomes obvious that $G \le \mathrm{Aut}(K_{m,n})$. The following result shows that G is in fact isomorphic to the automorphism group of an (m,n)-complete regular dessin.

Proposition 3.4. *Given an (m,n)-reciprocal pair of skew-morphisms (φ,φ^*), the triple $(G;a,b)$, where $a=\varphi\rho^*$ and $b=\varphi^*\rho$ are permutations acting on the disjoint union $\mathbb{Z}_m \cup \mathbb{Z}_m$, is an exact (m,n)-bicyclic triple. Furthermore, for the skew-morphisms induced by a and b in the triple $(G;a,b)$ we have $\varphi_a=\varphi$ and $\varphi_b=\varphi^*$.*

Proof. Let $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$ be an (m,n)-reciprocal pair of skew-morphisms. The definition of reciprocality requires $|\varphi|$ to divide m and $|\varphi^*|$ to divide n. Since $\varphi,\rho \in \mathrm{Sym}(\mathbb{Z}_n)$ and $\varphi^*,\rho^* \in \mathrm{Sym}(\mathbb{Z}_m)$ where $\mathbb{Z}_m \cap \mathbb{Z}_n = \emptyset$, we see that $[\varphi,\rho^*]=1$ and $[\varphi^*,\rho]=1$. It follows that the elements $a=\varphi\rho^*$ and $b=\varphi^*\rho$ have orders $|a|=m$ and $|b|=n$. Further, if $x \in \langle a\rangle \cap \langle b\rangle$, then $a^i = x = b^j$ for some integers i and j, so $(\varphi\rho^*)^i = (\varphi^*\rho)^j$. Thus $\varphi^i{\rho^*}^i = \rho^j{\varphi^*}^j$, and hence $\varphi^i=\rho^j$ and ${\rho^*}^i = {\varphi^*}^j$. Since $\varphi(0)=0$ and ρ is a full cycle, we have $n \mid j$ and $m \mid i$, and hence $x=1$. Therefore $\langle a\rangle \cap \langle b\rangle = \{1\}$.

Next we show that $\langle a\rangle\langle b\rangle$ is a subgroup of G. It is sufficient to verify that $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$. For this purpose we need to show that for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$ there exist numbers $\alpha(x)$, $\beta(x)$, $\alpha^*(y)$ and $\beta^*(y)$ such that the following commuting rules hold:

$$ab^x = b^{\alpha(x)}a^{\beta(x)} \quad \text{and} \quad ba^y = a^{\alpha^*(y)}b^{\beta^*(y)}. \tag{3.4}$$

Substituting $\varphi\rho^*$ and $\varphi^*\rho$ for a and b we see that the equations in (3.4) are equivalent to the following four equations:

$$\varphi\rho^x = \rho^{\alpha(x)}\varphi^{\beta(x)}, \qquad \rho^*{\varphi^*}^x = {\varphi^*}^{\alpha(x)}{\rho^*}^{\beta(x)}; \tag{3.5}$$

$$\varphi^*{\rho^*}^y = {\rho^*}^{\alpha^*(y)}{\varphi^*}^{\beta^*(y)}, \qquad \rho\varphi^y = \varphi^{\alpha^*(y)}\rho^{\beta^*(y)}. \tag{3.6}$$

Since φ and φ^* are skew-morphisms and π and π^* are the associated power functions, for all $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$ we have

$$\varphi\rho^x(i) = \varphi(x+i) = \varphi(x) + \varphi^{\pi(x)}(i) = \rho^{\varphi(x)}\varphi^{\pi(x)}(i);$$

$$\varphi^*{\rho^*}^y(j) = \varphi^*(y+j) = \varphi^*(y) + {\varphi^*}^{\pi^*(y)}(j) = {\rho^*}^{\varphi^*(y)}{\varphi^*}^{\pi^*(y)}(j).$$

These equations imply that the first equations in (3.5) and (3.6) hold if we set $\alpha(x) = \varphi(x)$, $\beta(x) = \pi(x)$, $\alpha^*(y) = \varphi^*(y)$ and $\beta^*(y) = \pi^*(y)$.

Employing induction, from the first equations in (3.5) and (3.6) we derive that

$$\varphi^k \rho^u = \rho^{\alpha^k(u)} \varphi^{\tau(u,k)} \quad \text{and} \quad {\varphi^*}^l {\rho^*}^v = {\rho^*}^{{\alpha^*}^l(v)} {\tau^*}^{\tau^*(v,l)},$$

where

$$\tau(u,k) = \sum_{i=1}^{k} \beta(\alpha^{i-1}(u)) \quad \text{and} \quad \tau^*(v,l) = \sum_{i=1}^{l} \beta^*({\alpha^*}^{i-1}(v)).$$

By inverting the identities we obtain

$$\rho^{-u} \varphi^{-k} = \varphi^{-\tau(u,k)} \rho^{-\alpha^k(u)} \quad \text{and} \quad {\rho^*}^{-v} {\varphi^*}^{-l} = {\varphi^*}^{-\tau^*(v,l)} {\rho^*}^{-{\alpha^*}^l(v)}.$$

In particular,

$$\rho \varphi^y = \varphi^{-\tau(-1,-y)} \rho^{-\alpha^{-y}(-1)} \quad \text{and} \quad \rho^* {\varphi^*}^x = {\varphi^*}^{-\tau^*(-1,-x)} {\rho^*}^{-{\alpha^*}^{-x}(-1)}.$$

Recall that

$$\beta(x) = \pi(x) = -{\varphi^*}^{-x}(-1) = -{\alpha^*}^{-x}(-1)$$

and

$$\beta^*(y) = \pi^*(y) = -\varphi^{-y}(-1) = -\alpha^{-y}(-1).$$

Thus the second equations in (3.5) and (3.6) will hold if

$$\alpha(x) = \varphi(x) \equiv -\tau^*(-1,-x) \pmod{|\varphi^*|}$$

and

$$\alpha^*(y) = \varphi^*(y) \equiv -\tau(-1,-y) \pmod{|\varphi|}.$$

Indeed, by Lemma 3.1(iii) we have $\tau^*(-1,|\varphi^*|) \equiv 0 \pmod{|\varphi^*|}$. Since

$$\begin{aligned}
\tau^*(-1,|\varphi^*|) &= \sum_{i=1}^{|\varphi^*|} \beta^*({\alpha^*}^{i-1}(-1)) = \sum_{i=1}^{|\varphi^*|} \pi^*({\varphi^*}^{i-1}(-1)) \\
&= \sum_{i=1}^{|\varphi^*|-x} \pi^*({\varphi^*}^{i-1}(-1)) + \sum_{i=|\varphi^*|-x+1}^{|\varphi^*|} \pi^*({\varphi^*}^{i-1}(-1)) \\
&= \tau^*(-1,-x) + \sum_{i=1}^{x} \pi^*({\varphi^*}^{-i}(-1)) \pmod{|\varphi^*|},
\end{aligned}$$

we obtain

$$-\sigma^*(-1,-x) \equiv \sum_{i=1}^{x} \pi^*({\varphi^*}^{-i}(-1)) \pmod{|\varphi^*|}.$$

On the other hand, since φ is a skew-morphism of $\mathbb{Z}_m$, we have $\varphi(z-1) = \varphi(z) + \varphi^{\pi(z)}(-1)$ for all $z \in \mathbb{Z}_n$, so $\varphi(z-1) - \varphi(z) = \varphi^{\pi(z)}(-1)$. By combining these identities we obtain

$$\begin{aligned}\varphi(x) &= -\big(\varphi(0)-\varphi(x)\big) = -\sum_{i=1}^{x}(\varphi(i-1)-\varphi(i)) = -\sum_{i=1}^{x}\varphi^{\pi(i)}(-1)\\ &= -\sum_{i=1}^{x}\varphi^{-\varphi^{*-i}(-1)}(-1) = \sum_{i=1}^{x}\pi^*(\varphi^{*-i}(-1)) \equiv -\sigma^*(-1,-x) \pmod{|\varphi^*|}.\end{aligned}$$

Thus we have shown that

$$\varphi(x) \equiv -\sigma^*(-1,-x) \equiv \sum_{i=1}^{x}\pi^*(\varphi^{*-i}(-1)) \pmod{|\varphi^*|}. \tag{3.7}$$

By using similar arguments we can prove that $\alpha^*(y) = \varphi^*(y) \equiv -\sigma(-1,-y) \pmod{|\varphi|}$. Thus, $\langle a\rangle\langle b\rangle$ is a subgroup of G, as claimed.

Finally, since $G = \langle a, b\rangle$, we have $G = \langle a\rangle\langle b\rangle$, so $(G; a, b)$ is an exact (m, n)-bicyclic triple. Note that $ab^x = b^{\alpha(x)}a^{\beta(x)}$ and $ba^y = a^{\alpha^*(y)}b^{\beta^*(y)}$ with $\alpha(x) = \varphi(x)$ and $\alpha^*(y) = \varphi^*(y)$. It follows that φ and φ^* are precisely the skew-morphisms induced by a and b in the triple $(G; a, b)$. □

Putting together Theorem 2.1, Proposition 3.3, and Proposition 3.4 we obtain a one-to-one correspondence between (m, n)-complete regular dessins, exact (m, n)-bicyclic triples, and (m, n)-reciprocal pairs of skew-morphisms.

Theorem 3.5. *For every pair of positive integers m and n there exists a one-to-one correspondence between any two sets of the following three types of objects:*

(i) *isomorphism classes of (m, n)-complete regular dessins,*

(ii) *equivalence classes of exact (m, n)-bicyclic triples, and*

(iii) *(m, n)-reciprocal pairs of skew-morphisms.*

Proof. The correspondence between the isomorphism classes of (m, n)-complete regular dessins and equivalence classes of exact (m, n)-bicyclic triples has been established in Theorem 2.1. It remains to prove that there is a one-to-one correspondence between equivalence classes of exact (m, n)-bicyclic triples and (m, n)-reciprocal pairs of skew-morphisms.

By Proposition 3.3, every exact (m, n)-bicyclic triple $(G; a, b)$ determines an (m, n)-reciprocal pair (φ, φ^*) of skew-morphisms of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Conversely, by Proposition 3.4, every (m, n)-reciprocal pair (φ, φ^*) of skew-morphisms determines an exact (m, n)-bicyclic triple $(G; a, b)$, and the pair of skew-morphisms induced by the elements a and b in this triple is identical to the original one. What remains to prove is the one-to-one correspondence.

If two (m, n)-reciprocal pairs (φ_1, φ_1^*) and (φ_2, φ_2^*) are identical, then clearly so will be the corresponding (m, n)-bicyclic triples. Conversely, let $(G_1; a_1, b_1)$ and $(G_2; a_2, b_2)$ be two equivalent exact (m, n)-bicyclic triples, and let (φ_1, φ_1^*) and (φ_2, φ_2^*) be the corresponding skew-morphisms. Since $(G_1; a_1, b_1)$ and $(G_2; a_2, b_2)$ are equivalent, the assignment $\theta\colon a_1 \mapsto a_2, b_1 \mapsto b_2$ extends to an isomorphism of G_1 to G_2; in particular, $|a_1| = |a_2|$ and $|b_1| = |b_2|$. Set $m = |a_1|$ and $n = |b_1|$. Recall that the skew-morphisms φ_1 and φ_2 induced by a_1 and a_2 are determined by the rules $a_1b_1^x = b_1^{\varphi_1(x)}a_1^{\pi_1(x)}$ and

$a_2 b_2^y = b_2^{\varphi_2(y)} a_2^{\pi_2(y)}$ where $x, y \in \mathbb{Z}_n$. If we apply the isomorphism θ to the first equation we obtain $a_2 b_2^x = \theta(a_1 b_1^x) = \theta(b_1^{\varphi_1(x)} a_1^{\pi_1(x)}) = b_2^{\varphi_1(x)} a_2^{\pi_1(x)}$, and combining this with the second equation we get $b_2^{\varphi_2(x)} a_2^{\pi_2(x)} = b_2^{\varphi_1(x)} a_2^{\pi_1(x)}$. Thus $\varphi_1 = \varphi_2$. Using similar arguments we can get $\varphi_1^* = \varphi_2^*$. Hence, $(\varphi_1, \varphi_1^*) = (\varphi_2, \varphi_2^*)$. □

In the course of the proof of Proposition 3.4 we have established the identity (3.7). The following corollary makes it explicit.

Corollary 3.6. *If (φ, φ^*) is an (m, n)-reciprocal pair of skew-morphisms, then φ and φ^* satisfy the following identities:*

$$\varphi(x) = \sum_{i=1}^{x} \pi^*(\varphi^{*-i}(-1)) \pmod{|\varphi^*|} \quad \textit{and} \quad \varphi^*(y) = \sum_{i=1}^{y} \pi(\varphi^{-i}(-1)) \pmod{|\varphi|}.$$

Next we offer two examples. The first of them deals with the standard (m, n)-complete dessins.

Example 3.7. Let us revisit the group $G = \langle a, b \mid a^m = b^n = [a, b] = 1 \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n$ considered in Example 2.2 and determine all reciprocal pairs of skew-morphisms arising from G. Obviously, G gives rise to only one equivalence class of bicyclic triples, so we only need to consider the pairs of skew-morphisms induced by a and b in the triple $(G; a, b)$. By checking the identities (3.2), we immediately see that the skew-morphisms are the identity automorphisms. Thus the only reciprocal pair of skew-morphisms arising from the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is $(\mathrm{id}_n, \mathrm{id}_m)$, where $\mathrm{id}_n \colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\mathrm{id}_m \colon \mathbb{Z}_m \to \mathbb{Z}_m$ denote the identity mappings. In other words, for every pair of positive integers m and n there exists only one complete dessin whose automorphism group is isomorphic to the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$, the standard (m, n)-complete dessin.

In the next example, which is extracted from [14], we present a complete list of pairs of reciprocal skew-morphisms of the cyclic groups $\mathbb{Z}_9$ and $\mathbb{Z}_{27}$.

Example 3.8. In order to list all reciprocal pairs (φ, φ^*) of skew-morphisms $\varphi \colon \mathbb{Z}_9 \to \mathbb{Z}_9$ and $\varphi^* \colon \mathbb{Z}_{27} \to \mathbb{Z}_{27}$ let us first observe that φ must be an automorphism. Indeed, the order of φ divides 27, so $|\varphi| = 1$ or $|\varphi| = 3$. If $|\varphi| = 1$, then φ is an identity automorphism. If φ has order 3 and is not an automorphism, then the power function of φ reduced to $\mathbb{Z}_3$ can take only two values 1 and 2, so the subgroup $\ker \varphi$ must have index 2 in $\mathbb{Z}_3$, which is impossible. This proves that φ is an automorphism.

Now, there are exactly 27 reciprocal pairs of skew-morphisms (φ, φ^*) of skew-morphisms $\varphi \colon \mathbb{Z}_9 \to \mathbb{Z}_9$ and $\varphi^* \colon \mathbb{Z}_{27} \to \mathbb{Z}_{27}$, falling into one of the following two types:

(i) *Both φ and φ^* are group automorphisms:* In this case $\varphi(x) \equiv ex \pmod 9$ and $\varphi^*(y) \equiv fy \pmod{27}$ where either $e = 1$ and $f \in \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$, or $e \in \{4, 7\}$ and $f \in \{1, 10, 19\}$. Thus there are $9 + 6 = 15$ reciprocal pairs of skew-morphisms of this type.

(ii) *φ is a group automorphism but φ^* is not:* In this case $\varphi(x) \equiv ex \pmod 9$ and $\varphi^*(y) \equiv y + 3t \sum_{i=1}^{y} \sigma(s, e^{i-1}) \pmod{27}$ where $e \in \{4, 7\}$ and $\sigma(s, e^{i-1}) = \sum_{j=1}^{e^{i-1}} s^{j-1}$ where $(s, t) = (4, 1), (7, 2), (4, 4), (7, 5), (4, 7)$ or $(7, 8)$. There are $2 \times 6 = 12$ reciprocal pairs of this type.

We remark that in [14, Theorem 14] all reciprocal pairs of skew-morphisms of cyclic groups are classified provided that one of the skew-morphisms is an automorphism.

The correspondence established in Theorem 3.5 implies that the second condition required in the definition of an (m, n)-reciprocal pair of skew-morphisms (see Definition 3.2) can be replaced with a simpler condition.

Corollary 3.9. *A pair* (φ, φ^*) *of skew-morphisms* $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ *and* $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$ *with power functions* π *and* π^**, respectively, is reciprocal if and only if the following two conditions are satisfied:*

(i) $|\varphi|$ *divides* m *and* $|\varphi^*|$ *divides* n*, and*

(ii) $\pi(x) = \varphi^{*x}(1)$ *and* $\pi^*(y) = \varphi^y(1)$.

Proof. It is sufficient to replace the original dessin, represented by an exact (m, n)-bicyclic triple $(G; a, b)$, with its mirror image, for which the corresponding bicyclic triple is $(G; a^{-1}, b^{-1})$, and use Theorem 3.5. □

4 The uniqueness theorem

We have seen in Example 3.7 that for each pair of positive integers m and n there exists, up to reciprocality and isomorphism, at least one complete regular dessin with the underlying graph $K_{m,n}$, namely, the standard (m, n)-complete dessin. In this section we determine all the pairs (m, n) for which the standard (m, n)-complete dessin is the only regular (m, n)-dessin.

A pair (m, n) of positive integers m and n will be called *singular* if

$$\gcd(m, \phi(n)) = \gcd(n, \phi(m)) = 1.$$

A positive integer n will be called *singular* if the pair (n, n) is singular, that is, if $\gcd(n, \phi(n)) = 1$. We now show that for each non-singular pair (m, n) of positive integers there exists a non-abelian exact (m, n)-bicyclic group.

Example 4.1. Let m and n be positive integers. First assume that $\gcd(n, \phi(m)) \neq 1$. It is well known that for $x \in \mathbb{Z}_m$ the assignment $1 \mapsto x$ extends to an automorphism of $\mathbb{Z}_m$ if and only if $\gcd(x, m) = 1$, and thus $|\operatorname{Aut}(\mathbb{Z}_m)| = \phi(m)$. Since $\gcd(n, \phi(m)) \neq 1$, there exists an integer r such that $r \not\equiv 1 \pmod{m}$ and $r^p \equiv 1 \pmod{m}$, where $p \mid \gcd(n, \phi(m))$. Define a group G with presentation

$$G = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle.$$

By Hölder's theorem [22, Chapter 7], G is a well-defined metacyclic group of order mn. Since $r \not\equiv 1 \pmod{m}$, the group G is non-abelian. If $\gcd(m, \phi(n)) \neq 1$, we proceed similarly. Thus, whenever (m, n) is non-singular, there always exists at least one non-abelian exact (m, n)-bicyclic group.

We remark that the argument used here is different from the one employed in the proof of Lemma 3.1 in [12].

We now apply our theory to proving the following theorem, which extends the validity of a result of Fan and Li [12] to all bicyclic groups, not just exact ones.

Theorem 4.2. *The following statements are equivalent for every pair of positive integers m and n:*

(i) *Every product of a cyclic group of order m with a cyclic group of order n is abelian.*

(ii) *The pair (m, n) is singular.*

Proof. If (i) holds, then by virtue of Example 4.1 the pair (m, n) must be singular. For the converse, assume that the pair (m, n) is singular and that G is an (m, n)-bicyclic group. We prove the statement by using induction on the size of $|G|$. By a result of Huppert [15] and Douglas [8] (see also [17, VI.10.1]), G is supersolvable, so for the largest prime factor p of $|G|$ the Sylow p-subgroup P of G is normal in (see [17, VI.9.1]). By the Schur-Zassenhaus theorem, G is a semidirect product of P by Q, where Q is a subgroup of order $|G/P|$ in G. To proceed we distinguish two cases.

Case 1. *p divides only one of m and n.* Without loss of generality we may assume that $p \mid m$ and $p \nmid n$. Let us write m in the form $m = p^e m_1$ where $p \nmid m_1$. Then the normal subgroup P is contained in the cyclic factor $A = \langle a \rangle$ of G of order m, so $P = \langle a^{m_1} \rangle$. The generator b of the cyclic factor $B = \langle b \rangle$ of order n induces an automorphism $a^{m_1} \mapsto (a^{m_1})^r$ of P by conjugation $b^{-1} a^{m_1} b = (a^{m_1})^r$ where r is an integer coprime to p. It follows that the multiplicative order $|r|$ of r in $\mathbb{Z}_{p^e}$ divides $|\operatorname{Aut}(P)| = \phi(p^e)$. On the other hand, $a^{m_1} = b^{-n} a^{m_1} b^n = (a^{m_1})^{r^n}$, so $r^n \equiv 1 \pmod{p^e}$, and hence $|r|$ also divides n. But $\phi(p^e)$ divides $\phi(m)$ and $\gcd(n, \phi(m)) = 1$, so $r \equiv 1 \pmod{p^e}$. Therefore P is contained in the centre of G, and hence $G = P \times Q$, where Q is an (m_1, n)-bicyclic group. It is evident that the pair (m_1, n) is also singular. By induction, Q is abelian, and therefore G is abelian.

Case 2. *p divides both m and n.* Since (m, n) is a singular pair, $p^2 \nmid m$ and $p^2 \nmid n$. Thus $m = pm_1$ and $n = pn_1$ where $p \nmid m_1$, $p \nmid n_1$ and $\gcd(m_1, p(p-1)) = \gcd(n_1, p(p-1)) = 1$. Since $|G| = |AB| = |A||B|/|A \cap B|$, the Sylow p-subgroup P of G is of order p or p^2. If p divides $|A \cap B|$, then $|P| = p$ and so $P \leq A \cap B$, which is central in G. Therefore, $G = P \times Q$, where Q is an (m_1, n_1)-bicyclic group, and the result follows by induction. Otherwise, $p \nmid |A \cap B|$, so $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$. We may view P as a 2-dimensional vector space over the Galois field $\mathbb{F}_p$. Let Ω be the set of 1-dimensional subspaces of P. Then $|\Omega| = p + 1$ and $\alpha = \langle a^{m_1} \rangle$ belongs to Ω. Consider the action of G on P by conjugation. The kernel of this action is $C_G(P)$, so $\overline{G} = G/C_G(P) \leq \mathrm{GL}(2, p)$ where $C_G(P)$ denotes the centraliser of P in G. Now we claim that $\overline{G} = 1$.

Suppose to the contrary that $\overline{G} \neq 1$. Since $G = \langle a, b \rangle$, we have $\overline{G} = \langle \overline{a}^p, \overline{b}^p \rangle$, where $\overline{a}^p = a^p C_G(P)$ and $\overline{b}^p = b^p C_G(P)$. Hence at least one of $\overline{a}^p$ and $\overline{b}^p$ is not the identity of $\overline{G}$, say $\overline{a}^p \neq 1$. Clearly, $|\overline{a}^p|$ divides m_1, the order of a^p in G.

Note that Ω is a complete block system of $\mathrm{GL}(2, p)$ on P and the induced action of $\mathrm{GL}(2, p)$ on Ω is transitive. By the Frattini argument, $|\mathrm{GL}(2, p)| = (p + 1)|\mathrm{GL}(2, p)_\alpha|$, and hence $|\mathrm{GL}(2, p)_\alpha| = p(p - 1)^2$ as $|\mathrm{GL}(2, p)| = p(p + 1)(p - 1)^2$. On the other hand, $\overline{a}^p$ fixes α as a fixes the subspace $\langle a \rangle$, implying that $\overline{a}^p \in \mathrm{GL}(2, p)_\alpha$. It follows that $|\overline{a^p}|$ divides $p(p - 1)^2$. Since $|\overline{a^p}|$ divides m_1 and $\gcd(m_1, p(p - 1)) = 1$, we have $|\overline{a^p}| = 1$, which is impossible because $\overline{a}^p \neq 1$. Thus $\overline{G} = 1$, as claimed.

Since $\overline{G} = 1$, we have $G = C_G(P)$, and hence $G = P \times Q$, where $Q = \langle a^p \rangle \langle b^p \rangle$ is an (m_1, n_1)-bicyclic group with the pair (m_1, n_1) being singular. The statement now follows by induction. □

The following result follows easily from Theorem 4.2.

Corollary 4.3. *Let m and n be positive integers. Then every group factorisable as an exact product of cyclic subgroups of orders m and n is abelian if and only if the pair (m, n) is singular.*

We summarize the results of this section in the following theorem.

Theorem 4.4. *The following statements are equivalent for any pair of positive integers m and n:*

(i) *The pair (m, n) is singular.*

(ii) *Every finite group factorisable as a product of two cyclic subgroups of orders m and n is abelian.*

(iii) *Every finite group factorisable as an exact product of two cyclic groups of orders m and n is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.*

(iv) *There is only one (m, n)-reciprocal pair of skew-morphisms $(\varphi, \varphi^*) = (\mathrm{id}_n, \mathrm{id}_m)$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$.*

(v) *Up to reciprocality, there is a unique isomorphism class of regular dessins whose underlying graph is the complete bipartite graph $K_{m,n}$.*

(vi) *There exists a unique isomorphism class of orientable edge-transitive embeddings of $K_{m,n}$.*

The proof of the equivalence between items (i), (iii) and (vi) of Theorem 4.4 can be found in [12, Theorem 1.1].

Remark 4.5. For a fixed positive integer x, it has been recently shown by Nedela and Pomerance [39] that the number of singular pairs (m, n) with $m, n \leq x$ is asymptotic to $z(x)^2$ where

$$z(x) = e^{\gamma} \frac{x}{\log \log \log x},$$

where γ is Euler's constant.

5 The symmetric case

Recall that a complete regular dessin $\mathcal{D} = (G; a, b)$ is symmetric if G has an automorphism transposing a and b. In this case the dessin $\mathcal{D}$ possesses an external symmetry transposing the colour-classes. If we ignore the vertex-colouring, the dessin can be regarded as an orientably regular map with underlying graph $K_{n,n}$. As a consequence of Theorem 3.5 we obtain the following correspondence between orientably regular embeddings of the complete bipartite graphs $K_{n,n}$ and symmetric skew-morphisms of $\mathbb{Z}_n$, partially indicated by Kwak and Kwon already in [34, Lemma 3.5].

Corollary 5.1. *The isomorphism classes of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ are in a one-to-one correspondence with the symmetric skew-morphisms of $\mathbb{Z}_n$.*

A complete classification of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ has already been accomplished by Jones et al. in a series of papers [9, 10, 11, 23, 25, 26, 40]. The methods used in the classification rely on the analysis of the structure of the associated exact bicyclic groups. A different approach to the classification can be taken on the basis of Corollary 5.1 via determining the corresponding symmetric skew-morphisms of $\mathbb{Z}_n$. In particular, we can reformulate Theorem A of [23] as follows:

Corollary 5.2. *The following statements are equivalent for every positive integer n:*

(i) *The integer n is singular.*

(ii) *Every finite group factorisable as a product of two cyclic subgroups of order n is abelian.*

(iii) *Every finite group factorisable as an exact product of two cyclic subgroups of order n is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$.*

(iv) *The cyclic group $\mathbb{Z}_n$ has only one symmetric skew-morphism.*

(v) *Up to isomorphism, the complete bipartite graph $K_{n,n}$ has a unique orientably regular embedding.*

Although skew-morphisms are implicitly present in the structure of the automorphism groups of the maps, how to find them explicitly is not at all clear. This leads us to formulating the following problems for future investigation.

Problem 5.3. Determine the symmetric skew-morphisms of cyclic groups by means of explicit formulae.

Problem 5.4. Classify all orientably regular embeddings of complete bipartite graphs $K_{n,n}$ in terms of the corresponding symmetric skew-morphisms.

The previous problem suggests the following natural question: under what conditions a symmetric skew-morphism is a group automorphism and what are the corresponding orientably regular maps? The following result determines these skew-morphisms explicitly.

Theorem 5.5. *Let $\varphi\colon x \mapsto rx$ be an automorphism of $\mathbb{Z}_n$ of order d, where* $\gcd(r,n) = 1$. *Then φ is a symmetric skew-morphism of $\mathbb{Z}_n$ if and only if $d \mid n$ and $r \equiv 1 \pmod{d}$.*

Proof. Note that the order of φ is equal to the multiplicative order of r in $\mathbb{Z}_n$. Since $|\operatorname{Aut}(\mathbb{Z}_n)| = \phi(n)$, we have $d \mid \phi(n)$. Since φ is an automorphism, the associated power function is $\pi(x) \equiv 1 \pmod{d}$ for all $x \in \mathbb{Z}_n$.

If φ is symmetric, then by Definition 3.2, $d \mid n$ and $\pi(x) = -\varphi^{-x}(-1) \pmod{d}$ for all $x \in \mathbb{Z}_n$. In particular, $1 = \pi(-1) \equiv -\varphi(-1) \equiv \varphi(1) \equiv r \pmod{d}$.

Conversely, assume that $d \mid n$ and $r \equiv 1 \pmod{d}$. By Definition 3.2, it suffices to show that $-\varphi^{-x}(-1)$ is a power function of φ where $x \in \mathbb{Z}_n$, that is, to show that $-\varphi^{-x}(-1) \equiv 1 \pmod{d}$. Since $r \equiv 1 \pmod{d}$, we have $-\varphi^{-x}(-1) = \varphi^{-x}(1) = r^{-x} \equiv 1 \pmod{d}$, as required. $\square$

The following example shows that there exist symmetric skew-morphisms of $\mathbb{Z}_n$ which are not automorphisms.

Example 5.6. The cyclic group $\mathbb{Z}_8$ has the total of six skew-morphisms, out of which four are automorphisms and two are proper skew-morphisms. The latter two are listed below along with the corresponding power functions:

$$\varphi = (0)(2)(4)(6)(1\,3\,5\,7), \qquad \pi_\varphi = [1][1][1][1][3\,3\,3\,3];$$
$$\psi = (0)(2)(4)(6)(1\,7\,5\,3), \qquad \pi_\psi = [1][1][1][1][3\,3\,3\,3].$$

Note that they are, in fact, antiautomorphisms in the sense of [43, 44]. It can be easily verified that all the six skew-morphisms are symmetric. It follows that they correspond to the six non-isomorphic orientably regular embeddings of $K_{8,8}$ described in [25, Table 1].

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Simultaneous current graph constructions for minimum triangulations and complete graph embeddings

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Abstract

The problems of calculating the genus of the complete graphs and of finding a minimum triangulation for each surface were both solved using the theory of current graphs, and each of them divided into twelve different cases, depending on the residue modulo 12 of the number of vertices. Cases 8 and 11 were of particular difficulty for both problems, with multiple families of current graphs developed to solve these cases. We solve these cases, in addition to Cases 6 and 9, in a unified manner, greatly simplifying previous constructions by Ringel, Youngs, Guy, and Jungerman. All these new constructions are index 3 current graphs sharing nearly all of the structure of the simple solution for Case 5 of the Map Color Theorem.

Keywords: Topological graph theory, current graphs, map coloring, triangulations.

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1 Introduction

In this paper, we only consider surfaces which are orientable. We let S_g denote the surface of genus g, i.e., the sphere with g handles. The *Heawood number* of the orientable surface S_g of genus g,

$$H(S_g) = \frac{7 + \sqrt{1 + 48g}}{2}$$

gives rise to two distinct problems which share many similarities. On one hand, the Heawood number is an *upper bound* on the chromatic number of the surface, and the celebrated Map Color Theorem of Ringel, Youngs, and others [17] proves that this inequality is tight

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(after rounding the Heawood number to the nearest integer) for all surfaces of genus $g \geq 1$ by determining the genus of the complete graphs. In the reverse direction, $H(S_g)$ is a *lower bound* on the minimum number of vertices needed to triangulate the surface with a simple graph. For $g \geq 1$, $g \neq 2$, this was also shown to be tight by Jungerman and Ringel [11].

Both of these problems break down into twelve cases, where "Case k" refers to the relevant graphs on $12s + k$ vertices. The main tool for constructing most of the required embeddings is the theory of current graphs [4]. At times, there is overlap—for example, the complete graph K_7 triangulates the torus, thereby demonstrating that the chromatic number of the torus and the smallest number of vertices needed to triangulate the torus is 7. However, many of the cases are solved separately, and furthermore, Jungerman and Ringel's [11] solution for the latter problem of *minimum triangulations*[1] often required multiple unrelated families of current graphs.

Our goal is a partial unification of both problems using index 3 current graphs, i.e., current graphs whose embeddings have three faces. The standard solutions for Cases 3 and 5 of the Map Color Theorem, i.e., the genus of the complete graphs on $12s+3$ and $12s+5$ vertices, respectively, used simple families of index 3 current graphs whose origins can be traced back to constructions for Steiner triple systems. However, other constructions employing index 3 current graphs, perhaps most notably Case 6 of the Map Color Theorem (see §9.3 of Ringel [16]), have not realized the same level of simplicity. For each of Cases 6, 8, 9, and 11, we present a single family of current graphs which solves both the complete graph and minimum triangulation problems except for a few small-order graphs or surfaces. Not only do these constructions improve upon past solutions in the literature, but the structure of the current graphs for the general case reuses all but a fixed part of the aforementioned current graphs used for Case 5.

2 Embeddings in surfaces and the Heawood numbers

For background in topological graph theory, see Gross and Tucker [3]. In a graph, possibly with self-loops or parallel edges, every edge has two *ends* that are each incident with a vertex. A *rotation* of a vertex is a cyclic permutation of its incident edge ends, and a *rotation system* of a graph is an assignment of a rotation to every vertex of the graph. The Heffter-Edmonds principle states that cellular embeddings of a graph are in one-to-one correspondence with rotation systems: each embedding in a surface defines a rotation system by considering the cyclic order of the edge ends emanating at each vertex, while in the reverse direction, the faces of the embedding can be traced out from the rotation system in a unique manner. Our convention will be that rotations define *clockwise* orderings, which induce *counterclockwise* orientations for faces. In the case of simple graphs, one can express a rotation in terms of the vertex's neighbors, so a rotation system can be represented as a table of vertices, where each row corresponds to a cyclic permutation of the neighbors of a specific vertex.

The *Euler polyhedral formula* states that for a cellular embedding $\phi\colon G \to S_g$, we have the expression

$$|V(G)| - |E(G)| + |F(G,\phi)| = 2 - 2g,$$

where g denotes the genus of the surface and $F(G,\phi)$ is the set of faces induced by the embedding. A standard consequence is the following inequality:

[1] Jungerman and Ringel [11] used the term *minimal triangulations*.

Proposition 2.1. *Let $\phi\colon G \to S_g$ be an embedding of a simple, connected graph G with at least 3 vertices. Then*

$$|E(G)| \leq 3|V(G)| - 6 + 6g,$$

where equality is achieved when the embedding is triangular*, i.e. when all its faces are triangular.*

The *(minimum) genus* of a graph G is the minimum genus over all cellular embeddings of G, and is denoted $\gamma(G)$. A *(minimum) genus embedding* of G is an embedding whose genus achieves this minimum.

Corollary 2.2. *For a simple, connected graph G with at least 3 vertices, its genus is at least*

$$\gamma(G) \geq \left\lceil \frac{|E(G)| - 3|V(G)| + 6}{6} \right\rceil.$$

We say that an embedding of a simple graph is *triangle-tight* if its genus equals this lower bound. If a triangle-tight embedding exists, it must necessarily be of minimum genus. From these relationships between the edge and vertex counts and the genus, one can derive the *Heawood number*

$$H(g) = \frac{7 + \sqrt{1 + 48g}}{2}$$

of the surface S_g, which serves as a rough measure of "maximum possible density" in the following two inequalities:

Proposition 2.3 (see Ringel [16, p. 63])**.** *For $g \geq 1$, the chromatic number $\chi(S_g)$ of the surface S_g, i.e., the maximum chromatic number over all graphs embeddable in S_g, satisfies*

$$\chi(S_g) \leq \lfloor H(S_g) \rfloor.$$

Let $MT(S_g)$ be the minimum number of vertices over all simple graphs G that have a triangular embedding in S_g.

Proposition 2.4 (Jungerman and Ringel [11])**.** *For each surface S_g with $g \geq 1$,*

$$MT(S_g) \geq \lceil H(S_g) \rceil.$$

Such an embedding in Proposition 2.4 is known as a *minimum triangulation* of S_g. We call a triangular embedding of a graph an *(n,t)-triangulation* if the graph has n vertices and $\binom{n}{2} - t$ edges, i.e. the graph is the complete graph on n vertices with t edges deleted. The tightness of the inequalities in Propositions 2.3 and 2.4 is proven via alternative formulations that emphasize the number of vertices:

Theorem 2.5 (Ringel and Youngs [17])**.** *The genus of the complete graph K_n is*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Theorem 2.6 (Jungerman and Ringel [11])**.** *For all pairs (n,t) of nonnegative integers $n \geq 3$, $t \leq n - 6$ satisfying*

$$(n-3)(n-4) \equiv 2t \pmod{12},$$

there exists an (n,t)-triangulation, except for $(n,t) = (9,3)$.

Another way of stating Theorem 2.5 is that every complete graph K_n, $n \geq 3$, has a triangle-tight embedding. In both problems, the proof breaks down into several cases, depending on the residue of the number of vertices $n \bmod 12$. We call the subcase concerning graphs with $n = 12s+k$ vertices *Case* k, for $k = 0, 1, \ldots, 11$, and we often reference the value s in our exposition. For example, if we speak of "Case 6, $s = 2$" of the Map Color Theorem, we are referring to the complete graph K_{30}. To differentiate between the two problems, we refer to "Case k-CG" and "Case k-MT" to denote Case k of the Map Color Theorem ("complete graph") and minimum triangulations problem, respectively.

The fact that there are 12 Cases depending on the number of vertices for both the Map Color Theorem and the minimum triangulations problem suggests a connection between the solutions of the two problems. Indeed, in several Cases, the current graphs used in the proof [17] of the Map Color Theorem for K_n have the dual purpose of also providing all the necessary minimum triangulations on the same number of vertices n. However, not all Cases have been combined in this manner.

In general, our constructions will proceed in the following way: using an index 3 current graph, we generate an (n, t)-triangulation. We wish to find other embeddings of graphs on the same number of vertices using the following operations:

- *handle subtraction*, which deletes edges from a triangular embedding to produce a triangular embedding on a lower-genus surface, and
- *additional adjacency*, which adds edges using extra handles and other local operations.

By subtracting handles, we obtain all the necessary (n, t')-triangulations, for $t' > t$, and over the course of the additional adjacency step for constructing a triangle-tight embedding of K_n, we construct the remaining (n, t'')-triangulations, for $t'' < t$.

3 Outline for additional adjacencies

The main goal for our additional adjacency steps is to utilize as little information about the embeddings as possible. For this reason, we present the additional adjacency solutions first, before describing any current graphs. Like in previous work, our additional adjacency solutions make use of three different operations for adding a handle, which are described in Constructions 3.1, 3.2, and 3.8 in primal form. Most of these constructions are already known, except Proposition 3.6 and Lemma 3.10. In prose, we describe the modifications to the embeddings in terms of rotation systems, so their correctness can be checked by tracing the faces and applying the Heffter-Edmonds principle. Our drawings, on the other hand, describe an alternative topological interpretation using surgery on the embedded surfaces. While these operations work more generally, we assume that all graphs in this section are simple and their embeddings are triangular.

Construction 3.1. *Modifying the rotation at vertex v from*

$$v.\quad x_1\quad \ldots\quad x_i\quad y_1\quad \ldots\quad y_j\quad z_1\quad \ldots\quad z_k$$

to

$$v.\quad x_1\quad \ldots\quad x_i\quad z_1\quad \ldots\quad z_k\quad y_1\quad \ldots\quad y_j,$$

as in Figure 1 increases the genus by 1 *and induces the* 9*-sided face*

$$[x_1, z_k, v, y_1, x_i, v, z_1, y_j, v].$$

Construction 3.2. *Modifying the rotation at vertex v from*

$$v. \quad x_1 \quad \ldots \quad x_i \quad y_1 \quad \ldots \quad y_j \quad z_1 \quad \ldots \quad z_k \quad w_1 \quad \ldots \quad w_\ell$$

to

$$v. \quad x_1 \quad \ldots \quad x_i \quad w_1 \quad \ldots \quad w_\ell \quad z_1 \quad \ldots \quad z_k \quad y_1 \quad \ldots \quad y_j$$

as in Figure 2 increases the genus by 1 *and induces the two* 6*-sided faces*

$$[x_1, w_\ell, v, z_1, y_j, v] \quad \textit{and} \quad [w_1, z_k, v, y_1, x_i, v].$$

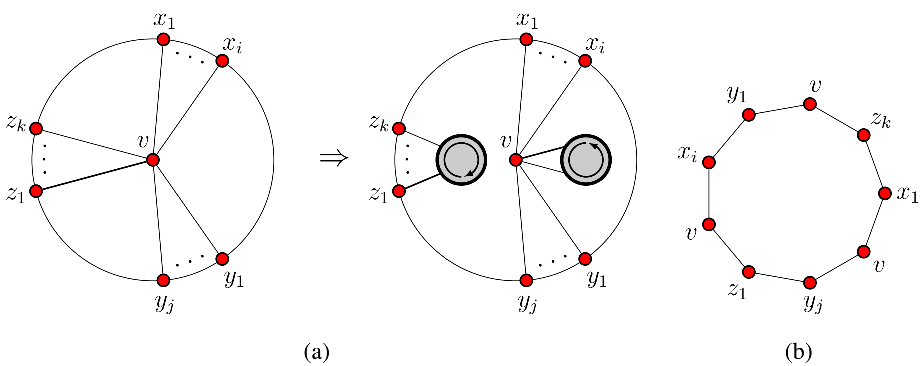


Figure 1: Rearranging the rotation at vertex v (a) increases the genus and creates room (b) to add new edges.

Remark 3.3. While the drawings in Figures 1 and 2 are drawn asymmetrically, the operations are in fact invariant under cyclic shifts of the subsets $x_1, \ldots, x_i; y_1, \ldots, y_j$, etc.

Several Cases of the Map Color Theorem are solved by first finding triangular embeddings of $K_n - K_3$. The first consequence of Construction 3.1 is to transform such an embedding into a genus embedding of a complete graph.

Proposition 3.4 (Ringel [15])**.** *If there exists a triangular embedding of $K_n - K_3$ in the surface S_g, then there exists a genus embedding of K_n in the surface S_{g+1}.*

Before showing how this follows from the above constructions, we first argue that all the embeddings of complete graphs we construct are in fact of minimum genus.

Proposition 3.5. *Suppose we have a triangular embedding of a graph $K_n - H_e$, where H_e is a graph on e edges, $e < 6$. If we add the missing e edges by using one handle, the resulting embedding of K_n is triangle-tight.*

Proof. One can verify that the difference between the genus of $K_n - H_e$, as given by Proposition 2.1, and the genus of K_n is exactly 1. □

Proof of Proposition 3.4. If the three nonadjacent vertices are $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, pick any other vertex v and apply Construction 3.1 with $x_1 = \boldsymbol{a}, y_1 = \boldsymbol{b}, z_1 = \boldsymbol{c}$. In the resulting nontriangular face, the nonadjacent vertices can be connected like in Figure 3(a). □

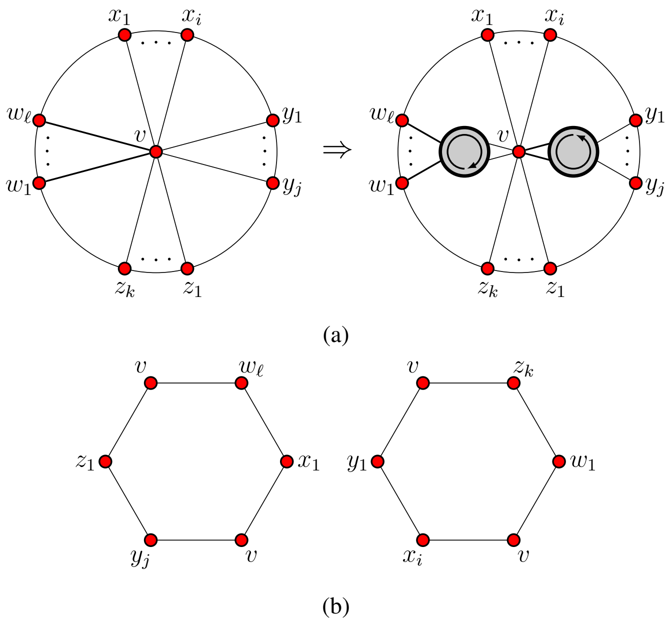


Figure 2: Rearranging four groups of neighbors (a) yields two hexagonal faces (b).

For Cases 8 and 11, we will construct triangular embeddings of the graph $K_n - K_{1,4}$. These missing edges can be added in using one handle if the embedding satisfies an additional constraint:

Proposition 3.6. *Let $K_n - K_{1,4}$ be a complete graph with the edges $(u, q_1), \dots, (u, q_4)$ deleted. If there exists a triangular embedding $\phi\colon (K_n - K_{1,4}) \to S_g$ with a vertex v with rotation*

$$v.\quad \dots \quad q_1 \quad q_2 \quad \dots \quad q_3 \quad q_4 \quad \dots,$$

then there exists a genus embedding of K_n in the surface S_{g+1}.

Proof. Note that vertices u and v are adjacent, so assume without loss of generality that u appears in the rotation of v in between q_4 and q_1. Apply Construction 3.1 with

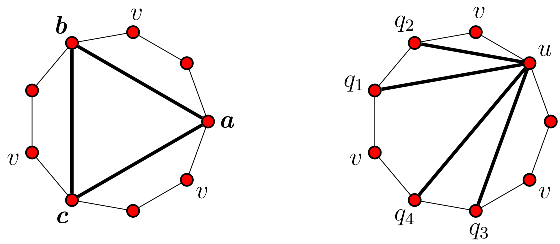
$$x_i = q_1, \quad y_1 = q_2, \quad y_j = q_3, \quad z_1 = q_4, \quad z_k = u$$

and connect the missing edges in the 9-sided face, as in Figure 3(b). □

This constraint is relatively easy to satisfy, since there are a few possible permutations for $q_1, \dots, q_4$, in addition to the fact that v is an arbitrary vertex. In fact, when we only need to add back three edges, this is always possible:

Corollary 3.7 (Ringel *et al.* [5, 19])**.** *If there exists a triangular embedding of $K_n - K_{1,3}$ in the surface S_g, then there exists a genus embedding of K_n in the surface S_{g+1}.*

Proof. One can always find such a vertex v by choosing a vertex on one of the triangles incident with, say, the edge (q_1, q_2). □



(a) (b)

Figure 3: Two possibilities for adding edges after invoking Construction 3.1: a K_3 subgraph (a), and a $K_{1,4}$ subgraph (b).

A third type of handle operation is to merge two faces with a handle without modifying the rotations at any vertices. To do this, we excise a disk from two faces and identify the resulting boundaries. In Figure 4, adding the handle between faces F_1 and F_2 causes the embedding to become noncellular, as the resulting region is an annulus. However, once we start adding edges between the two boundary components of the annulus, the embedding becomes cellular again.

Construction 3.8. *Let $F_1 = [u_1, u_2, \dots, u_i]$ and $F_2 = [v_1, v_2, \dots, v_j]$ be two faces. Inserting the edge (u_1, v_1) in the following way*

$$\begin{array}{lllll} u_1. & \dots & u_i & u_2 & \dots \\ v_1. & \dots & v_j & v_2 & \dots \end{array} \quad \Rightarrow \quad \begin{array}{llllll} u_1. & \dots & u_i & v_1 & u_2 & \dots \\ v_1. & \dots & v_j & u_1 & v_2 & \dots \end{array}$$

as in Figure 4 increases the genus by 1 *and induces the* $(i + j + 2)$*-sided face*

$$[u_1, u_2, \dots, u_i, u_1, v_1, v_2, \dots, v_j, v_1].$$

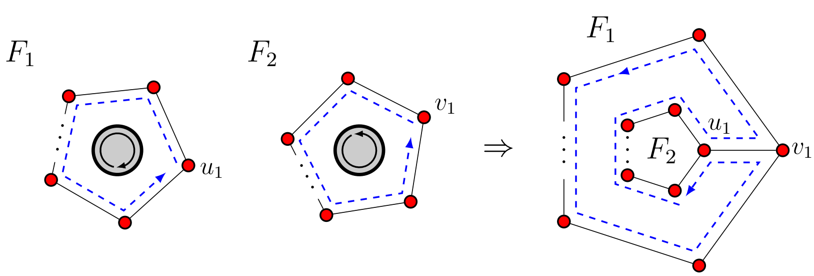


Figure 4: Adding a handle between two faces, then adding an edge to transform the annulus into a cell. Note that the order of vertices of one of the faces becomes reversed as we traverse one of the (oriented) boundaries the annulus.

The most elementary operation one can do is to simply add one edge to create a genus embedding:

Proposition 3.9. *If there exists a triangular embedding $\phi\colon K_n - K_2 \to S_g$, then there exists a genus embedding of K_n in the surface S_{g+1}.*

The forthcoming additional adjacency solutions are to be applied on triangular embeddings of graphs of the form $K_n - K_\ell$, which is the graph formed by taking the complete graph K_n and deleting all the pairwise adjacencies between ℓ vertices. We label the vertices missing adjacencies with bold letters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots, \boldsymbol{h}$. The remaining vertices will be assigned numbers and are represented here as unadorned letters $(u, v, p_i, \dots)$. We apply the traditional method of adding handles to supply all the missing edges—in Section 3.1, we give an alternative viewpoint that aims to demystify the specific choices of added edges.

Lemma 3.10. *If there exists a triangular embedding of $K_n - K_5$ with numbered vertices u and v whose rotations are of the form*

$$u. \quad \dots \quad \boldsymbol{a} \quad p_1 \quad \boldsymbol{b} \quad p_2 \quad \boldsymbol{c} \quad p_3 \quad \boldsymbol{d} \quad p_4 \quad \boldsymbol{e} \quad \dots$$

and

$$v. \quad \dots \quad p_{\sigma(1)} \quad p_{\sigma(2)} \quad \dots \quad p_{\sigma(3)} \quad p_{\sigma(4)} \quad \dots,$$

where $\sigma\colon \{1, \dots, 4\} \to \{1, \dots, 4\}$ is some permutation, then there exist $(n, 10)$- and $(n, 4)$-triangulations and a triangle-tight embedding of K_n.

Proof. The initial embedding is an $(n, 10)$-triangulation. First, delete the edges (u, p_1), $(u, \boldsymbol{b})$, (u, p_2) in exchange for $(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{a}, \boldsymbol{c})$, $(\boldsymbol{b}, \boldsymbol{c})$ and apply edge flips on (u, p_3) and (u, p_4) to obtain $(\boldsymbol{c}, \boldsymbol{d})$ and $(\boldsymbol{d}, \boldsymbol{e})$, as in Figure 5(a). If we merge the faces $[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}]$ and $[u, \boldsymbol{e}, \boldsymbol{d}]$ with a handle, we can recover the deleted edge $(u, \boldsymbol{b})$ and add in the remaining edges between lettered vertices following Figure 5(b). The missing edges $(u, p_1), \dots, (u, p_4)$ in this $(n, 4)$-triangulation can be reinserted with one handle using Proposition 3.6, setting $p_{\sigma(i)} = q_i$, to get a triangle-tight embedding of K_n. □

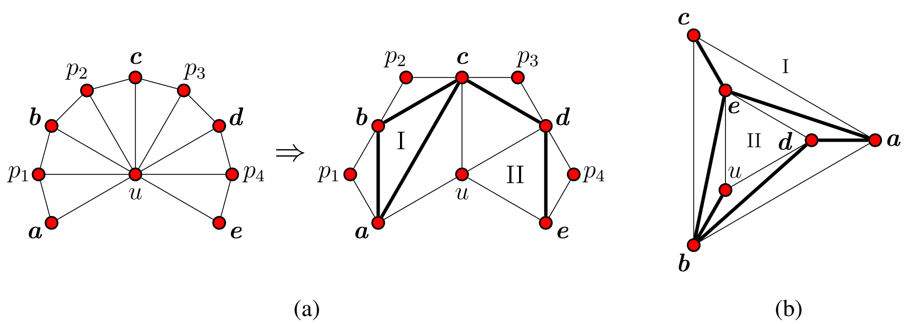


Figure 5: Various edge flips are applied in the neighborhood of vertex u (a) so that one handle suffices for connecting all the lettered vertices.

Lemma 3.11 (Guy and Ringel [5])**.** *If there exists a triangular embedding of $K_n - K_6$ with a numbered vertex u whose rotation is of the form*

$$u. \quad \dots \quad \boldsymbol{a} \quad p_1 \quad \boldsymbol{b} \quad \dots \quad \boldsymbol{c} \quad p_2 \quad \boldsymbol{d} \quad \dots \quad \boldsymbol{e} \quad p_3 \quad \boldsymbol{f} \quad \dots,$$

then there exist $(n, 15)$-, $(n, 9)$-, and $(n, 3)$-triangulations and a triangle-tight embedding of K_n.

Proof. We first modify the embedding near vertex u using edge flips to gain the edges $(\boldsymbol{a},\boldsymbol{b})$, $(\boldsymbol{c},\boldsymbol{d})$, and $(\boldsymbol{e},\boldsymbol{f})$, as in Figure 6(a). If we apply Construction 3.1 to vertex u, we obtain a 9-sided face incident with all six vertices $\boldsymbol{a},\boldsymbol{b},\dots,\boldsymbol{f}$. In Figure 6(b) and (c), we give one way to insert the twelve missing edges between these lettered vertices with the help of a handle. The embeddings before and after adding the handle are $(n,9)$- and $(n,3)$-triangulations, respectively.

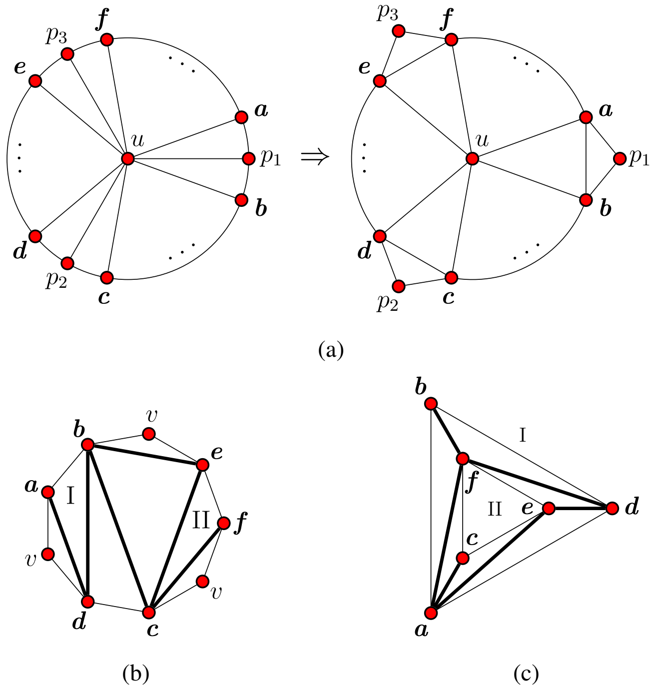


Figure 6: Three pairs of lettered vertices are connected with some edge flips (a), after which a handle adds some of the missing adjacencies (b). The remaining edges between lettered vertices are added using another handle merging faces I and II (c).

The missing edges (u,p_1), (u,p_2), (u,p_3) can be added back using Corollary 3.7, yielding a triangle-tight embedding of K_n. □

Lemma 3.12. *If there exists a triangular embedding of $K_n - K_8$ with numbered vertices u and v whose rotations are of the form*

$$u.\quad \dots\quad \boldsymbol{a}\quad p_1\quad \boldsymbol{b}\quad \dots\quad \boldsymbol{c}\quad p_2\quad \boldsymbol{d}\quad \dots\quad \boldsymbol{e}\quad p_3\quad \boldsymbol{f}\quad \dots\quad \boldsymbol{g}\quad p_4\quad \boldsymbol{h}\quad \dots$$

and

$$v.\quad \dots\quad p_{\sigma(1)}\quad p_{\sigma(2)}\quad \dots\quad p_{\sigma(3)}\quad p_{\sigma(4)}\quad \dots,$$

where $\sigma\colon \{1,\dots,4\}\to\{1,\dots,4\}$ is some permutation, then there exist $(n,28)$-, $(n,22)$-, $(n,16)$-, $(n,10)$-, and $(n,4)$-triangulations and a triangle-tight embedding of K_n.

Proof. The first four handles of our additional adjacency approach is the same as that of Ringel and Youngs' solution for Case 2-CG [19] (also see Ringel [16, §7.5]), with different vertex names. We perform an edge flip on each edge (u, p_i) for $i = 1, \dots, 4$, gaining the edges $(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{c}, \boldsymbol{d})$, $(\boldsymbol{e}, \boldsymbol{f})$, and $(\boldsymbol{g}, \boldsymbol{h})$. Now, the rotation at vertex u is of the form

$$u. \quad \dots \quad \boldsymbol{a} \quad \boldsymbol{b} \quad \dots \quad \boldsymbol{c} \quad \boldsymbol{d} \quad \dots \quad \boldsymbol{e} \quad \boldsymbol{f} \quad \dots \quad \boldsymbol{g} \quad \boldsymbol{h} \quad \dots$$

These edge flips are depicted in Figure 7. Applying Construction 3.2 to this resulting rotation yields two nontriangular faces

$$[\boldsymbol{h}, \boldsymbol{g}, v, \boldsymbol{d}, \boldsymbol{c}, v] \quad \text{and} \quad [\boldsymbol{f}, \boldsymbol{e}, v, \boldsymbol{b}, \boldsymbol{a}, v].$$

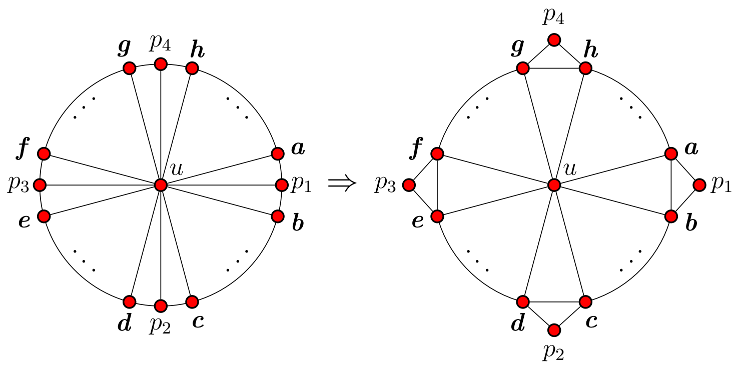


Figure 7: Initial edge flips to join some of the vortex letters.

In these faces, we induce two quadrilateral faces by adding the edges $(\boldsymbol{d}, \boldsymbol{g})$, $(\boldsymbol{c}, \boldsymbol{h})$, $(\boldsymbol{b}, \boldsymbol{e})$, and $(\boldsymbol{a}, \boldsymbol{f})$, as in Figure 8(a). Three more handles are used to add all the remaining edges between lettered vertices $\boldsymbol{a}, \dots, \boldsymbol{h}$ as shown in Figure 8(bc). At this point, the embedding is of the graph $K_n - K_{1,4}$ and is still triangular, so we add back the deleted edges (u, p_i) with one handle using Proposition 3.6 to obtain a triangle-tight embedding of K_n.

The embeddings after adding the second through fourth handles are all triangular and hence are minimum triangulations. After adding only the first handle, the two quadrilateral faces in Figure 8(a) can be triangulated arbitrarily to form an $(n, 22)$-triangulation. □

We note some recurring themes in these additional adjacency solutions, which one could view as another form of unification between Cases. The "chord" edges and subsequent handle for connecting five vortices in Lemma 3.10 reappear in Lemma 3.11. Proposition 3.6 is invoked in both Lemma 3.10 and 3.12. As mentioned earlier, most of the construction in Lemma 3.12 was applied to Case 2-CG by Ringel and Youngs [19].

3.1 Recasting handle operations

Additional adjacency solutions are traditionally presented as a sequence of handles, which has the benefit of constructing some of the requisite minimum triangulations. However, when several handles are involved, it is not immediately apparent how such a construction was derived—Ringel [16] described the solution for Case 2-CG, which is largely identical to the one we used in Lemma 3.12, as "adventurous" and "much easier to understand than to discover." We can instead interpret parts of these additional adjacency solutions as surgical operations that glue together existing embeddings, akin to the diamond sum operation of

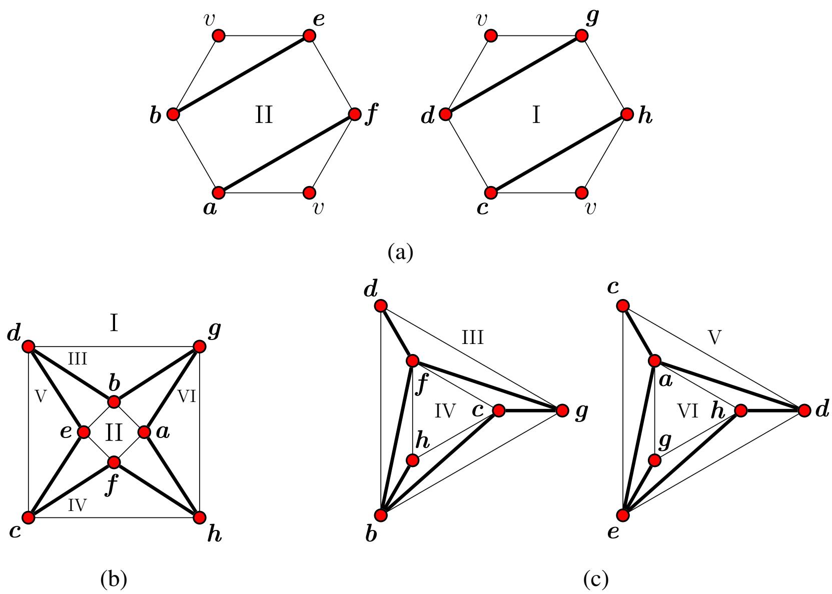


Figure 8: After connecting some of the lettered vertices with a handle (a), another handle can be introduced in between the faces I and II (b). Using faces generated from this handle (III and IV, V and VI), we can add all the remaining edges using two additional handles (c).

Bouchet [1] or the inductive constructions found in Ringel [16, §10]. In our case, we make use of the embedding of K_6 in S_1 formed by deleting a vertex from the triangular embedding of K_7, and a genus embedding of K_8 in S_2 where the two quadrilateral faces are incident with disjoint sets of vertices. An example of the latter embedding appears in Ringel [16, p. 79] and is reproduced in Appendix C.

Recall that in Lemma 3.12, the second, third, and fourth handles add all the remaining missing edges between lettered vertices, where all of the activity takes place inside of the two quadrilateral faces formed from the first handle. Let $\phi\colon G \to S_g$ be the embedding of the graph after the first handle in Lemma 3.12. Combining the next three handles into one step is equivalent to the following procedure, which is sketched in Figure 9:

- Excise the interiors of the quadrilateral faces of ϕ and the aforementioned embedding $K_8 \to S_2$.
- Identify the two embedded surfaces at their boundaries so that the two disjoint sets of four vertices become identified and the resulting surface is orientable.

Hence the three handles are equivalent to a construction of a genus embedding of K_8. We may also apply the same idea to reinterpret the constructions in Lemma 3.10 and 3.11 using the embedding of K_6. If, for example, we remove the edges $(\boldsymbol{b},\boldsymbol{c})$, $(\boldsymbol{b},\boldsymbol{d})$, and $(\boldsymbol{c},\boldsymbol{e})$ from Figure 6, we have the hexagonal face $[\boldsymbol{a},\boldsymbol{d},\boldsymbol{c},\boldsymbol{f},\boldsymbol{e},\boldsymbol{b}]$. The goal of the last handle of the additional adjacency step in Lemma 3.11 is to add all the remaining edges between the lettered vertices, which we may accomplish by attaching the embedding of K_6 along this hexagonal face, as shown in Figure 10.

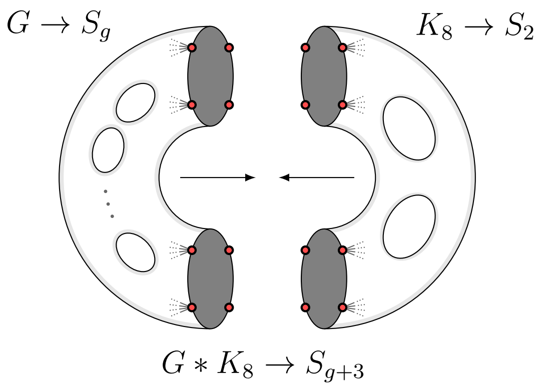


Figure 9: Adding adjacencies between eight vertices with an embedding of K_8. Note that the genus increases by 3 since two boundary components are identified.

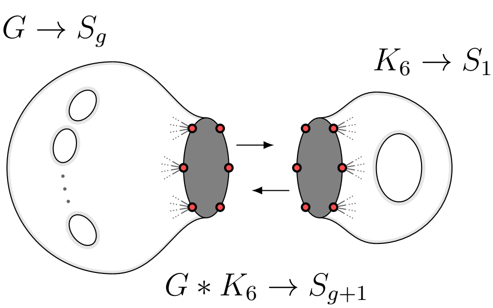


Figure 10: An alternative way of adding the edges between six vertices using one handle.

4 Index 3 current graphs

We assume familiarity with current graphs, especially §9 of Ringel [16]. An *index k current graph* is a triple (D, ϕ, α), where D is a directed graph, $\phi\colon D \to S$ is a cellular k-face embedding of D in an orientable surface S and $\alpha\colon E(D) \to \Gamma$ is a labeling of each arc of D with an element of a group Γ. These arc labels are called *currents*, and Γ is referred to as the *current group*. In this paper, we only consider index 3 current graphs that are labeled with elements from cyclic current groups $\Gamma = \mathbb{Z}_{3m}$ for some integer m. Its three face boundary walks, which we call *circuits*, are labeled [0], [1], and [2].

The *excess* of a vertex is the sum of the incoming currents minus the sum of the outgoing currents, and we say a vertex satisfies *Kirchhoff's current law* (KCL) if its excess is 0. Vertices of degree 3 which do not satisfy KCL are called *vortices*, which are each labeled with a lowercase letter. The *log* of a circuit records the currents encountered along the walk in the following manner: if we traverse arc e along its orientation, we write down $\alpha(e)$; otherwise, we write down $-\alpha(e)$; if we encounter a vortex, we record its label. If the order 2 element $\gamma \in \mathbb{Z}_{3m}$ is the current of an arc incident with a vertex of degree 1, it appears twice consecutively in the log of the incident circuit. We discard one of those instances so that the embedded graph is simple. Our drawings of current graphs which have such arcs follow the convention where the degree 1 vertex is omitted.

All of our index 3 current graphs with current groups $\mathbb{Z}_{3m}$ satisfy the following additional "construction principles", which are effectively the same as those in Ringel [16, §9.1]:

- (E1) Each vertex is of degree 3 or 1.
- (E2) The embedding has three circuits labeled [0], [1], [2].
- (E3) The log of each circuit consists of each nonzero element of $\mathbb{Z}_{3m}$ exactly once and any number of vortex letters.
- (E4) KCL is satisfied at every vertex of degree 3, except vortices, which are labeled with letters.
- (E5) Every vortex is incident with all three circuits and has an excess which generates the subgroup of $\mathbb{Z}_{3m}$ consisting of the multiples of 3.
- (E6) If circuit $[a]$ traverses arc e along its orientation and circuit $[b]$ traverses e in the opposite direction, then $\alpha(e) \equiv b - a \pmod{k}$.
- (E7) The current on every arc incident with a vertex of degree 1 is of order 2 or 3 in $\mathbb{Z}_{3m}$.

If all the construction principles are satisfied, the current graph generates a triangular embedding of the graph $K_{3m} + \overline{K_\ell}$, where $G + H$ is the graph join operation, $\overline{G}$ is the edge-complement of G, and ℓ is the number of vortices. Each element of $\mathbb{Z}_{3m}$ corresponds to a vertex in the complete graph K_{3m}, and each of the vortices provides an additional vertex, which is adjacent to all elements of $\mathbb{Z}_{3m}$, but none of the other vortex vertices. It is more common to think of the resulting graph instead as $K_{3m+\ell} - K_\ell$, which highlights the total number of vertices and the number of missing edges needed to form a complete graph. An example of an index 3 current graph is given in Figure 11. The logs of its circuits are:

[0].	1	***a***	8	5	9	4	13	12	14	***b***	7	10	6	11	2	3
[1].	14	2	6	4	13	9	11	5	12	7	10	3	8	***b***	1	***a***
[2].	1	13	9	11	2	6	4	10	3	8	5	12	7	***a***	14	***b***

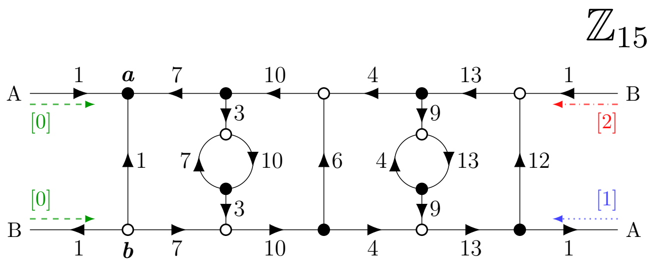


Figure 11: A current graph for $K_{17} - K_2$. Solid and hollow vertices correspond to clockwise and counterclockwise rotations, respectively.

To generate the embedding from the logs of these circuits, for each element $\gamma \in \mathbb{Z}_{3m}$ in the group, the rotation at vertex γ is found by taking the log of circuit $[\gamma \bmod k]$ and adding γ to each of its non-letter elements. The rotations at the numbered vertices thus read:

0.	1	***a***	8	5	9	4	13	12	14	***b***	7	10	6	11	2	3
1.	0	3	7	5	14	10	12	6	13	8	11	4	9	***b***	2	***a***
2.	3	0	11	13	4	8	6	12	5	10	7	14	9	***a***	1	***b***
3.	4	***a***	11	8	12	7	1	0	2	***b***	10	13	9	14	5	6
4.	3	6	10	8	2	13	0	9	1	11	14	7	12	***b***	5	***a***
5.	6	3	14	1	7	11	9	0	8	13	10	2	12	***a***	4	***b***
6.	7	***a***	14	11	0	10	4	3	5	***b***	13	1	12	2	8	9
7.	6	9	13	11	5	1	3	12	4	14	2	10	0	***b***	8	***a***
8.	9	6	2	4	10	14	12	3	11	1	13	5	0	***a***	7	***b***
9.	10	***a***	2	14	3	13	7	6	8	***b***	1	4	0	5	11	12
10.	9	12	1	14	8	4	6	0	7	2	5	13	3	***b***	11	***a***
11.	12	9	5	7	13	2	0	6	14	4	1	8	3	***a***	10	***b***
12.	13	***a***	5	2	6	1	10	9	11	***b***	4	7	3	8	14	0
13.	12	0	4	2	11	7	9	3	10	5	8	1	6	***b***	14	***a***
14.	0	12	8	10	1	5	3	9	2	7	4	11	6	***a***	13	***b***

The rotation around each lettered vertex is "manufactured" so that the entire embedding is triangular and orientable. To facilitate this process, we make use of the following characterization of triangular embeddings:

Proposition 4.1 (e.g., Ringel [16, §2.3]). *An embedding of a simple graph G is triangular if and only if for all vertices i, j, k, if the rotation at vertex i is of the form*

$$i. \quad \ldots \quad j \quad k \quad \ldots,$$

then the rotation at vertex j is of the form

$$j. \quad \ldots \quad k \quad i \quad \ldots$$

From the partial rotation system we have built up so far, we can determine the rotations at the remaining vortex vertices:

a.	0	1	2	9	10	11	3	4	5	12	13	14	6	7	8
b.	0	14	13	6	5	4	12	11	10	3	2	1	9	8	7

The final embedding is a triangular one of $K_{17} - K_2$, which is a $(17, 1)$-triangulation. It can be augmented into a genus embedding of K_{17} using Proposition 3.9.

The group we use for most of our constructions, including all infinite families, is $\mathbb{Z}_{12s+3}$. By combining construction principles (E6) and (E7), we find that in order to have a degree 1 vertex using this group, it must be the case that $s \equiv 2 \pmod 3$. Thus, we only make use of degree 1 vertices and principle (E7) in a few constructions deferred to Appendix B.

The increased flexibility acquired from using index 3 current graphs is crucial. Since vortices have the same degree as other vertices, one can tweak the number of vortices while keeping the number of total vertices and edges fixed, i.e., one cannot rule out the existence of such current graphs using just divisibility conditions on the numbers of vertices and edges in the current graph. Furthermore, the conditions in Lemma 3.10, i.e., having all five vortices lined up nearly consecutively, is only possible for current graphs with index at least 3. For indices 1 and 2, such a configuration would violate a "global" KCL condition.

A sketch of the standard proof of Case 5-CG (see Ringel [16, §9.2] or Youngs [23]) is given first, as we reuse significant parts of its structure for our current graphs. The case $s = 1$ was given earlier in Figure 11, and the higher order cases are given in Figures 12 and 13. The construction also works trivially for $s = 0$ as well.

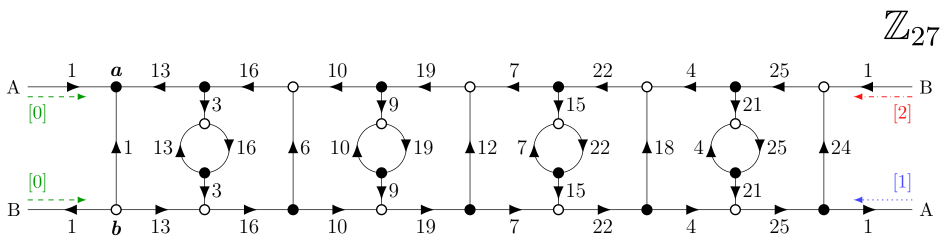


Figure 12: A current graph for $K_{29} - K_2$.

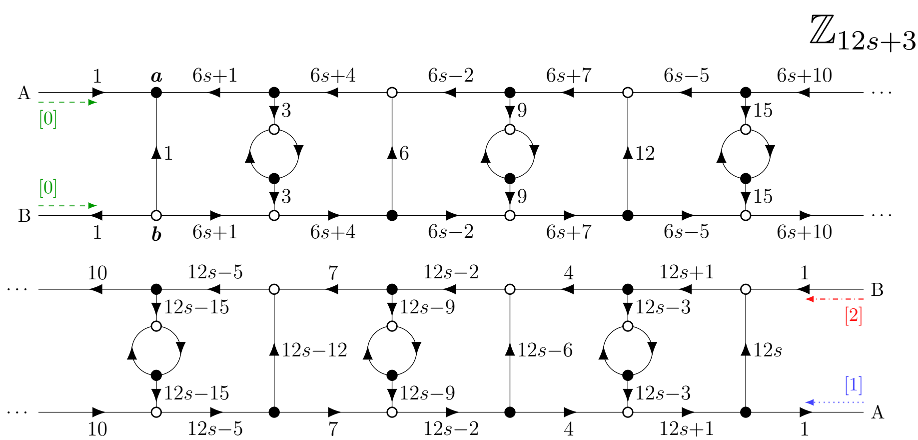


Figure 13: The family of current graphs for $K_{12s+5} - K_2$, for general s. The omitted current on a circular arc is the same as those on the horizontal arcs above and below it.

The general shape of the family of current graphs is a long ladder whose "rungs" alternate between simple vertical arcs and so-called "globular rungs," where the two additional vertices have a pair of parallel edges between them. As we parse from left to right, the vertical arcs, except for the arc connecting the two vortices, alternate in direction and form an arithmetic sequence consisting of the nonzero multiples of 3 in $\mathbb{Z}_{12s+3}$. The zigzag pattern induced on the horizontal arcs is essentially the canonical graceful labeling of a path graph on $4s+1$ vertices (see, e.g., Goddyn *et al.* [2] for more information on this connection), where the vertical arcs correspond to the edge labels on the path graph. The horizontal arcs come in pairs that share the same current and are oriented in opposite directions. The currents on these arcs exhaust all the elements of the form $3k+1$ in $\mathbb{Z}_{12s+3}$.

Infinite families of current graphs typically consist of

- a *fixed portion*, which contains vortices and some salient currents for additional adjacency solutions. The underlying directed graph stays the same, while the currents may vary as a function of s, and
- a *varying portion*, which subsumes all remaining currents not present in the fixed portion. The size of this ingredient varies as a function of s, and the currents are arranged in a straightforward pattern.

In the construction for Case 5, we might consider the vortices and its incident edge ends as the fixed portion, and the rest of the graph (see Figure 14) as the varying portion. The elegant solutions for Cases 3 and 5 of the Map Color Theorem were first described in Youngs [23], improving upon similar ideas of Ringel [15] and Gustin [4]. We consider this varying portion, which we call the *Youngs ladder*, to be the best possible choice for index 3 current graphs.

The approach of Youngs *et al.* [5, 6, 23] first finalizes the fixed portion and then solve auxiliary labeling problems for the varying portion. We tackle the problem in reverse, opting to massage the fixed portion around a preset varying portion, which we choose to be a contiguous subset of the Youngs ladder. Starting with the arc labeled 1 that runs between the two vortices, we successively peel off rungs of the Youngs ladder until we have enough material for our desired fixed portion.

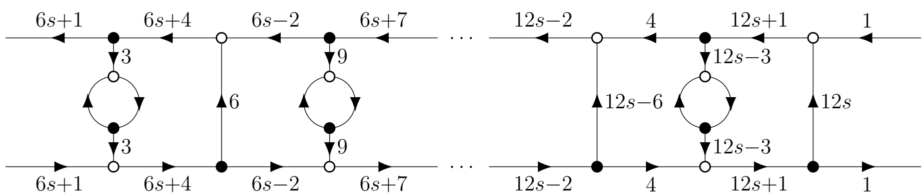


Figure 14: The Youngs ladder is essentially the current graphs for Case 5 with two vertices deleted.

We expect this procedure to become more difficult as the number of vortices increases– not only do we need appropriate currents that feed into the vortices, but there becomes an imbalance between the currents which are not divisible by 3 and those which are. Each vortex will use three currents of the former type, leaving a surplus of those of the latter type. The gadget in Figure 15, which we call the *double bubble*, accounts for this effect. By tracing out the partial circuits and invoking construction principle (E6), we find that all six currents entering the highest and lowest vertices must be divisible by 3, while the four remaining arcs may be labeled with an element not divisible by 3 depending on which circuits touch this gadget. The double bubble and its generalization have appeared in other work regarding current graphs of index greater than 1, such as Korzhik and Voss [12] and Pengelley and Jungerman [14].

In all of our current graph constructions, we use the cyclic group $\mathbb{Z}_{12s+3}$ unless we specify otherwise. While we often simplify the labels by reversing the directions of some arcs, e.g. replacing a label like $12s+1$ with 2, the ends which connect to the Youngs ladder are kept unchanged, i.e., as a current which is congruent to $1 \pmod 3$.

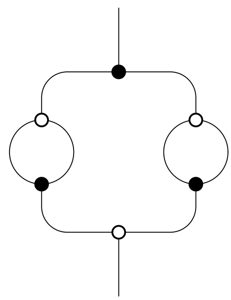


Figure 15: The "double bubble" motif appears in all of our general constructions.

5 Handle subtraction for minimum triangulations

The forthcoming embeddings $K_{12s+3+k} - K_k$ and the embeddings *en route* to constructing a genus embedding of $K_{12s+3+k}$ already constitute minimum triangulations, namely

$$\left(12s+3+k, \binom{k}{2} - 6h\right)\text{-triangulations,}$$

where h is a nonnegative integer less than the number of added handles. To construct minimum triangulations on the same number of vertices, but with more missing edges, we turn to the main idea of Jungerman and Ringel [11]: we enforce a specific structure in the current graph that allows us to "subtract" handles. The fragment shown in Figure 16 is what we refer as an *arithmetic* 3*-ladder*. If the step size h in the arithmetic sequence is divisible by 3 (more generally, divisible by the index of the current graph), then it is possible to find triangular embeddings in smaller-genus surfaces in the following manner:

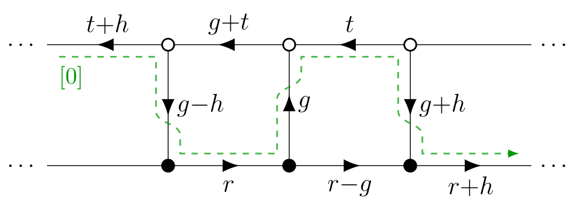


Figure 16: An arithmetic 3-ladder and a circuit passing through it.

Lemma 5.1 (Jungerman and Ringel [11]). *Let (D, ϕ, α) be an index 3 current graph with current group $\mathbb{Z}_{3m}$ that satisfies all construction principles. Suppose further that it has an arithmetic 3-ladder with step size divisible by 3. If the derived embedding of the current graph has $|V|$ vertices and $|E|$ edges, then for each $k = 0, \dots, m$, there exists a triangular embedding of a graph with $|V|$ vertices and $|E| - 6k$ edges.*

Proof. Following Figure 16, the rotation at vertices 0 and h are of the form

$$\begin{array}{lcccccccc}
0. & \dots & -t{-}h & g{-}h & r & g & -t & g{+}h & r{+}h & \dots \\
h. & \dots & -t & g & r{+}h & g{+}h & & & & \dots
\end{array}$$

Here we used the fact that h is divisible by 3. We may infer, by repeated application of Proposition 4.1, the following partial rotation system, for $i = 0, 1, \dots, m$:

$$\begin{array}{rcccccc}
0. & \dots & g & -t & g{+}h & r{+}h & \dots \\
g. & \dots & r{+}h & h & -t & 0 & \dots \\
r{+}h. & \dots & 0 & g{+}h & h & g & \dots \\
\\
h. & \dots & -t & g & r{+}h & g{+}h & \dots \\
-t. & \dots & g{+}h & 0 & g & h & \dots \\
g{+}h. & \dots & h & r{+}h & 0 & -t & \dots
\end{array} \tag{5.1}$$

If we delete the middle two columns, the rotation system becomes

$$\begin{array}{rcccc}
0. & \dots & g & r{+}h & \dots \\
g. & \dots & r{+}h & 0 & \dots \\
r{+}h. & \dots & 0 & g & \dots \\
\\
h. & \dots & -t & g{+}h & \dots \\
-t. & \dots & g{+}h & h & \dots \\
g{+}h. & \dots & h & -t & \dots
\end{array}$$

This new embedding has six fewer edges, and is still triangular by Proposition 4.1, hence it must be a triangular embedding on a surface with one fewer handle by Proposition 2.1.

More generally, we obtain other handles that can be subtracted in the same manner, using the additivity rule. That is, we can find another subtractible handle by adding a multiple of 3 to every element of (5.1). The six edges from each of m handles can be deleted simultaneously, as none of the handles share any faces. □

One way to visualize this operation is to interpret it as the reverse of Construction 3.8, like in Figure 17. One can check that in all instances in this paper, the number of handles we can subtract in a given embedding is greater than the number needed to realize the minimum triangulation with the fewest number of edges, i.e., the (n, t)-triangulation where $t \approx n-6$.

6 The current graph constructions

6.1 Comparison with existing literature

Our utilization of index 3 current graphs is rooted in Jungerman and Ringel's [11] solution for Case 5-MT as a straightforward modification of the current graphs used for Case 5-CG.

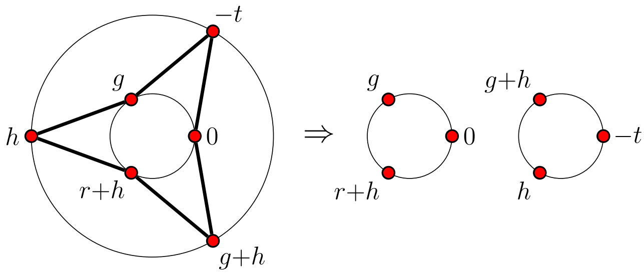


Figure 17: The six deleted edges form a cycle that is, roughly speaking, surrounded by two triangles.

We make no improvement here, but use a variation of their construction as an example of the infinite families of current graphs we seek.

The standard approach to Case 6-CG is to use index 3 current graphs to first obtain a triangular embedding of $K_{12s+6}-K_3$. The general solution Ringel [16, §9.3] chose to present works for all $s \geq 4$, and for $s = 2$, a current graph that makes use of construction principle (E7) is shown. Jungerman and Ringel [11] solved the remaining minimum triangulations using two families of index 1 current graphs. For $s = 1$, the case of $(18, 3)$-triangulations is particularly difficult—Jungerman [8] found a triangular embedding of $K_{18} - K_3$ using computer search, and we believe that such an embedding cannot be constructed with index 3 or lower current graphs (see the discussion in Section 6.3 and Appendix A). In [21], the author starts with an $(18, 9)$-triangulation due to Jungerman and Ringel [10] and produces a $(18, 3)$-triangulation and a genus embedding of K_{18}. The $(18, 3)$-triangulation is of the graph $K_{18} - 3K_2$.

Index 1 embeddings of $K_{12s+8} - K_5$ were apparently known to Ringel and Youngs (see Ringel [16, p. 86]), though they were unable to extend these embeddings to genus embeddings of K_{12s+8}. Instead, Jungerman and Ringel [11] used them for most of the minimum triangulations on $12s + 8$ vertices, i.e., the $(12s + 8, 10 + 6h)$-triangulations for nonnegative h. For the remaining $(12s + 8, 4)$-triangulation case, they found two families of index 2 current graphs whose derived embeddings could be modified into an embedding of $K_{12s+8} - (K_2 \cup P_3)$.

The best solution for Case 9-CG is a beautiful construction of Jungerman, but it does not construct minimum triangulations except for the exceptional surface S_2. For the general case, a family of current graphs found by Guy and Ringel [5][2] produced minimum triangulations for all $s \geq 5$. Jungerman and Ringel [11] supplied the remaining cases via a variety of approaches, primarily using an inductive construction where some triangular embeddings are glued to one another.

The only previously known solution for the genus of K_{12s+11} for $s \geq 1$ is that of Ringel and Youngs [18] for $s \geq 2$ and the asymmetric embedding of Mayer [13] for $s = 1$. In the general case, Ringel and Youngs start with an embedding of $K_{12s+11} - K_5$, where the missing edges are added using a highly tailored additional adjacency step. The same current graph yields minimum triangulations of type $(12s + 11, 10 + 6h)$ for $h \geq 0$, but the troublesome case of $(12s + 11, 4)$-triangulations, like in Case 8-MT, was resolved via

[2]There are two errors in Figure 1 of [5]: the top left current should be "$6s + 1$" and the vertex between "x" and "z" should be a vortex labeled "w."

two complicated families of index 2 current graphs.

It seems that nowhere in the literature, including in the original proof of the Map Color Theorem, is there a construction of a genus embedding of K_n derived from an $(n, 4)$-triangulation. Even though we outlined a natural approach in Proposition 3.6 for converting an $(n, 4)$-triangulation to a genus embedding of K_n, no prior such unification was known.

Our approach gives a unified construction for both the Map Color Theorem and the minimum triangulations problem for Cases 6, 8, 9, and 11. The infinite families of current graphs cover all $s \geq 2$ for Cases 6, 8, and 9, and $s \geq 3$ for Case 11. In all these solutions, we use families of index 3 current graphs whose varying portions are a part of the Youngs ladder. One attractive property of using index 3 current graphs is that we are able to give a solution that does not break into two parts depending on the parity of s, as was the case in Jungerman and Ringel's [11] current graphs for Case 6-, 8-, and 11-MT. For Cases 9 and 11, we supply additional constructions for smaller values of s. Of particular interest is the case of $n = 23$, for which we give the first current graph construction for a genus embedding of K_{23}.

We present the constructions in increasing difficulty of the additional adjacency solution. In particular, Case 9, which has six vortices, is ultimately simpler than Case 8 because of the additional constraint needed in Lemma 3.10.

6.2 Case 5

As a warmup, let us consider how to find minimum triangulations for Case 5. The original solution in Figure 13 does not have any arithmetic 3-ladders, but we can modify it by swapping two of the rungs in the Youngs ladder, namely the two with vertical arcs labeled 6 and $12s-3$, as in Figure 18. In this drawing and all forthcoming figures, we only describe the fixed portion of the family of current graphs—at the ellipses, we complete the picture by attaching the corresponding segment of the Youngs ladder, as mentioned earlier. Exactly where to truncate the Youngs ladder is determined by the currents at the ends of the fixed portion.

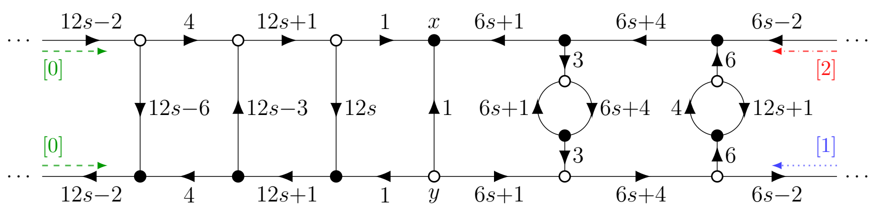


Figure 18: A slight modification to the Youngs ladder that produces minimum triangulations.

The idea of pairing the rungs is crucial in Youngs' method [23] for constructing index 3 current graphs. In their proof of minimum triangulations for Case 5, Jungerman and Ringel [11] took this idea to the extreme and switched all pairs of rungs so that all of the globular rungs appeared on one side of the ladder, but as seen in our example, implementing all these exchanges is not necessary.

We note that to the left of the vortices in our drawing in Figure 18, the directions of the arcs are inverted from that of Figure 13. Most of our infinite families (except the alternate Case 6-CG construction in Appendix A) involve attaching a Youngs ladder with a "Möbius twist," i.e., the final current graph is a long ladder-like graph whose top-left and bottom-left ends become identified with the bottom-right and top-right ends, respectively.

6.3 Case 6

The family of current graphs in Figure 19 applies for all $s \geq 2$ and has an arithmetic 3-ladder, giving a simpler and more unified construction for Case 6-CG (after applying Proposition 3.4), in addition to providing a single family of current graphs, irrespective of parity, for Case 6-MT. The case $s = 1$ is particularly pesky—in the original proof of the Map Color Theorem, the minimum genus embedding of K_{18} was found using purely *ad hoc* methods by Mayer [13]. An exhaustive computer search suggests that there are no index 3 current graphs for generating triangular embeddings of $K_{18} - K_3$. In Appendix A, we present another solution for Case 6-CG, $s \geq 2$, that almost achieves the 18-vertex case.

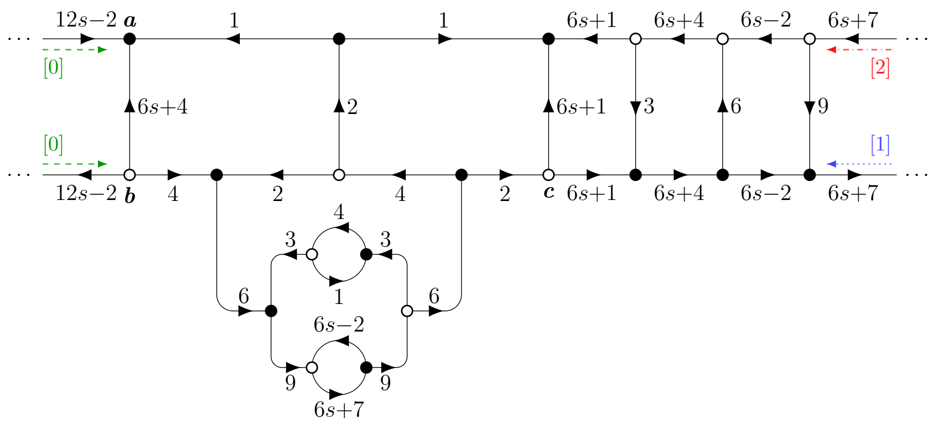


Figure 19: A current graph for $K_{12s+6} - K_3$ for $s \geq 2$.

6.4 Case 9

We improve on the construction of Guy and Ringel [5] with the family of index 3 current graphs seen in Figure 20. These current graphs produce triangular embeddings of $K_{12s+9} - K_6$ for all $s \geq 2$, and the vertical rungs labeled $3, 6, 9$ form an arithmetic 3-ladder. The circuits [1] and [2] have the six vortices packed as close together as possible. In particular, the log of circuit [1] reads

$$[1]. \quad \dots \quad \boldsymbol{a} \quad 4 \quad \boldsymbol{b} \quad \dots \quad \boldsymbol{c} \quad 1 \quad \boldsymbol{d} \quad \dots \quad \boldsymbol{e} \quad 12s{+}1 \quad \boldsymbol{f} \quad \dots,$$

so we may apply Lemma 3.11 with, e.g., $u = 1$, to obtain $(12s + 9, 9)$- and $(12s + 9, 3)$-triangulations and a genus embedding of K_{12s+9}.

For the case $s = 1$, Appendix B contains an index 3 current graph with an arithmetic 3-ladder that yields a triangular embedding of $K_{21} - K_3$. The remaining case $s = 0$ is the lone exception to Theorem 2.6. Huneke [7] proved that no triangulation of the surface S_2 has 9 vertices, so the embedding of K_8 in S_2 with its quadrilateral faces subdivided (see Appendix C) is a minimum triangulation on 10 vertices. Adding an edge between these two subdivision vertices with Construction 3.8 and subsequently contracting that edge results in a genus embedding of K_9.

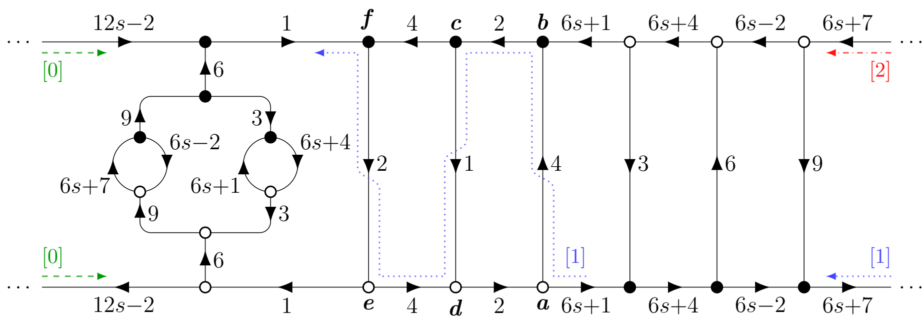


Figure 20: A current graph for $K_{12s+9} - K_6$ for $s \geq 2$. Additional fragments of circuits besides the guidelines at the left and right ends indicate components used in the additional adjacency solution.

6.5 Case 8

The family of current graphs in Figure 21 yields triangular embeddings of $K_{12s+8} - K_5$ and has an arithmetic 3-ladder. The logs of this current graph are of the form

$$[0]. \quad \ldots \quad 6s{+}1 \quad 12s \quad \ldots \quad 12s{-}3 \quad 6s{-}2 \quad \ldots$$
$$[2]. \quad \ldots \quad \boldsymbol{a} \quad 6s{+}2 \quad \boldsymbol{b} \quad 12s{+}1 \quad \boldsymbol{c} \quad 6s{-}1 \quad \boldsymbol{d} \quad 12s{-}2 \quad \boldsymbol{e} \quad \ldots$$

These translate, by additivity, to the rotations

$$3. \quad \ldots \quad 6s{+}4 \quad 0 \quad \ldots \quad 12s \quad 6s{+}1 \quad \ldots$$
$$2. \quad \ldots \quad \boldsymbol{a} \quad 6s{+}4 \quad \boldsymbol{b} \quad 0 \quad \boldsymbol{c} \quad 6s{+}1 \quad \boldsymbol{d} \quad 12s \quad \boldsymbol{e} \quad \ldots$$

By applying Lemma 3.10 with $u = 2$, $v = 3$, $(p_1, p_2, p_3, p_4) = (6s + 4, 0, 6s + 1, 12s)$, we can construct a $(12s + 8, 4)$-triangulation and a genus embedding of K_{12s+8}.

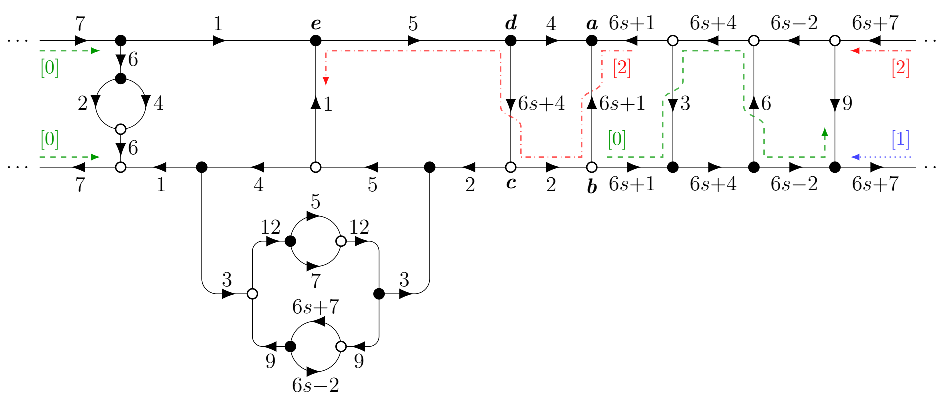


Figure 21: A family of index 3 current graphs for $K_{12s+8} - K_5$, $s \geq 2$.

Remark 6.1. Our additional adjacency solution makes use of some of the arcs forming the arithmetic 3-ladder. However, there is no conflict since handle subtraction and additional adjacency operations are not applied simultaneously.

6.6 Case 11

For $s \geq 3$, we introduce the family of current graphs in Figure 22 that generate triangular embeddings of $K_{12s+11} - K_8$. On the bottom right is an arithmetic 3-ladder with labels $9, 12, 15$. By examining circuit [1], we obtain the rotations

$$\begin{array}{l} 1. \quad \dots \ \boldsymbol{a} \ \ 6s{+}8 \ \ \boldsymbol{b} \ \dots \ \boldsymbol{c} \ \ 5 \ \ \boldsymbol{d} \ \dots \ \boldsymbol{e} \ \ 12s{-}1 \ \ \boldsymbol{f} \ \dots \ \boldsymbol{g} \ \ 6s{+}2 \ \ \boldsymbol{h} \ \dots \\ \qquad\qquad 12s{+}1. \quad \dots \ 6s{+}8 \ \ 6s{+}2 \ \dots \ 5 \ \ 12s{-}1 \ \dots \end{array}$$

Applying Lemma 3.12 with $u = 1, v = 12s{+}1, (p_1, p_2, p_3, p_4) = (6s{+}8, 5, 12s{-}1, 6s{+}2)$ yields the remaining minimum triangulations and a genus embedding of K_{12s+11}, $s \geq 3$.

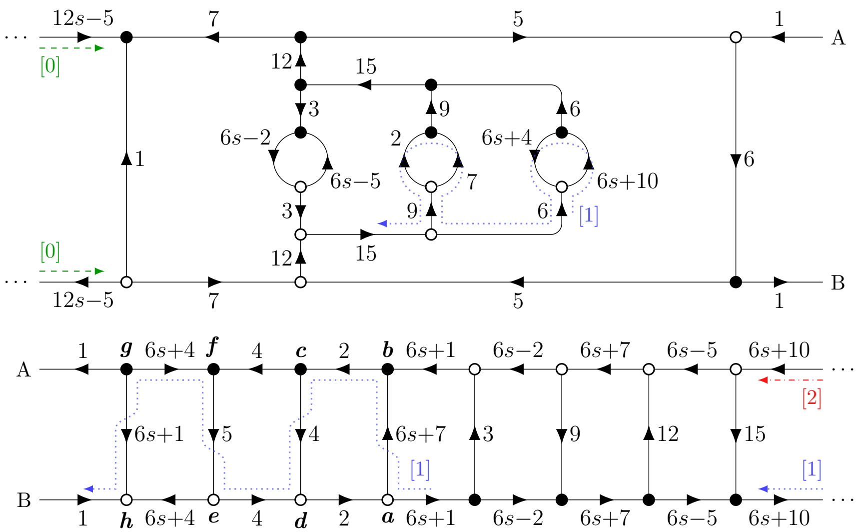


Figure 22: Index 3 current graphs for $K_{12s+11} - K_8$, $s \geq 3$.

The special cases $s = 1, 2$ have current graphs found in Appendix B, and a rotation system for $s = 0$ is given in Appendix C.

7 Conclusion

We found index 3 constructions that produced simultaneous solutions to the genus of the complete graphs and to minimum triangulations of surfaces, for Cases 6, 8, 9, and 11:

- Two constructions were presented for Case 6, $s \geq 2$ of the Map Color Theorem. Prior to the present paper, the only previously known current graph for $s = 2$ was not generalizable to higher values of s due to its use of construction principle (E7).
- A significantly simpler solution was found for $K_{12s+9} - K_6$ than that of Guy and Ringel [5] that also works for $s = 2, 3$.
- We gave unified constructions for Cases 8 and 11. For the latter, they are the first known triangular embeddings of $K_{12s+11} - K_8$ for $s \geq 3$, and the case $s = 1$ for Case 11-CG now has a solution using current graphs. The additional adjacency

solution for Case 11 (Lemma 3.12) is more straightforward than the original construction by Ringel and Youngs [18], especially in light of the interpretation given in Section 3.1.

As mentioned earlier, index 3 current graphs allow for changing the number of vortices without violating divisibility conditions necessary for the existence of current graphs. We expect that for fixed $k > 1$ and sufficiently large s, there exist appropriate current graphs for triangular embeddings of $K_{12s+3+k} - K_k$. The results of this paper extend the applicability of index 3 current graphs to roughly half of both of the Map Color Theorem and the minimum triangulations problem, and we believe that a complete solution for a sufficiently large number of vertices is possible by extending the results presented here.

We made use of the current group $\mathbb{Z}_{12s+3}$ in our infinite families of current graphs, reserving the group $\mathbb{Z}_{12s+6}$ for the special cases presented in the Appendix B. We were unable to find triangular embeddings of $K_{12s+9} - K_6$ and $K_{12s+11} - K_8$ for small values of s, so we resorted to a different approach for these cases. An open problem would be to find an analogue of the Youngs ladder for the latter group—one tricky aspect is incorporating the order 2 element $6s + 3$ into such a pattern. A desirable application of such a method would be a unified construction for all $s \geq 1$ for Case 11. Our current graph for $s = 1$, the first known current graph construction for finding a genus embedding of K_{23}, is a step towards that goal.

Some recent unifications were found by the author in the context of index 1 current graphs. Originally, these constructions were meant to improve Case 0-CG [22] and Case 1-CG [20], but these current graphs also have arithmetic 3-ladders and hence also constitute unified constructions that improve upon those found in Jungerman and Ringel [11]. At present, Case 2 is the least unified of the residues. Triangular embeddings of $K_{12s+2} - K_2$ for all $s \geq 1$ were found by Jungerman [9], which by Construction 3.8 yields genus embeddings of K_{12s+2}. The remaining minimum triangulations were found by an entirely different construction by Jungerman and Ringel [11]. It seems plausible that lifting to index 3 current graphs may help, as it did with K_{20} (see [20]) and K_{23}.

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Appendix A An alternate family of current graphs for Case 6-CG

In Figure 23, we give another index 3 construction for triangular embeddings of $K_{12s+6} - K_3$ using as much of the Youngs ladder as possible. The corresponding segment of the Youngs ladder has $4s - 5$ rungs—if we had a family of current graphs where the varying portion was part of a Youngs ladder with one more rung, then an index 3 current graph would exist for $s = 1$ (with 0 rungs from the Youngs ladder). Thus, we argue that this construction, combined with our experimental results showing nonexistence for $s = 1$, maximizes the length of the Youngs ladder fragment used. As a side note, this family of current graphs uses the same building blocks known to Ringel *et al.*

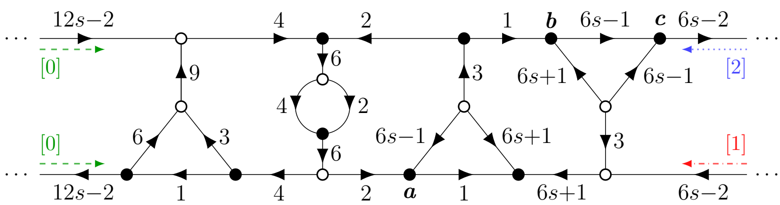

Figure 23: Another construction for triangular embeddings of $K_{12s+6} - K_3$.

Appendix B Small current graphs, Cases 9 and 11

B.1 Case 9

For $s = 1$ we use the special current graph in Figure 24. It is essentially one of the inductive constructions used by Jungerman and Ringel [11], with the additional observation that the current graph used has an arithmetic 3-ladder.

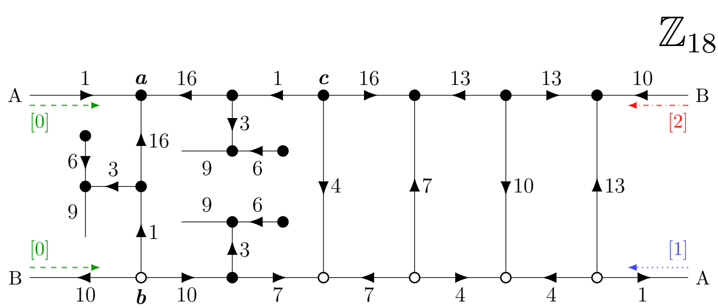

Figure 24: A current graph for $K_{21} - K_3$ with an arithmetic 3-ladder.

B.2 Case 11

For $s = 1, 2$, we first find a current graph with group $\mathbb{Z}_{12s+6}$ that generates a triangular embedding of $K_{12s+11} - K_5$. For $s = 1$, consider the index 3 current graph in Figure 25.

The rotations at vertices 1 and 12 are of the form

$$\begin{array}{lllllllllll} 1. & \ldots & \boldsymbol{a} & 3 & \boldsymbol{b} & 5 & \boldsymbol{c} & 9 & \boldsymbol{d} & 8 & \boldsymbol{e} \quad \ldots \\ 12. & \ldots & 5 & 8 & \ldots & 3 & 9 & \ldots, \end{array}$$

so applying Lemma 3.10 with $u = 1, v = 12, (p_1, p_2, p_3, p_4) = (3, 5, 9, 8)$ yields $(23, 10)$- and $(23, 4)$-triangulations, and a genus embedding of K_{23}. For $s = 2$, the current graph in Figure 26 generates a triangular embedding of $K_{35} - K_5$. Similar to the $s = 1$ case, we use the rotations

$$\begin{array}{lllllllllll} 2. & \ldots & \boldsymbol{a} & 10 & \boldsymbol{b} & 6 & \boldsymbol{c} & 7 & \boldsymbol{d} & 3 & \boldsymbol{e} \quad \ldots \\ 19. & \ldots & 7 & 10 & \ldots & 6 & 3 & \ldots, \end{array}$$

and Lemma 3.10 to find the $(35, 10)$- and $(35, 4)$ triangulations, and a genus embedding of K_{35}. The remaining minimum triangulations can be found using the arithmetic 3-ladder.

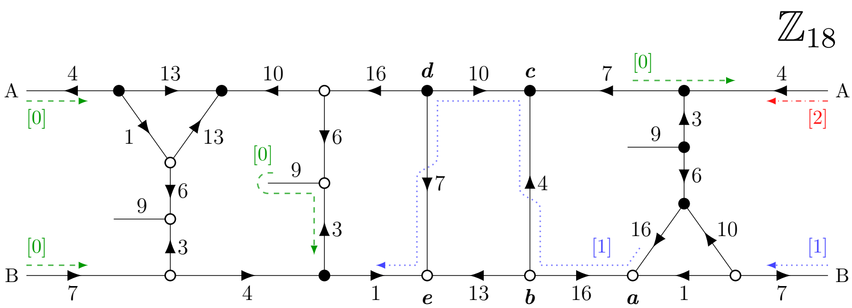

Figure 25: An index 3 current graph for $K_{23} - K_5$.

An embedding is said to be *nearly triangular* if it has at most one nontriangular face. The following result relates nearly triangular embeddings to minimum triangulations:

Proposition B.1. *Suppose there exists a triangle-tight embedding of K_n in a surface S_g with exactly one nontriangular face. If the boundary of the nontriangular face contains no repeated vertices, then there exists a minimum triangulation of S_g on $n + 1$ vertices.*

Proof. The bounds derived from Heawood numbers $H(g)$ (Propositions 2.3 and 2.4) show that $MT(g) \geq n + 1$ (as $H(g)$ is not an integer). Subdividing the nontriangular face of the embedding with a new vertex and connecting it to all the vertices along the face yields the desired triangulation. □

In particular, the aforementioned nonexistence result for $(9, 3)$-triangulations due to Huneke [7] was used to show that K_8 does not have a nearly triangular embedding in S_2 [20]. We use the nearly triangular genus embedding of K_{22} given in [20] to construct the remaining $(23, 16)$-triangulation.

Finally, a unification of the 11-vertex case using an asymmetric embedding is given in Appendix C.

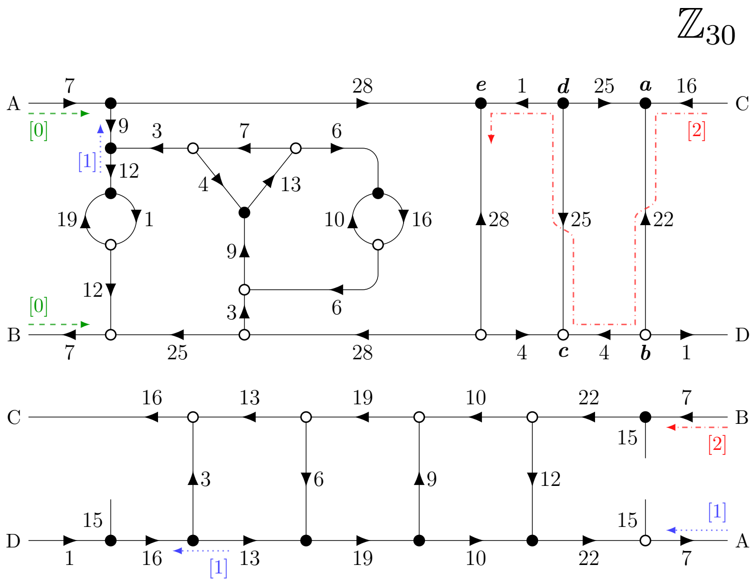

Figure 26: An index 3 current graph for $K_{35} - K_5$.

Appendix C Some small embeddings

We collect a few special embeddings in this section. The first such embedding, found in Ringel [16, p. 79], is of K_8 with two additional subdivision vertices:

0.	2	7	3	1	4	5	6	q_0
2.	4	1	5	3	6	7	0	q_0
4.	6	3	7	5	0	1	2	q_0
6.	0	5	1	7	2	3	4	q_0
1.	7	6	5	2	4	0	3	q_1
3.	1	0	7	4	6	2	5	q_1
5.	3	2	1	6	0	4	7	q_1
7.	5	4	3	0	2	6	1	q_1
q_0.	6	4	2	0				
q_1.	1	3	5	7				

This embedding was used in several ways: it is a minimum triangulation of S_2, it is a genus embedding of K_9 after amalgamating q_0 and q_1, and three of the handles of Lemma 3.12 can be thought of as gluing this embedding at two quadrilateral faces.

Known $(11, 4)$-triangulations and genus embeddings of K_{11} do not follow naturally from current graph constructions. To lessen the load of having to verify these special embeddings, we give a triangular embedding of $K_{11} - C_4$:

0.	1	10	8	4	2	9	7	5	3	6
1.	0	6	4	8	5	9	3	7	2	10
2.	0	4	10	1	7	6	5	8	3	9
3.	0	5	10	4	7	1	9	2	8	6
4.	0	8	1	6	9	5	7	3	10	2
5.	0	7	4	9	1	8	2	6	10	3
6.	0	3	8	10	5	2	7	9	4	1
7.	0	9	6	2	1	3	4	5		
8.	0	10	6	3	2	5	1	4		
9.	0	2	3	1	5	4	6	7		
10.	0	1	2	4	3	5	6	8		

The missing edges are $(7, 8)$, $(8, 9)$, $(9, 10)$, and $(10, 7)$, which can be added with one handle using Construction 3.8 as in Figure 27. Note that this construction does not really make use of any specific structure in the embedding, as we can always find a face incident with a given edge. We thus formulate this additional adjacency approach more generally:

Proposition C.1. *If there exists a triangular embedding of $K_n - C_4$, then there exists a triangle-tight embedding of K_n.*

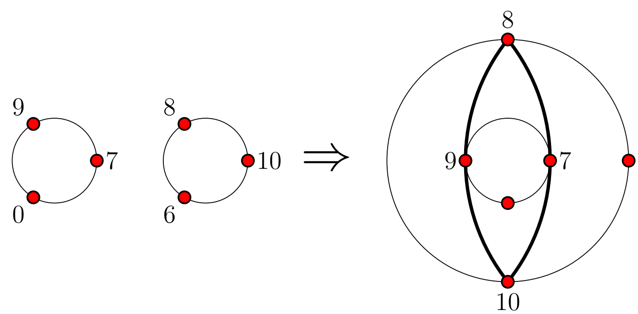

Figure 27: A generic method for adding a C_4 with one handle, applied to the triangular embedding of $K_{11} - C_4$.

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The thickness of the Kronecker product of graphs*

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Abstract

The thickness of a graph G is the minimum number of planar subgraphs whose union is G. In this paper, we present sharp lower and upper bounds for the thickness of the Kronecker product $G \times H$ of two graphs G and H. We also give the exact thickness numbers for the Kronecker product graphs $K_n \times K_2$, $K_{m,n} \times K_2$ and $K_{n,n,n} \times K_2$.

Keywords: Thickness, Kronecker product graph, planar decomposition.

Math. Subj. Class. (2020): 05C10

1 Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. It is a measurement of the planarity of a graph, the graph with $\theta(G) = 1$ is a planar graph; it also has important application in VLSI design [15]. Since W. T. Tutte [16] inaugurated the thickness problem in 1963, the thickness of some classic types of graphs have been obtained by various authors, such as [1, 3, 4, 13, 17, 19] etc. In recent years, some authors focus on the thickness of the graphs which are obtained by operating on two graphs, such as the Cartesian product graph [8, 20] and join graph [7]. In this paper, we are concerned with the Kronecker product graph.

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The *Kronecker product* (also called as tensor product, direct product, categorical product) $G \times H$ of graphs G and H is the graph whose vertex set is $V(G \times H) = V(G) \times V(H)$ and edge set is $E(G \times H) = \{(g,h)(g',h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}$. Figure 1 shows the Kronecker product graph $K_5 \times K_2$ in which $\{u_1, \dots, u_5\}$ and $\{v_1, v_2\}$ are the vertex sets of the complete graphs K_5 and K_2, respectively. Many authors did research on various topics of the Kronecker product graph, such as for its planarity [2, 10], connectivity [18], coloring [9, 12] and application [14] etc.

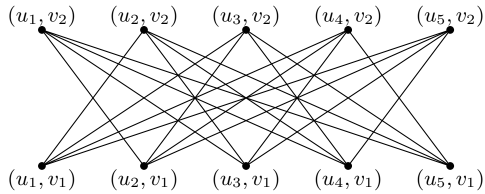


Figure 1: The Kronecker product graph $K_5 \times K_2$.

The *complete graph* K_n is the graph on n vertices in which any two vertices are adjacent. The *complete bipartite graph* $K_{m,n}$ is the graph whose vertex set can be partitioned into two parts X and Y, $|X| = m$ and $|Y| = n$, every edge has its ends in different parts and every two vertices in different parts are adjacent. The *complete tripartite graph* $K_{l,m,n}$ is defined analogously.

In this paper, we present lower and upper bounds for the thickness of the Kronecker product of two graphs in Section 2, in which the lower bound comes from Euler's formula and the upper bound is derived from the structure of the Kronecker product graph. Then we study the thickness of the Kronecker product of a graph with K_2. There are two reasons why we interested in it. One reason is that the upper bound for the thickness of the Kronecker product of two graphs we will provide relies on that of the Kronecker product of a graph with K_2. Another reason is that the planarity of the Kronecker product of two graphs have been characterized in [10], but a graph with K_2 is one of its missing cases. It's a difficult case, because there exist non-planar graphs whose Kronecker product with K_2 are planar graphs, see Figures 1 and 2 in [2] for example. In Sections 3 and 4, we provide the exact thickness numbers for the Kronecker product graphs $K_n \times K_2$, $K_{m,n} \times K_2$ and $K_{n,n,n} \times K_2$.

For undefined terminology, see [5].

2 Thickness of the Kronecker product graph $G \times H$

A *k-edge-coloring* of a graph G is a mapping $f \colon E(G) \to S$, where S is a set of k colors. A k-edge-coloring is proper if incident edges have different colors. A graph is *k-edge-colorable* if it has a proper k-edge-coloring. The *edge chromatic number* $\chi'(G)$ of a graph G is the least k such that G is k-edge-colorable.

Theorem 2.1. *Let G and H be two simple graphs on at least two vertices, then*

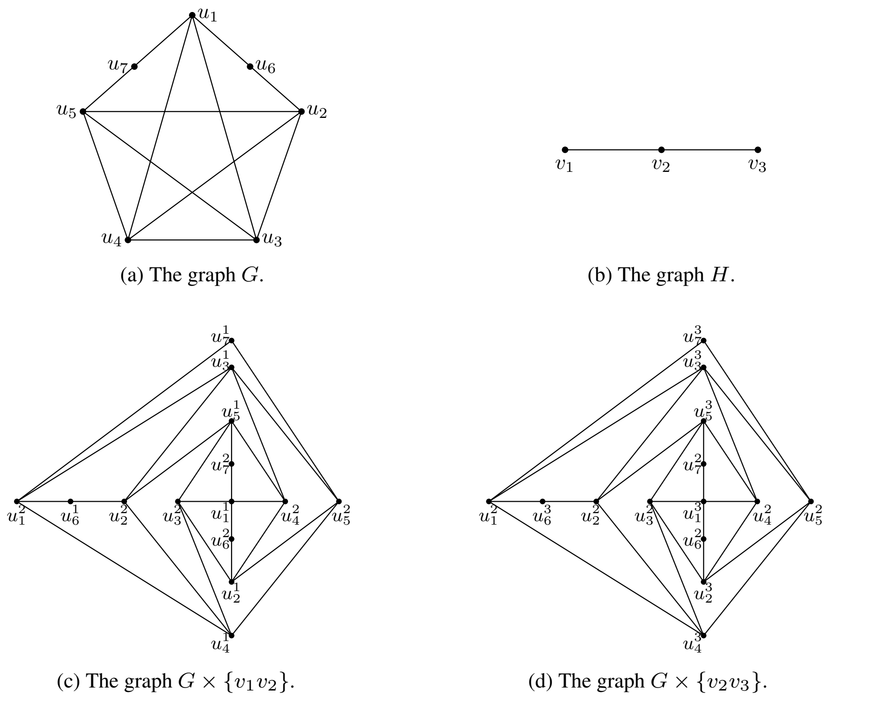
$$\left\lceil \frac{2|E(G)||E(H)|}{3|V(G)||V(H)| - 6} \right\rceil \leq \theta(G \times H) \leq \min\{\chi'(H)\theta(G \times K_2), \chi'(G)\theta(H \times K_2)\},$$

in which $\chi'(H)$ and $\chi'(G)$ are edge chromatic number of H and G respectively.

Proof. It is easy to observe that the number of edges in $G \times H$ is $|E(G \times H)| = 2|E(G)||E(H)|$ and the number of vertices in $G \times H$ is $|V(G \times H)| = |V(G)||V(H)|$. From the Euler's Formula, the planar graph with $|V(G)||V(H)|$ vertices, has at most $3|V(G)||V(H)| - 6$ edges, the lower bound follows.

The $\chi'(H)$-edge-coloring of H can be seen as a partition $\{M_1, \ldots, M_{\chi'(H)}\}$ of $E(H)$, in which M_i denotes the set of edges assigned color i ($1 \leq i \leq \chi'(H)$). Then M_i is a matching and $E(H) = M_1 \cup \cdots \cup M_{\chi'(H)}$. Because $G \times H = \cup_{i=1}^{\chi'(H)}(G \times M_i)$ and $\theta(G \times M_i) = \theta(G \times K_2)$, we have $\theta(G \times H) \leq \chi'(H)\theta(G \times K_2)$. With the same argument, we have $\theta(G \times H) \leq \chi'(G)\theta(H \times K_2)$. The upper bound can be derived. $\square$

In the following, we will give examples to show both the lower and upper bound in Theorem 2.1 are sharp. Let G and H be the graphs as shown in Figure 2(a) and (b) respectively. Figure 2(c) illustrates a planar embedding of the graph $G \times \{v_1v_2\}$, in which we denote the vertex (u_i, v_j) by u_i^j, $1 \leq i \leq 7$, $1 \leq j \leq 2$. So the thickness of $G \times \{v_1v_2\}$ is one which meets the lower bound in Theorem 2.1. Figure 2(d) illustrates a planar embedding of the graph $G \times \{v_2v_3\}$ which is isomorphic to $G \times \{v_1v_2\}$. Because $G \times H = G \times \{v_1v_2\} \cup G \times \{v_2v_3\}$, we get a planar subgraph decomposition of $G \times H$ with two subgraphs, which shows the thickness of $G \times H$ is not more than two. On the other hand, the graph $G \times H$ contains a subdivision of K_5 which is exhibited in Figure 2(e), so $G \times H$ is not a planar graph, its thickness is greater than one. Therefore, the thickness of $G \times H$ is two which meets the upper bound in Theorem 2.1.



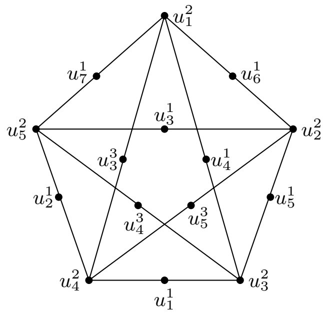
(a) The graph G.

(b) The graph H.

(c) The graph $G \times \{v_1v_2\}$.

(d) The graph $G \times \{v_2v_3\}$.

Figure 2: An example to show both lower and upper bounds in Theorem 2.1 are sharp.



(e) A subgraph of $G \times H$.

Figure 2: An example to show both lower and upper bounds in Theorem 2.1 are sharp.

The graph $G \times H$ has a triangle if and only if both G and H have triangles. If $G \times H$ does not contain any triangles, from the Euler's Formula, the planar graph with $|V(G)||V(H)|$ vertices, has at most $2|V(G)||V(H)| - 4$ edges, a tighter lower bound can be derived.

Theorem 2.2. *Let G and H be two simple graphs on at least two vertices. If $G \times H$ does not contain any triangles, then*

$$\left\lceil \frac{|E(G)||E(H)|}{|V(G)||V(H)| - 2} \right\rceil \leq \theta(G \times H) \leq \min\{\chi'(H)\theta(G \times K_2), \chi'(G)\theta(H \times K_2)\}.$$

3 The thickness of $K_n \times K_2$ and $K_{m,n} \times K_2$

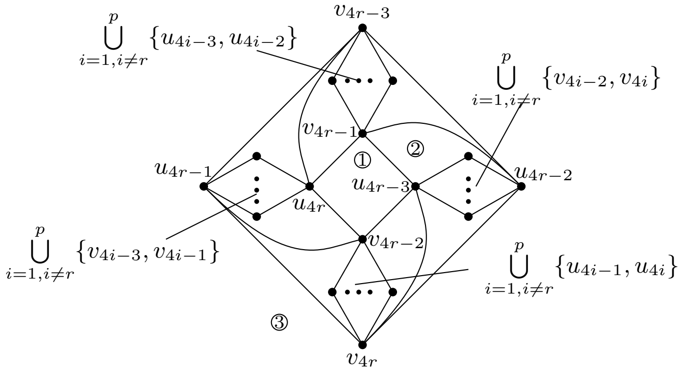
In this section, by making use of the thickness number of $K_{n,n}$ and a known planar decomposition of $K_{n,n}$ as shown in Lemmas 3.1 and 3.2 respectively, we will obtain the exact thickness numbers of $K_n \times K_2$ and $K_{m,n} \times K_2$.

Let G be a simple graph with n vertices, $V(G) = \{v_1, \ldots, v_n\}$ and $V(K_2) = \{1, 2\}$. Then $G \times K_2$ is a bipartite graph, the two vertex parts are $\{(v_i, 1) \mid 1 \leq i \leq n\}$ and $\{(v_i, 2) \mid 1 \leq i \leq n\}$, so $G \times K_2$ is a subgraph of $K_{n,n}$ which shows that $\theta(G \times K_2) \leq \theta(K_{n,n})$. Although the thickness of the complete bipartite $K_{m,n}$ have not been solved completely, when $m = n$, the following result is known.

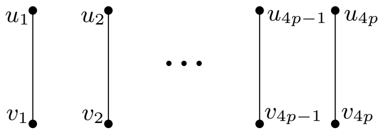
Lemma 3.1 ([4]). *The thickness of the complete bipartite graph $K_{n,n}$ is*

$$\theta(K_{n,n}) = \left\lceil \frac{n+2}{4} \right\rceil.$$

When $n = 4p$ ($p \geq 1$), Chen and Yin [8] gave a planar subgraphs decomposition of $K_{4p,4p}$ with $p+1$ planar subgraphs $G_1, \ldots, G_{p+1}$. Denote the two vertex parts of $K_{4p,4p}$ by $U = \{u_1, \ldots, u_{4p}\}$ and $V = \{v_1, \ldots, v_{4p}\}$, Figure 3 shows their planar subgraphs decomposition of $K_{4p,4p}$, in which for each G_r ($1 \leq r \leq p$), both v_{4r-3} and v_{4r-1} join to each vertex in set $\bigcup_{i=1, i \neq r}^{p} \{u_{4i-3}, u_{4i-2}\}$, both v_{4r-2} and v_{4r} join to each vertex in set $\bigcup_{i=1, i \neq r}^{p} \{u_{4i-1}, u_{4i}\}$, both u_{4r-1} and u_{4r} join to each vertex in set $\bigcup_{i=1, i \neq r}^{p} \{v_{4i-3}, v_{4i-1}\}$, and both u_{4r-3} and u_{4r-2} join to each vertex in set $\bigcup_{i=1, i \neq r}^{p} \{v_{4i-2}, v_{4i}\}$. Notice that G_{p+1} is a perfect matching of $K_{4p,4p}$, the edge set of it is $\{u_i v_i \mid 1 \leq i \leq 4p\}$.



(a) The graph G_r $(1 \leq r \leq p)$.



(b) The graph G_{p+1}.

Figure 3: A planar decomposition of $K_{4p,4p}$.

Lemma 3.2 ([8]). *Suppose $K_{n,n}$ is a complete bipartite graph with two vertex parts $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. When $n = 4p$, there exists a planar subgraphs decomposition of $K_{4p,4p}$ with $p+1$ planar subgraphs $G_1, \ldots, G_{p+1}$ in which G_{p+1} is a perfect matching of $K_{4p,4p}$ with edge set $\{u_i v_i \mid 1 \leq i \leq 4p\}$.*

Theorem 3.3. *The thickness of the Kronecker product of K_n and K_2 is*

$$\theta(K_n \times K_2) = \left\lceil \frac{n}{4} \right\rceil.$$

Proof. Suppose that the vertex sets of K_n and K_2 are $\{x_1, \ldots, x_n\}$ and $\{1, 2\}$ respectively. The graph $K_n \times K_2$ is a bipartite graph whose two vertex parts are $\{(x_i, 1) \mid 1 \leq i \leq n\}$ and $\{(x_i, 2) \mid 1 \leq i \leq n\}$, and edge set is $\{(x_i, 1)(x_j, 2) \mid 1 \leq i, j \leq n, i \neq j\}$. For $1 \leq i \leq n$, $1 \leq k \leq 2$, we denote the vertex (x_i, k) of $K_n \times K_2$ by x_i^k for simplicity.

Since $|E(K_n \times K_2)| = n(n-1)$ and $|V(K_n \times K_2)| = 2n$, from Theorem 2.2, we have

$$\theta(K_n \times K_2) \geq \left\lceil \frac{n(n-1)}{4n-4} \right\rceil = \left\lceil \frac{n}{4} \right\rceil. \tag{3.1}$$

In the following, we will construct planar decompositions of $K_n \times K_2$ with $\lceil \frac{n}{4} \rceil$ subgraphs to complete the proof.

Case 1. When $n = 4p$.
Suppose that $K_{n,n}$ is a complete bipartite graph with vertex partition (X^1, X^2) in which $X^1 = \{x_1^1, \ldots, x_n^1\}$ and $X^2 = \{x_1^2, \ldots, x_n^2\}$. The graph G_{p+1} is a perfect matching of $K_{4p,4p}$ whose edge set is $\{x_i^1 x_i^2 \mid 1 \leq i \leq n\}$, then $K_n \times K_2 = K_{n,n} - G_{p+1}$. From Lemma 3.2, there exists a planar decomposition $\{G_1, \ldots, G_p\}$ of $K_n \times K_2$ in which G_r $(1 \leq r \leq p)$ is isomorphic to the graph in Figure 3(a). Therefore, $\theta(K_{4p} \times K_2) \leq p$.

Case 2. When $n = 4p + 2$.
When $p \geq 1$, we draw a graph G'_{p+1} as shown in Figure 4, then $\{G_1, \dots, G_p, G'_{p+1}\}$ is a planar decomposition of $K_{4p+2} \times K_2$ with $p+1$ subgraphs, so we have $\theta(K_{4p+2} \times K_2) \leq p+1$. When $n = 2$, $K_2 \times K_2 = 2K_2$ is a planar graph.

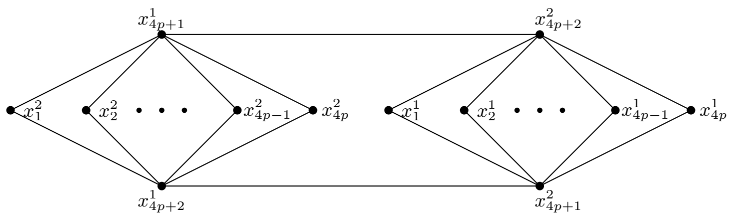


Figure 4: The graph G'_{p+1}.

Case 3. When $n = 4p + 1$ and $n = 4p + 3$.
Because $K_{4p+1} \times K_2$ is a subgraph of $K_{4p+2} \times K_2$, we have $\theta(K_{4p+1} \times K_2) \leq \theta(K_{4p+2} \times K_2) = p + 1$. Similarly, when $n = 4p + 3$, we have $\theta(K_{4p+3} \times K_2) \leq \theta(K_{4(p+1)} \times K_2) = p + 1$.

Summarizing Cases 1, 2 and 3, we have

$$\theta(K_n \times K_2) \leq \left\lceil \frac{n}{4} \right\rceil . \tag{3.2}$$

Theorem follows from inequalities (3.1) and (3.2). □

Theorem 3.4. *Let G be a simple graph on n ($n \geq 2$) vertices, then*

$$\left\lceil \frac{E(G)}{2n-2} \right\rceil \leq \theta(G \times K_2) \leq \left\lceil \frac{n}{4} \right\rceil .$$

Proof. Because $G \times K_2$ is a subgraph of $K_n \times K_2$, we have $\theta(G \times K_2) \leq \theta(K_n \times K_2)$. Combining it with Theorems 2.2 and 3.3, the theorem follows. □

Lemma 3.5 ([10]). *The Kronecker product of $K_{m,n}$ and $K_{p,q}$ is a disjoint union $K_{mp,nq} \cup K_{mq,np}$.*

Theorem 3.6. *The thickness of the Kronecker product of $K_{m,n}$ and $K_{p,q}$ is*

$$\theta(K_{m,n} \times K_{p,q}) = \max\{\theta(K_{mp,nq}), \theta(K_{mq,np})\}.$$

Proof. From Lemma 3.5, the proof is straightforward. □

Because K_2 is also $K_{1,1}$, the following corollaries are easy to get, from Theorem 3.6 and Lemma 3.1.

Corollary 3.7. *The thickness of the Kronecker product of $K_{m,n}$ and K_2 is*

$$\theta(K_{m,n} \times K_2) = \theta(K_{m,n}).$$

Corollary 3.8. *The thickness of the Kronecker product of $K_{n,n}$ and K_2 is*

$$\theta(K_{n,n} \times K_2) = \left\lceil \frac{n+2}{4} \right\rceil .$$

4 The thickness of the Kronecker product graph $K_{n,n,n} \times K_2$

Let (X, Y, Z) be the vertex partition of the complete tripartite graph $K_{l,m,n}$ $(l \leq m \leq n)$ in which $X = \{x_1, \ldots, x_l\}$, $Y = \{y_1, \ldots, y_m\}$, $Z = \{z_1, \ldots, z_n\}$. Let $\{1, 2\}$ be the vertex set of K_2. We denote the vertex (v, k) of $K_{l,m,n} \times K_2$ by v^k in which $v \in V(K_{l,m,n})$ and $k \in \{1, 2\}$. For $k = 1, 2$, we denote $X^k = \{x_1^k, \ldots, x_l^k\}$, $Y^k = \{y_1^k, \ldots, y_m^k\}$ and $Z^k = \{z_1^k, \ldots, z_n^k\}$. In Figure 5, we draw a sketch of the graph $K_{l,m,n} \times K_2$, in which the edge joining two vertex set indicates that each vertex in one vertex set is adjacent to each vertex in another vertex set. Suppose $G(X^1, Y^2)$ is the graph induced by the vertex sets X^1 and Y^2 of $K_{l,m,n} \times K_2$, then $G(X^1, Y^2)$ is isomorphic to $K_{l,m}$, the graphs $G(Y^1, Z^2)$, $G(Z^1, X^2)$, $G(X^2, Y^1)$, $G(Y^2, Z^1)$ and $G(Z^2, X^1)$ are defined analogously. We define

$$G^1 = G(X^1, Y^2) \cup G(Y^1, Z^2) \cup G(Z^1, X^2)$$

and

$$G^2 = G(X^2, Y^1) \cup G(Y^2, Z^1) \cup G(Z^2, X^1),$$

then $K_{l,m,n} \times K_2 = G^1 \cup G^2$.

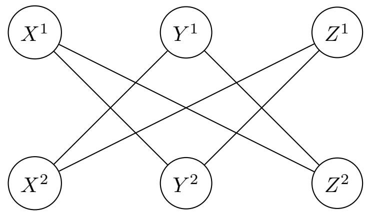


Figure 5: The graph $K_{l,m,n} \times K_2$.

Theorem 4.1. *The thickness of the Kronecker product graph $K_{l,m,n} \times K_2$ $(l \leq m \leq n)$ satisfies the inequality*

$$\left\lceil \frac{lm + ln + mn}{2(l + m + n) - 2} \right\rceil \leq \theta(K_{l,m,n} \times K_2) \leq 2\theta(K_{m,n}).$$

Proof. From Theorem 3.4, one can get the lower bound in this theorem easily. Any two graphs of $G(X^1, Y^2)$, $G(Y^1, Z^2)$ and $G(Z^1, X^2)$ are disjoint with each other and $l \leq m \leq n$, so we have

$$\theta(G^1) \leq \max\{\theta(G(X^1, Y^2), \theta(G(Y^1, Z^2), \theta(G(Z^1, X^2)\} = \theta(K_{m,n}).$$

Similarly, we have

$$\theta(G^2) \leq \max\{\theta(G(X^2, Y^1), \theta(G(Y^2, Z^1), \theta(G(Z^2, X^1))\} = \theta(K_{m,n}).$$

Due to the graph $K_{l,m,n} \times K_2 = G^1 \cup G^2$, we have $\theta(K_{l,m,n} \times K_2) \leq 2\theta(K_{m,n})$. Summarizing the above, the theorem is obtained. $\square$

In the following, we will construct planar decompositions of $K_{n,n,n} \times K_2$ when $n = 4p, 4p+1, 4p+3$ in Lemmas 4.2, 4.4 and 4.5 respectively. Then combining these lemmas with Theorem 2.2, we will get the thickness of $K_{n,n,n} \times K_2$ and we will see when $n = 4p+2$, the upper and lower bound in Theorem 4.1 are equal, so both bounds in Theorem 4.1 are sharp.

Lemma 4.2. *When $n = 4p$, there exists a planar decomposition of the Kronecker product graph $K_{n,n,n} \times K_2$ with $2p+1$ subgraphs.*

Proof. Because $|X^k| = |Y^k| = |Z^k| = n$ $(k = 1, 2)$, all the graphs $G(X^1, Y^2)$, $G(Y^1, Z^2), G(Z^1, X^2), G(X^2, Y^1), G(Y^2, Z^1), G(Z^2, X^1)$ are isomorphic to $K_{n,n}$.

Let $\{G_1, \dots, G_{p+1}\}$ be the planar decomposition of $K_{n,n}$ as shown in Figure 3. For $1 \le r \le p+1$, G_r is a bipartite graph, so we also denote it by $G_r(V, U)$. In $G_r(V, U)$, we replace the vertex set V by X^1, U by Y^2, i.e., for each $1 \le i \le n$, replace the vertex v_i by x_i^1, and u_i by y_i^2, then we get graph $G_r(X^1, Y^2)$. Analogously, we obtain graphs $G_r(Y^1, Z^2), G_r(Z^1, X^2), G_r(X^2, Y^1), G_r(Y^2, Z^1)$ and $G_r(Z^2, X^1)$.

For $1 \le r \le p+1$, let

$$G_r^1 = G_r(X^1, Y^2) \cup G_r(Y^1, Z^2) \cup G_r(Z^1, X^2)$$

and

$$G_r^2 = G_r(X^2, Y^1) \cup G_r(Y^2, Z^1) \cup G_r(Z^2, X^1).$$

Because $G_r(X^1, Y^2), G_r(Y^1, Z^2), G_r(Z^1, X^2)$ are all planar graphs and they are disjoint with each other, G_r^1 is a planar graph. For the same reason, we have that G_r^2 is also a planar graph.

Let graph G_{p+1} be the graph $G_{p+1}^1 \cup G_{p+1}^2$. We have

$$\begin{aligned} G_{p+1} &= G_{p+1}^1 \cup G_{p+1}^2 \\ &= \left\{ \bigcup_{i=1}^{n} (x_i^1 y_i^2 \cup y_i^1 z_i^2 \cup z_i^1 x_i^2) \right\} \cup \left\{ \bigcup_{i=1}^{n} (x_i^2 y_i^1 \cup y_i^2 z_i^1 \cup z_i^2 x_i^1) \right\} \\ &= \bigcup_{i=1}^{n} (x_i^1 y_i^2 z_i^1 x_i^2 y_i^1 z_i^2 x_i^1). \end{aligned}$$

It is easy to see G_{p+1} consists of n disjoint cycles of length 6, hence G_{p+1} is a planar graph.

Because

$$G(X^1, Y^2) = \bigcup_{r=1}^{p+1} G_r(X^1, Y^2), \qquad G(Y^1, Z^2) = \bigcup_{r=1}^{p+1} G_r(Y^1, Z^2),$$

$$G(Z^1, X^2) = \bigcup_{r=1}^{p+1} G_r(Z^1, X^2), \qquad G(X^2, Y^1) = \bigcup_{r=1}^{p+1} G_r(X^2, Y^1),$$

and

$$G(Y^2, Z^1) = \bigcup_{r=1}^{p+1} G_r(Y^2, Z^1), \qquad G(Z^2, X^1) = \bigcup_{r=1}^{p+1} G_r(Z^2, X^1),$$

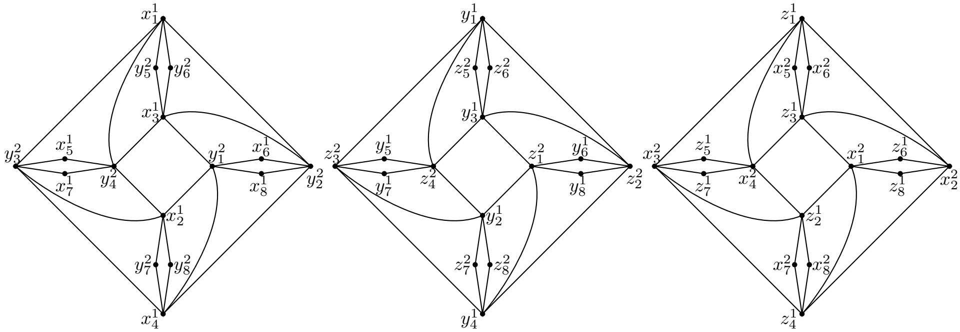
we have

$$\begin{aligned} K_{n,n,n} \times K_2 &= G^1 \cup G^2 \\ &= \bigcup_{r=1}^{p+1} (G_r^1 \cup G_r^2) \\ &= \bigcup_{r=1}^{p} (G_r^1 \cup G_r^2) \cup G_{p+1}. \end{aligned}$$

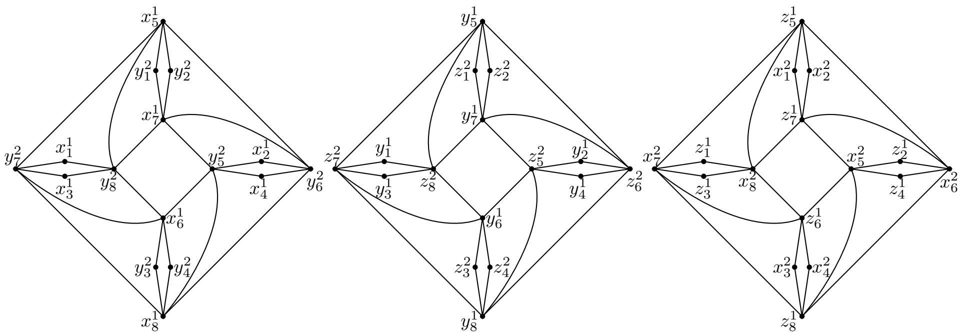
So we get a planar decomposition of $K_{4p,4p,4p} \times K_2$ with $2p+1$ subgraphs $G_1^1, \dots, G_p^1$, $G_1^2, \dots, G_p^2, G_{p+1}$. The proof is completed. $\square$

We draw the planar decomposition of $K_{8,8,8} \times K_2$ as shown in Figure 6.

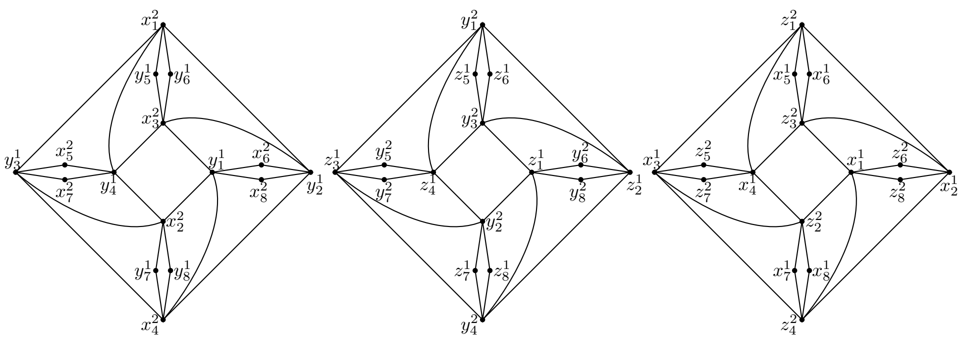
Lemma 4.3 ([5]). *Let G be a planar graph, and let f be a face in some planar embedding of G. Then G admits a planar embedding whose outer face has the same boundary as f.*



(a) The graph G_1^1.

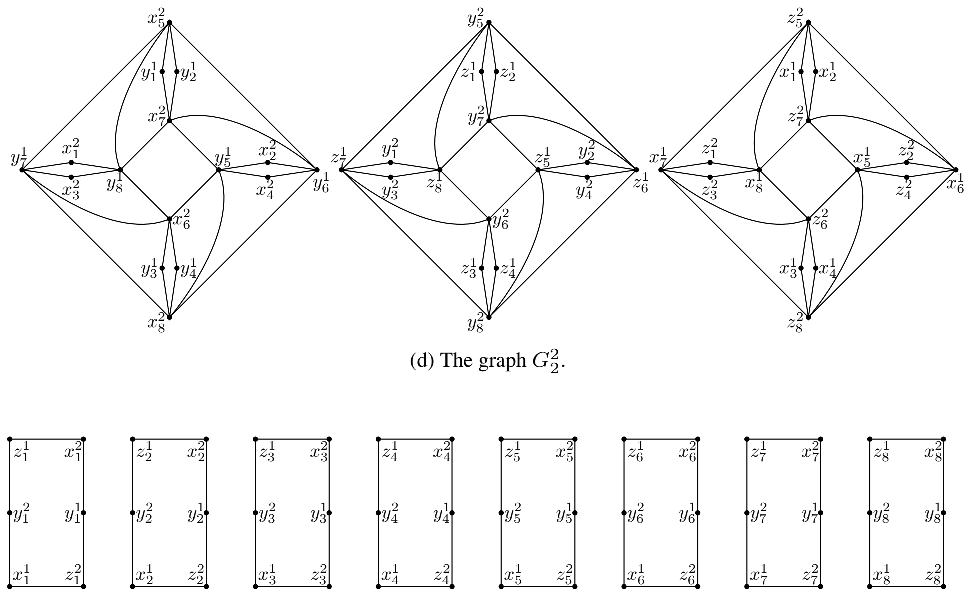


(b) The graph G_2^1.



(c) The graph G_1^2.

Figure 6: A planar decomposition of $K_{8,8,8} \times K_2$.



(d) The graph G_2^2.

(e) The graph G_3.

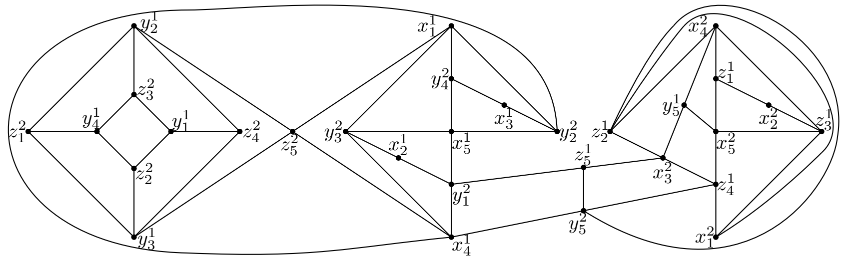
Figure 6: A planar decomposition of $K_{8,8,8} \times K_2$.

Lemma 4.4. *When $n = 4p + 1$, there exists a planar decomposition of the Kronecker product graph $K_{n,n,n} \times K_2$ with $2p + 1$ subgraphs.*

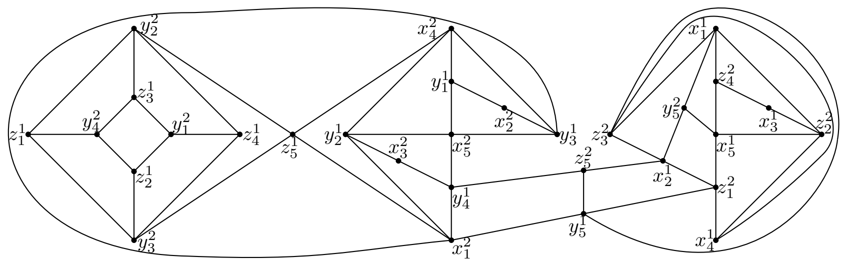
Proof. **Case 1.** When $p \leq 1$.
When $p = 0$, the Kronecker product graph $K_{1,1,1} \times K_2$ is a cycle of length 6, so $K_{1,1,1} \times K_2$ is a planar graph. When $p = 1$, as shown in Figure 7, we give a planar decomposition of $K_{5,5,5} \times K_2$ with three subgraphs A, B and C.

Case 2. When $p \geq 2$.
Suppose that $\{G_1^1, \ldots, G_p^1, G_1^2, \ldots, G_p^2, G_{p+1}\}$ is the planar decomposition of $K_{4p,4p,4p} \times K_2$ as provided in the proof of Lemma 4.2. By adding vertices $x_{4p+1}^1, x_{4p+1}^2, y_{4p+1}^1, y_{4p+1}^2, z_{4p+1}^1, z_{4p+1}^2$ to each graph in this decomposition, and some modifications of adding and deleting edges to these graphs, a planar decomposition of $K_{4p+1,4p+1,4p+1} \times K_2$ will be obtained.

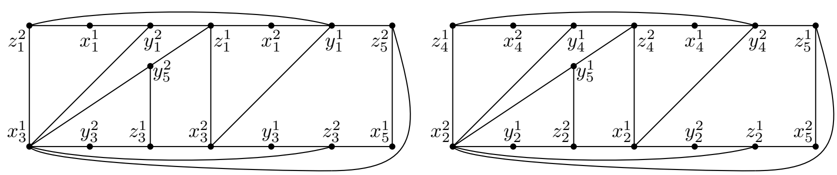
For convenience, in Figure 3(a) we label some faces of G_r ($1 \leq r \leq p$) with face $1, 2$ and 3. As indicated in Figure 3(a), the face 1 is bounded by $v_{4r-1}u_{4r-3}v_{4r-2}u_{4r}$, the face 3 is its outer face, bounded by $v_{4r-3}u_{4r-2}v_{4r}u_{4r-1}$. The face 2 is bounded by $u_{4r-3}v_{4r-1}u_{4r-2}v_j$ in which vertex v_j can be any vertex of $\bigcup_{i=1,i\neq r}^{p}\{v_{4i-2}, v_{4i}\}$. Because u_{4r-3} and u_{4r-2} in G_r ($1 \leq r \leq p$) is joined by $2p-2$ edge-disjoint paths of length two that we call parallel paths, we can change the order of these parallel paths without changing the planarity of G_r. Analogously, we can change the order of parallel paths between u_{4r-1} and u_{4r}, v_{4r-3} and v_{4r-1}, v_{4r-2} and v_{4r}. In addition, the subscripts of all the vertices are taken module $4p$, except that of the new added vertices $x_{4p+1}^1, x_{4p+1}^2, y_{4p+1}^1, y_{4p+1}^2, z_{4p+1}^1$ and z_{4p+1}^2.



(a) The graph A.



(b) The graph B.



(c) The graph C.

Figure 7: A planar decomposition of $K_{5,5,5} \times K_2$.

Step 1: Add the vertices x^1_{4p+1} and y^2_{4p+1} to graph $G_r(X^1, Y^2)$.
Place vertices x^1_{4p+1} and y^2_{4p+1} in face 1 and face 2 of $G_r(X^1, Y^2)$, respectively. Join x^1_{4p+1} to vertices y^2_{4r-3} and y^2_{4r}. Change the order of the parallel paths between y^2_{4r-2} and y^2_{4r-3}, such that $x^1_{4r+2} \in \bigcup_{i=1,i\neq r}^{p}\{x^1_{4i-2}, x^1_{4i}\}$ are incident with the face 2, and join y^2_{4p+1} to both x^1_{4r-1} and x^1_{4r+2}.

Step 2: Add the vertices x^2_{4p+1} and y^1_{4p+1} to graph $G_r(X^2, Y^1)$.
Similar to step 1, place x^2_{4p+1} and y^1_{4p+1} in face 1 and face 2 of $G_r(X^2, Y^1)$, respectively. Join x^2_{4p+1} to both y^1_{4r-3} and y^1_{4r}, join y^1_{4p+1} to both x^2_{4r-1} and $x^2_{4r+2} \in \bigcup_{i=1,i\neq r}^{p}\{x^2_{4i-2}, x^2_{4i}\}$.

Step 3: Add the vertices y^1_{4p+1} and z^2_{4p+1} to graph $G_r(Y^1, Z^2)$.
Place y^1_{4p+1} in face 3 of $G_r(Y^1, Z^2)$ and join it to vertices z^2_{4r-2} and z^2_{4r-1}. Place z^2_{4p+1} in face 1 of $G_r(Y^1, Z^2)$ and join it to vertices y^1_{4r-2} and y^1_{4r-1}.

Step 4: Add the vertices y^2_{4p+1} and z^1_{4p+1} to graph $G_r(Y^2, Z^1)$.
Place y^2_{4p+1} in face 3 of $G_r(Y^2, Z^1)$ and join it to vertices z^1_{4r-2} and z^1_{4r-1}. Place z^1_{4p+1} in face 1 of $G_r(Y^2, Z^1)$ and join it to vertices y^2_{4r-2} and y^2_{4r-1}.

Step 5: Add the vertices z^1_{4p+1} and x^2_{4p+1} to graph $G_r(Z^1, X^2)$.
Place z^1_{4p+1} in face 1 of $G_r(Z^1, X^2)$ and join it to vertices x^2_{4r-3} and x^2_{4r}. Place x^2_{4p+1} in face 3 of $G_r(Z^1, X^2)$ and join it to vertices z^1_{4r-3} and z^1_{4r}.

Step 6: Add the vertices z^2_{4p+1} and x^1_{4p+1} to graph $G_r(Z^2, X^1)$.
Place z^2_{4p+1} in face 1 of $G_r(Z^2, X^1)$ and join it to vertices x^1_{4r-3} and x^1_{4r}. Place x^1_{4p+1} in face 3 of $G_r(Z^2, X^1)$ and join it to vertices z^2_{4r-3} and z^2_{4r}.

We denote the above graphs we obtain from Steps 1–6 by $\widehat{G}_r(X^1, Y^2), \widehat{G}_r(X^2, Y^1), \widehat{G}_r(Y^1, Z^2), \widehat{G}_r(Y^2, Z^1), \widehat{G}_r(Z^1, X^2)$ and $\widehat{G}_r(Z^2, X^1)$ respectively.

Let

$$\widehat{G}^1_r = \widehat{G}_r(X^1, Y^2) \cup \widehat{G}_r(Y^1, Z^2) \cup \widehat{G}_r(Z^1, X^2)$$

and

$$\widehat{G}^2_r = \widehat{G}_r(X^2, Y^1) \cup \widehat{G}_r(Y^2, Z^1) \cup \widehat{G}_r(Z^2, X^1).$$

Step 7: Add the edges $z^1_{4r}x^2_{4r}, y^1_{4r-1}z^2_{4r-1}, z^1_{4r-2}y^2_{4r-2}, x^1_{4r-3}z^2_{4r-3}$ and $z^2_{4r}x^1_{4r}, y^2_{4r-1}z^1_{4r-1}, z^2_{4r-2}y^1_{4r-2}, x^2_{4r-3}z^1_{4r-3}$ to graphs $\widehat{G}^1_r$ and $\widehat{G}^2_r$ respectively, $1 \leq r \leq p$.
For graph $\widehat{G}_r(Y^1, Z^2) \subset \widehat{G}^1_r$, we delete the edge $y^1_{4r-3}z^2_{4r}$ and join the vertex y^1_{4r-1} to vertex z^2_{4r-1}, then we get a planar graph $\widetilde{G}_r(Y^1, Z^2)$. According to Lemma 4.3, the graph $\widetilde{G}_r(Y^1, Z^2)$ has a planar embedding whose outer face has the same boundary as face 2, then the vertex z^2_{4r-3} is on the boundary of this outer face.

For graph $\widehat{G}_r(Z^1, X^2) \subset \widehat{G}^1_r$, delete the edge $z^1_{4r-2}x^2_{4r-1}$ and join z^1_{4r} to x^2_{4r}, then we get a planar graph $\widetilde{G}_r(Z^1, X^2)$. According to Lemma 4.3, the graph $\widetilde{G}_r(Z^1, X^2)$ has a planar embedding whose outer face has boundary as $z^1_{4r}x^2_{4r}z^1_{4r-2}x^2_i z^1_{4r}$ ($x^2_i \in \bigcup^p_{i=1, i\neq r}\{x^2_{4i-1}, x^2_{4i}\}$), then the vertex z^1_{4r-2} is on the boundary of this outer face.

Since the vertices x^1_{4r-3} and y^2_{4r-2} are on the boundary of the outer face of the embedding of $\widehat{G}_r(X^1, Y^2) \subset \widehat{G}^1_r$, we can join x^1_{4r-3} to z^2_{4r-3}, y^2_{4r-2} to z^1_{4r-2} without edge crossing. Then we get a planar graph $\widetilde{G}^1_r$.

With the same process, for the graph G^2_r, we delete edges $y^2_{4r-3}z^1_{4r}$ and $z^2_{4r-2}x^1_{4r-1}$, join y^2_{4r-1} to z^1_{4r-1}, join z^2_{4r} to x^1_{4r}, join x^2_{4r-3} to z^1_{4r-3} and join y^1_{4r-2} to z^2_{4r-2}, then we get a planar graph $\widetilde{G}^2_r$.

Table 1 shows the edges that we add to G^1_r and G^2_r ($1 \leq r \leq p$) in Steps 1–7.

Step 8: The remaining edges form a planar graph $\widetilde{G}_{p+1}$.
The edges that belong to $K_{4p+1,4p+1,4p+1} \times K_2$ but not to any $\widetilde{G}^1_r, \widetilde{G}^2_r$ ($1 \leq r \leq p$) are shown in Table 2, in which the edges in the last two rows list the edges deleted in Step 7. The remaining edges form a graph, denote by $\widetilde{G}_{p+1}$. We draw a planar embedding of $\widetilde{G}_{p+1}$ in Figure 8, so $\widetilde{G}_{p+1}$ is a planar graph.

Table 1: The edges we add to G_r^1 and G_r^2 $(1 \le r \le p)$.

Edges		Subscript
$x_{4p+1}^1 y_i^2, x_{4p+1}^2 y_i^1,$ $x_{4p+1}^1 z_i^2, x_{4p+1}^2 z_i^1,$	$z_{4p+1}^1 x_i^2, z_{4p+1}^2 x_i^1,$ $x_i^1 z_i^2, x_i^2 z_i^1,$	$i = 4r-3, 4r.$
$y_{4p+1}^1 z_i^2, y_{4p+1}^2 z_i^1,$ $y_{4p+1}^1 x_i^2, y_{4p+1}^2 x_i^1,$	$z_{4p+1}^1 y_i^2, z_{4p+1}^2 y_i^1,$ $y_i^1 z_i^2, y_i^2 z_i^1,$	$i = 4r-2, 4r-1.$

Table 2: The edges of $\widetilde{G}_{p+1}$.

Edges	Subscript $(1 \le r \le p)$
$x_{4p+1}^1 y_i^2, x_{4p+1}^2 y_i^1, \quad z_{4p+1}^1 x_i^2, z_{4p+1}^2 x_i^1,$ $x_{4p+1}^1 z_i^2, x_{4p+1}^2 z_i^1, \quad x_i^1 z_i^2, x_i^2 z_i^1,$	$i = 4r-2, 4r-1.$
$y_{4p+1}^1 z_i^2, y_{4p+1}^2 z_i^1, \quad z_{4p+1}^1 y_i^2, z_{4p+1}^2 y_i^1,$ $y_{4p+1}^1 x_i^2, y_{4p+1}^2 x_i^1, \quad y_i^1 z_i^2, y_i^2 z_i^1,$	$i = 4r-3, 4r.$
$x_i^1 y_i^2, x_i^2 y_i^1,$	$i = 4r-3, 4r-2, 4r-1, 4r.$
$x_i^1 y_i^2, y_i^2 z_i^1, z_i^1 x_i^2, x_i^2 y_i^1, y_i^1 z_i^2, z_i^2 x_i^1,$	$i = 4p+1.$
$y_i^1 z_j^2, y_i^2 z_j^1,$	$i = 4r-3, j = 4r.$
$z_i^1 x_j^2, z_i^2 x_j^1,$	$i = 4r-2, j = 4r-1.$

Therefore $\{\widetilde{G}_1^1, \ldots, \widetilde{G}_p^1, \widetilde{G}_1^2, \ldots, \widetilde{G}_p^2, \widetilde{G}_{p+1}\}$ is a planar decomposition of $K_{4p+1,4p+1,4p+1} \times K_2$, the Lemma follows. □

Figure 9 illustrates a planar decomposition of $K_{9,9,9} \times K_2$ with five subgraphs.

A graph G is said to be thickness *t-minimal*, if $\theta(G) = t$ and every proper subgraphs of it have a thickness less than t.

Lemma 4.5. *When $n = 4p+3$, there exists a planar decomposition of Kronecker product graph $K_{4p+3,4p+3,4p+3} \times K_2$ with $2p+2$ subgraphs.*

Proof. **Case 1.** When $p = 0$.
As shown in Figure 10, we give a planar decomposition of $K_{3,3,3} \times K_2$ with 2 subgraphs.

Case 2. When $p \ge 1$.
The graph $K_{4p+3,4p+3}$ is a thickness $(p+2)$-minimal graph. Hobbs, Grossman [11] and Bouwer, Broere [6] proved it independently, by giving two different planar subgraphs decompositions $\{H_1, \ldots, H_{p+2}\}$ of $K_{4p+3,4p+3}$ in which H_{p+2} contains only one edge. Suppose that the two vertex parts of $K_{n,n}$ is $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_n\}$, the only one edge in the H_{p+2} is $v_a u_b$ (the edge is $v_1 u_1$ in [11] and $v_{4p+3} u_{4p-1}$ in [6]). For $1 \le i \le p+2$, H_i is a bipartite graph, so we also denote it by $H_i(V, U)$.

Because $K_{n,n,n} \times K_2 = G^1 \cup G^2$ in which $G^1 = G(X^1, Y^2) \cup G(Y^1, Z^2) \cup G(Z^1, X^2)$ and $G^2 = G(X^2, Y^1) \cup G(Y^2, Z^1) \cup G(Z^2, X^1)$, $|X^i| = |Y^i| = |Z^i| = n$ $(i = 1, 2)$, all the graphs $G(X^1, Y^2), G(Y^1, Z^2), G(Z^1, X^2), G(X^2, Y^1), G(Y^2, Z^1)$ and $G(Z^2, X^1)$ are isomorphic to $K_{n,n}$.

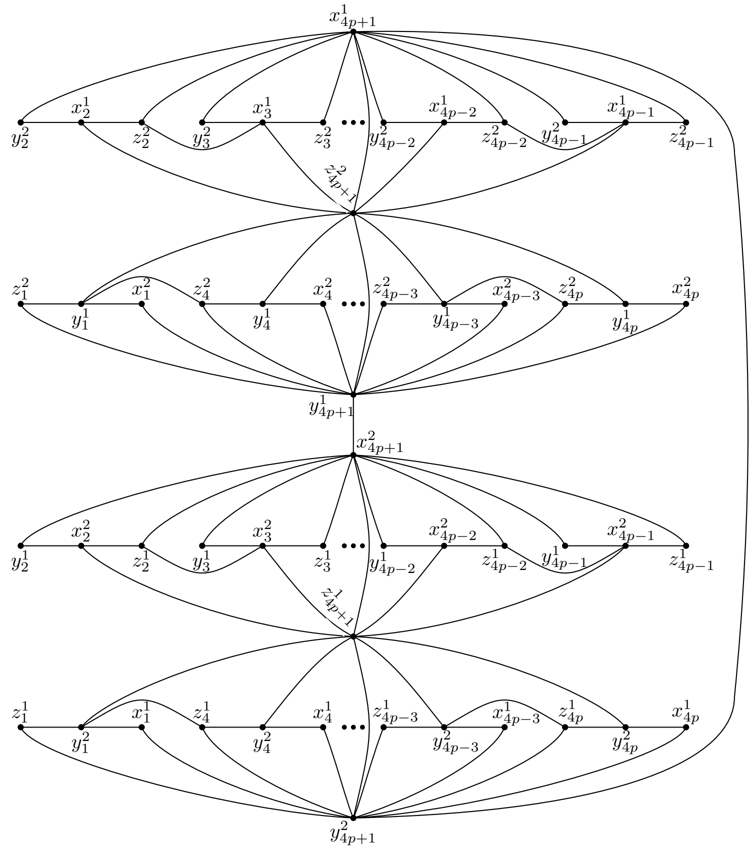
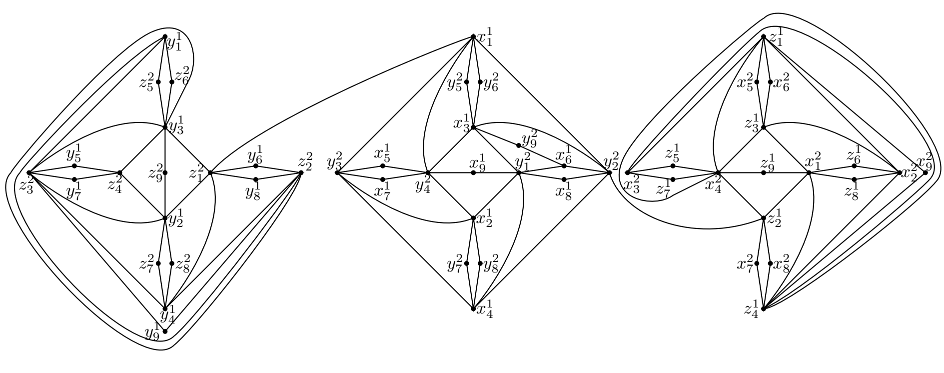
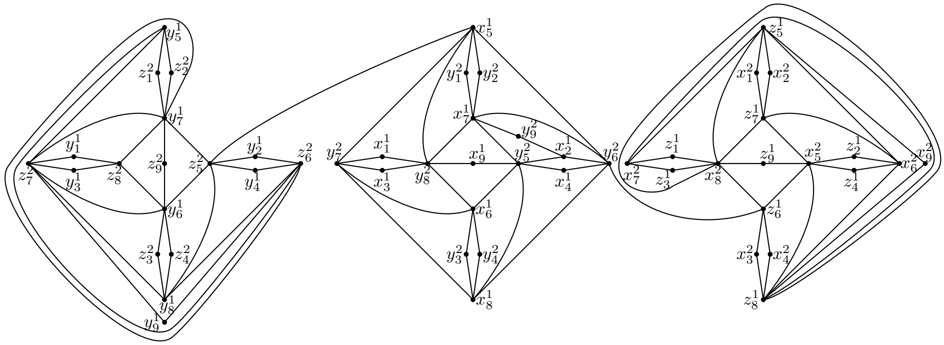


Figure 8: The graph $\widetilde{G}_{p+1}$.

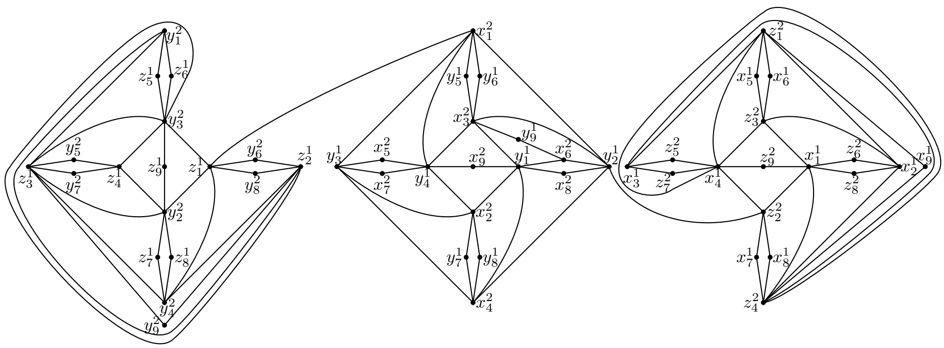


(a) The graph $\widetilde{G}^1_1$.

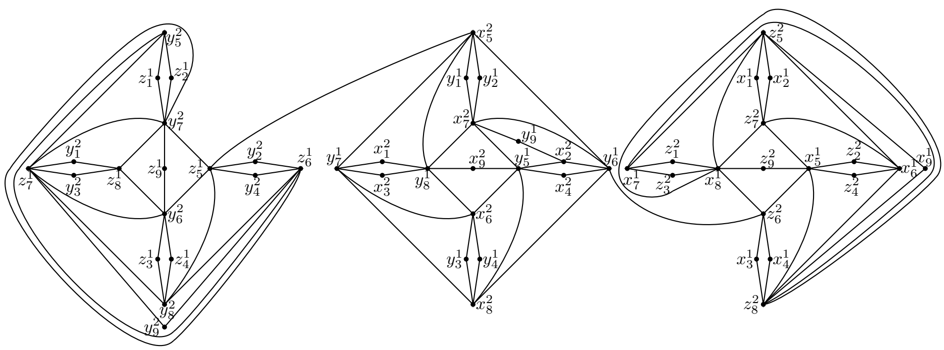
Figure 9: A planar decomposition of $K_{9,9,9} \times K_2$.



(b) The graph $\widetilde{G}_2^1$.

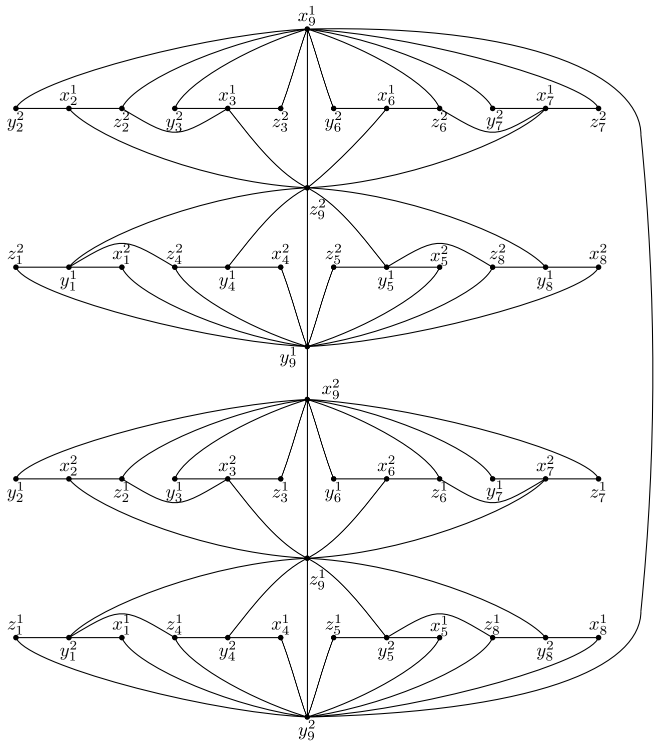


(c) The graph $\widetilde{G}_1^2$.



(d) The graph $\widetilde{G}_2^2$.

Figure 9: A planar decomposition of $K_{9,9,9} \times K_2$.



(e) The graph $\widetilde{G}_3$.

Figure 9: A planar decomposition of $K_{9,9,9} \times K_2$.

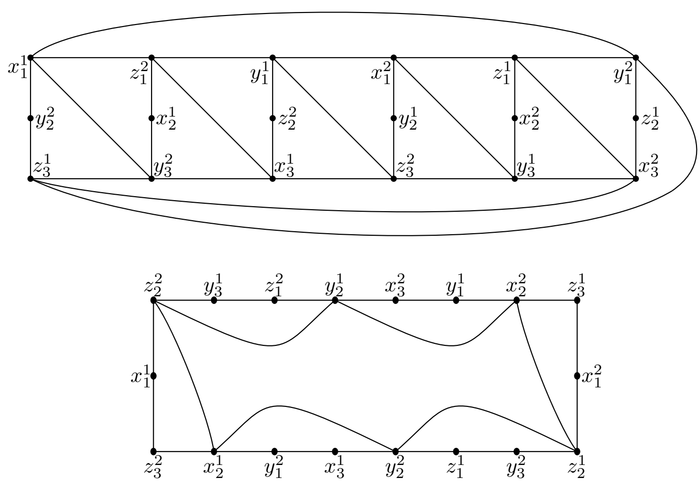


Figure 10: A planar decomposition of $K_{3,3,3} \times K_2$.

For graph $H_i(V,U)$ ($1 \le i \le p+2$), we replace the vertex set V by X^1, U by Y^2, i.e., for each $1 \le t \le n$, replace the vertex v_t by x_t^1, and u_t by y_t^2, then we get a graph $H_i(X^1, Y^2)$. Analogously, we can obtain graphs $H_i(Y^1, Z^2)$, $H_i(Z^1, X^2)$, $H_i(X^2, Y^1)$, $H_i(Y^2, Z^1)$ and $H_i(Z^2, X^1)$. For $1 \le i \le p+2$, let

$$H_i^1 = H_i(X^1, Y^2) \cup H_i(Y^1, Z^2) \cup H_i(Z^1, X^2),$$

then H_i^1 is a planar graph, because $H_i(X^1, Y^2)$, $H_i(Y^1, Z^2)$, $H_i(Z^1, X^2)$ are disjoint with each other. For the same reason, the graph

$$H_i^2 = H_i(X^2, Y^1) \cup H_i(Y^2, Z^1) \cup H_i(Z^2, X^1)$$

is also a planar graph, $1 \le i \le p+2$. And we have

$$K_{4p+3,4p+3,4p+3} \times K_2 = G^1 \cup G^2 = \bigcup_{i=1}^{p+2} (H_i^1 \cup H_i^2),$$

in which $E(H_{p+2}^1) = \{x_a^1 y_b^2, y_a^1 z_b^2, z_a^1 x_b^2\}$ and $E(H_{p+2}^2) = \{x_a^2 y_b^1, y_a^2 z_b^1, z_a^2 x_b^1\}$.

In the following, we will add edges in $E(H_{p+2}^1)$ to graphs H_1^2 and H_2^2, add edges in $E(H_{p+2}^2)$ to graphs H_1^1 and H_1^2 to complete the proof. From Lemma 4.3, there exists a planar embedding of $H_1(Y^1, Z^2)$ such that vertex z_a^2 on the boundary of its outer face, exists a planar embedding of $H_1(X^1, Y^2)$ such that x_b^1 on the boundary of its outer face. Then we join z_a^2 to x_b^1 without edge crossing. Suppose y_b^1 is on the boundary of inner face F of the embedding of $H_1(Y^1, Z^2)$, put the embedding of $H_1(Z^1, X^2)$ in face F with x_a^2 on the boundary of its outer face, then we join x_a^2 to y_b^1 without edge crossing. After adding both $x_a^2 y_b^1$ and $z_a^2 x_b^1$ to H_1^1 without edge crossing, we get a planar graph $\widetilde{H}_1^1$. With the same process, we add both $x_a^1 y_b^2$ and $z_a^1 z_b^2$ to H_1^2 without edge crossing, then we get a planar graph $\widetilde{H}_1^2$. From Lemma 4.3, we can also add $y_a^2 z_b^1$ to H_2^1, and $y_a^1 z_b^2$ to H_2^2 without edge crossing, then we get planar graphs $\widetilde{H}_2^1$ and $\widetilde{H}_2^2$ respectively.

Then we get a planar decomposition

$$\left\{ \widetilde{H}_1^1, \widetilde{H}_2^1, H_3^1, \ldots, H_{p+1}^1, \widetilde{H}_1^2, \widetilde{H}_2^2, H_3^2, \ldots, H_{p+1}^2 \right\}$$

of $K_{4p+3,4p+3,4p+3} \times K_2$ with $2p+2$ subgraphs.

Summarizing Cases 1 and 2, the lemma follows. □

Theorem 4.6. *The thickness of the Kronecker product of $K_{n,n,n}$ and K_2 is*

$$\theta(K_{n,n,n} \times K_2) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Because of $E(K_{n,n,n} \times K_2) = 6n^2$ and $V(K_{n,n,n} \times K_2) = 6n$, from Theorem 2.2, we have

$$\theta(K_{n,n,n} \times K_2) \ge \left\lceil \frac{6n^2}{2(6n)-4} \right\rceil = \left\lceil \frac{n}{2} + \frac{n}{6n-2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil. \tag{4.1}$$

When $n = 4p+2$, because $K_{4p+2,4p+2,4p+2} \times K_2$ is a subgraph of $K_{4p+3,4p+3,4p+3} \times K_2$, we have $\theta(K_{4p+2,4p+2,4p+2} \times K_2) \le \theta(K_{4p+3,4p+3,4p+3} \times K_2)$. Combining this fact with Lemmas 4.2, 4.4 and 4.5, we have

$$\theta(K_{n,n,n} \times K_2) \le \left\lceil \frac{n+1}{2} \right\rceil. \tag{4.2}$$

From inequalities (4.1) and (4.2), the theorem is obtained. □

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On an annihilation number conjecture*

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Abstract

Let $\alpha(G)$ denote the cardinality of a maximum independent set, while $\mu(G)$ be the size of a maximum matching in the graph $G = (V, E)$. If $\alpha(G) + \mu(G) = |V|$, then G is a *König-Egerváry graph*. If $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of G, then the *annihilation number* $a(G)$ of G is the largest integer k such that $\sum_{i=1}^{k} d_i \leq |E|$. A set $A \subseteq V$ satisfying $\sum_{v \in A} \deg(v) \leq |E|$ is an *annihilation set*; if, in addition, $\deg(x) + \sum_{v \in A} \deg(v) > |E|$, for every vertex $x \in V(G) - A$, then A is a *maximal annihilation set* in G.

In 2011, Larson and Pepper conjectured that the following assertions are equivalent:

(i) $\alpha(G) = a(G)$;

(ii) G is a König-Egerváry graph and every maximum independent set is a maximal annihilating set.

It turns out that the implication "(i) $\Longrightarrow$ (ii)" is correct.

In this paper, we show that the opposite direction is not valid, by providing a series of generic counterexamples.

Keywords: Maximum independent set, maximum matching, König-Egerváry graph, annihilation set, annihilation number.

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1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V = V(G)$ of cardinality $|V(G)| = n(G)$, and edge set $E = E(G)$ of size $|E(G)| = m(G)$. If $X \subset V(G)$, then $G[X]$ is the subgraph of G induced by X. By $G - v$ we mean the subgraph $G[V(G) - \{v\}]$, for $v \in V(G)$. $K_n, K_{m,n}, P_n, C_n$ denote respectively, the complete graph on $n \geq 1$ vertices, the complete bipartite graph on $m, n \geq 1$ vertices, the path on $n \geq 1$ vertices, and the cycle on $n \geq 3$ vertices, respectively.

The *disjoint union* of the graphs G_1, G_2 is the graph $G_1 \cup G_2$ having the disjoint union of $V(G_1), V(G_2)$ as a vertex set, and the disjoint union of $E(G_1), E(G_2)$ as an edge set. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$. Let $\Omega(G)$ denote the family of all maximum independent sets.

A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that no two edges of M share a common vertex. A matching of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one saturating all vertices of G.

It is known that $\lfloor n(G)/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n(G) \leq \alpha(G) + 2\mu(G)$ hold for every graph G [6]. If $\alpha(G) + \mu(G) = n(G)$, then G is called a König-Egerváry graph [11, 36]. For instance, each bipartite graph is a König-Egerváry graph [13, 20]. Various properties of König-Egerváry graphs can be found in [3, 4, 5, 16, 17, 18, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 35].

Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of a graph G. Pepper [33, 34] defined the annihilation number of G, denoted $a(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. In other words, $a(G)$ is precisely the largest integer k such that $\sum_{i=1}^{k} d_i \leq m(G)$.

Clearly, $a(G) = n(G)$ if and only if $m(G) = 0$. If $m(G) = 1$, then $a(G) = n(G) - 1$. The converse is not true; e.g., the graph $K_{1,p}$ has $a(K_{1,p}) = m(K_{1,p}) = p = n(K_{1,p}) - 1$, while p may be greater than one.

For $A \subseteq V(G)$, let $\deg(A) = \sum_{v \in A} \deg(v)$. Every $A \subseteq V(G)$ satisfying $\deg(A) \leq m(G)$ is an *annihilating set*. Clearly, every independent set is annihilating. An annihilating set A is *maximal* if $\deg(A \cup \{x\}) > m(G)$, for every vertex $x \in V(G) - A$, and it is *maximum* if $|A| = a(G)$ [33]. For example, if $G = K_{p,q} = (A, B, E)$ and $p > q$, then A is a maximum annihilating set, while B is a maximal annihilating set.

Theorem 1.1 ([33]). *For every graph G,*

$$a(G) \geq \max\left\{ \left\lfloor \frac{n(G)}{2} \right\rfloor, \alpha(G) \right\}.$$

For instance,

$$a(C_7) = \alpha(C_7) = \left\lfloor \frac{n(C_7)}{2} \right\rfloor, \qquad a\left(\overline{P_5}\right) = 3 > \alpha\left(\overline{P_5}\right) = \left\lfloor \frac{n\left(\overline{P_5}\right)}{2} \right\rfloor,$$

$$a(K_{2,3}) = \alpha(K_{2,3}) > \left\lfloor \frac{n(K_{2,3})}{2} \right\rfloor, \quad \text{while} \quad a\left(\overline{C_6}\right) = \left\lfloor \frac{n\left(\overline{C_6}\right)}{2} \right\rfloor > \alpha\left(\overline{C_6}\right).$$

The relation between the annihilation number and various parameters of a graph were studied in [1, 2, 7, 8, 9, 10, 12, 14, 15, 19, 32, 33].

Theorem 1.2 ([24]). *For a graph G with $a(G) \geq \frac{n(G)}{2}$, $\alpha(G) = a(G)$ if and only if G is a König-Egerváry graph and every $S \in \Omega(G)$ is a maximum annihilating set.*

All the maximum independent sets of the cycle C_5 are maximum annihilating. Moreover, $a(C_5) = \alpha(C_5)$. Nevertheless, C_5 is not a König-Egerváry graph. In other words, the condition $a(G) \geq \frac{n(G)}{2}$ in Theorem 1.2 is necessary.

Actually, Larson and Pepper [24] proved a stronger result that reads as follows.

Theorem 1.3. *Let G be a graph with $a(G) \geq \frac{n(G)}{2}$. Then the following are equivalent:*

(i) $\alpha(G) = a(G)$;

(ii) G is a König-Egerváry graph and every $S \in \Omega(G)$ is a maximum annihilating set;

(iii) G is a König-Egerváry graph and some $S \in \Omega(G)$ is a maximum annihilating set.

Along these lines, it was conjectured that the impacts of maximum and maximal annihilating sets are the same.

Conjecture 1.4 ([24]). *Let G be a graph with $a(G) \geq \frac{n(G)}{2}$. Then the following assertions are equivalent:*

(i) $\alpha(G) = a(G)$;

(ii) G is a König-Egerváry graph and every $S \in \Omega(G)$ is a maximal annihilating set.

One can easily infer that every maximum annihilating set is also a maximal annihilating set, since the sum of the $a+1$ smallest entries from the degree sequence $D = (d_1 \leq d_2 \leq \cdots \leq d_n)$ is greater than $m(G)$, then the same is true for every $a+1$ entries of D. Thus the "(i) $\Longrightarrow$ (ii)" part of Conjecture 1.4 is valid, in accordance with Theorem 1.2.

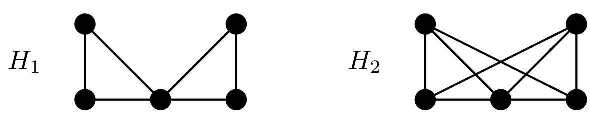

Figure 1: Non-König-Egerváry graphs with $a(H_1) = 3$ and $a(H_2) = 2$.

Consider the graphs from Figure 1. The graph H_1 has $a(H_1) > \alpha(H_1)$ and none of its maximum independent sets is a maximal or a maximum annihilating set. The graph H_2 has $a(H_2) = \alpha(H_2)$ and each of its maximum independent sets is both a maximal and a maximum annihilating set. Notice that $a(H_1) > \frac{n(H_1)}{2}$, while $a(H_2) < \frac{n(H_2)}{2}$.

Consider the graphs from Figure 2. The graph G_1 has $\alpha(G_1) = \frac{n(G_1)}{2} < a(G_1)$ and each of its maximum independent sets is neither a maximal nor a maximum annihilating set. The graph G_2 has $a(G_2) = \alpha(G_2) = \frac{n(G_2)}{2}$, every of its maximum independent sets is both a maximal and a maximum annihilating set, and it has a maximal independent set that is a maximal non-maximum annihilating set, namely $\{u, v\}$. The graph G_3 has $a(G_3) = \alpha(G_3) > \frac{n(G_3)}{2}$ and every of its maximum independent sets is both a maximal

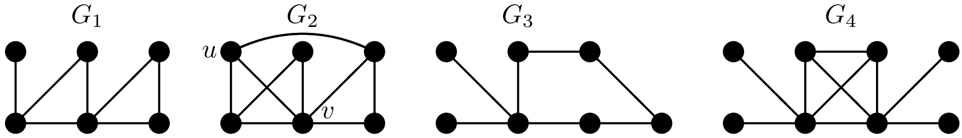

Figure 2: König-Egerváry graphs with $a(G_1) = a(G_3) = 4$, $a(G_2) = 3$, $a(G_4) = 6$.

and a maximum annihilating set. The graph G_4 has $a(G_4) > \alpha(G_4) > \frac{n(G_4)}{2}$ and none of its maximum independent sets is a maximal or a maximum annihilating set.

In this paper we invalidate the "(ii) $\Longrightarrow$ (i)" part of Conjecture 1.4, by providing some generic counterexamples. Let us notice that, if G is a König-Egerváry graph, and $H = qK_1 \cup G$, then H inherits this property. Moreover, the relationship between the independence numbers and annihilation numbers of G and H remains the same, because $\alpha(H) = \alpha(G) + q$ and $a(H) = a(G) + q$. Therefore, it is enough to construct only connected counterexamples. Finally, we prove that Conjecture 1.4 is true for graphs with independence number equal to three.

2 An infinite family of counterexamples

In what follows, we present a series of counterexamples to the opposite direction of Conjecture 1.4. All these graphs have unique maximum independent sets.

Lemma 2.1. *The graph $H_k, k \geq 0$, from Figure 3 is a connected König-Egerváry graph that has a unique maximum independent set, namely, $S_k = \{x_k, \ldots, x_1, a_4, a_3, a_2, a_1\}$, where $H_0 = H_k - \{x_j, y_j : j = 1, 2, \ldots, k\}$ and $S_0 = \{a_4, a_3, a_2, a_1\}$.*

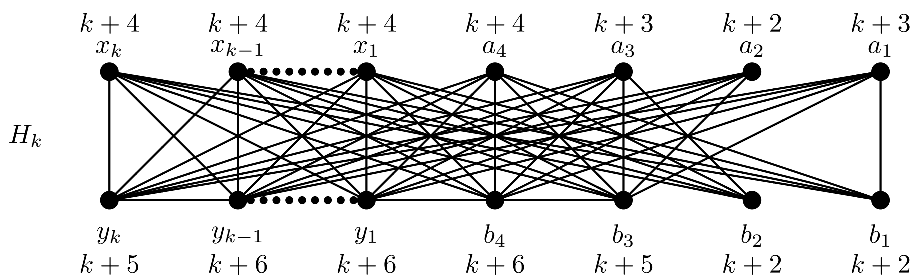

Figure 3: H_k is a König-Egerváry graph with $\alpha(H_k) = k + 4, k \geq 0$.

Proof. Notice that the graph H_k from Figure 3 can be defined as follows:

$$V(H_k) = V(K_{k+4,k+4}) = \{x_i, y_i : i = 1, \ldots, k\} \cup \{a_1, a_2, a_3, a_4\} \cup \{b_1, b_2, b_3, b_4\},$$
$$E(H_k) = E(K_{k+4,k+4}) \cup \{y_k y_{k-1}, \ldots, y_2 y_1, y_1 b_4, b_4 b_3\} - \{a_3 b_1, a_2 b_2, a_2 b_1, a_1 b_2\}.$$

Clearly, $S_k = \{x_k, \ldots, x_1, a_4, a_3, a_2, a_1\}$ is an independent set and

$$\{x_j y_j : j = 1, 2, ..., k\} \cup \{a_4 b_4, a_3 b_2, a_3 b_3, a_1 b_1\}$$

is a perfect matching of H_k. Hence, we get

$$|V_k| = 2\mu(H_k) = |S_k| + \mu(H_k) \leq \alpha(H_k) + \mu(H_k) \leq |V_k|,$$

which implies $\alpha(H_k) + \mu(H_k) = |V_k|$, i.e., H_k is a König-Egerváry graph, and $\alpha(H_k) = k + 4 = |S_k|$.

Let $L_k = H_k[X_k \cup Y_k]$, $k \geq 1$, and $L_0 = H_k[A \cup B]$, where

$$X_k = \{x_j : j = 1, \ldots, k\}, \qquad Y_k = \{y_j : j = 1, \ldots, k\},$$
$$A = \{a_1, a_2, a_3, a_4\} \quad \text{and} \quad B = \{b_1, b_2, b_3, b_4\}.$$

Since L_k has, on the one hand, $K_{k,k}$ as a subgraph, and, on the other hand,

$$y_k y_{k-1}, y_{k-1} y_{k-2}, \ldots, y_2 y_1 \in E(L_k),$$

it follows that X_k is the unique maximum independent set of L_k.

The graph L_0 has A as a unique independent set, because

$$C_8 + b_3b_4 = (A \cup B, \{a_1b_4, b_4a_2, a_2b_3, b_3a_3, a_3b_2, b_2a_4, a_4b_1, b_1a_1, b_3b_4\})$$

has A as a unique maximum independent set, and L_0 can be obtained from $C_8 + b_3b_4$ by adding a number of edges.

Since H_k can be obtained from the union of L_k and L_0 by adding some edges, and $S_k = X_k \cup A$ is independent in H_k, it follows that H_k has S_k as a unique maximum independent set. □

Corollary 2.2. *The graph $G_k, k \geq 0$, from Figure 4 is a connected König-Egerváry graph that has a unique independent set, namely, $S_k = \{x_i : i = 1, \ldots, k\} \cup \{a_i : i = 1, \ldots, 5\}$, where $G_0 = G_k - \{x_j, y_j : j = 1, 2, \ldots, k\}$ and $S_0 = \{a_i : i = 1, \ldots, 5\}$.*

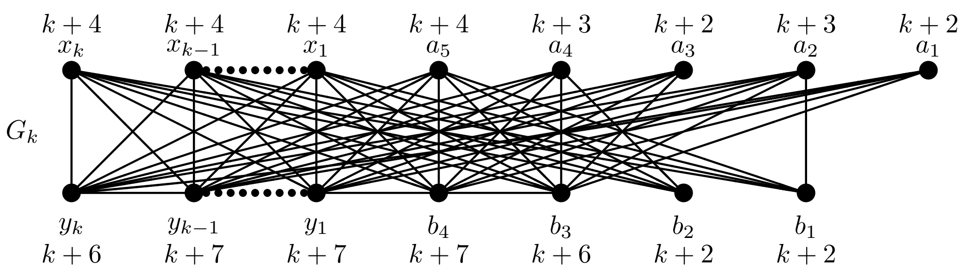

Figure 4: G_k is a König-Egerváry graph with $\alpha(G_k) = k + 5, k \geq 0$.

Proof. Notice that the graph G_k from Figure 4 can be defined as follows:

$$\begin{aligned} V(G_k) &= V(K_{k+5,k+4}) \\ &= \{x_i, y_i : i = 1, \ldots, k\} \cup \{a_1, a_2, a_3, a_4, a_5\} \cup \{b_1, b_2, b_3, b_4\}, \\ E(G_k) &= E(K_{k+4,k+4}) \cup \{y_k y_{k-1}, \ldots, y_2 y_1, y_1 b_4, b_4 b_3\} \\ &\qquad - \{a_3b_2, a_3b_1, a_2b_2, a_1b_2, a_1b_1\}. \end{aligned}$$

According to Lemma 2.1, $G_k - a_1$ is a König-Egerváry graph with a unique maximum independent set, namely, $W_k = \{x_i : i = 1, \ldots, k\} \cup \{a_i : i = 1, \ldots, 4\}$. Since $S_k = W_k \cup \{a_1\}$ is an independent set and $\mu(G_k) = \mu(G_k - a_1) = k + 4$, it follows that G_k is a König-Egerváry graph and S_k is its unique maximum independent set. □

Theorem 2.3. *For every $k \geq 0$, there exists a connected non-bipartite König-Egerváry graph $H_k = (V_k, E_k)$, of order $2k+8$, satisfying the following:*

- $a(H_k) > \frac{n(H_k)}{2} = \alpha(H_k)$,
- *each $S \in \Omega(H_k)$ is a maximal annihilating set.*

Proof. Let $H_k = (V_k, E_k), k \geq 0$, be the graph from Figure 3 (in the bottom and the top lines are written the degrees of its vertices), where $H_0 = H_k - \{x_1, \dots, x_k, y_1, \dots, y_k\}$. Clearly, every H_k is non-bipartite.

By Lemma 2.1, each $H_k, k \geq 0$, is a König-Egerváry graph with a unique maximum independent set, namely, $S_k = \{x_k, \dots, x_1, a_4, a_3, a_2, a_1\}$, where $S_0 = \{a_4, a_3, a_2, a_1\}$.

Case 1. $k = 0$. Since $m(H_0) = 13$ and the degree sequence $(2,2,2,3,3,4,5,5)$, we infer that $a(H_0) = 5 > 4 = \alpha(H_0)$. In addition, $\deg(S_0) = m(H_0) - 1$, i.e., each maximum independent set of H_0 is a maximal non-maximum annihilating set.

Case 2. $k \geq 1$. Clearly, H_k has $m(G_k) = k^2 + 9k + 13$ and its degree sequence is

$$k+2, k+2, k+2, k+3, k+3, \underbrace{k+4, \dots, k+4}_{k+1}, k+5, k+5, \underbrace{k+6, \dots, k+6}_{k}.$$

Since the sum of the first $k+6$ degrees of the sequence satisfies

$$k^2 + 10k + 16 > m(H_k),$$

we infer that the annihilation number $a(H_k) \leq k+6$. The sum $12 + 4(x-5) + kx$ of the first $x \geq 5$ degrees of the sequence satisfies $12 + 4(x-5) + kx \leq m(H_k)$ for $x \leq \frac{k^2+9k+21}{k+4}$. This implies

$$a(H_k) = \left\lfloor \frac{k^2+9k+21}{k+4} \right\rfloor = k+5 > k+4 = \alpha(H_k),$$

i.e., H_k has no maximum annihilating set belonging to $\Omega(H_k)$. Since its unique maximum independent set $S_k = \{a_1, a_2, a_3, a_4, x_1, x_2, \dots, x_k\}$ has

$$\deg(S_k) = k^2 + 8k + 12 < m(H_k),$$

while

$$\deg(S_k) + \min\{\deg(v) : v \in V_k - S\} = \left(k^2 + 8k + 12\right) + (k+2) > m(H_k),$$

we infer that S_k is a maximal annihilating set. □

Theorem 2.4. *For every $k \geq 0$, there exists a connected non-bipartite König-Egerváry graph $G_k = (V_k, E_k)$, of order $2k+9$, satisfying the following:*

- $a(G_k) > \left\lceil \frac{n(G_k)}{2} \right\rceil = \alpha(G_k)$,
- *each $S \in \Omega(G_k)$ is a maximal annihilating set.*

Proof. Let $G_k = (V_k, E_k), k \geq 1$, be the graph from Figure 4 (in the bottom and the top lines are written the degrees of its vertices), and $G_0 = G_k - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$.

Corollary 2.2 claims that $G_k, k \geq 0$, is a König-Egerváry graph with a unique maximum independent set, namely $S_k = \{x_1, \ldots, x_k, a_1, \ldots, a_5\}, k \geq 1$, and $S_0 = \{a_1, \ldots, a_5\}$.

Case 1. The non-bipartite König-Egerváry graph G_0 has $m(G_0) = 15$ and the degree sequence $(2, 2, 2, 2, 3, 3, 4, 6, 6)$. Hence, $a(G_0) = 6 > 5 = \alpha(G_0)$. In addition, $\Omega(G_0) = \{S_0\}$, and $\deg(S_0) = 14$, i.e., each maximum independent set of G_0 is a maximal non-maximum annihilating set.

Case 2. $k \geq 1$. Clearly, G_k has $m(G_k) = k^2 + 10k + 15$ and its degree sequence is

$$k+2, k+2, k+2, k+2, k+3, k+3, \underbrace{k+4, \ldots, k+4}_{k+1}, k+6, k+6, \underbrace{k+7, \ldots, k+7}_{k}.$$

Since the sum of the first $k+7$ degrees of the sequence satisfies

$$k^2 + 11k + 18 > m(G_k),$$

we infer that the annihilation number $a(G_k) \leq k+6$. The sum $14 + 4(x-5) + kx$ of the first $x \geq 6$ degrees of the sequence satisfies $14 + 4(x-6) + kx \leq m(G_k)$ for $x \leq \frac{k^2+10k+25}{k+4}$. This implies

$$a(G_k) = \left\lfloor \frac{k^2 + 10k + 25}{k+4} \right\rfloor = k + 6 > k + 5 = \alpha(G_k),$$

i.e., G_k has no maximum annihilating set belonging to $\Omega(G_k)$. Since its unique maximum independent set S_k has

$$\deg(S_k) = k^2 + 9k + 14 < m(G_k),$$

while

$$\deg(S_k) + \min\{\deg(v) : v \in V_k - S_k\} = \left(k^2 + 9k + 14\right) + (k+2) > m(G_k),$$

we infer that S_k is a maximal annihilating set. □

3 Conclusions

If G is a König-Egerváry graph with $\alpha(G) \in \{1, 2\}$, then $\alpha(G) = a(G)$ and each maximum independent set is maximal annihilating, since the list of such König-Egerváry graphs reads as follows:

$$\{K_1, K_2, K_1 \cup K_1, K_1 \cup K_2, K_2 \cup K_2, P_3, P_4, C_4, K_3 + e, K_4 - e\}.$$

Consequently, Conjecture 1.4 is correct for König-Egerváry graphs with $\alpha(G) \leq 2$.

Let G be a disconnected König-Egerváry graph with $\alpha(G) = 3$.

- If $\alpha(G) = a(G)$, then

$$G \in \left\{ \begin{array}{c} 3K_1, 2K_1 \cup K_2, K_1 \cup 2K_2, 3K_2, K_1 \cup P_3, K_1 \cup P_4, \\ K_1 \cup C_4, K_1 \cup (K_3 + e), K_1 \cup (K_4 - e), K_2 \cup P_3, K_2 \cup C_4 \end{array} \right\},$$

  while every $S \in \Omega(G)$ is a maximal annihilating set.

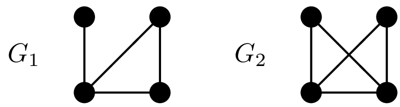

Figure 5: $G_1 = K_3 + e$ and $G_2 = K_4 - e$.

- If $\alpha(G) < a(G)$, then $G \in \{K_2 \cup P_4, K_2 \cup (K_3 + e), K_2 \cup (K_4 - e)\}$, while for every such G, there exists a maximum independent set, which is a not a maximal annihilating set. Moreover, for $K_2 \cup (K_3 + e)$ and $K_2 \cup (K_4 - e)$ all maximum independent sets are not maximal annihilating.

Thus Conjecture 1.4 is true for disconnected König-Egerváry graphs with $\alpha(G) = 3$.

We have already mentioned in Introduction that the "(i) $\Longrightarrow$ (ii)" part of Conjecture 1.4 is true.

Proposition 3.1. *Let G be a graph with $a(G) \geq \frac{n(G)}{2}$. If G is a connected König-Egerváry graph with $\alpha(G) = 3$, and every $S \in \Omega(G)$ is a maximal annihilating set, then $\alpha(G) = a(G)$.*

Proof. In Figure 6 we present all connected König-Egerváry graphs with $\alpha(G) = 3$ having $n(G) \in \{4, 5\}$. For these graphs $\alpha(G) = a(G)$, which means that Conjecture 1.4 is true.

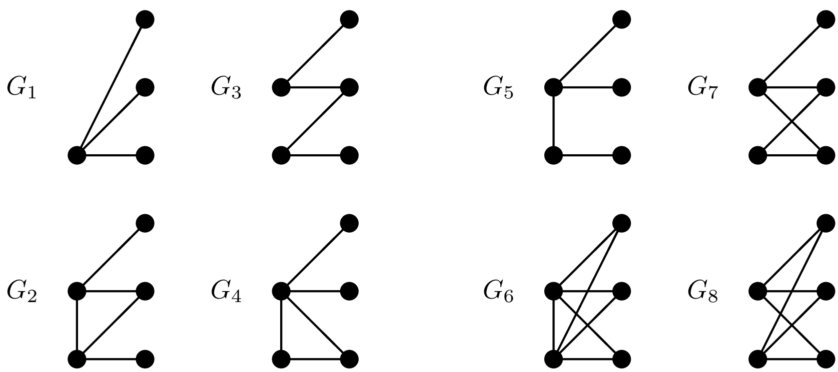

Figure 6: König-Egerváry graphs with $\alpha(G) = 3 = a(G)$ and $n(G) \leq 5$.

Now, we may assume that $n(G) = 6$, since $\alpha(G) \geq \mu(G)$ holds for each König-Egerváry graph.

Let $d_1 \leq d_2 \leq \cdots \leq d_6$ be the degree sequence of G.

It is known that $\alpha(G) \leq a(G)$ (Theorem 1.1). Thus we have only three cases with $3 = \alpha(G) < a(G)$ to cover, namely, $a(G) \in \{4, 5, 6\}$.

Case 1. $a(G) = 4$. Then, by definition, $d_1 + d_2 + d_3 + d_4 \leq m(G) \leq d_5 + d_6$ and $d_1 + d_2 + d_3 + d_4 + d_5 > m(G) > d_6$.

Let q be the number of edges in G joining the vertices v_5, v_6 with the vertices v_1, v_2, v_3, v_4. At least two vertices from the set $\{v_1, v_2, v_3, v_4\}$ must be joined by an edge, otherwise, $\alpha(G) \geq 4 > 3$. Assume that $v_3 v_4 \in E(G)$. Hence, $v_5 v_6 \in E(G)$, otherwise,

$$d_5 + d_6 = q < q + 2 \leq d_1 + d_2 + d_3 + d_4,$$

in contradiction with $d_1 + d_2 + d_3 + d_4 \leq d_5 + d_6$. Similarly, there are no more edges but v_3v_4 joining vertices from the set $\{v_1, v_2, v_3, v_4\}$, otherwise

$$d_5 + d_6 = q + 2 < q + 4 \leq d_1 + d_2 + d_3 + d_4,$$

in contradiction with $d_1 + d_2 + d_3 + d_4 \leq d_5 + d_6$. Therefore, $\{v_1, v_2, v_3\}$ is a maximum independent set of G, since $\alpha(G) = 3$. On the other hand, $\{v_1, v_2, v_3\}$ is not a maximal annihilating set, because $d_1 + d_2 + d_3 + d_4 \leq m(G)$.

Case 2. $a(G) = 5$. By definition, it follows that $d_1 + d_2 + d_3 + d_4 + d_5 \leq m(G) \leq d_6$. Hence, the set $\{v_1, v_2, v_3, v_4, v_5\}$ is independent, in contradiction with the fact that $\alpha(G) = 3$.

Case 3. $a(G) = 6$. This means that G has no edges, which is not possible, because $\alpha(G) = 3$. □

To complete the picture, Theorems 2.3 and 2.4 present various counterexamples to the "(ii) $\Longrightarrow$ (i)" part of Conjecture 1.4 for every independence number greater than three. Our intuition tells us that the real obstacle for the "(i) $\Longrightarrow$ (ii)" part Conjecture 1.4 not to be true is the size of the annihilation number. It motivates the following.

Conjecture 3.2. *If G is a König-Egerváry graph with $a(G) \geq \frac{3}{4}n(G)$, and every $S \in \Omega(G)$ is a maximal annihilating set, then $\alpha(G) = a(G)$.*

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The complete bipartite graphs which have exactly two orientably edge-transitive embeddings*

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Abstract

In 2018, Fan and Li classified the complete bipartite graph $\mathbf{K}_{m,n}$ that has a unique orientably edge-transitive embedding. In this paper, we extend this to give a complete classification of $\mathbf{K}_{m,n}$ which have exactly two orientably edge-transitive embeddings.

Keywords: Bipartite graphs, edge-transitive embeddings.

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1 Introduction

Let $\mathcal{M} = (V, E, F)$ be an orientable map with vertex set V, edge set E and face set F, that is, $\mathcal{M}$ is a 2-cell embedding of the underlying graph $\Gamma = (V, E)$ in an orientable surface. A permutation of $V \cup E \cup F$ which preserves V, E, F, and their incidence relations is called an *automorphism* of $\mathcal{M}$. All automorphisms of $\mathcal{M}$ form the *automorphism group* $\mathrm{Aut}\,\mathcal{M}$ under composition.

A map $\mathcal{M} = (V, E, F)$ is said to be *G-edge-transitive* if $G \leq \mathrm{Aut}\,\mathcal{M}$ is transitive on E; if in addition G also preserves the orientation of the supporting surface, then $\mathcal{M}$ is called *orientably edge-transitive*. Similarly, *orientably arc-transitive* maps are defined.

It is a main aim of topological graph theory to determine and enumerate all the 2-cell embeddings of a given class of graphs, see [2, 3, 7, 10, 11, 12] for arc-transitive maps, and [5, 8, 9, 13] for edge-transitive maps.

Although each map has a unique underlying graph, a graph may have many non-isomorphic 2-cell embeddings usually. For example, $\mathbf{K}_{3,2}$ has two edge transitive embeddings that have automorphism groups $\mathbb{Z}_6$ and S_3, respectively. As a special case, the complete bipartite graphs that has a unique edge-transitive embedding has been received much attention. For instance, Jones, Nedela and Škoviera [10] proved that $\mathbf{K}_{n,n}$ has a unique orientably arc-transitive embedding if and only if $\gcd(n, \phi(n)) = 1$, where $\phi(n)$ is the Euler *phi-function*. Fan and Li [4] showed that $\mathbf{K}_{m,n}$ have a unique edge-transitive embedding if and only if $\gcd(m, \phi(n)) = 1 = \gcd(n, \phi(m))$. For convenience, we call the pair (m, n) *singular* if $\gcd(m, \phi(n)) = 1 = \gcd(n, \phi(m))$ in the following.

The aim of this paper is to consider the analogous problem for the complete bipartite graph $\mathbf{K}_{m,n}$, and we give a complete classification of $\mathbf{K}_{m,n}$ which have exactly two orientably edge-transitive embeddings. To state the theorem, we need some notations. For an integer n and a prime p, let $n = n_p n_{p'}$ such that n_p is a p-power and $\gcd(n_p, n_{p'}) = 1$. The main theorem of this paper is now stated as follows.

Theorem 1.1. *A complete bipartite graph $\mathbf{K}_{m,n}$ has exactly two orientably edge-transitive embeddings if and only if, interchanging m and n if necessary, one of the following holds:*

(i) $(m, n) = (4, 2)$;

(ii) $m = p^e$ *with p odd,* $n = 2n_{2'}$, *and* $(m, n_{2'})$ *is a singular pair;*

(iii) $m = p^e$ *with* $p \equiv 3 \pmod 4$, $n = 2^f n_{2'}$ *with* $f \geq 2$, *and* $(m, n_{2'})$ *is a singular pair;*

(iv) $m = 2p^e$ *with p odd, and* $n = 2$.

This solved the problem in [4] to determine complete bipartite graphs which have exactly two non-isomorphic orientably edge-transitive embeddings.

Particularly, the following corollary about $\mathbf{K}_{n,n}$ is easily observed.

Corollary 1.2. *There exists no complete bipartite graph $\mathbf{K}_{n,n}$ $(n \geq 2)$ that has exactly two non-isomorphic orientably edge-transitive embeddings.*

2 Complete bipartite edge-transitive maps

Let m, n be positive integers, and let $\Gamma = (V, E)$ be a complete bipartite graph $\mathbf{K}_{m,n}$. Let $\mathcal{M}$ be an orientable map with underlying graph $\Gamma = \mathbf{K}_{m,n}$. Let $\mathrm{Aut}^+\,\mathcal{M}$ consist of automorphisms of $\mathcal{M}$ which preserves the biparts of Γ, and let $\mathrm{Aut}^{\mathrm{O}}\,\mathcal{M}$ be the subgroup of $\mathrm{Aut}\,\mathcal{M}$ which preserves the orientation of the supporting surface. Let

$$\mathrm{Aut}^{\oplus}\,\mathcal{M} = \mathrm{Aut}^{+}\,\mathcal{M} \cap \mathrm{Aut}^{\mathrm{O}}\,\mathcal{M}.$$

Then $\mathrm{Aut}^{\oplus}\,\mathcal{M}$ contains all elements of $\mathrm{Aut}\,\mathcal{M}$ which preserve the orientation of the supporting surface, and fixes the biparts of the underlying graph. It is clear that isomorphic embeddings of $\mathbf{K}_{m,n}$ have isomorphic automorphism groups.

Orientable edge-transitive embeddings of $\mathbf{K}_{m,n}$ have automorphism groups being *bicyclic*, which is defined as follows.

Definition 2.1. A group G is called *bicyclic* if $G = \langle a\rangle\langle b\rangle$ for some elements $a, b \in G$. If $|a| = m$ and $|b| = n$, then G is said to be of *order* $\{m, n\}$. If in addition $\langle a\rangle \cap \langle b\rangle = 1$, then G is called an *exact bicyclic group*, and $\{a, b\}$ is called an *exact bicyclic pair* of order $\{m, n\}$.

It is known that orientable edge-transitive embeddings of $\mathbf{K}_{m,n}$ precisely correspond to exact bicyclic pairs of order $\{m, n\}$. We denote by

$$\mathcal{M}(G, a, b)$$

the edge-transitive embedding of $\mathbf{K}_{m,n}$ corresponding to a bicyclic group G associated with a bicyclic pair $\{a, b\}$. For convenience, (a, b) is called an *edge-regular pair* for G. Moreover, $\mathcal{M}(G, a, b)$ is called an *abelian embedding* if G is abelian, and *non-abelian embedding* otherwise.

The following lemma is well-known and easy to prove, see [11] or [6].

Lemma 2.2. *Let G be an exact bicyclic group of order $\{m, n\}$, and let $a, b \in G$ be a bicyclic pair. Then there is an edge-transitive orientable embedding $\mathcal{M} = \mathcal{M}(G, a, b)$ of $\mathbf{K}_{m,n}$ such that $\mathrm{Aut}^{\oplus}\,\mathcal{M} = G$ is edge-regular on $\mathcal{M}$, and for any bicyclic pair $x, y \in G$, $\mathcal{M}(G, a, b) \cong \mathcal{M}(G, x, y)$ if and only if there is an automorphism σ of $\mathrm{Aut}(G)$ such that $(a, b)^\sigma = (x, y)$.*

Since there exists an abelian bicyclic group $\mathbb{Z}_m \times \mathbb{Z}_n$ for any positive integers m and n, the graph $\Gamma = \mathbf{K}_{m,n}$ has a unique orientable edge-regular embedding $\mathcal{M}$ such that $\mathrm{Aut}^{\oplus}\,\mathcal{M} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, see [4, Lemma 2.3]. Moreover, it is known that if $\{m, n\}$ is a singular pair of integers then each exact bicyclic group of order $\{m, n\}$ is abelian, see [4, Lemma 3.3]. This leads to the following observation.

Lemma 2.3. *Let m, n be positive integers for which $\mathbf{K}_{m,n}$ has exactly two non-isomorphic edge-transitive embeddings. Then there exists a unique non-abelian exact bicyclic group of order $\{m, n\}$.*

In the next section, we work out a classification of integer pairs $\{m, n\}$ for which there is only one non-abelian exact bicyclic group.

3 Uniqueness of groups

Let (m, n) be a pair of integers such that there is a unique non-abelian exact bicyclic group of order $\{m, n\}$. Then (m, n) is not a singular pair. So there exist divisors $m_p \mid m$, and $n_q \mid n$ such that (m_p, n_q) is not a singular pair.

The first lemma determines (m_p, n_p) for the same prime p.

Lemma 3.1. *If a prime* $p \mid \gcd(m,n)$*, then*

(i) $\{m_p, n_p\} = \{4, 2\}$*, or*

(ii) $\{m_p, n_p\} = \{p^e, p\}$ *with* p *an odd prime and* $e \geq 2$*, or*

(iii) $\{m_p, n_p\} = \{p^2, p^2\}$ *with* p *an odd prime.*

Proof. To prove the lemma, we may assume that $m_p \geq n_p$ and $m_p = p^e$ with $e \geq 2$.

Assume first that $n_p = p$. If $p = 2$ and $e \geq 3$, then there are 3 non-isomorphic groups $G_i = \mathbb{Z}_m{:}\mathbb{Z}_n = \mathbb{Z}_{m_{2'}} \times \mathbb{Z}_{n_{2'}} \times P_i$, where $i = 1, 2$ or 3, and $P_i = \langle x_2\rangle\langle y_2\rangle = \mathbb{Z}_{2^e}{:}\mathbb{Z}_2$ as below:

$$\begin{aligned} P_1 &= \langle x_2, y_2 \mid x_2^{y_2} = x_2^{-1}\rangle, \\ P_2 &= \langle x_2, y_2 \mid x_2^{y_2} = x_2^{2^{e-1}+1}\rangle, \\ P_3 &= \langle x_2, y_2 \mid x_2^{y_2} = x_2^{2^{e-1}-1}\rangle. \end{aligned}$$

This contradiction shows that either $p = 2$ and $e = 2$, that is $(m_p, n_p) = (4, 2)$, or p is odd and $(m_p, n_p) = (p^e, p)$ with $e \geq 2$.

Next, assume that $n_p = p^f$ with $f \geq 2$. Suppose further that $e \geq 3$. Then there exist at least 2 non-isomorphic groups $G_i = \mathbb{Z}_m{:}\mathbb{Z}_n = \mathbb{Z}_{m_{p'}} \times \mathbb{Z}_{n_{p'}} \times P_i$, where $i = 1$ or 2, and $P_i = \langle x_p\rangle\langle y_p\rangle = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{p^f}$ as below:

$$\begin{aligned} P_1 &= \langle x_p, y_p \mid x_p^{y_p} = x_p^{1+p^{e-1}}\rangle, \\ P_2 &= \langle x_p, y_p \mid x_p^{y_p} = x_p^{1+p^{e-2}}\rangle. \end{aligned}$$

This is a contradiction. Thus $e = 2$, and $f = 2$.

Suppose further that $p = 2$, there are two non-isomorphic groups $G_i = \mathbb{Z}_m\mathbb{Z}_n = \mathbb{Z}_{m_{2'}} \times \mathbb{Z}_{n_{2'}} \times P_i$, where $i = 1$ or 2, and $P_i = \langle x_2\rangle\langle y_2\rangle = \mathbb{Z}_4\mathbb{Z}_4$ is non-abelian as below:

$$\begin{aligned} P_1 &= \langle x_2, y_2 \mid x_2^4 = y_2^4 = 1, x_2^{y_2} = x_2^{-1}\rangle, \\ P_2 &= \langle x_2, y_2 \mid x_2^4 = y_2^4 = [x_2^2, y_2] = [x_2, y_2^2] = 1, [y_2, x_2] = x_2^2 y_2^2\rangle. \end{aligned}$$

So $\{m_p, n_p\} = \{p^2, p^2\}$ with p an odd prime. □

The next lemma determines the relation m_p and n_q for distinct primes p, q.

Lemma 3.2. *Assume that* $m_p = p^e$ *and* $n_q = q^f$*, where* $q \mid (p-1)$*. Then either* $f = 1$*, or* $q^2 \nmid (p-1)$*; equivalently,* $\gcd(n_q, \phi(m_p)) = q$*.*

Proof. Suppose that $f > 1$ and q^2 divides $p - 1$. Then there exist at least 2 non-abelian groups

$$G_i = \mathbb{Z}_m{:}\mathbb{Z}_n = \mathbb{Z}_{m_{p'}} \times \mathbb{Z}_{n_{q'}} \times H_i,$$

where $i = 1$ or 2, and $H_i = \langle x_p\rangle{:}\langle y_q\rangle = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ are as below:

$$\begin{aligned} H_1 &= \langle x_p, y_q \mid x_p^{y_q} = x_p^i\rangle, \text{ where } i \neq 1 \text{ and } i^q \equiv 1 \pmod{p^e}; \\ H_2 &= \langle x_p, y_q \mid x_p^{y_q} = x_p^j\rangle, \text{ where } j^q \not\equiv 1 \pmod{p^e}. \end{aligned}$$

This contradiction shows that either $f = 1$, or $q^2 \nmid (p-1)$, as stated. □

Remark on Lemma 3.2. Interchange m and n, if $p \mid (q-1)$, then either $e = 1$, or $p^2 \nmid (q-1)$; equivalently, $\gcd(m_p, \phi(n_q)) = p$.

Now we are ready to state the main result of this section.

Theorem 3.3. *Given a pair of integers $\{m, n\}$. Then the following two statements are equivalent:*

(a) there is a unique non-abelian exact bicyclic group (up to isomorphism) of order $\{m, n\}$,

(b) there exist exactly one prime $p \mid m$ and exactly one prime $q \mid n$ such that (m_p, n_q) is not a singular pair, and either

  (i) $\gcd(\phi(m_p), n_q) = q$, and $\gcd(m_p, \phi(n_q)) = 1$, or

  (ii) $\gcd(m_p, \phi(n_q)) = p$, and $\gcd(\phi(m_p), n_q) = 1$.

If further $p = q$, then $\{m_p, n_p\} = \{4, 2\}$, or $\{p^2, p^2\}$ with p an odd prime, or $\{p^e, p\}$ with p a prime and $e \geq 3$.

Proof. First, assume (a) holds. Let $G = \langle a\rangle\langle b\rangle$ be the unique exact nonabelian bicyclic group of order $\{m, n\}$, where $|a| = m$ and $|b| = n$. Then (m, n) is not a singular pair. So there exist at least one prime $p \mid m$, and at least one prime $q \mid n$, such that (m_p, n_q) is not a singular pair.

Suppose that p_1, p_2 are prime divisors of m and q_1, q_2 are prime divisors of n such that $\gcd(n_{q_i}, \phi(m_{p_i})) \neq 1$ with $i = 1$ or 2, and either $p_1 \neq p_2$ or $q_1 \neq q_2$. Then there are 2 non-isomorphic nonabelian exact bicyclic groups of the form $G = \langle a\rangle{:}\langle b\rangle = \mathbb{Z}_m{:}\mathbb{Z}_n$:

$$\langle a\rangle{:}\langle b\rangle = \langle a_{p_1'} a_{p_1}\rangle{:}\langle b_{q_1'} b_{q_1}\rangle = \langle a_{p_1'}\rangle \times \langle b_{q_1'}\rangle \times (\langle a_{p_1}\rangle{:}\langle b_{q_1}\rangle),$$
$$\langle a\rangle{:}\langle b\rangle = \langle a_{p_2'} a_{p_2}\rangle{:}\langle b_{q_2'} b_{q_2}\rangle = \langle a_{p_2'}\rangle \times \langle b_{q_2'}\rangle \times (\langle a_{p_2}\rangle{:}\langle b_{q_2}\rangle).$$

This is a contradiction.

Similarly, interchanging m and n, suppose that p_1, p_2 are prime divisors of m and q_1, q_2 are prime divisors of n such that $\gcd(m_{p_i}, \phi(n_{q_i})) \neq 1$ with $i = 1$ or 2, and either $p_1 \neq p_2$ or $q_1 \neq q_2$. Then there are 2 non-isomorphic nonabelian exact bicyclic groups of the form $G = \langle b\rangle{:}\langle a\rangle = \mathbb{Z}_n{:}\mathbb{Z}_m$:

$$\langle b\rangle{:}\langle a\rangle = \langle b_{p_1'} b_{p_1}\rangle{:}\langle a_{q_1'} a_{q_1}\rangle = \langle a_{p_1'}\rangle \times \langle b_{q_1'}\rangle \times (\langle b_{q_1}\rangle{:}\langle a_{p_1}\rangle),$$
$$\langle b\rangle{:}\langle a\rangle = \langle b_{p_2'} b_{p_2}\rangle{:}\langle a_{q_2'} a_{q_2}\rangle = \langle a_{p_2'}\rangle \times \langle b_{q_2'}\rangle \times (\langle b_{q_2}\rangle{:}\langle a_{p_2}\rangle).$$

This is a contradiction.

Now, suppose that p_1, p_2 are prime divisors of m and q_1, q_2 are prime divisors of n such that $\gcd(n_{q_1}, \phi(m_{p_1}) \neq 1$ and $\gcd(m_{p_2}, \phi(n_{q_2})) \neq 1$. Then there are 2 non-isomorphic nonabelian exact bicyclic groups G of the form:

$$\langle a\rangle{:}\langle b\rangle = \langle a_{p_1'} a_{p_1}\rangle{:}\langle b_{q_1'} b_{q_1}\rangle = \langle a_{p_1'}\rangle \times \langle b_{q_1'}\rangle \times (\langle a_{p_1}\rangle{:}\langle b_{q_1}\rangle),$$
$$\langle b\rangle{:}\langle a\rangle = \langle b_{q_2'} b_{q_2}\rangle{:}\langle a_{p_2'} a_{p_2}\rangle = \langle b_{q_2'}\rangle \times \langle a_{p_2'}\rangle \times (\langle b_{q_2}\rangle{:}\langle a_{p_2}\rangle).$$

This is a contradiction.

We thus conclude that there is exactly one prime $p \mid m$ and exactly one prime $q \mid n$ such that (m_p, n_q) is not a singular pair.

Assume that $\gcd(\phi(m_p), n_q) \neq 1$ and $\gcd(m_p, \phi(n_q)) \neq 1$ such that $p \neq q$. By Lemma 3.2, $\gcd(\phi(m_p), n_q) = q$ and $\gcd(m_p, \phi(n_q)) = p$, which implies that $q \mid (p-1)$ and $p \mid (q-1)$, this is not possible. Thus either part (b)(i) or part (b)(ii) holds. Moreover, if $p = q$, then by Lemma 3.1, we have $\gcd(m_p, \phi(n_p)) = p$, or $\gcd(\phi(m_p), n_p) = p$, which implies that $\{m_p, n_p\} = \{4, 2\}$, or $\{p^2, p^2\}$ with p an odd prime, $\{p^e, p\}$ with p a prime and $e \geq 3$.

Conversely, let m, n be integers satisfying condition (b). We claim that both $(m_{p'}, n)$ and $(m, n_{q'})$ are singular pairs. In fact, suppose to the contrary that one of $(m_{p'}, n)$ and $(m, n_{q'})$, say $(m_{p'}, n)$, is not singular. Then there is a prime $p_1 \neq p$ of m, and a prime q_1 of n, such that the pair (m_{p_1}, n_{q_1}) is not singular, which contradicts with the unique choice of the prime p.

Now let $G = \langle a\rangle\langle b\rangle$ such that $\langle a\rangle = \mathbb{Z}_m$, $\langle b\rangle = \mathbb{Z}_n$ and $\langle a\rangle \cap \langle b\rangle = 1$. Then G is supersoluble by [1]. Further let $G = \langle a\rangle\langle b\rangle = \langle a_p a_{p'}\rangle\langle b_q b_{q'}\rangle = G_p G_{p'}$, then $G_p = \langle a_p\rangle = \mathbb{Z}_{m_p}$ and $G_{p'} = \langle a_{p'}\rangle(\langle b_q\rangle \times \langle b_{q'}\rangle) = \mathbb{Z}_{m_{p'}}\mathbb{Z}_n$. As $\langle a_{p'}\rangle \cap \langle b\rangle = 1$, we have that $G_{p'}$ is an exact bicyclic group of order $\{m_{p'}, n\}$. By [4, Lemma 3.3], $G_{p'}$ is abelian. So $G_{p'} = (\langle a_{p'}\rangle \times \langle b_{q'}\rangle) \times \langle b_q\rangle$. Similarly, $G_{q'} = (\langle a_{p'}\rangle \times \langle b_{q'}\rangle) \times \langle a_p\rangle$. Thus a Hall subgroup $G_{\{p,q\}'}$ is abelian and centralizes both G_p and G_q, and so $G = G_{\{p,q\}'} \times G_{\{p,q\}}$, where $G_{\{p,q\}'} = \langle a_{p'}\rangle \times \langle b_{q'}\rangle = \mathbb{Z}_{m'_p} \times \mathbb{Z}_{n_{q'}}$ and $G_{\{p,q\}} = \langle a_p\rangle\langle b_q\rangle = \mathbb{Z}_{m_p}\mathbb{Z}_{n_q}$ is nonabelian. Moreover, assume that (b)(i) hold, that is $\gcd(\phi(m_p), n_q) = q$ and $\gcd(m_p, \phi(n_q)) = 1$, we have $a_p^{b_q} = a_p^{\lambda}$, where $\lambda \neq 1$ and $\lambda^q \equiv 1 \pmod{m_p}$. So the group $G_{\{p,q\}} = \langle a_p\rangle{:}\langle b_q\rangle$. We claim that the group $G_{\{p,q\}}$ is unique up to isomorphism. In fact, assume that $\mu \neq \lambda$ such that $H = \langle x, y \mid x^y = x^{\mu}, \mu \neq 1, \mu^q \equiv 1 \pmod{p^e}\rangle$. Then $\langle\lambda\rangle$ and $\langle\mu\rangle$ are both subgroups of order q in $\mathbb{Z}_{p^e}^*$, where $\mathbb{Z}_{p^e}^*$ is the multiplicative group consisting of all the unites in the ring $\mathbb{Z}_{p^e}$. Since $\mathbb{Z}_{p^e}^* \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{p^{e-1}}$, which has a unique subgroup of order q, we have $\langle\lambda\rangle = \langle\mu\rangle$. Thus $\lambda = \mu^k \pmod{p^e}$ for some integer k. Let $z = y^k$. Then $z \in H$ and $x^z = x^{u^k} = x^{\lambda}$. Hence $H \cong G_{\{p,q\}}$. Similarly, assume that (b)(ii) hold, we have the group $G_{\{p,q\}} = \langle b_q\rangle{:}\langle a_p\rangle$, $b_q^{a_p} = b_q^{\lambda}$, where $\lambda \neq 1$ and $\lambda^p \equiv 1 \pmod{n_q}$, which is unique up to isomorphism. Therefore, there is only one non-abelian exact bicyclic group of order $\{m, n\}$.

In particular, assume that $p = q$. Then $G = \langle a\rangle\langle b\rangle = G_p G_{p'}$, where $G_p = \langle a_p\rangle\langle b_p\rangle \cong \mathbb{Z}_{m_p}\mathbb{Z}_{n_p}$ and $G_{p'} = \langle a_{p'}\rangle\langle b_{p'}\rangle \cong \mathbb{Z}_{m_{p'}}\mathbb{Z}_{n_{p'}}$. By the assumption, there exists exactly one prime $p \mid \gcd(m, n)$ such that (m_p, n_p) is not a singular pair. We conclude that $(m_{p'}, n_{p'})$, $(m_{p'}, n)$ and $(m, n_{p'})$ are all singular pairs. Since $\langle a_{p'}\rangle \cap \langle b_{p'}\rangle \subseteq \langle a\rangle \cap \langle b\rangle = 1$, by [4, Lemma 3.3], $G_{p'}$ is abelian. Similarly, from $\langle a_{p'}\rangle \cap \langle b\rangle = 1$ and $\langle a\rangle \cap \langle b_{p'}\rangle = 1$, it follows that $\langle a_{p'}\rangle\langle b\rangle = G_{p'}\langle b_p\rangle$ and $\langle a\rangle\langle b_{p'}\rangle = G_{p'}\langle a_p\rangle$ are abelian. That is to say $G_{p'}$ centralizes both $\langle a_p\rangle$ and $\langle b_p\rangle$. Thus $G = G_{p'} \times G_p$, where $G_{p'} = \langle a_{p'}\rangle \times \langle b_{p'}\rangle$ and $G_p = \langle a_p\rangle{:}\langle b_p\rangle$, which is unique discussed as above. These prove (a) holds. $\square$

4 Proof of the main theorem

Let m, n be integers for which $\mathbf{K}_{m,n}$ only has one non-abelian edge-transitive embedding. Then there is only one non-abelian exact bicyclic group $G = \mathbb{Z}_m\mathbb{Z}_n$. By Theorem 3.3, interchanging m and n if necessary, there are a unique prime divisor p of m and a unique prime divisor q of n such that $G = G_{\{p,q\}'} \times G_{\{p,q\}}$, and $G_{\{p,q\}} = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ is nonabelian.

We give a basic fact at first which is used repeatedly in the following.

Lemma 4.1. *Suppose* $H = \langle a\rangle{:}\langle b\rangle = \mathbb{Z}_k{:}\mathbb{Z}_l$ *is a split extension such that* $a^b = a^\lambda$*, where* $\lambda \neq 1$*,* $\lambda^l \equiv 1 \pmod{k}$ *and* l *is odd. Then the following map of* H*:*

$$\sigma\colon a \mapsto a^i,\ b \mapsto b^{-1},$$

where $\gcd(i,k) = 1$*, is not an automorphism of* H*.*

Proof. Supppose to the contrary. Then $\sigma(a)^{\sigma(b)} = (a^i)^{b^{-1}} = ba^ib^{-1} = \sigma(a^\lambda) = a^{\lambda i} = b^{-1}a^ib$, and so $a^ib^2 = b^2a^i$. Since $o(b^2) = o(b)$ and $o(a^i) = o(a)$, it follows that $H = \langle a^i, b^2\rangle$ is abelian, which is a contradiction. □

Now we determine the $\{p^e, q^f\}$ when $p = q$.

Lemma 4.2. *If* $p = q$*, then* $\{p^e, p^f\} = \{4, 2\}$*.*

Proof. Since a group of order p^2 is abelian, without loss of generality, we may assume that $e \geq 2$, and $e \geq f$.

Suppose that p is odd. Then there exists a non-abelian metacyclic group $G_{\{p,q\}} = \langle x_p\rangle{:}\langle y_p\rangle$ such that $x_p^{y_p} = x_p^\lambda$ where $\lambda \neq 1$ and $\lambda^q \equiv 1 \pmod{p^e}$. Let

$$G = G_{\{p,q\}'} \times G_{\{p,q\}} = \langle x_{p'}\rangle \times \langle y_{p'}\rangle \times (\langle x_p\rangle{:}\langle y_p\rangle) = \langle x_{p'}x_p\rangle{:}\langle y_{p'}y_p\rangle \cong \mathbb{Z}_m{:}\mathbb{Z}_n.$$

Then the pairs $(x_{p'}x_p, y_{p'}y_p)$ and $(x_{p'}x_p, y_{p'}y_p^{-1})$ are not equivalent under $\mathrm{Aut}(G)$ by Lemma 4.1, and thus $\mathbf{K}_{m,n}$ has at least 3 non-isomorphic orientably edge-transitive embeddings, which is a contradiction.

We thus conclude that $p = 2$, and so by Theorem 3.3, $\{p^e, p^f\} = \{4, 2\}$. □

Next we determine the $\{p^e, q^f\}$ when $p \neq q$.

Lemma 4.3. *If* $p \neq q$*, then* $q = 2$*, and either* $q^f = 2$*, or* $q^f \geq 4$ *and* $p \equiv 3 \pmod{4}$*.*

Proof. Assume $p \neq q$. Since $G_{\{p,q\}} = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ is nonabelian, q divides $p-1$. Suppose that q is odd. Let $\langle x'\rangle = \mathbb{Z}_{m_{p'}}$ and $\langle y'\rangle = \mathbb{Z}_{n_{q'}}$, and let

$$G = \langle x'\rangle \times \langle y'\rangle \times (\langle x_p\rangle{:}\langle y_q\rangle) = \langle x_px'\rangle{:}\langle y_qy'\rangle = \mathbb{Z}_m{:}\mathbb{Z}_n.$$

Then (x_px', y_qy') and $(x_px', y_q^{-1}y')$ are not equivalent under $\mathrm{Aut}(G)$ by Lemma 4.1, and so $\mathbf{K}_{m,n}$ has at least 3 non-isomorphic orientably edge-transitive embeddings, which is a contradiction.

We thus have that $q = 2$, and $\gcd(n_2, \phi(m_p)) = 2$ by Lemma 3.2, that is, either $q^f = 2$, or $q^f \geq 4$ and $p \equiv 3 \pmod{4}$. □

Now we are ready to produce a list of groups for G.

Lemma 4.4. *The unique nonabelian exact bicyclic group* $G = \mathbb{Z}_m\mathbb{Z}_n$ *satisfies one of the following, where* p *is a prime:*

(i) $G = \mathrm{D}_8$*;*

(ii) $G = \mathrm{D}_{2p^e} \times \mathbb{Z}_{n_{2'}}$*, where* $(m, n_{2'})$ *is a singular pair;*

(iii) $G = (\mathbb{Z}_{p^e}{:}\mathbb{Z}_{2^f}) \times \mathbb{Z}_{n_{2'}}$*, where* $p \equiv 3 \pmod{4}$ *and* $(m, n_{2'})$ *is a singular pair;*

(iv) $G = \mathrm{D}_{4p^e}$ *with* p *odd.*

Proof. Noting that the group $G = G_{\{p,q\}'} \times G_{\{p,q\}}$, and $G_{\{p,q\}} = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ is nonabelian. By Lemma 4.3, $q = 2$, and if $m_2 > 2$ is even, then $(m_2, n_2) = (4, 2)$ by Lemma 4.2. We conclude that $(m, n) = (4, 2)$, and the corresponding group $G = \mathrm{D}_8$, as in part (i).

Assume now that $m_2 = 1$. It follows that $m = p^e$ and either $q^f = 2$, or $q^f = 2^f \geq 4$ and $p \equiv 3 \pmod 4$ by Lemma 4.3. If $n_2 = 2$, then $G = \mathbb{Z}_{n_{2'}} \times (\mathbb{Z}_{p^e}{:}\mathbb{Z}_2) = \mathbb{Z}_{n_{2'}} \times \mathrm{D}_{2p^e}$ such that $(m, n_{2'})$ is a singular pair, as in part (ii). If $n_2 \geq 4$, then $G = (\mathbb{Z}_{p^e}{:}\mathbb{Z}_{2^f}) \times \mathbb{Z}_{n_{2'}}$, where $p \equiv 3 \pmod 4$ and $(m, n_{2'})$ is a singular pair, as in part (iii).

Finally assume that $m_2 = 2$. Then $m = 2p^e$ with p odd, and $n = 2$. So the corresponding group $G = \mathrm{D}_{4p^e}$, as in part (iv). This completes the proof. $\square$

To complete the proof of Theorem 1.1, we need to prove that for each group G listed in Lemma 4.4, all edge regular pairs are equivalent.

Lemma 4.5. *Let* $G = \mathrm{D}_{2m}$ *be dihedral. Then all edge-regular pairs for* G *on* $\mathbf{K}_{m,2}$ *are equivalent.*

Proof. Let (x, y) be an edge-regular pair for G acting on $\mathbf{K}_{m,2}$. Then $|x| = m$, $|y| = 2$, and $x^y = x^{-1}$. Let x', y' be another edge-regular pair such that $G = \langle x'\rangle\langle y'\rangle$. Then $|x'| = m$, $|y'| = 2$, and $(x')^{y'} = (x')^{-1}$. Clearly, there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that

$$\sigma\colon x \mapsto x',\ y \mapsto y',$$

so all regular pairs for G on $\mathbf{K}_{m,2}$ are equivalent. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1.1. The necessity is easily found from Lemma 4.4.

To prove the sufficiency, we need prove that for each group $G = \mathbb{Z}_m\mathbb{Z}_n$ listed in Lemma 4.4, there is exactly one non-abelian orientably edge-transitive embedding of $\mathbf{K}_{m,n}$. If G is a dihedral group, then the proof follows from Lemma 4.5. Thus we assume that G is not a dihedral group.

Assume that $m = p^e$ and $n = 2n_{2'}$. Then the only exact bicyclic group of order $\{m, n\}$ is $G = (\langle a\rangle{:}\langle b_2\rangle) \times \langle b_{2'}\rangle$, where $\langle a\rangle{:}\langle b_2\rangle = \mathrm{D}_{2p^e}$ and $|b_{2'}| = n_{2'}$. Let (x_1, y_1) and (x_2, y_2) be two edge-regular pairs from G. Then

$$\begin{aligned} x_1 &= a^{i_1}, & y_1 &= b_2 b_{2'}^{j_1},\\ x_2 &= a^{i_2}, & y_2 &= b_2 b_{2'}^{j_2}, \end{aligned}$$

where i_1, i_2 are coprime to p, and j_1, j_2 are coprime to $n_{2'}$. There is an automorphism $\sigma \in \mathrm{Aut}(\langle a\rangle{:}\langle b_2\rangle)$ which sends a^{i_1} to a^{i_2}; there is an automorphism $\tau \in \mathrm{Aut}(\langle b_{2'}\rangle)$ which sends b^{j_1} to b^{j_2}. Then (σ, τ) is an automorphism of G which maps (x_1, y_1) to (x_2, y_2).

Assume that $G = (\mathbb{Z}_{p^e}{:}\mathbb{Z}_{2^f}) \times \mathbb{Z}_{n_{2'}}$, where $m = p^e$ with $p \equiv 3 \pmod 4$, and $\gcd(\phi(n), m) = 1$ and $\gcd(n, \phi(m)) = 2$. Then

$$G = (\langle a\rangle{:}\langle b_2\rangle) \times \langle b_{2'}\rangle,$$

where $|a| = m = p^e$ and $n = 2^f n_{2'}$, and $a^{b_2} = a^{-1}$. Let (x, y) and (x', y') be edge-regular pairs for G on $\mathbf{K}_{m,n}$ such that $|x| = |x'| = m$ and $|y| = |y'| = n$. Then

$$x = a^i, \qquad y = b_2^j b_{2'}^k, \quad \text{and}$$
$$x' = a^{i'}, \qquad y' = b_2^{j'} b_{2'}^{k'},$$

where $p \nmid ii'$, jj' is odd and $\gcd(kk', n_{2'}) = 1$. It is easily shown that there are automorphisms $\sigma \in \mathrm{Aut}(\langle a\rangle{:}\langle b_2\rangle)$ and $\tau \in \mathrm{Aut}(\langle b_{2'}\rangle)$ such that

$$\sigma\colon a^i \mapsto a^{i'},\ b_2^j \mapsto b_2^{j'}$$
$$\tau\colon b_{2'}^k \mapsto b_{2'}^{k'}.$$

It follows that (σ, τ) is an automorphism of G which sends (x, y) to (x', y'). Thus all edge-regular pairs for G on $\mathbf{K}_{m,n}$ are equivalent. □

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The expansion of a chord diagram and the Genocchi numbers

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Abstract

A chord diagram E is a set of chords of a circle such that no pair of chords has a common endvertex. Let $v_1, v_2, \ldots, v_{2n}$ be a sequence of vertices arranged in clockwise order along a circumference. A chord diagram $\{v_1v_{n+1}, v_2v_{n+2}, \ldots, v_nv_{2n}\}$ is called an n-crossing and a chord diagram $\{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$ is called an n-necklace. For a chord diagram E having a 2-crossing $S = \{x_1x_3, x_2x_4\}$, the expansion of E with respect to S is to replace E with $E_1 = (E \setminus S) \cup \{x_2x_3, x_4x_1\}$ or $E_2 = (E \setminus S) \cup \{x_1x_2, x_3x_4\}$. Beginning from a given chord diagram E as the root, by iterating chord expansions in both ways, we have a binary tree whose all leaves are nonintersecting chord diagrams. Let $\mathcal{NCD}(E)$ be the multiset of the leaves. In this paper, the multiplicity of an n-necklace in $\mathcal{NCD}(E)$ is studied. Among other results, it is shown that the multiplicity of an n-necklace generated from an n-crossing equals the Genocchi number when n is odd and the median Genocchi number when n is even.

Keywords: Chord diagram, chord expansion, Genocchi number, Seidel triangle.

Math. Subj. Class. (2020): 05A15, 05A10

1 Introduction

A set of chords of a circle is called a *chord diagram*, if they have no common endvertex. If a chord diagram consists of a set of n mutually crossing chords, it is called an n*-crossing*. A 2-crossing is simply called a crossing as well. If a chord diagram contains no crossing, it is called *nonintersecting*.

Let V be a set of $2n$ vertices on a circle, and let E be a chord diagram of order n, where each chord has endvertices of V. In this situation, V is called a *support* of

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E. We denote the family of all chord diagrams having V as a support by $\mathcal{CD}(V)$. Let $x_1, x_2, x_3, x_4 \in V$ be placed on a circle in clockwise order. Let $E \in \mathcal{CD}(V)$. For a crossing $S = \{x_1x_3, x_2x_4\} \subset E$, let $S_1 = \{x_2x_3, x_4x_1\}$, and $S_2 = \{x_1x_2, x_3x_4\}$. The *expansion* of E with respect to S is defined as a replacement of E with $E_1 = (E \setminus S) \cup S_1$ or $E_2 = (E \setminus S) \cup S_2$ (see Figure 1).

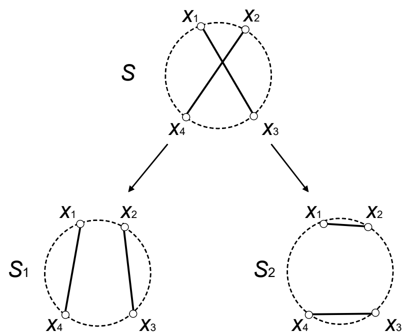


Figure 1: The expansion of a chord diagram with respect to a 2-crossing S. Other chords except those in S are not shown.

Let $E \in \mathcal{CD}(V)$ be a chord diagram. Form a binary tree as follows. Begin with E as the root, arbitrarily choose a crossing of E, and expand E in both ways, adding the results as children of E. Choose crossings in each child if any exists, expand them each in both ways, and repeat the procedure until all leaves are nonintersecting. This procedure terminates and the multiset of leaves is independent of the choices made at each step ([14]). Let us denote the multiset of nonintersecting chord diagrams generated from E by $\mathcal{NCD}(E)$. For a chord diagram $E \in \mathcal{CD}(V)$, let us define the *chord expansion number* $f(E)$ as the cardinality of $\mathcal{NCD}(E)$ as a multiset.

For a chord diagram E, the *circle graph*, also called the *interlace graph* G_E of E, is a graph such that a vertex of G_E corresponds to a chord of E and two vertices of G_E are joined by an edge if their corresponding chords of E are mutually crossing. We say that two chord diagrams E_1 and E_2 with a common support are isomorphic if G_{E_1} and G_{E_2} are isomorphic as graphs. It is proved that $f(E)$ equals $t(G_E; 2, -1)$, where $t(G; x, y)$ is the *Tutte polynomial* of a graph G ([15]).

In the case E is an n-crossing C_n, its associated circle graph is a complete graph K_n with n vertices. In [13], Merino proved that $t(K_n; 2, -1) = Eul_{n+1}$ for $n \geq 1$, where $(Eul)_{n\geq 1} = (1, 1, 2, 5, 16, 61, 272, \ldots)$ is the *Euler number*. Hence, we have $f(C_n) = Eul_{n+1}$ for $n \geq 1$. See also [12] for the evaluation of $t(G; 2, -1)$ for a graph G.

For two nonnegative integers k and n with $k \leq n$, we define $A(n, k)$ as a chord diagram of order $n + 1$, in which there is an n-crossing E_0 with an extra chord e such that e crosses exactly k chords of E_0. (See Figure 2.) Note that $A(n-1, n-1)$ is simply an n-crossing, and that $A(n, 0)$ is a union of an n-crossing and an isolated chord.

Let us denote $\{1, 2, \ldots, n\}$ by $[n]$. A permutation σ on $[n]$ is called an *alternating permutation* if $(\sigma(i) - \sigma(i-1))(\sigma(i+1) - \sigma(i)) < 0$ for $2 \leq i \leq n-1$. An alternating permutation σ is called an *up-down permutation* (resp. *down-up permutation*) if $\sigma(1) < \sigma(2)$ (resp. $\sigma(1) > \sigma(2)$). For $0 \leq k \leq n$, the *Entringer number* $Ent_{n,k}$ is defined as the number of down-up permutations on $[n+1]$ with the first term $k+1$ ([11]). For $n \geq 1$,

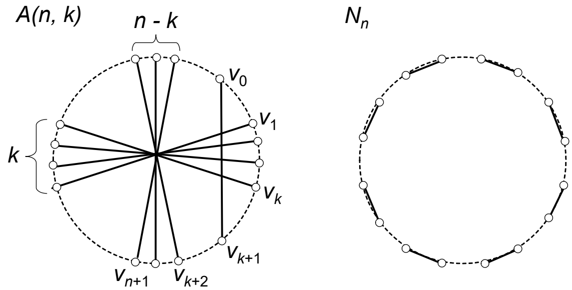


Figure 2: $A(n, k)$ with $n = 7$ and $k = 4$ (left), and N_n with $n = 8$ (right).

$Ent_{n+1,1}$ equals Eul_n, the number of all down-up permutations on $[n]$. In [14], it is proved that $f(A(n, k)) = Ent_{n+2,k+1}$.

For a chord diagram E and for a nonintersecting chord diagram F with a common support, let us denote the *multiplicity* of F in $\mathcal{NCD}(E)$ by $m(E, F)$. For a nonintersecting chord diagram E, a chord $e \in E$ is called an *ear*, if there is no other chord of E on at least one side of e. In [15], it is shown that for an n-crossing C_n and a nonintersecting chord diagram F with a common support, $m(C_n, F) = 1$ if and only if F has at most 3 ears. A nonintersecting chord diagram E with n chords is called an *n-necklace*, denoted by N_n, if all chords of E are ears. (See Figure 2.) The main purpose of the paper is to show that $m(C_n, N_n)$ equals the *Genocchi number* when n is odd and the *median Genocchi number* when n is even. The Genocchi numbers and the median Genocchi numbers will be introduced in the following section.

Recently, Bigeni showed a relation between a weight system of sl_2 of chord diagrams and the median Genocchi numbers ([2]). In Definition 1 of [2], followed from [3], a weight system of sl_2 is defined inductively by applying an operation for chord diagrams. The operation and the chord expansion are closely related to each other, although our main results in the paper do not seem directly followed from the results in [2].

The rest of this paper is organized as follows. In Section 2, the Genocchi numbers and the median Genocchi numbers are introduced. In Section 3, the main results of the paper are proved. In Section 4, another combinatorial interpretation for the multiplicity of n-necklaces is exhibited. Finally, in Section 5, some open problems are discussed.

2 The Genocchi numbers and the median Genocchi numbers

According to [10], but with slightly different indices, let us recursively define the entry $S(n, k)$ in row $n \geq 1$ and column $k \geq 0$ of the *Seidel triangle* ([17]):

$$\begin{aligned}
S(1,1) &= 1, \\
S(n,k) &= 0 \quad \text{for } k = 0 \text{ or } n \leq 2(k-1), \\
S(2n,k) &= \sum_{i \geq k} S(2n-1,i) \quad \text{for } 1 \leq k \leq n, && (2.1) \\
S(2n+1,k) &= \sum_{i \leq k} S(2n,i) \quad \text{for } 1 \leq k \leq n+1. && (2.2)
\end{aligned}$$

Table 1: The Seidel triangle $S(n, k)$.

$n \setminus k$	1	2	3	4	5
1	1				
2	1				
3	1	1			
4	2	1			
5	2	3	3		
6	8	6	3		
7	8	14	17	17	
8	56	48	34	17	
9	56	104	138	155	155
10	608	552	448	310	155

(See Table 1.) By the equations (2.1) and (2.2), we have the following recurrence relations.

$$S(2n, k) = S(2n-1, k) + S(2n, k+1) \quad \text{for } 1 \le k \le n, \tag{2.3}$$

$$S(2n+1, k) = S(2n, k) + S(2n+1, k-1) \quad \text{for } 1 \le k \le n+1. \tag{2.4}$$

The *Genocchi numbers* (or Genocchi numbers of the first kind) $G(2n)$ are defined as $S(2n-1, n)$, the numbers on the right edge of the Seidel triangle, and the *median Genocchi numbers* (or Genocchi numbers of the second kind) $H(2n+1)$ are defined as $S(2n+2, 1)$, the numbers on the left edge of the Seidel triangle. Note that $(G(2n))_{n\ge1} = (1, 1, 3, 17, 155, \dots)$ and $(H(2n+1))_{n\ge0} = (1, 2, 8, 56, 608, \dots)$.

Combinatorial properties of the Genocchi numbers have been extensively studied ([1, 4, 5, 6, 7, 8, 9, 10, 16, 19]). It is known that the Genocchi number $G(2n)$ counts the number of permutations σ on $[2n-1]$ such that $\sigma(i) < \sigma(i+1)$ if $\sigma(i)$ is odd, and $\sigma(i) > \sigma(i+1)$ if $\sigma(i)$ is even ([6]). It is also known that the median Genocchi number $H(2n+1)$ counts the number of permutations σ on $[2n+1]$ such that $\sigma(i) > i$ if i is odd and $i \ne 2n+1$, and $\sigma(i) < i$ if i is even ([6]).

In the on-line encyclopedia of integer sequences [18], we can find more information for the sequences A001469 (Genocchi numbers), A005439 (median Genocchi numbers), A099960 (An interleaving of the Genocchi numbers of the first and second kind) and A014781 (Seidel triangle).

3 Main results

Our aim is to show a new combinatorial interpretation for the values of the Seidel triangle by using chord expansions.

Let $v_0, v_1, \dots, v_{2n+1}$ be a sequence of vertices in clockwise order along a circumference. Let $V = \{v_i : 0 \le i \le 2n+1\}$. As one of chord diagrams $E \in \mathcal{CD}(V)$ isomorphic to $A(n, k)$, introduced in the previous section, we have $E = \{v_0v_{k+1}\} \cup \{v_iv_{n+i+1} : 1 \le i \le k\} \cup \{v_iv_{n+i} : k+2 \le i \le n+1\}$. (See Figure 2.) Now let us define $(n+1)$-necklaces $N^+_{n+1,k}$ and $N^-_{n+1,k} \in \mathcal{CD}(V)$ such that $N^+_{n+1,k}$ contains an ear v_kv_{k+1} and $N^-_{n+1,k}$ contains an ear $v_{k+1}v_{k+2}$. The values of $m(A(n, k), N^+_{n+1,k})$ for n and k small are shown in Table 2.

Table 2: $m(A(n,k), N^+_{n+1,k})$ for $0 \le k \le n \le 8$.

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	1	1						
3	1	1	2	2					
4	2	2	3	3	3				
5	3	3	6	6	8	8			
6	8	8	14	14	17	17	17		
7	17	17	34	34	48	48	56	56	
8	56	56	104	104	138	138	155	155	155

Let us define $b^+_{n,k} = m(A(n,k), N^+_{n+1,k})$ and $b^-_{n,k} = m(A(n,k), N^-_{n+1,k})$. We also simply denote $b^+_{n,k}$ by $b_{n,k}$. The main result of the paper is the following theorem.

Theorem 3.1. *Let $n \ge 1$. Then we have*

$$b_{2n-1,k} = S(2n, n - \lfloor k/2 \rfloor) \quad \textit{for } 0 \le k \le 2n-1, \tag{3.1}$$

and

$$b_{2n,k} = S(2n+1, \lfloor k/2 \rfloor + 1) \quad \textit{for } 0 \le k \le 2n. \tag{3.2}$$

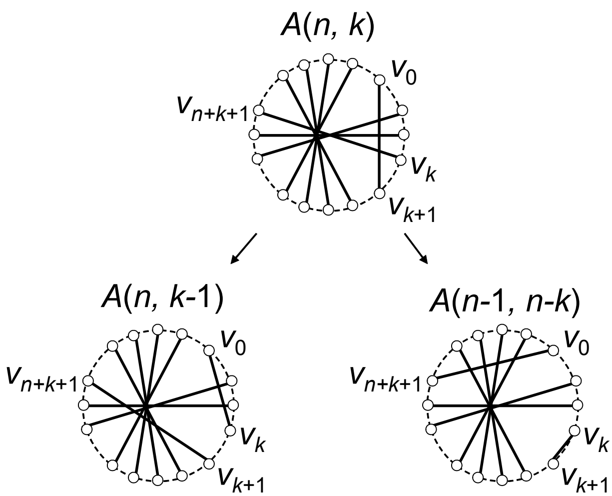


Figure 3: A chord expansion of $A(n,k)$ with respect to $\{v_0 v_{k+1}, v_k v_{n+k+1}\}$ with $n = 7$ and $k = 3$.

Firstly, we show a relation between $b^-_{n,k}$ and $b^+_{n,k}$.

Lemma 3.2. $b^-_{n,k} = b^+_{n,k-1}$ *for $1 \le k \le n$.*

Proof. Let E be a chord diagram isomorphic to $A(n,k)$, as shown in Figure 3. By the chord expansion of E with respect to $\{v_0v_{k+1}, v_kv_{n+k+1}\}$, we have two successors E_1 and E_2, which are isomorphic to $A(n,k-1)$ and $A(n-1,n-k)$, respectively. Since E_2 contains a chord v_kv_{k+1}, it does not generate $N^-_{n+1,k}$. Furthermore, since $N^-_{n+1,k}$ is a necklace having a chord $v_{k-1}v_k$, we have $b^-_{n,k} = m(A(n,k), N^-_{n+1,k}) = m(A(n,k-1), N^+_{n+1,k-1}) = b^+_{n,k-1}$, as required. □

In order to prove Theorem 3.1, let us show a recurrence relation for $b_{n,k}$.

Lemma 3.3. *We have $b_{0,0} = 1$ and for $n \geq 1$, we have*

$$\begin{aligned} &b_{n,0} = b_{n,1} = b_{n-1,n-1}, \\ &b_{n,k} = \begin{cases} b_{n,k-2} + b_{n-1,n-k} & \text{for } 2 \leq k \leq n \text{ and } n \text{ is odd}, \\ b_{n,k-2} + b_{n-1,n-k-1} & \text{for } 2 \leq k \leq n-1 \text{ and } n \text{ is even}, \end{cases} \\ &b_{n,n} = b_{n,n-2} \quad \text{for } n \text{ is even}. \end{aligned}$$

Proof. When $k = 0, 1$ or n, equations $b_{n,0} = b_{n,1} = b_{n-1,n-1}$ can be proved easily. Let us consider the case $2 \leq k \leq n$. As in the proof of Lemma 3.2, we use the expansion of $A(n,k)$ with respect to $\{v_0v_{k+1}, v_kv_{n+k+1}\}$.

If n is odd, we have

$$\begin{aligned} b^+_{n,k} &= b^-_{n,k-1} + b^+_{n-1,n-k} \\ &= b^+_{n,k-2} + b^+_{n-1,n-k}. \end{aligned}$$

If n is even and $k < n$, we have

$$\begin{aligned} b^+_{n,k} &= b^-_{n,k-1} + b^-_{n-1,n-k} \\ &= b^+_{n,k-2} + b^+_{n-1,n-k-1}. \end{aligned}$$

Finally, if n is even and $k = n$, since $b^-_{n-1,0} = 0$, we have

$$\begin{aligned} b^+_{n,n} &= b^-_{n,n-1} + b^-_{n-1,0} \\ &= b^+_{n,n-2}, \end{aligned}$$

as needed. □

Proof of Theorem 3.1. We proceed by induction on n and k. For (3.1) with $n = 1$, we have $b_{1,0} = 1$ and $b_{1,1} = 1$. On the other hand, we have $S(2,1) = 1$. For (3.2) with $n = 1$, we have $b_{2,0} = 1$, $b_{2,1} = 1$ and $b_{2,2} = 1$. On the other hand, we have $S(3,1) = S(3,2) = 1$.

Let $n \geq 2$. For $k = 0$, we have

$$\begin{aligned} b_{2n-1,0} &= b_{2n-2,2n-2} \\ &= S(2n-1,n) \\ &= S(2n,n), \end{aligned}$$

and

$$\begin{aligned} b_{2n,0} &= b_{2n-1,2n-1} \\ &= S(2n,1) \\ &= S(2n+1,1). \end{aligned}$$

For $k = 1$, we have

$$\begin{aligned} b_{2n-1,1} &= b_{2n-1,0} \\ &= S(2n,n), \end{aligned}$$

and

$$\begin{aligned} b_{2n,1} &= b_{2n,0} \\ &= S(2n+1,1). \end{aligned}$$

For (3.1) with $2 \le k \le 2n-1$, we have

$$\begin{aligned} b_{2n-1,k} &= b_{2n-1,k-2} + b_{2n-2,2n-1-k} \\ &= S(2n, n - \lfloor (k-2)/2 \rfloor) + S(2n-1, \lfloor (2n-1-k)/2 \rfloor + 1) \\ &= S(2n, n+1-\lfloor k/2 \rfloor) + S(2n-1, n - \lfloor k/2 \rfloor) \\ &= S(2n, n - \lfloor k/2 \rfloor), \end{aligned}$$

and for (3.2) with $2 \le k \le 2n-1$, we have

$$\begin{aligned} b_{2n,k} &= b_{2n,k-2} + b_{2n-1,2n-1-k} \\ &= S(2n+1, \lfloor (k-2)/2 \rfloor + 1) + S(2n, n - \lfloor (2n-1-k)/2 \rfloor) \\ &= S(2n+1, \lfloor k/2 \rfloor) + S(2n, 1 + \lfloor k/2 \rfloor) \\ &= S(2n+1, 1 + \lfloor k/2 \rfloor), \end{aligned}$$

and for (3.2) with $k = 2n$, we have

$$\begin{aligned} b_{2n,2n} &= b_{2n,2n-2} \\ &= S(2n+1,n) \\ &= S(2n+1,n+1). \end{aligned}$$ □

By Theorem 3.1, we have the following corollary.

Corollary 3.4. *$m(C_{2n}, N_{2n}) = H(2n-1)$ and $m(C_{2n-1}, N_{2n-1}) = G(2n)$ for $n \ge 1$.*

Proof. By Theorem 3.1, we have $m(C_{2n}, N_{2n}) = b_{2n-1,2n-1} = S(2n,1) = H(2n-1)$, and $m(C_{2n-1}, N_{2n-1}) = b_{2n-2,2n-2} = S(2n-1,n) = G(2n)$. □

4 Multiplicity of an N-necklace and the number of perfect matchings of an associated graph

In this section, we will exhibit a combinatorial interpretation of $m(E, N_n)$ for a given chord diagram E. For a set V of vertices on the circumference, $\mathcal{C}(V)$ denotes the set of all chords whose endvertices are in V. A *Ptolemy weight* w on $\mathcal{C}(V)$ is defined as a function that satisfies

$$w(x_1x_3)w(x_2x_4) = w(x_2x_3)w(x_1x_4) + w(x_1x_2)w(x_3x_4) \tag{4.1}$$

for all vertices $x_1, x_2, x_3, x_4 \in V$ placed along the circle. If $w(e)$ is the Euclidean length of a chord e, then (4.1) holds by the Ptolemy's theorem in Euclidean geometry. Let w be a Ptolemy weight on $\mathcal{C}(V)$. If a chord diagram $E \in \mathcal{CD}(V)$ has a 2-crossing S, by the chord expansion of E with respect to S, we have two successors E_1 and E_2. Then by (4.1), we have

$$\prod_{e\in E} w(e) = \prod_{e\in E_1} w(e) + \prod_{e\in E_2} w(e). \tag{4.2}$$

We denote the left-hand side of (4.2) by $w(E)$. By iterating chord expansions with (4.2), we have

$$w(E) = \sum_{F\in\mathcal{NCD}(E)} w(F). \tag{4.3}$$

Let $V = \{v_1, v_2, \dots, v_{2n}\}$, where $v_1, v_2, \dots, v_{2n}$ are placed along the circumference in this order. A Ptolemy weight w on $\mathcal{C}(V)$ is called *rectilinear* if $w(v_iv_j) = \sum_{i\le k<j} w(v_kv_{k+1})$ for all $1 \le i < j \le 2n$. For example, if the vertices are placed on a straight line and the weight $w(v_iv_j)$ is defined as the Euclidean distance between v_i and v_j, then w is indeed a rectilinear Ptolemy weight.

In order to analyze $m(E, N_n)$, let us consider the rectilinear Ptolemy weight w on $\mathcal{C}(V)$ such that $w(v_{2k-1}v_{2k}) = x_k$ for $1 \le k \le n$ and $w(v_{2k}v_{2k+1}) = 0$ for $1 \le k \le n-1$. In this weight, since for every chord e, $w(e)$ corresponds to a first degree polynomial of a multiple variables $x_1, x_2, \dots, x_n$ or $w(e) = 0$, for all chord diagrams E, $w(E)$ is a homogeneous polynomial of degree n or $w(E) = 0$. From this point until the end of this section, we fix this weight. Let us define an n-necklace $N_n = \{v_{2k-1}v_{2k} : 1 \le k \le n\}$.

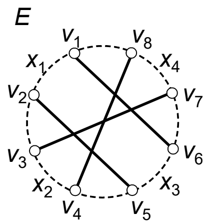
Lemma 4.1. *In the rectilinear Ptolemy weight w as defined in the above, for a chord diagram E, $m(E, N_n)$ equals the coefficient of $x_1x_2 \dots x_n$ of the polynomial $w(E)$.*

Proof. Since $w(N_n) = x_1x_2 \dots x_n$, what we need to show is that if $F \in \mathcal{NCD}(E)\setminus\{N_n\}$, a polynomial $w(F)$ contains no monomial $x_1x_2 \dots x_n$. Suppose to a contradiction that $F \in \mathcal{NCD}(E) \setminus \{N_n\}$ and F has a monomial $x_1x_2 \dots x_n$. Since $F \ne N_n$, there exists a chord $v_{2k-1}v_{2\ell}$ of F with $1 \le k < \ell \le n$ such that $\ell - k \ge 1$ is maximal. Then the two variables x_k and x_ℓ do not appear together in the weight of any chord of F, otherwise such a chord would either intersect $v_{2k-1}v_{2\ell}$ or contradict $\ell - k$ being maximal. It follows that the product x_kx_ℓ never appears in $w(F)$. This contradicts to that $w(F)$ contains a monomial $x_1x_2 \dots x_n$. □

For a chord diagram E having n chords $e_1, e_2, \dots, e_n$ with the rectilinear Ptolemy weight w as defined in the above, let us define a balanced bipartite graph $G(E, X)$ with partite sets $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ as follows. For $1 \le i \le n$ and $1 \le j \le n$, a_i and b_j are adjacent if and only if a polynomial $w(e_i)$ contains a monomial x_j.

Theorem 4.2. *For a chord diagram E with n chords and its associated balanced bipartite graph $G(E, X)$ as defined in the above, $m(E, N_n)$ equals the number of perfect matchings of $G(E, X)$.*

Proof. We have $w(E) = \prod_{e \in E} w(e)$, and for all chords e, $w(e) = 0$ or $w(e) = x_i + x_{i+1} + \cdots + x_j$ for some $1 \leq i \leq j \leq n$. Hence, the coefficient of $x_1 x_2 \ldots x_n$ of $w(E)$, which is $m(E, N_n)$ by Lemma 4.1, is the number of possible combinations to choose a variable $x \in X$ from each $w(e)$ without repetition. This is the number of perfect matchings of $G(E, X)$. □



G(E, X)

	x_1	x_2	x_3	x_4
e_1	1	1	1	0
e_2	0	1	0	0
e_3	0	1	1	0
e_4	0	0	1	1

Figure 4: A chord diagram E (left) and its biadjacency matrix of a corresponding bipartite graph $G(E, X)$ (right).

Example 4.3. Let $n = 4$. Let $V = \{v_i : 1 \leq i \leq 2n\}$, where $v_1, v_2, \ldots, v_{2n}$ are placed on the circumference in this order. Let us consider a rectilinear Ptolemy weight w on $\mathcal{C}(V)$ such that $w(v_{2i-1}v_{2i}) = x_i$ for $1 \leq i \leq n$ and $w(v_{2i}v_{2i+1}) = 0$ for $1 \leq i \leq n-1$. Let $E = \{e_i : 1 \leq i \leq 4\}$ be a chord diagram, where $e_1 = v_1v_6, e_2 = v_2v_5, e_3 = v_3v_7, e_4 = v_4v_8$. (See Figure 4.) Since

$$w(E) = \prod_{1 \leq i \leq n} w(e_i) = (x_1 + x_2 + x_3)x_2(x_2 + x_3)(x_3 + x_4),$$

the coefficient of $x_1x_2x_3x_4$ of $w(E)$ is 1, and the number of perfect matchings of $G(E, X)$ is also 1. Hence, we have $m(E, N_n) = 1$.

By Corollary 3.4 and Theorem 4.2 for n-crossings C_n, we have the following corollary.

Corollary 4.4. *The number of perfect matchings of the following bipartite graphs G and H corresponds to Genocchi numbers $G(2n)$ and median Genocchi numbers $H(2n-1)$ as follows:*

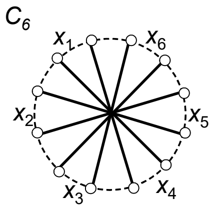
$$\begin{aligned}
&V(G) = E \cup X, \quad \textit{where } E = \{e_1, e_2, \ldots, e_{2n-1}\}, X = \{x_1, x_2, \ldots, x_{2n-1}\},\\
&E(G) = \{e_i x_j : 1 \leq i \leq 2n-1, \lfloor i/2 \rfloor + 1 \leq j \leq \lfloor (i-1)/2 \rfloor + n\}.\\
&V(H) = E \cup X, \quad \textit{where } E = \{e_1, e_2, \ldots, e_{2n}\}, X = \{x_1, x_2, \ldots, x_{2n}\},\\
&E(H) = \{e_i x_j : 1 \leq i \leq 2n, \lfloor i/2 \rfloor + 1 \leq j \leq \lfloor i/2 \rfloor + n\}.
\end{aligned}$$

Example 4.5. As shown in Figure 5,

$$w(C_6) = (x_1 + x_2 + x_3)(x_2 + x_3 + x_4)^2(x_3 + x_4 + x_5)^2(x_4 + x_5 + x_6).$$

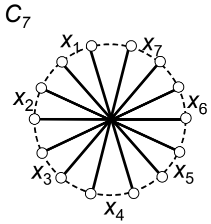
The coefficient of $x_1x_2x_3x_4x_5x_6$ of $w(C_6)$ is 8, and the number of perfect matchings of $G(C_6, X)$ is also 8. Hence, we have $H(5) = m(C_6, N_6) = 8$.

$$\begin{aligned}
w(C_7) = {}&(x_1 + x_2 + x_3 + x_4)(x_2 + x_3 + x_4)(x_2 + x_3 + x_4 + x_5)(x_3 + x_4 + x_5)\\
&(x_3 + x_4 + x_5 + x_6)(x_4 + x_5 + x_6)(x_4 + x_5 + x_6 + x_7).
\end{aligned}$$



$G(C_6, X)$

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	1	0	0	0
e_2	0	1	1	1	0	0
e_3	0	1	1	1	0	0
e_4	0	0	1	1	1	0
e_5	0	0	1	1	1	0
e_6	0	0	0	1	1	1



$G(C_7, X)$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e_1	1	1	1	1	0	0	0
e_2	0	1	1	1	0	0	0
e_3	0	1	1	1	1	0	0
e_4	0	0	1	1	1	0	0
e_5	0	0	1	1	1	1	0
e_6	0	0	0	1	1	1	0
e_7	0	0	0	1	1	1	1

Figure 5: n-crossings (upper left, lower left) and their biadjacency matrices of corresponding bipartite graphs $G(C_n, X)$ (upper right, lower right).

The coefficient of $x_1x_2x_3x_4x_5x_6x_7$ of $w(C_7)$ is 17, and the number of perfect matchings of $G(C_7, X)$ is also 17. Hence, we have $G(8) = m(C_7, N_7) = 17$.

5 Further discussions

There are a lot of unknown things for the multiplicity in $\mathcal{NCD}(E)$. One ambitious problem is to find a formula for $m(E, F)$ in general.

In Section 4, we represent $m(E, N_n)$ by the number of perfect matchings of a corresponding bipartite graph. It is interesting if we can find an efficient method to calculate the number of perfect matchings in a graph of this kind.

As is shown in [15], there is a relation between the chord expansion number and the evaluation of the Tutte polynomial at the point $(2, -1)$. As a future research subject, it is considered to find a relation between the multiplicity $m(E, F)$ in general, or $m(E, N_n)$, and some counting polynomials of graphs.

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