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ON THE SINGULAR TWO-PARAMETER EIGENVALUE PROBLEM*

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Abstract. In the 1960s Atkinson introduced an abstract algebraic setting for multiparameter eigenvalue problems. He showed that a nonsingular multiparameter eigenvalue problem is equivalent to the associated system of generalized eigenvalue problems. Many theoretical results and numerical methods for nonsingular multiparameter eigenvalue problems are based on this relation.

We extend the above relation to singular two-parameter eigenvalue problems and show that the simple finite regular eigenvalues of both problems agree. This enables one to solve a singular two-parameter eigenvalue problem by computing the common regular eigenvalues of the associated system of two singular generalized eigenvalue problems.

Key words. Singular two-parameter eigenvalue problem, operator determinant, Kronecker canonical form, minimal reducing subspace

AMS subject classifications. 15A18, 15A69, 65F15.

1. Introduction.

We consider *the algebraic two-parameter eigenvalue problem*

$$\begin{aligned} W_1(\lambda,\mu)x_1 &:= (A_1 + \lambda B_1 + \mu C_1)x_1 = 0, \\ W_2(\lambda,\mu)x_2 &:= (A_2 + \lambda B_2 + \mu C_2)x_2 = 0, \end{aligned} \tag{1.1}$$

where A_i, B_i, and C_i are $n_i \times n_i$ matrices over $\mathbb{C}$, $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$. A pair (λ,μ) is an *eigenvalue* if it satisfies (1.1) for nonzero vectors x_1, x_2, and the tensor product $x_1 \otimes x_2$ is the corresponding *(right) eigenvector*. Similarly, $y_1 \otimes y_2$ is the corresponding *left eigenvector* if $y_i \neq 0$ and $y_i^* W_i(\lambda,\mu) = 0$ for $i = 1, 2$.

The eigenvalues of (1.1) are the roots of the following system of two bivariate polynomials

$$\begin{aligned} p_1(\lambda,\mu) &:= \det(W_1(\lambda,\mu)) = 0, \\ p_2(\lambda,\mu) &:= \det(W_2(\lambda,\mu)) = 0. \end{aligned} \tag{1.2}$$

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A two-parameter eigenvalue problem can be expressed as two coupled generalized eigenvalue problems. On the tensor product space $S := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of the dimension $N := n_1 n_2$ we define *operator determinants*

$$
\begin{aligned}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2,\\
\Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2,\\
\Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2,
\end{aligned} \tag{1.3}
$$

for details see, e.g., [1]. The problem (1.1) is then related to a coupled pair of generalized eigenvalue problems

$$
\begin{aligned}
\Delta_1 z &= \lambda \Delta_0 z,\\
\Delta_2 z &= \mu \Delta_0 z,
\end{aligned} \tag{1.4}
$$

for decomposable tensors $z \in S$, $z = x \otimes y$.

Usually we assume that the two-parameter eigenvalue problem (1.1) is *nonsingular*, i.e., the corresponding operator determinant Δ_0 is nonsingular. In this case the matrices $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute and the eigenvalues of (1.1) agree with the eigenvalues of (1.4). By applying this relation, a nonsingular two-parameter eigenvalue problem can be numerically solved using standard tools for the generalized eigenvalue problems, for an algorithm see, e.g., [8].

However, several applications lead to singular two-parameter eigenvalue problems where Δ_0 is singular and (1.4) is a pair of singular generalized eigenvalue problems. Two such examples are the model updating [3] and the quadratic two-parameter eigenvalue problem [12]. Apart from a few theoretical results and numerical methods in [3] and [12], which only cover very specific examples, the singular two-parameter eigenvalue problem is still open. We extend Atkinson's results from [1] and show that simple eigenvalues of the singular two-parameter eigenvalue problem (1.1) can be computed from the eigenvalues of the corresponding pair of singular generalized eigenvalue problems (1.4). This opens new possibilities in the study of singular two-parameter eigenvalue problems. The new results justify that the numerical method presented in [12] can be applied not only to the linearization of a quadratic two-parameter eigenvalue problem but also to a general singular two-parameter eigenvalue problem.

Definition 1.1. The *normal rank* of a two-parameter matrix pencil $W_i(\lambda,\mu)$ is

$$\mathrm{nrank}(W_i(\lambda,\mu)) = \max_{\lambda,\mu\in\mathbb{C}} \mathrm{rank}(W_i(\lambda,\mu))$$

for $i = 1,2$. A pair $(\lambda_*,\mu_*) \in \mathbb{C}^2$ is a *finite regular eigenvalue* of the two-parameter eigenvalue problem (1.1) if $\mathrm{rank}(W_i(\lambda_*,\mu_*)) < \mathrm{nrank}(W_i(\lambda,\mu))$ for $i = 1,2$. The *geometric multiplicity* of the eigenvalue (λ_*, μ_*) is equal to

$$\prod_{i=1}^{2} \Big(\mathrm{nrank}(W_i(\lambda,\mu)) - \mathrm{rank}(W_i(\lambda_*,\mu_*)) \Big).$$

Throughout this paper we assume that the two-parameter eigenvalue problem (1.1) is *regular*, which means that both matrix pencils $W_1(\lambda,\mu)$ and $W_2(\lambda,\mu)$ have full normal rank, i.e., $\mathrm{nrank}(W_i(\lambda,\mu)) = n_i$ for $i = 1, 2$. This is equivalent to the condition that both polynomials p_1 and p_2 are not identical to zero. We also assume that p_1 and p_2 do not have a nontrivial common divisor, because this would lead to infinitely many eigenvalues. If the greatest common divisor of p_1 and p_2 is a scalar, then (1.1) has (counting with multiplicities) $k \le N$ finite regular eigenvalues, where $k \le \mathrm{rank}(\Delta_0)$. If the problem (1.1) is nonsingular, then $k = N$; if (1.1) is singular, then $k < N$ and the remaining $N - k$ eigenvalues, which belong to the singular part, can be considered as *infinite eigenvalues*.

In addition, we assume that neither p_1 nor p_2 has a factor that depends on exactly one of the variables λ or μ. If this is not true then we can fix it by applying a linear substitution $\begin{bmatrix} \lambda' \\ \mu' \end{bmatrix} = M \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$ on the variables of (1.1), where M is a nonsingular 2×2 matrix. As a result, for each λ there exists a μ (and vice-versa) such that $p_i(\lambda,\mu) \neq 0$ for $i = 1, 2$.

Definition 1.2. The *normal rank* of a square matrix pencil $A - \lambda B$ is

$$\mathrm{nrank}(A - \lambda B) = \max_{\lambda \in \mathbb{C}} \mathrm{rank}(A - \lambda B).$$

A scalar $\lambda_* \in \mathbb{C}$ is a *finite regular eigenvalue* of the matrix pencil if $\mathrm{rank}(A - \lambda_* B) < \mathrm{nrank}(A - \lambda B)$. The *geometric multiplicity* of λ_* is $\mathrm{nrank}(A - \lambda B) - \mathrm{rank}(A - \lambda_* B)$.

In the next section we introduce the Kronecker canonical form and other auxiliary results. In Section 3 we show that all simple eigenvalues of a singular two-parameter eigenvalue problem (1.1) agree with the finite regular eigenvalues of the associated pair of generalized eigenvalue problems (1.4). In Section 4 we give examples of small two-parameter eigenvalue problems that support the theory and in the final section we review how to numerically solve a singular two-parameter eigenvalue problem using the algorithm for the computation of the common regular subspace of a singular matrix pencil, presented in [12].

2. Auxiliary results. In this section we review the Kronecker canonical form and Kronecker chains of a matrix pencil. More about the Kronecker canonical form and its numerical computation can be found in, e.g., [4], [5], [7], and [14].

Definition 2.1. Let $A-\lambda B\in\mathbb{C}^{m\times n}$ be a matrix pencil. Then there exist nonsingular matrices $P\in\mathbb{C}^{m\times m}$ and $Q\in\mathbb{C}^{n\times n}$ such that

$$P^{-1}(A-\lambda B)Q=\widetilde{A}-\lambda\widetilde{B}=\mathrm{diag}(A_1-\lambda B_1,\ldots,A_k-\lambda B_k)$$

is the *Kronecker canonical form*. Each block $A_i-\lambda B_i$, $i=1,\ldots,k$, must be of one of the following forms: $J_j(\alpha),N_j,L_j$, or L_j^T, where blocks

$$J_p(\alpha)=\begin{bmatrix}\alpha-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & \alpha-\lambda\end{bmatrix},\quad N_p=\begin{bmatrix}1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda\\ & & & 1\end{bmatrix},$$

$$L_p=\begin{bmatrix}-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1\end{bmatrix},\quad L_p^T=\begin{bmatrix}-\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda\\ & & 1\end{bmatrix},$$

represent a finite regular block, an infinite regular block, a right singular block, and a left singular block, respectively.

To each Kronecker block of the matrix pencil $A-\lambda B$ we can associate a Kronecker chain of linearly independent vectors as follows (see, e.g., [11]):

a) A finite regular block $J_p(\alpha)$ is associated with vectors $u_1,\ldots,u_p$ that satisfy

$$\begin{aligned}(A-\alpha B)u_1&=0,\\ (A-\alpha B)u_{i+1}&=Bu_i,\quad i=1,\ldots,p-1.\end{aligned}$$

b) An infinite regular block N_p is associated with vectors $u_1,\ldots,u_p$ that satisfy

$$\begin{aligned}Bu_1&=0,\\ Bu_{i+1}&=Au_i,\quad i=1,\ldots,p-1.\end{aligned}$$

c) A right singular block L_p is associated with vectors $u_1,\ldots,u_{p+1}$ that satisfy

$$\begin{aligned}Au_1&=0,\\ Au_{i+1}&=Bu_i,\quad i=1,\ldots,p,\\ 0&=Bu_{p+1}.\end{aligned}$$

d) A left singular block L_p^T, $p\ge 1$, is associated with vectors $u_1,\ldots,u_p$ that satisfy

$$Au_i=Bu_{i+1},\quad i=1,\ldots,p-1.$$

The union of the Kronecker chains for all Kronecker blocks is a basis for $\mathbb{C}^n$. We say that a subspace $\mathcal{M} \subset \mathbb{C}^n$ is a *reducing subspace* for the pencil $A - \lambda B$ if $\dim(A\mathcal{M} + B\mathcal{M}) = \dim(\mathcal{M}) - s$, where s is the number of the right singular blocks L_p in the Kronecker canonical form for $A - \lambda B$. The vectors from the Kronecker chains of all right singular blocks L_p form a basis for the *minimal reducing subspace* $\mathcal{R}(A, B)$, which is a subset of all reducing subspaces. The minimal reducing subspace is unique and can be numerically computed in a stable way from the generalized upper triangular form (GUPTRI), see, e.g., [4, 5].

Definition 2.2. Let λ_* be a finite regular eigenvalue of a matrix pencil $A - \lambda B$ and let $(A - \lambda_* B)z = 0$. We say that z is a *regular eigenvector* if z does not belong to the minimal reducing subspace $\mathcal{R}(A, B)$. In the other case, we say that z is a *singular eigenvector.*

It is trivial to construct a basis for the kernel of the tensor product $A \otimes D$ from the kernels of A and D. The task is much harder if we take a difference of two tensor products, which is the form of the operator determinants (1.3). Košir showed in [11] that the kernel of an operator determinant $\Delta = A \otimes D - B \otimes C$ can be constructed from the Kronecker chains of matrix pencils $A - \lambda B$ and $C - \mu D$.

Theorem 2.3 ([11, Theorem 4]). *A basis for the kernel of $\Delta = A \otimes D - B \otimes C$ is the union of sets of linearly independent vectors that belong to the following pairs of Kronecker blocks of pencils $A - \lambda B$ and $C - \mu D$, respectively:*

a) $J_{p_1}(\alpha)$ and $J_{p_2}(\alpha)$,
b) N_{p_1} and N_{p_2},
c) L_{p_1} and L_{p_2},
d) L_{p_1} and $L_{p_2}^T$, where $p_1 < p_2$,
e) $L_{p_1}^T$ and L_{p_2}, where $p_1 > p_2$,
f) L_{p_1} and $J_{p_2}(\alpha)$,
g) $J_{p_1}(\alpha)$ and L_{p_2}.

Each pair is associated with a set of linearly independent vectors. The construction for a pair from points a) and b) is as follows: If $u_1, \ldots, u_{p_1}$ and $v_1, \ldots, v_{p_2}$ are the associated Kronecker chains for the pencils $A - \lambda B$ and $C - \mu D$, respectively, then the vectors

$$z_j = \sum_{i=1}^{j} u_1 \otimes v_{j+1-i}, \quad j = 1, \ldots, \min(p_1, p_2),$$

form a set of linearly independent vectors for this pair of Kronecker blocks.

In the above theorem we omitted the constructions for all pairs of Kronecker blocks that are not relevant for our case. For a complete description see [11].

We will also need the following relations between a submatrix of a matrix and its complementary submatrix in the inverse matrix.

LEMMA 2.4. *Let A be a nonsingular $n \times n$ matrix, and let A and A^{-1} be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where A_{11} and B_{11} are $r \times r$ matrices, $1 \le r < n$. Then

a) $\dim(\mathrm{Ker}(A_{11})) = \dim(\mathrm{Ker}(B_{22}))$,
b) $\det(A_{11}) = \det(B_{22})\det(A)$.

Proof. Point a) is a well-known nullity theorem by Fiedler and Markham [6].

If A_{11} is singular, then point a) yields that B_{22} is singular as well. So $\det(A_{11}) = \det(B_{22}) = 0$ and point b) is true. If A_{11} is nonsingular, then we can write

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A_{11}. The proof now follows from the relations $\det(A) = \det(A_{11})\det(S)$ and $B_{22} = S^{-1}$. □

3. Delta matrices and simple eigenvalues. In the nonsingular case the eigenvalues of (1.1) agree with the eigenvalues of the associated pair of generalized eigenvalue problems (1.4), see, e.g., [1]. In this section we show that in a similar way the finite regular eigenvalues of (1.1) are related to the finite regular eigenvalues of (1.4).

DEFINITION 3.1. A pair $(\lambda_*, \mu_*) \in \mathbb{C}^2$ is a *finite regular eigenvalue* of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if the following is true:

a) λ_* is a finite regular eigenvalue of $\Delta_1 - \lambda\Delta_0$,
b) μ_* is a finite regular eigenvalue of $\Delta_2 - \mu\Delta_0$,
c) there exists a common regular eigenvector z, i.e., $z \neq 0$ such that $(\Delta_1 - \lambda_*\Delta_0)z = 0$, $(\Delta_2 - \mu_*\Delta_0)z = 0$, and $z \notin \mathcal{R}(\Delta_i, \Delta_0)$ for $i = 1, 2$.

The *geometric multiplicity* of (λ_*, μ_*) is $\dim(\mathcal{N}) - \dim(N \cap \mathcal{R}(\Delta_1, \Delta_0) \cap \mathcal{R}(\Delta_2, \Delta_0))$, where $\mathcal{N} = \mathrm{Ker}(\Delta_1 - \lambda_*\Delta_0) \cap \mathrm{Ker}(\Delta_2 - \mu_*\Delta_0)$.

In order to obtain a general result, instead of the linear two-parameter eigenvalue problem (1.1) we consider a nonlinear two-parameter eigenvalue problem

$$\begin{aligned} T_1(\lambda, \mu)x_1 &= 0 \\ T_2(\lambda, \mu)x_2 &= 0, \end{aligned} \tag{3.1}$$

where $T_i(.,.) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^{n_i \times n_i}$ is differentiable for $i = 1, 2$. If (3.1) is satisfied for nonzero vectors x_1 and x_2, then (λ, μ) is an eigenvalue and $x_1 \otimes x_2$ is the corresponding right eigenvector. The corresponding left eigenvector is $y_1 \otimes y_2$ such that $y_i \neq 0$ and $y_i^* T_i(\lambda, \mu) = 0$ for $i = 1, 2$.

The following proposition is a two-parameter generalization of the one-parameter version from [13].

PROPOSITION 3.2. *Let (λ_*, μ_*) be an algebraically and geometrically simple eigenvalue of the nonlinear two-parameter eigenvalue problem (3.1) and let $x_1 \otimes x_2$ and $y_1 \otimes y_2$ be the corresponding right and left eigenvector. Then the matrix*

$$M_0 := \begin{bmatrix} y_1^* \frac{\partial T_1}{\partial \lambda}(\lambda_*, \mu_*) x_1 & y_1^* \frac{\partial T_1}{\partial \mu}(\lambda_*, \mu_*) x_1 \\ y_2^* \frac{\partial T_2}{\partial \lambda}(\lambda_*, \mu_*) x_2 & y_2^* \frac{\partial T_2}{\partial \mu}(\lambda_*, \mu_*) x_2 \end{bmatrix} \tag{3.2}$$

is nonsingular.

Proof. For $i = 1, 2$, let us define

$$S_i(\lambda, \mu) = \begin{bmatrix} T_i(\lambda, \mu) & y_i \\ x_i^* & 0 \end{bmatrix}.$$

One can see from $\dim(\mathrm{Ker}(T_i(\lambda_*, \mu_*))) = 1$ that $S_i(\lambda_*, \mu_*)$ is nonsingular. Let us denote the element in the lower right corner of $S_i^{-1}(\lambda, \mu)$ by $\alpha_i(\lambda, \mu)$, i.e.,

$$\alpha_i(\lambda, \mu) = e_{n_i+1}^T S_i(\lambda, \mu)^{-1} e_{n_i+1}. \tag{3.3}$$

Let $r_i(\lambda, \mu) = \det(T_i(\lambda, \mu))$ and $q_i(\lambda, \mu) = \det(S_i(\lambda, \mu))$. Lemma 2.4 yields that

$$r_i(\lambda, \mu) = \alpha_i(\lambda, \mu) \cdot q_i(\lambda, \mu). \tag{3.4}$$

We know that $r_i(\lambda_*, \mu_*) = 0$ and $q_i(\lambda_*, \mu_*) \neq 0$, therefore $\alpha_i(\lambda_*, \mu_*) = 0$ for $i = 1, 2$.

By differentiating the expressions (3.3) and (3.4) at (λ_*, μ_*) we obtain

$$\frac{\partial \alpha_i}{\partial \lambda}(\lambda_*, \mu_*) = -y_i^* \frac{\partial T_i}{\partial \lambda}(\lambda_*, \mu_*) x_i \tag{3.5}$$

$$\frac{\partial \alpha_i}{\partial \mu}(\lambda_*, \mu_*) = -y_i^* \frac{\partial T_i}{\partial \mu}(\lambda_*, \mu_*) x_i \tag{3.6}$$

and

$$\frac{\partial r_i}{\partial \lambda}(\lambda_*, \mu_*) = \frac{\partial \alpha_i}{\partial \lambda}(\lambda_*, \mu_*) \cdot q_i(\lambda_*, \mu_*) \tag{3.7}$$

$$\frac{\partial r_i}{\partial \mu}(\lambda_*, \mu_*) = \frac{\partial \alpha_i}{\partial \mu}(\lambda_*, \mu_*) \cdot q_i(\lambda_*, \mu_*) \tag{3.8}$$

for $i = 1, 2$. We join equations (3.5), (3.6), (3.7), and (3.8) into

$$\begin{bmatrix} \frac{\partial r_1}{\partial \lambda}(\lambda_*, \mu_*) & \frac{\partial r_1}{\partial \mu}(\lambda_*, \mu_*) \\ \frac{\partial r_2}{\partial \lambda}(\lambda_*, \mu_*) & \frac{\partial r_2}{\partial \mu}(\lambda_*, \mu_*) \end{bmatrix} = - \begin{bmatrix} q_1(\lambda_*, \mu_*) & \\ & q_2(\lambda_*, \mu_*) \end{bmatrix} M_0. \tag{3.9}$$

The matrix on the left hand side of (3.9) is nonsingular as it is the Jacobian matrix of the nonlinear system $r_1(\lambda,\mu)=0$, $r_2(\lambda,\mu)=0$ at a simple root $(\lambda,\mu)=(\lambda_*,\mu_*)$. This implies that M_0 is nonsingular. □

COROLLARY 3.3. *Let (λ_*,μ_*) be an algebraically simple eigenvalue of the two-parameter eigenvalue problem (1.1) and let $x_1\otimes x_2$ and $y_1\otimes y_2$ be the corresponding right and left eigenvector. It follows that*

$$(y_1\otimes y_2)^*\Delta_0(x_1\otimes x_2)=\begin{vmatrix} y_1^*B_1x_1 & y_1^*C_1x_1\\ y_2^*B_2x_2 & y_2^*C_2x_2\end{vmatrix}\neq 0. \tag{3.10}$$

Let us remark that the result in Corollary 3.3 was obtained for the nonsingular multiparameter eigenvalue problem by Košir in [10, Lemma 3]. The connection (3.9) between the Jacobian matrix of the polynomial system (1.2) and the matrix

$$\begin{bmatrix} y_1^*B_1x_1 & y_1^*C_1x_1\\ y_2^*B_2x_2 & y_2^*C_2x_2\end{bmatrix}$$

was established for the nonsingular right definite two-parameter eigenvalue problem in [9, Proposition 13].

LEMMA 3.4. *Let $A(.):\mathbb{C}\to\mathbb{C}^{n\times n}$ be a matrix polynomial and let λ_* be its eigenvalue, i.e., $\det(A(\lambda_*))=0$, with the corresponding right and left eigenvector x and y, respectively. If*

$$y^*A'(\lambda_*)x\neq 0 \tag{3.11}$$

then λ_* is a finite regular eigenvalue.

Proof. It suffices to show that the eigenvector x is not part of the singular subspace of $A(\lambda)$, which is composed of vector polynomials $u(\lambda)$ such that $A(\lambda)u(\lambda)=0$ for all λ.

Suppose that there exists a polynomial $u(\lambda)$ such that

$$A(\lambda)u(\lambda)=0 \tag{3.12}$$

for all λ and $u(\lambda_*)=x$. If we differentiate (3.12) at $\lambda=\lambda_*$ then we obtain

$$A(\lambda_*)u'(\lambda_*)+A'(\lambda_*)x=0.$$

When we multiply this equality by y^* we get

$$y^*A'(\lambda_*)x=0,$$

which contradicts the assumption (3.11). So, such a polynomial $u(\lambda)$ does not exist and x is not in the singular subspace. Therefore, the rank of $A(\lambda)$ drops at $\lambda=\lambda_*$ and λ_* is a finite regular eigenvalue. □

THEOREM 3.5. *Let us assume that all finite eigenvalues of a regular two-parameter eigenvalue problem (1.1) are algebraically simple. If* (λ_*, μ_*) *is a finite regular eigenvalue of (1.1) then* (λ_*, μ_*) *is a finite regular eigenvalue of the associated pair of generalized eigenvalue problems (1.4).*

Proof. Let (λ_*, μ_*) be a finite regular eigenvalue of (1.1) and let $z = x_1 \otimes x_2$ and $w = y_1 \otimes y_2$ be the corresponding right and left eigenvector, respectively. It follows from Corollary 3.3 that $w^*\Delta_0 z \neq 0$. Now we can apply Lemma 3.4 to conclude that λ_* is a finite regular eigenvalue of $\Delta_1 - \lambda\Delta_0$.

We can show the same for the eigenvalue μ_* of the matrix pencil $\Delta_2 - \mu\Delta_0$. It follows that (λ_*, μ_*) is a finite regular eigenvalue of pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ with the common regular eigenvector z. □

In order to establish a bidirectional link between the eigenvalues of the two-parameter eigenvalue problem (1.1) and the eigenvalues of the associated pair of generalized eigenvalue problems (1.4), we have to prove the relation in the opposite direction as well. We do this in the following theorem.

THEOREM 3.6. *Let us assume that all finite eigenvalues of a regular two-parameter eigenvalue problem (1.1) are algebraically simple. Let* (λ_*, μ_*) *be a finite regular eigenvalue of the associated pair of generalized eigenvalue problems (1.4) such that* λ_* *and* μ_* *are nondefective eigenvalues of the pencils* $\Delta_1 - \lambda\Delta_0$ *and* $\Delta_2 - \mu\Delta_0$, *respectively. Then* (λ_*, μ_*) *is a finite regular eigenvalue of the regular two-parameter eigenvalue problem (1.1).*

Proof. Let z be a common regular eigenvector for the eigenvalue (λ_*, μ_*). We can write

$$\Delta_1 - \lambda_*\Delta_0 = W_1(\lambda_*, 0) \otimes C_2 - C_1 \otimes W_2(\lambda_*, 0). \tag{3.13}$$

It follows from Theorem 2.3 and (3.13) that z is a linear combination of vectors associated with appropriate pairs of Kronecker blocks of pencils $W_1(\lambda_*, 0) - \alpha_1 C_1$ and $W_2(\lambda_*, 0) - \alpha_2 C_2$. Since z is regular, at least one of this vectors has to be regular.

Among all combinations of Kronecker blocks that are listed in Theorem 2.3, the possible ones are either of type a) $(J_{p_1}(\alpha), J_{p_2}(\alpha))$ or b) (N_{p_1}, N_{p_2}). Namely, in all other combinations one of the Kronecker blocks is of type L_p. Suppose, without loss of generality, that L_p belongs to the pencil $W_1(\lambda_*, 0) - \alpha_1 C_1$. Then there exists a polynomial vector $u(\alpha)$ of order p such that

$$W_1(\lambda_*, -\alpha)u(\alpha) = (W_1(\lambda_*, 0) - \alpha C_1)u(\alpha) = 0$$

for all α. But, then $\det(W_1(\lambda, -\alpha)) = 0$ for all α, which contradicts the assumption that for each λ there exists a μ such that $\det(W_1(\lambda, \mu)) \neq 0$.

Suppose that the pencil $W_i(\lambda_*,0)-\alpha_i C_i$ has a Kronecker block N_{p_i} for $i=1,2$. The two Kronecker blocks are associated to the Kronecker chains $u_1,\ldots,u_{p_1}$ and $v_1,\ldots,v_{p_2}$ such that

$$\begin{aligned} C_1u_1 &= 0,\\ C_1u_{i+1} &= W_1(\lambda_*,0)u_i, \quad i=1,\ldots,p_1-1, \end{aligned}$$

and

$$\begin{aligned} C_2v_1 &= 0,\\ C_2v_{i+1} &= W_2(\lambda_*,0)v_i, \quad i=1,\ldots,p_2-1. \end{aligned}$$

If follows from Theorem 2.3 that the vectors $z_1,\ldots,z_p$, where $p=\min(p_1,p_2)$ and

$$z_j=\sum_{i=1}^{j} u_i\otimes v_{j+1-i}, \quad j=1,\ldots,p,$$

are in the basis for $\mathrm{Ker}(\Delta_1-\lambda_*\Delta_0)$. For $z_1=u_1\otimes v_1$ it is easy to see that

$$\Delta_1 z_1=\Delta_0 z_1=0. \tag{3.14}$$

Therefore, z_1 is not a regular eigenvector of $\Delta_1-\lambda\Delta_0$ at $\lambda=\lambda_*$. If $p>1$ then

$$\begin{aligned} \Delta_0 z_j &= (B_1\otimes C_2-C_1\otimes B_2)\sum_{i=1}^{j} u_i\otimes v_{j+1-i}\\ &= B_1\otimes W_2(\lambda_*,0)\sum_{i=1}^{j-1} u_i\otimes v_{j-i}-W_1(\lambda_*,0)\otimes B_2\sum_{i=2}^{j} u_{i-1}\otimes v_{j+1-i} \qquad (3.15)\\ &= \Big(B_1\otimes W_2(\lambda_*,0)-W_1(\lambda_*,0)\otimes B_2\Big)\sum_{i=1}^{j-1} u_i\otimes v_{j-i}=-\Delta_2 z_{j-1}. \end{aligned}$$

Suppose that the basis for the kernel of $\Delta_1-\lambda_*\Delta_0$ is a union of sets of vectors that belong to m pairs of Kronecker blocks of the type (N_{p_1},N_{p_2}) only. Then it follows from (3.14) and (3.15) that there exist vectors $z_{k1},\ldots,z_{kp_k}$ for $k=1,\ldots,m$ such that

$$\Delta_0 z_{k1}=0 \tag{3.16}$$

and

$$\Delta_0 z_{kj}=-\Delta_2 z_{k,j-1} \tag{3.17}$$

for $j=2,\ldots,p_k$. By Theorem 2.3 we can expand the common regular eigenvector $z\in \mathrm{Ker}(\Delta_1-\lambda_*\Delta_0)\cap\mathrm{Ker}(\Delta_2-\mu_*\Delta_0)$ in this basis as

$$z=\sum_{j=1}^{m}\sum_{k=1}^{p_j}\xi_{jk}z_{jk}.$$

Using the relation (3.17), we define a chain of vectors $z^{(0)},\ldots,z^{(P)}$ such that

$$(\Delta_2-\mu_*\Delta_0)z=\Delta_2\left(\sum_{j=1}^{m}\left(\sum_{k=1}^{p_j}\xi_{jk}z_{jk}+\mu_*\sum_{k=2}^{p_j}\xi_{jk}z_{j,k-1}\right)\right)=\Delta_2 z^{(0)}=0$$

$$\Delta_0 z^{(0)}=-\Delta_2\left(\sum_{j=1}^{m}\left(\sum_{k=2}^{p_j}\xi_{jk}z_{j,k-1}+\mu_*\sum_{k=3}^{p_j}\xi_{jk}z_{j,k-2}\right)\right)=-\Delta_2 z^{(1)}$$

$$\vdots$$

$$\Delta_0 z^{(P-1)}=-\Delta_2\left(\sum_{j=1}^{m}\left(\sum_{k=P+1}^{p_j}\xi_{jk}z_{j,k-P}+\mu_*\sum_{k=P+2}^{p_j}\xi_{jk}z_{j,k-P-1}\right)\right)=-\Delta_2 z^{(P)},$$

where $P=\max\{k-1:\ \xi_{jk}\neq 0,\ j=1,\ldots,m,\ k=1,\ldots,p_j\}$. The chain ends with $\Delta_0 z^{(P)}=0$, which follows from (3.16) and

$$z^{(P)}=\sum_{\substack{j=1\\ p_j\geq P+1}}^{m}\xi_{jP}z_{j1}.$$

The relations $\Delta_2 z^{(0)}=0,\ \Delta_0 z^{(0)}=-\Delta_2 z^{(1)},\ \ldots\ ,\Delta_0 z^{(P-1)}=-\Delta_2 z^{(P)},\ \Delta_0 z^{(P)}=0$ show that $z^{(0)},\ldots,z^{(P)}$ belong to the right singular subspace of the pencil $\Delta_2-\mu\Delta_0$ (see, e.g., [7, Section 12.3]). It follows that z, which is a linear combination of the vectors $z^{(0)},\ldots,z^{(P)}$, belongs to the singular part of $\Delta_2-\mu\Delta_0$. We conclude that by vectors solely from the combinations (N_{p_1},N_{p_2}) it is not possible to write down the common regular eigenvector z.

From the above discussion we see that the only option for the existence of a common regular eigenvector is that each pencil $W_i(\lambda_*,0)-\alpha_i C_i$ has a Kronecker block $J_{p_i}(\alpha)$ for $i=1,2$. Then $\det(W_i(\lambda_*,-\alpha))=0$ for $i=1,2$ and $(\lambda_*,-\alpha)$ is a finite eigenvalue of the two-parameter eigenvalue problem (1.1). As we assume that all finite eigenvalues of (1.1) are algebraically simple, $p_1=p_2=1$ and it follows from Theorem 3.5 that $(\lambda_*,-\alpha)$ is an eigenvalue of the associated pair of generalized eigenvalue problems (1.4).

Let m_1 be the geometric multiplicity of λ_* as an eigenvalue of the pencil $\Delta_1-\lambda\Delta_0$. Then there exist $\alpha_1,\ldots,\alpha_{m_1}$ and linearly independent vectors $x_{11}\otimes x_{21},\ldots,x_{1m_1}\otimes x_{2m_1}$ such that $(\lambda_*,-\alpha_i)$ is an eigenvalue of (1.1) and $x_{1i}\otimes x_{2i}$ is the corresponding eigenvector for $i=1,\ldots,m_1$.

By repeating the above steps for the pencil $\Delta_2-\mu\Delta_0$ we can show that there exist $\beta_1,\ldots,\beta_{m_2}$ and linearly independent vectors $u_{11}\otimes u_{21},\ldots,u_{1m_2}\otimes u_{2m_2}$ such that (β_i,μ_*) is an eigenvalue of (1.1) and $u_{1i}\otimes u_{2i}$ is the corresponding eigenvector for $i = 1, \ldots, m_2$, where m_2 is the geometric multiplicity of μ_* as an eigenvalue of $\Delta_2 - \mu\Delta_0$.

Suppose that $\lambda_* \neq \beta_i$ for $i = 1, \ldots, m_2$. It follows that the vectors $x_{11} \otimes x_{21}, \ldots, x_{1m_1} \otimes x_{2m_1}, u_{11} \otimes u_{21}, \ldots, u_{1m_2} \otimes u_{2m_2}$ are linearly independent and they are not elements of the minimal reducing subspace $\mathcal{R}(\Delta_1, \Delta_0)$. There exists a basis $\mathcal{B}$ for S that contains all vectors $x_{1i} \otimes x_{2i}$ and $u_{1j} \otimes u_{2j}$ for $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$.

On one hand, z is a regular eigenvector of $\Delta_1 - \lambda\Delta_0$ for the eigenvalue λ_*. If we expand z in the basis $\mathcal{B}$ then the coefficient at $u_{1j} \otimes u_{2j}$, which is a regular eigenvector for $\beta_j \neq \lambda_*$, is zero for $j = 1, \ldots, m_2$. As λ_* is nondefective, it follows that

$$z = a_1 x_{11} \otimes x_{21} + \cdots + a_{m_1} x_{1m_1} \otimes x_{2m_1} + s_1, \tag{3.18}$$

where not all of $a_1, \ldots, a_{m_1}$ are zero and $s_1 \in \mathrm{Ker}(\Delta_1 - \lambda_*\Delta_0) \cap \mathcal{R}(\Delta_1, \Delta_0)$.

On the other hand, z is a regular eigenvector of $\Delta_2 - \mu\Delta_0$ for the eigenvalue μ_*. It follows that $z = b_1 u_{11} \otimes u_{21} + \cdots + b_{m_2} u_{1m_2} \otimes u_{2m_2} + s_2$, where not all of $b_1, \ldots, b_{m_2}$ are zero and $s_2 \in \mathrm{Ker}(\Delta_2 - \mu_*\Delta_0) \cap \mathcal{R}(\Delta_2, \Delta_0)$. This contradicts the expansion (3.18) that does not include the terms $u_{11} \otimes u_{21}, \ldots, u_{1m_2} \otimes u_{2m_2}$. It follows that there exists a pair i, j, $1 \leq i \leq m_i$, $1 \leq j \leq m_2$, such that $x_{1i} \otimes x_{2i}$ and $u_{1j} \otimes u_{2j}$ are linearly dependent, $\beta_j = \lambda_*$, and $\alpha_i = -\mu_*$. Therefore, (λ_*, μ_*) is really a finite regular eigenvalue of (1.1). □

From the above proof some interesting properties of the pencil $\Delta_1 - \lambda\Delta_0$ (same applies to $\Delta_2 - \mu\Delta_0$) can be deduced. We collect them in the following corollary.

Corollary 3.7.

a) If the pencil $\Delta_1 - \lambda\Delta_0$ is singular, then it contains at least one block L_0.

b) Suppose that λ_* is not a regular eigenvalue of the pencil $\Delta_1 - \lambda\Delta_0$, let $W_i(\lambda_*, 0)$ be nonsingular and let the Kronecker canonical form of the pencil $W_i(\lambda_*, 0) - \alpha_i C_i$ contain infinite regular blocks $N_{p_1}, \ldots, N_{p_{k_i}}$ for $i = 1, 2$. Then

$$\mathrm{rank}(\Delta_1 - \lambda_*\Delta_0) = N - \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \min(p_i, p_j) \tag{3.19}$$

and the Kronecker canonical form of the pencil $\Delta_1 - \lambda\Delta_0$ contains at least $k_1 k_2$ blocks L_0.

From (3.19) one can compute the normal rank of the pencil $\Delta_1 - \lambda\Delta_0$ without working with the large matrices Δ_0 and Δ_1 explicitly.

Remark 3.8. A natural question is whether we can extend the relations in Theorem 3.5 and Theorem 3.6 to multiparameter eigenvalue problems with more than two parameters?

One direction is simple. It is straightforward to extend Proposition 3.2 and Corollary 3.3 to cover problems with more than two parameters. This allows us to generalize Theorem 3.5 and show that simple eigenvalues of a singular multiparameter eigenvalue problem are common regular eigenvalues of the associated system of singular generalized eigenvalue problems.

But, it is not clear how to prove the connection in the other direction. The proof of Theorem 3.6 relies on Theorem 2.3 which is only available for 2×2 operator determinants. So, in order to generalize Theorem 3.6 to more than two parameters, a different approach is required and this problem is still open.

4. Examples. In this section we present some small two-parameter eigenvalue problems that illustrate the theory from the previous sections.

EXAMPLE 4.1. If we take

$$\begin{aligned} W_1(\lambda,\mu) &= (A_1+\lambda B_1+\mu C_1) = \begin{bmatrix} -\lambda-\mu & -1 \\ -1 & 1 \end{bmatrix}, \\ W_2(\lambda,\mu) &= (A_2+\lambda B_2+\mu C_2) = \begin{bmatrix} -2\lambda+\mu & -1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

then

$$\begin{aligned} p_1(\lambda,\mu) &= \det(W_1(\lambda,\mu)) = -\lambda-\mu+1 \\ p_2(\lambda,\mu) &= \det(W_2(\lambda,\mu)) = -4\lambda+2\mu+1 \end{aligned}$$

and the problem is clearly regular. Its only eigenvalue is $(\lambda,\mu)=(\frac{1}{2},\frac{1}{2})$. The corresponding operator determinants are

$$\Delta_0 = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & -2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Pencils $\Delta_1-\lambda\Delta_0$ and $\Delta_2-\mu\Delta_0$ have the same Kronecker structure which contains the following blocks: $L_0, 2N_1, L_0^T$, and $J_1(\frac{1}{2})$. The minimal reducing subspace is $\mathcal{R}(\Delta_1,\Delta_0)=\mathcal{R}(\Delta_2,\Delta_0)=\mathrm{span}(e_4)$. The corresponding subspace for the two blocks N_1 is $\mathrm{span}(e_2,e_3)$.

The assumptions of Theorems 3.5 and 3.6 are satisfied. A common regular right eigenvector for the eigenvalue $(\lambda,\mu)=(\frac{1}{2},\frac{1}{2})$ is $[2 \quad 1 \quad 2 \quad 1]^T+\alpha e_4$ for an arbitrary $\alpha\in\mathbb{C}$. If we take $\alpha=0$, then $x=[2 \quad 1 \quad 2 \quad 1]^T=[1 \quad 1]^T\otimes[2 \quad 1]^T$ is a decomposable regular eigenvector. As all matrices are symmetric and the eigenvalue is real, x is also a regular left eigenvector. One can see that $x^*\Delta_0 x=-6$ is nonzero as predicted by Proposition 3.3.

EXAMPLE 4.2. In [12] we proved that one can solve a quadratic two-parameter eigenvalue problem by linearizing it as a singular two-parameter eigenvalue problem. In Appendix of [12] we provide a linearization of an arbitrary polynomial two-parameter eigenvalue problem as a singular two-parameter eigenvalue problem. The new theory presented in this paper shows that all simple eigenvalues can be computed from the above linearization.

For an example we take the following system of two bivariate polynomials

$$
\begin{aligned}
p_1(\lambda,\mu) &= 1+2\lambda+3\mu+4\lambda^2+5\lambda\mu+6\mu^2+7\lambda^3+8\lambda^2\mu+9\lambda\mu^2+10\mu^3=0\\
p_2(\lambda,\mu) &= 10+9\lambda+8\mu+7\lambda^2+6\lambda\mu+5\mu^2+4\lambda^3+3\lambda^2\mu+2\lambda\mu^2+\mu^3=0.
\end{aligned} \tag{4.1}
$$

Following [12] we linearize the above system as a singular two-parameter eigenvalue problem, where

$$
W_1(\lambda,\mu)=\begin{bmatrix}
1 & 2 & 3 & 4+7\lambda & 5+8\lambda & 6+9\lambda+10\mu\\
\lambda & -1 & 0 & 0 & 0 & 0\\
\mu & 0 & -1 & 0 & 0 & 0\\
0 & \lambda & 0 & -1 & 0 & 0\\
0 & 0 & \lambda & 0 & -1 & 0\\
0 & 0 & \mu & 0 & 0 & -1
\end{bmatrix},
$$

$$
W_2(\lambda,\mu)=\begin{bmatrix}
10 & 9 & 8 & 7+4\lambda & 6+3\lambda & 5+2\lambda+\mu\\
\lambda & -1 & 0 & 0 & 0 & 0\\
\mu & 0 & -1 & 0 & 0 & 0\\
0 & \lambda & 0 & -1 & 0 & 0\\
0 & 0 & \lambda & 0 & -1 & 0\\
0 & 0 & \mu & 0 & 0 & -1
\end{bmatrix}.
$$

One can check that $\det(W_i(\lambda,\mu))=p_i(\lambda,\mu)$ for $i=1,2$. The obtained two-parameter eigenvalue problem has 9 finite regular eigenvalues, which are all simple. The only real eigenvalue is $(\lambda,\mu)=(-2.4183,1.8542)$ while the remaining 8 eigenvalues appear in conjugate pairs. All eigenvalues agree with the roots of the system (4.1).

In a similar way an arbitrary system of two bivariate polynomials could be linearized as a singular two-parameter eigenvalue problem. This gives a new approach for the numerical computation of roots of such systems. The dimension of the matrices of the linearized two-parameter eigenvalue problem is large, but they are also very sparse. Therefore, the new approach is most likely not competitive to advanced numerical methods that compute all solutions of polynomial systems, for instance, to the homotopy method PHCPack [15]. But, combined with a Jacobi-Davidson approach, it might be an alternative when one is interested only in part of the roots that are close to a given target. We plan to explore this in our further research.

EXAMPLE 4.3. For this example we take

$$W_1(\lambda,\mu)=\begin{bmatrix}1+\lambda+\mu & 0 & 0\\ 0 & 1+\lambda+\mu & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$W_2(\lambda,\mu)=\begin{bmatrix}2+4\lambda+6\mu & 0\\ 0 & 1\end{bmatrix}.$$

Now $p_1(\lambda,\mu)=(1+\lambda+\mu)^2$, $p_2(\lambda,\mu)=2+4\lambda+6\mu$, and the problem has a double eigenvalue $(\lambda,\mu)=(-2,1)$ with geometric multiplicity 2. Its right (and left) eigenvectors lie in the span of $e_1\otimes e_1$ and $e_2\otimes e_1$. Although the assumptions of Proposition 3.3 are not satisfied we obtain $x^*\Delta_0 x=2\neq 0$ for $x=e_1\otimes e_1$ and (3.10) is satisfied. It follows from Lemma 3.4 that $(-2,1)$ is a regular finite eigenvalue of matrix pencils $\Delta_1-\lambda\Delta_0$ and $\Delta_2-\mu\Delta_0$.

EXAMPLE 4.4. We take

$$\begin{aligned}W_1(\lambda,\mu)&=\begin{bmatrix}2+\lambda & 1+2\lambda & \lambda\\ \lambda & 2+2\lambda+2\mu & \mu\\ \mu & 1+2\mu & 2+\mu\end{bmatrix},\\ W_2(\lambda,\mu)&=\begin{bmatrix}1+\lambda & 1+2\lambda & \lambda\\ \lambda & 1+2\lambda+2\mu & \mu\\ \mu & 1+2\mu & 1+\mu\end{bmatrix}.\end{aligned}\tag{4.2}$$

Now

$$\begin{aligned}p_1(\lambda,\mu)&=\lambda^2+6\mu\lambda+10\lambda+\mu^2+10\mu+8\\ p_2(\lambda,\mu)&=(\lambda+\mu+1)^2\end{aligned}$$

and the problem has a quadruple eigenvalue $(\lambda,\mu)=(-\frac{1}{2},-\frac{1}{2})$ that is geometrically simple.

The pencils $\Delta_1-\lambda\Delta_0$ and $\Delta_2-\mu\Delta_0$ have the same Kronecker structure with the following blocks: $L_0, 2N_2, L_0^T$, and $J_4(-\frac{1}{2})$. The minimal reducing subspaces are

$$\begin{aligned}\mathcal{R}(\Delta_1,\Delta_0)&=\operatorname{span}\left([0\quad 1\quad -2]^T\otimes[0\quad 1\quad -2]^T\right)\\ \mathcal{R}(\Delta_2,\Delta_0)&=\operatorname{span}\left([2\quad -1\quad 0]^T\otimes[2\quad -1\quad 0]^T\right).\end{aligned}$$

A common regular eigenvector for pencils $\Delta_1-\lambda\Delta_0$ and $\Delta_2-\mu\Delta_0$ for the eigenvalue $(-\frac{1}{2},-\frac{1}{2})$ is

$$z_1=[0\quad 1\quad 0]^T\otimes[1\quad -1\quad 1]^T.$$

The corresponding common left eigenvector is $y=[1\quad 2\quad 1]^T\otimes[1\quad 0\quad 1]^T$. This gives $y^*\Delta_0 z_1=0$ and the condition (3.10) is not satisfied. The ascent of the eigenvalue

$(-\frac{1}{2},-\frac{1}{2})$ is 4. If we take vectors

$$
\begin{aligned}
z_2 &= [\,1 \quad 0 \quad -1\,]^T \otimes [\,1 \quad -1 \quad 1\,]^T,\\
z_3 &= [\,1 \quad 0 \quad 1\,]^T \otimes [\,1 \quad -1 \quad 1\,]^T + [\,0 \quad 1 \quad 0\,]^T \otimes [\,0 \quad 1 \quad 0\,]^T,\\
z_4 &= [\,1 \quad 0 \quad -1\,]^T \otimes [\,0 \quad 1 \quad 0\,]^T + [\,0 \quad 2 \quad 0\,]^T \otimes [\,1 \quad 0 \quad -1\,]^T
\end{aligned}
$$

and define subspaces $\mathcal{S}_i = \mathrm{span}(z_1,\dots,z_i)$ for $i = 1,\dots,4$, then $(\Delta_1 + \frac{1}{2}\Delta_0)\mathcal{S}_i \subset \Delta_0\mathcal{S}_{i-1}$ and $(\Delta_2 + \frac{1}{2}\Delta_0)\mathcal{S}_i \subset \Delta_0\mathcal{S}_{i-1}$ for $i = 2,3,4$. Vectors z_1, z_2, z_3, and z_4 thus form a basis for the common root subspace of the eigenvalue $(-\frac{1}{2},-\frac{1}{2})$.

Although the assumptions of Theorems 3.5 and 3.6 are not satisfied, we see that also in this case the finite regular eigenvalues of the two-parameter eigenvalue problem (4.2) agree with the finite regular eigenvalues of the associated system (1.4). We obtained the same for many other numerical examples with multiple eigenvalues and this indicates that the theory could probably be extended to cover a wider class of singular two-parameter eigenvalue problems.

5. Numerical methods. There are several numerical methods for two-parameter eigenvalue problems, see for instance [8] and references therein, but, most of the methods require that the problem is nonsingular. There are some exceptions, for instance, we can apply the Newton method [2] on (1.2), but this method requires a good initial approximation and computes only one eigenvalue. All methods that can compute all the eigenvalues require that the problem is nonsingular.

In [12] we presented a numerical algorithm for the computation of the common regular eigenvalues of a pair of singular matrix pencils. The details can be found in [12], let us just mention that the algorithm is based on the staircase algorithm for one matrix pencil from [14]. The algorithm returns matrices Q and U with unitary columns that define matrices $\widetilde{\Delta}_i = Q^*\Delta_i U$ of size $k \times k$ for $i = 0,1,2$ such that $\widetilde{\Delta}_0$ is nonsingular and the k common finite regular eigenvalues of (1.4) are the eigenvalues of the projected regular matrix pencils $\widetilde{\Delta}_1 - \lambda\widetilde{\Delta}_0$ and $\widetilde{\Delta}_2 - \mu\widetilde{\Delta}_0$. This algorithm can be applied to compute the eigenvalues of a general singular two-parameter eigenvalue problem.

When N is very large, it is not feasible anymore to apply the algorithm from [12] because of its complexity. For such problems, in particular when the matrices are sparse, one could apply a Jacobi–Davidson method [8]. The only adjustment is that as it might happen that the smaller projected problem is singular, the routine for the solution of a smaller projected two-parameter eigenvalue problem should be replaced by a method that can handle singular problems.

6. Conclusion and acknowledgments. The results in this paper prove that for a large class of singular two-parameter eigenvalue problems (1.1) one could compute all eigenvalues by computing the common regular eigenvalues of the associated coupled pair of generalized eigenvalue problems (1.4). The theory guarantees that this works for all algebraically simple eigenvalues. Various numerical results suggest that this approach is correct for all finite regular eigenvalues of a singular two-parameter eigenvalue problem.

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