GREEN'S FUNCTION FOR AN ELASTIC LAYER LOADED HARMONICALLY ON ITS SURFACE

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Abstract

The Green's function in surface displacement plays an important role in soil structure interaction. In evaluating the Green's function, several difficulties occur because it is formulated in the infinite integral form. This paper outlines a method of analyzing the steady-state dynamic response of an elastic layer subjected to general point load excitation. It is assumed that the load is applied at the surface. The application Hankel integral transform, to the governing differential equations and boundary conditions yields the response displacements at the surface in integral representation. It will be shown that these semi-infinite integrals can be reduced to the integral with the finite range of integration, which can be efficiently taken numerically. The numerical results are presented, which show the efficiency of the developed procedure.

Keywords

Soil-Structure Interaction, Layered Half-Space, Elastodynamics, Green's Function

1 INTRODUCTION

The fundamental dynamic solutions for homogeneous half-space as well as for a layered one are studied in depth and known in the literature. They are given in two fundamentally different mathematical forms. On one hand, there are approximate solutions, e.g. the ingenuous thin layer method introduced by Kausel [1], and on the other hand analytical methods leading to the solutions in form of integrals with infinite or semi-infinite path of integration, e.g. Vostroukhov [2], Jin and Liu [3], and others. The sufficiently accurate evaluation of these integrals, as needed in practical engineering problems primarily in dynamic soil-structure interaction, is time consuming if not tedious. Kobayashi [4] made a step forward showing that in the case of a homogeneous half-space the integrals of semi-infinite extension representing displacements could be transformed to the integrals with the finite path of integration. The numerical evaluation of these integrals is then easy and straightforward. The drawback of these techniques is that it applies to homogeneous half-space only. In order to extend the Kobayashi [4] method also to the layered half-spaces, the authors first show that the Kobayashi [4] method can be applied to a homogeneous layer, where it is understood that a homogeneous half-space is a special example of such a layer only, i.e. a layer of semi-infinite depth. This is the topic of this paper. In the forthcoming papers, the solutions for the layers will be combined leading to a solution for a layered half-space expressed in the form of integrals with the finite integration path.

2 METHOD OF ANALYSIS

Let us consider an elastic layer subjected to general point load excitation, which can be represented by two components, the vertical and the horizontal one (Fig.1).

The model is analyzed under the following assumptions:

- The load varies in time harmonically.
- A general point load is applied at the surface of the medium.
- The material constants of an elastic layer are the shear modulus μ, Poisson's ratio ν, the mass density ρ and the damping coefficient $\hat{\mu}$.
- Material damping in the elastic layer is introduced in accordance with Voigt's rheological model.

With these assumptions in mind the problem at hand has a steady-state solution, which varies in time in the same manner as the load, namely as $e^{i \cdot \omega \cdot t}$.

The equation of motion [5]:

$$\mu \cdot \nabla^2 \vec{U} + (\lambda + \mu) \cdot \vec{\nabla} \cdot (\vec{\nabla} \bullet \vec{U}) + \rho \cdot \vec{F} - \rho \cdot \ddot{\vec{U}} = 0, \quad (1)$$

which is well known as Navier equation of motion, serves as the starting point.

The system of equations (1) presents a disadvantageous feature as it couples three displacement components. Of course, we can uncouple this system of equations by eliminating two of three displacement components through two of three equations, but this results in partial differential equations of the sixth order. A far more convenient approach is to express the components of the displacement vector in terms of derivatives of potentials. These potentials satisfy uncoupled wave equations.

So, to formulate the problem mathematically we employ the displacement potentials by means of which the displacement vector $\vec{U}$ of the homogeneous half-space may be decomposed as [6]:

$$\vec{U} = \vec{\nabla} \cdot \varphi + \vec{\nabla} \times \vec{\psi}. \quad (2)$$

The above equation has in the cylindrical coordinate system r, ϑ and z the following form:

$$u_r = \frac{\partial \varphi_r}{\partial r} + \frac{1}{r} \cdot \frac{\partial \psi_z}{\partial \vartheta} - \frac{\partial \psi_\vartheta}{\partial z} \quad (3)$$

$$u_\vartheta = \frac{1}{r} \cdot \frac{\partial \varphi_\vartheta}{\partial \vartheta} + \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r} \quad (4)$$

$$u_z = \frac{\partial \varphi_z}{\partial z} + \frac{1}{r} \cdot \frac{\partial (\psi_\vartheta \cdot r)}{\partial r} - \frac{1}{r} \cdot \frac{\partial \psi_r}{\partial \vartheta}, \quad (5)$$

with φ and $\vec{\psi} = (\psi_r, \psi_\vartheta, \psi_z)$, i.e. the scalar and the vector Helmholtz potentials that satisfy the following wave equations in the absence of body forces:
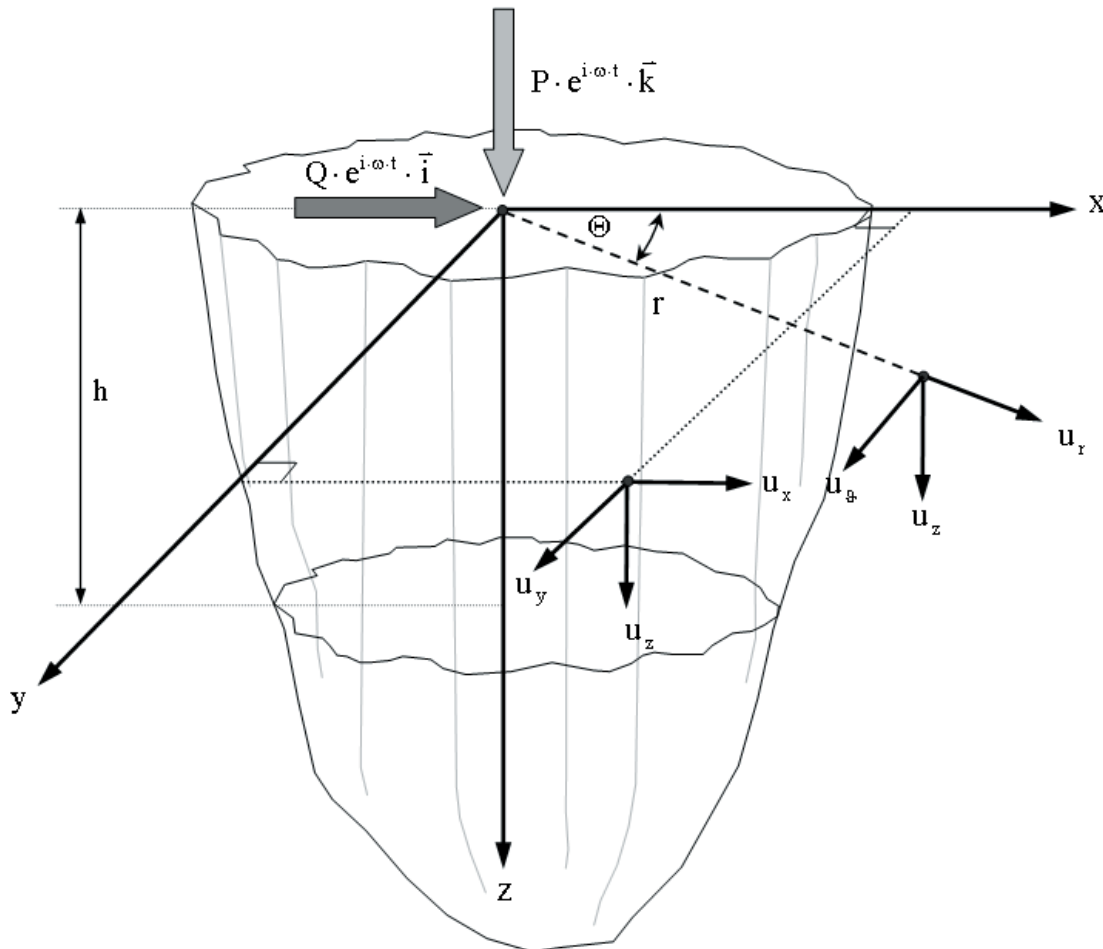

Figure 1. An elastic layer subjected to the surface with a general harmonic point load.

$$\nabla^2 \varphi = \frac{1}{c_L^2} \cdot \frac{\partial^2 \varphi}{\partial t^2} \ , \quad (6)$$

$$\nabla^2 \psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2} \cdot \frac{\partial \psi_\theta}{\partial \theta} = \frac{1}{c_T^2} \cdot \frac{\partial^2 \psi_r}{\partial t^2} \quad (7)$$

$$\nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2} \cdot \frac{\partial \psi_r}{\partial \theta} = \frac{1}{c_T^2} \cdot \frac{\partial^2 \psi_\theta}{\partial t^2} \quad (8)$$

$$\nabla^2 \psi_z = \frac{1}{c_T^2} \cdot \frac{\partial^2 \psi_z}{\partial t^2} \ , \quad (9)$$

where the Laplacian ∇^2 is defined as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \ . \quad (10)$$

and:

$$c_L = \frac{\omega}{k_L} = \sqrt{\frac{\lambda + 2 \cdot \mu}{\rho}} \quad (11)$$

$$c_T = \frac{\omega}{k_T} = \sqrt{\frac{\mu}{\rho}} \quad (12)$$

are the velocity of the dilatational (P-waves) and the velocity of the shear waves (S-waves).

It should be noted that Eq. (2) relates the three components of the displacement vector to four other functions, i.e. the scalar potential and the three components of the vector potential. This indicates that φ and the $\vec{\psi}$ should be subjected to an additional constraint condition.

Generally the components of $\vec{\psi}$ are taken to be related in some way. Usually, but not always, the relation:

$$\vec{\nabla} \cdot \vec{\psi} = 0 \quad (13)$$

is taken as an additional constraint condition. This relation has the advantage that it is consistent with the Helmholtz decomposition of a vector.

In the case of the vertical point load, the corresponding elastodynamic problem is axi-symmetrical. Displacement components are independent of coordinate ϑ and are given by:

$$\vec{u} = \left\{ \begin{array}{c} u_r(r,t) \\ 0 \\ u_z(r,t) \end{array} \right\} = \left\{ \begin{array}{c} \dfrac{\partial \varphi_r}{\partial r} - \dfrac{\partial \psi_\vartheta}{\partial z} \\ 0 \\ \dfrac{\partial \varphi_r}{\partial z} + \dfrac{1}{r} \cdot \dfrac{\partial (r \cdot \psi_\vartheta)}{\partial r} \end{array} \right\} \ . \quad (14)$$

The two scalar wave potentials have to satisfy the partial differential equations (6) and (8), the boundary conditions at the top of the layer:

$$\sigma_z = -\frac{P \cdot H(t) \cdot \delta(r)}{2 \cdot \pi \cdot r} \quad (15)$$

$$\tau_{zr} = 0 \ , \quad (16)$$

and any axi-symmetric conditions on $z = h$.

After application of the Hankel transform $r \to \xi$ to the above stated problem and the solution of the resulting ordinary differential equation we obtain:

$$\overline{w}(r,0,\omega) = -\frac{P}{2 \cdot \pi} \cdot \frac{k_T^2}{\mu} \cdot \int_0^\infty \frac{\xi \cdot \sqrt{\xi^2 - k_L^2}}{D(\xi)} \cdot J_0(\xi r) \cdot d\xi$$

$$-2 \cdot k_T^2 \cdot \int_0^\infty \frac{\xi \cdot (\xi^2 - k_T^2) \cdot \sqrt{\xi^2 - k_L^2}}{D(\xi)} \cdot C_1(\xi) \cdot J_0(\xi r) \cdot d\xi \quad (17a)$$

$$-4 \cdot k_T^2 \cdot \int_0^\infty \frac{\xi^2 \cdot \sqrt{\xi^2 - k_L^2} \cdot \sqrt{\xi^2 - k_T^2}}{D(\xi)} \cdot C_3(\xi) \cdot J_0(\xi r) \cdot d\xi$$

$$\overline{u}(r,0,\omega) = -\frac{P}{2 \cdot \pi \cdot \mu} \cdot \int_0^\infty \frac{\xi^2 \cdot \left(2 \cdot \xi^2 - k_T^2 - 2 \cdot \sqrt{\xi^2 - k_L^2} \cdot \sqrt{\xi^2 - k_T^2}\right)}{D(\xi)} \cdot J_1(\xi r) \cdot d\xi$$

$$-4 \cdot k_T^2 \cdot \int_0^\infty \frac{\xi^2 \cdot \sqrt{\xi^2 - k_L^2} \cdot \sqrt{\xi^2 - k_T^2}}{D(\xi)} \cdot C_1(\xi) \cdot J_1(\xi r) \cdot d\xi \quad (18a)$$

$$-2 \cdot k_T^2 \cdot \int_0^\infty \frac{\xi \cdot (\xi^2 - k_T^2) \cdot \sqrt{\xi^2 - k_L^2}}{D(\xi)} \cdot C_3(\xi) \cdot J_1(\xi r) \cdot d\xi$$

where $D(\eta)$ is defined as:

$$D(\xi) = \left(2 \cdot \xi^2 - k_T^2\right)^2 - 4 \cdot \xi^2 \sqrt{\xi^2 - k_L^2} \cdot \sqrt{\xi^2 - k_T^2} \quad . \quad (19)$$

For a half-space, which is understood as a layer of semi-infinite depth, the undetermined constants C_1 and C_3 in (17a) and (18a) can be set to be equal zero. This gives:

$$\overline{w}(r,0,\omega) = -\frac{P}{2 \cdot \pi} \cdot \frac{k_T^2}{\mu} \cdot \int_0^\infty \frac{\xi \cdot \sqrt{\xi^2 - k_L^2}}{D(\xi)} \cdot J_0(\xi r) \cdot d\xi \quad (17)$$

$$\overline{u}(r,0,\omega) = -\frac{P}{2 \cdot \pi \cdot \mu} \cdot \int_0^\infty \frac{\xi^2 \cdot \left(2 \cdot \xi^2 - k_T^2 - 2 \cdot \sqrt{\xi^2 - k_L^2} \cdot \sqrt{\xi^2 - k_T^2}\right)}{D(\xi)} \cdot J_1(\xi r) \cdot d\xi$$
$$(18)$$

The above solutions are analogous to these given by Achenbach [5] for the Laplace domain.

The integrals of semi-infinite integration range in equations (17) and (18), which are to be reduced to the finite integration range, can be clearly identified.

Now we turn our attention to a layer loaded on its upper surface with a concentrated horizontal point load Without loss of generality, we can assume that it acts on positive x-axis direction.

The relevant stress-displacement relations for an elastic layer may be written as:

$$\sigma_z = \lambda \cdot \left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \cdot \frac{\partial u_\vartheta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right] + 2 \cdot \mu \cdot \frac{\partial u_z}{\partial z} \quad (20)$$

$$\sigma_{\vartheta z} = \mu \cdot \left[\frac{1}{r} \cdot \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\vartheta}{\partial z} \right] \quad (21)$$

$$\sigma_{zr} = \mu \cdot \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \quad . \quad (22)$$

For the problem at hand we again assume the harmonic time dependence:

$$a(r,\vartheta,z,t) = \overline{a}(r,\vartheta,z,\omega) \cdot e^{i \cdot \omega \cdot t} \quad (23)$$

thus we obtain:

$$\overline{u}_r = \frac{\partial \overline{\varphi}_r}{\partial r} + \frac{1}{r} \cdot \frac{\partial \overline{\psi}_z}{\partial \vartheta} - \frac{\partial \overline{\psi}_\vartheta}{\partial z} \quad (24)$$

$$\overline{u}_\vartheta = \frac{1}{r} \cdot \frac{\partial \overline{\varphi}_\vartheta}{\partial \vartheta} + \frac{\partial \overline{\psi}_r}{\partial z} - \frac{\partial \overline{\psi}_z}{\partial r} \quad (25)$$

$$\overline{u}_z = \frac{\partial \overline{\varphi}_z}{\partial z} + \frac{1}{r} \cdot \frac{\partial \left(\overline{\psi}_\vartheta \cdot r \right)}{\partial r} - \frac{1}{r} \cdot \frac{\partial \overline{\psi}_r}{\partial \vartheta} \quad , \quad (26)$$

$$\overline{\sigma}_z = \lambda \cdot \left[\frac{\partial \overline{u}_r}{\partial r} + \frac{\overline{u}_r}{r} + \frac{1}{r} \cdot \frac{\partial \overline{u}_\vartheta}{\partial \theta} + \frac{\partial \overline{u}_z}{\partial z} \right] + 2 \cdot \mu \cdot \frac{\partial \overline{u}_z}{\partial z} \quad (27)$$

$$\overline{\sigma}_{\vartheta z} = \mu \cdot \left[\frac{1}{r} \cdot \frac{\partial \overline{u}_z}{\partial \theta} + \frac{\partial \overline{u}_\vartheta}{\partial z} \right] \quad (28)$$

$$\overline{\sigma}_{zr} = \mu \cdot \left[\frac{\partial \overline{u}_r}{\partial z} + \frac{\partial \overline{u}_z}{\partial r} \right] \quad (29)$$

$$\nabla^2 \overline{\varphi} = -\frac{\omega^2}{c_L^2} \cdot \overline{\varphi} \quad , \quad (30)$$

$$\nabla^2 \overline{\psi}_r - \frac{\overline{\psi}_r}{r^2} - \frac{2}{r^2} \cdot \frac{\partial \overline{\psi}_\theta}{\partial \theta} = -\frac{\omega^2}{c_T^2} \cdot \overline{\psi}_r \quad (31)$$

$$\nabla^2 \overline{\psi}_\theta - \frac{\overline{\psi}_\theta}{r^2} + \frac{2}{r^2} \cdot \frac{\partial \overline{\psi}_r}{\partial \theta} = -\frac{\omega^2}{c_T^2} \cdot \overline{\psi}_\theta \quad (32)$$

$$\nabla^2 \overline{\psi}_z = -\frac{\omega^2}{c_T^2} \cdot \overline{\psi}_z \quad , \quad (33)$$

$$\frac{\partial \left(\overline{\psi}_r \cdot r \right)}{\partial r} + \frac{\partial \overline{\psi}_\theta}{\partial \theta} + r \cdot \frac{\partial \overline{\psi}_z}{\partial z} = 0 \quad . \quad (34)$$

The loading conditions in Cartesian coordinate system are given by:

$$F_x = \frac{\overline{Q}(\omega)}{2 \cdot \pi \cdot r} \cdot \delta(r) \quad (35)$$

Their transformation to the cylindrical coordinates yields:

$$F_r = F_x \cdot \cos(\vartheta) \quad (36)$$

$$F_\vartheta = -F_x \cdot \sin(\vartheta) \quad (37)$$

$$F_z = 0 \quad . \quad (38)$$

Using the above results, the boundary conditions on the surface z=0 of an elastic layer can be written as:

$$\overline{\sigma}_{rz}(r,\vartheta,0,\omega) = -\frac{\overline{Q}(\omega) \cdot \delta(r)}{2 \cdot \pi \cdot r} \cdot \cos(\vartheta) \quad (39)$$

$$\overline{\sigma}_{\vartheta z}(r,\vartheta,0,\omega) = \frac{\overline{Q}(\omega) \cdot \delta(r)}{2 \cdot \pi \cdot r} \cdot \sin(\vartheta) \quad (40)$$

$$\overline{\sigma}_{zz}(r,\vartheta,0,\omega) = 0 \quad (41)$$

and they complete the statement of the problem.

The ϑ-dependence of the loading as given by equations (36) to (38) and equations (39) to (41) respectively and the geometry, which is axi-symmetric permit us to seek the solution to the problem in the following form

$$\overline{\varphi}(r,\vartheta,z,\omega)=\overline{\overline{\varphi}}(r,z,\omega)\cdot\cos(\vartheta) \quad (42)$$

$$\overline{\psi}_r(r,\vartheta,z,\omega)=\overline{\overline{\psi}}_r(r,z,\omega)\cdot\sin(\vartheta) \quad (43)$$

$$\overline{\psi}_\vartheta(r,\vartheta,z,\omega)=\overline{\overline{\psi}}_\vartheta(r,z,\omega)\cdot\cos(\vartheta) \quad (44)$$

$$\overline{\psi}_z(r,\vartheta,z,\omega)=\overline{\overline{\psi}}_z(r,z,\omega)\cdot\sin(\vartheta)\,, \quad (45)$$

which correspond for the following ϑ-dependencies of the displacements and stresses:

$$\overline{u}_r(r,\vartheta,z,\omega)=\overline{\overline{u}}_r(r,z,\omega)\cdot\cos(\vartheta) \quad (46)$$

$$\overline{u}_\vartheta(r,\vartheta,z,\omega)=\overline{\overline{u}}_\vartheta(r,z,\omega)\cdot\sin(\vartheta) \quad (47)$$

$$\overline{u}_z(r,\vartheta,z,\omega)=\overline{\overline{u}}_z(r,z,\omega)\cdot\cos(\vartheta) \quad (48)$$

$$\overline{\sigma}_{rz}(r,\vartheta,z,\omega)=\overline{\overline{\sigma}}_{rz}(r,z,\omega)\cdot\cos(\vartheta) \quad (49)$$

$$\overline{\sigma}_{\vartheta z}(r,\vartheta,z,\omega)=\overline{\overline{\sigma}}_{\vartheta z}(r,z,\omega)\cdot\sin(\vartheta) \quad (50)$$

$$\overline{\sigma}_{zz}(r,\vartheta,z,\omega)=\overline{\overline{\sigma}}_{zz}(r,z,\omega)\cdot\cos(\vartheta)\,. \quad (51)$$

The substitution of Eqs. (42)-(45) into wave Eqs. (30)-(33) and Eq. (34) yields:

$$\left[\frac{\partial^2\overline{\overline{\varphi}}}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\overline{\overline{\varphi}}}{\partial r}-\frac{1}{r^2}\cdot\overline{\overline{\varphi}}+\frac{\partial^2\overline{\overline{\varphi}}}{\partial z^2}\right]+k_L^2\cdot\overline{\overline{\varphi}}=0 \quad (52)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{2}{r^2}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\psi}}_r+\frac{2}{r^2}\cdot\overline{\overline{\psi}}_\vartheta+k_T^2\cdot\overline{\overline{\psi}}_r=0 \quad (53)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{2}{r^2}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\psi}}_\theta+\frac{2}{r^2}\cdot\overline{\overline{\psi}}_r+k_T^2\cdot\overline{\overline{\psi}}_\theta=0 \quad (54)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\psi}}_z+k_T^2\cdot\overline{\overline{\psi}}_z=0 \quad (55)$$

$$\overline{\overline{\psi}}_r-\overline{\overline{\psi}}_\theta+r\cdot\left[\frac{\partial\overline{\overline{\psi}}_r}{\partial r}+\frac{\partial\overline{\overline{\psi}}_z}{\partial z}\right]=0\,, \quad (56)$$

where:

$$c_L=\frac{\omega}{k_L} \quad (57)$$

$$c_T=\frac{\omega}{k_T}\,. \quad (58)$$

Eqs. (53) and (54) are coupled. To decouple these equations, it is customary to introduce the new potentials $\overline{\overline{\chi}}$ and $\overline{\overline{\kappa}}$:

$$\overline{\overline{\chi}}=\overline{\overline{\psi}}_r+\overline{\overline{\psi}}_\vartheta \qquad \overline{\overline{\kappa}}=\overline{\overline{\psi}}_r-\overline{\overline{\psi}}_\vartheta \quad (59)$$

$$\overline{\overline{\psi}}_r=\frac{1}{2}\cdot\left(\overline{\overline{\chi}}+\overline{\overline{\kappa}}\right) \qquad \overline{\overline{\psi}}_\vartheta=\frac{1}{2}\cdot\left(\overline{\overline{\chi}}-\overline{\overline{\kappa}}\right)\,. \quad (60)$$

In terms of these newly introduced potentials wave equations (52)-(55), Eq. (56) and displacements, Eqs. (24)-(26), can be rewritten as:

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\varphi}}+k_L^2\cdot\overline{\overline{\varphi}}=0 \quad (61)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\chi}}+k_T^2\cdot\overline{\overline{\chi}}=0 \quad (62)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{4}{r^2}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\kappa}}+k_T^2\cdot\overline{\overline{\kappa}}=0 \quad (63)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right]\cdot\overline{\overline{\psi}}_z+k_T^2\cdot\overline{\overline{\psi}}_z=0 \quad (64)$$

$$\overline{\overline{\kappa}}+r\cdot\left[\frac{1}{2}\cdot\frac{\partial\left(\overline{\overline{\chi}}+\overline{\overline{\kappa}}\right)}{\partial r}+\frac{\partial\overline{\overline{\psi}}_z}{\partial z}\right]=0\,. \quad (65)$$

$$\overline{\overline{u}}_r=\frac{\partial\overline{\overline{\varphi}}}{\partial r}+\frac{1}{r}\cdot\overline{\overline{\psi}}_z-\frac{1}{2}\cdot\frac{\partial\left(\overline{\overline{\chi}}-\overline{\overline{\kappa}}\right)}{\partial z} \quad (66)$$

$$\overline{\overline{u}}_\vartheta=-\frac{1}{r}\cdot\overline{\overline{\varphi}}+\frac{1}{2}\cdot\frac{\partial\left(\overline{\overline{\chi}}+\overline{\overline{\kappa}}\right)}{\partial z}-\frac{\partial\overline{\overline{\psi}}_z}{\partial r} \quad (67)$$

$$\overline{\overline{u}}_z=\frac{\partial\overline{\overline{\varphi}}}{\partial z}-\frac{1}{r}\cdot\overline{\overline{\kappa}}+\frac{1}{2}\cdot\frac{\partial\left(\overline{\overline{\chi}}-\overline{\overline{\kappa}}\right)}{\partial r}\,. \quad (68)$$

The general solutions of wave equations (61)-(64) is found by applying the Hankel transform method with respect to the radial coordinate. This transform is defined as in [8]:

$$f^{H_n}(\xi) = \int_0^\infty r \cdot f(r) \cdot J_n(\xi r) \cdot dr \quad (69)$$

$$f(r) = \int_0^\infty \xi \cdot f^{H_n}(\xi) \cdot J_n(\xi r) \cdot d\xi , \quad (70)$$

where n is the order of the transform and $J_n(\xi r)$ is the ordinary Bessel function of order n.

For Eqs. (61) and (64) the order of the transform should be equal to 1, for Eq. (62) to 0 and for Eq. (63) to 2. Applying these transform to Eqs. (61)-(64) one obtains:

$$\frac{d^2 \overline{\breve{\varphi}}^{H_1}(\xi)}{\partial z^2} - \alpha^2 \cdot \overline{\breve{\varphi}}^{H_1}(\xi) = 0 \quad (71)$$

$$\frac{d^2 \overline{\overline{\breve{\chi}}}^{H_0}(\xi)}{\partial z^2} - \beta^2 \cdot \overline{\overline{\breve{\chi}}}^{H_0}(\xi) = 0 \quad (72)$$

$$\frac{d^2 \overline{\breve{\kappa}}^{H_2}(\xi)}{\partial z^2} - \beta^2 \cdot \overline{\breve{\kappa}}^{H_2}(\xi) = 0 \quad (73)$$

$$\frac{d^2 \overline{\breve{\psi}}_z^{H_1}(\xi)}{\partial z^2} - \beta^2 \cdot \overline{\breve{\psi}}_z^{H_1}(\xi) = 0 , \quad (74)$$

with:

$$\alpha = \sqrt{\xi^2 - \frac{\omega^2}{c_L^2}} = \sqrt{\xi^2 - k_L^2} \quad (75)$$

$$\beta = \sqrt{\xi^2 - \frac{\omega^2}{c_T^2}} = \sqrt{\xi^2 - k_T^2} . \quad (76)$$

The general solutions of Eqs. (71)-(74) can be found easily as:

$$\overline{\breve{\varphi}}^{H_1} = C_1 \cdot e^{\alpha \cdot z} + C_2 \cdot e^{-\alpha \cdot z} \quad (77)$$

$$\overline{\overline{\breve{\chi}}}^{H_0} = C_3 \cdot e^{\beta \cdot z} + C_4 \cdot e^{-\beta \cdot z} \quad (78)$$

$$\overline{\breve{\kappa}}^{H_2} = C_5 \cdot e^{\beta \cdot z} + C_6 \cdot e^{-\beta \cdot z} \quad (79)$$

$$\overline{\breve{\psi}}_z^{H_1} = C_7 \cdot e^{\beta \cdot z} + C_8 \cdot e^{-\beta \cdot z} . \quad (80)$$

We continue the analysis by expressing the relationship given Eq. (65) through the transformed potentials. They are defined of as the Hankel transforms, expression for of $\overline{\overline{\varphi}}$, $\overline{\overline{\chi}}$, $\overline{\overline{\kappa}}$ and $\overline{\overline{\psi}}_z$ and are given as:

$$\overline{\overline{\varphi}}(r,z,\omega) = \int_0^\infty \xi \cdot \overline{\breve{\varphi}}^{H_1}(\xi,z,\omega) \cdot J_1(\xi r) \cdot d\xi \quad (81)$$

$$\overline{\overline{\chi}}(r,z,\omega) = \int_0^\infty \xi \cdot \overline{\overline{\breve{\chi}}}^{H_0}(\xi,z,\omega) \cdot J_0(\xi r) \cdot d\xi \quad (82)$$

$$\overline{\overline{\kappa}}(r,z,\omega) = \int_0^\infty \xi \cdot \overline{\breve{\kappa}}^{H_2}(\xi,z,\omega) \cdot J_2(\xi r) \cdot d\xi \quad (83)$$

$$\overline{\overline{\psi}}_z(r,z,\omega) = \int_0^\infty \xi \cdot \overline{\breve{\psi}}_z^{H_1}(\xi,z,\omega) \cdot J_1(\xi r) \cdot d\xi \quad (84)$$

The substitution of these expressions into Eq. (65) yields:

$$\xi \cdot \overline{\breve{\kappa}}^{H_2}(\xi,z,\omega) - \xi \cdot \overline{\overline{\breve{\chi}}}^{H_0}(\xi,z,\omega) + 2 \cdot \frac{\partial \overline{\breve{\psi}}_z^{H_1}(\xi,z,\omega)}{\partial z} = 0 \quad (85)$$

which establishes r-independent relationship between the transformed components of the vector-potential.

Having taken into account relationship (85) and using recurrent relation between the Bessel functions [8], the substitution of expressions (81)-(84) into transformed displacements (66)-(68) yields:

$$\overline{\overline{u}}_r(r,z,\omega) = \frac{1}{2} \cdot \int_0^\infty \xi \cdot \left\{ \left[\xi \, \overline{\breve{\varphi}}^{H_1}(\xi,z,\omega) + \xi \cdot \overline{\breve{\psi}}_z^{H_1}(\xi,z,\omega) + \right. \right.$$

$$\left. - \frac{\partial \overline{\breve{\kappa}}^{H_2}(\xi,z,\omega)}{\partial z} - \frac{2}{\xi} \cdot \frac{\partial^2 \overline{\breve{\psi}}_z^{H_1}(\xi,z,\omega)}{\partial z^2} \right] \cdot J_0(\xi r)$$

$$\left. + \left[-\xi \cdot \overline{\breve{\varphi}}^{H_1}(\xi,z,\omega) + \xi \cdot \overline{\breve{\psi}}_z^{H_1}(\xi,z,\omega) + \frac{\partial \overline{\breve{\kappa}}^{H_2}(\xi,z,\omega)}{\partial z} \right] \cdot J_2(\xi r) \right\} \cdot d\xi$$

$$(86)$$

$$\overline{\overline{u}}_\vartheta(r,z,\omega) = \frac{1}{2}\cdot\int_0^\infty \xi \cdot \left\{ -\left[\xi\cdot\overset{\smile}{\overline{\varphi}}{}^{H_1}(\xi,z,\omega) - \frac{\partial\overset{\smile}{\overline{\kappa}}{}^{H_2}(\xi,z,\omega)}{\partial z} - \frac{2}{\xi}\cdot\frac{\partial^2\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)}{\partial z^2} + \right. \right.$$

$$\left. + \xi\cdot\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)\right]\cdot J_0(\xi r) + \tag{87}$$

$$\left. + \left[-\xi\cdot\overset{\smile}{\overline{\varphi}}{}^{H_1}(\xi,z,\omega) + \frac{\partial\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)}{\partial z} + \xi\cdot\overset{\smile}{\overline{\psi}}_z{}^{H_1}(\xi,z,\omega)\right]\cdot J_2(\xi r)\right\}\cdot d\xi$$

$$\overline{\overline{u}}_z(r,z,\omega) = \int_0^\infty \xi\cdot\left[\frac{\partial\overset{\smile}{\overline{\overline{\varphi}}}{}^{H_1}(\xi,z,\omega)}{\partial z} - \frac{\partial\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)}{\partial z} - \xi\cdot\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)\right]\cdot J_1(\xi r)\cdot d\xi \tag{88}$$

where we can see, that the displacements in Eqs. (86)-(88) are written only with three of four potentials. Substituting the expressions for transformed displacements into expressions for stresses (27)-(29) yields:

$$\overline{\overline{\sigma}}_{rz}(r,z,\omega) = \frac{\mu}{2}\cdot\left\{ \int_0^\infty \xi\cdot\left\{ \left[2\cdot\xi\cdot\frac{\partial\overset{\smile}{\overline{\overline{\varphi}}}{}^{H_1}}{\partial z}(\xi,z,\omega) - \frac{2}{\xi}\cdot\frac{\partial^3\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)}{\partial z^3} + \right. \right. \right.$$

$$\left. - \frac{\partial^2\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)}{\partial z^2} - \xi^2\cdot\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)\right]\cdot J_0(\xi r) + $$

$$+ \left[-2\cdot\xi\cdot\frac{\partial\overset{\smile}{\overline{\overline{\varphi}}}{}^{H_1}(\xi,z,\omega)}{\partial z} + 2\cdot\xi\cdot\frac{\partial\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)}{\partial z} + \right. \tag{89}$$

$$\left. \left. \left. + \frac{\partial^2\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)}{\partial z^2} + \xi^2\cdot\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)\right]\cdot J_2(\xi r)\right\}\cdot d\xi\right\}$$

$$\overline{\overline{\sigma}}_{z\vartheta}(r,z,\omega) = \frac{\mu}{2}\cdot\left\{ \int_0^\infty \xi\cdot\left\{ \left[-2\cdot\xi\cdot\frac{\partial\overset{\smile}{\overline{\overline{\varphi}}}{}^{H_1}(\xi,z,\omega)}{\partial z} + \xi^2\cdot\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega) + \right. \right. \right.$$

$$\left. + \frac{2}{\xi}\cdot\frac{\partial^3\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)}{\partial z^3} + \frac{\partial^2\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)}{\partial z^2}\right]\cdot J_0(\xi r) + $$

$$\tag{90}$$

$$+ \left[-2\cdot\xi\cdot\frac{\partial}{\partial z}\overset{\smile}{\overline{\overline{\varphi}}}{}^{H_1}(\xi,z,\omega) + \frac{\partial^2\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)}{\partial z^2} + \right.$$

$$\left. \left. \left. + 2\cdot\xi\cdot\frac{\partial\overset{\smile}{\overline{\overline{\psi}}}_z{}^{H_1}(\xi,z,\omega)}{\partial z} + \xi^2\cdot\overset{\smile}{\overline{\overline{\kappa}}}{}^{H_2}(\xi,z,\omega)\right]\cdot J_2(\xi r)\right\}\cdot d\xi\right\}$$

$$\bar{\bar{\sigma}}_{zz}(r,z,\omega) = \mu \cdot \int_0^\infty \xi \cdot \left\{ \left[\left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) \cdot \left| \frac{\partial^2 \bar{\breve{\bar{\varphi}}}^{H_1}(\xi,z,\omega)}{\partial z^2} - \xi^2 \cdot \bar{\breve{\varphi}}^{H_1}(\xi,z,\omega) \right| \right. \right.$$

$$+ 2 \cdot \xi^2 \cdot \bar{\breve{\varphi}}^{H_1}(\xi,z,\omega) - 2 \cdot \xi \cdot \frac{\partial}{\partial z} \bar{\breve{\kappa}}^{H_2}(\xi,z,\omega) + \tag{91}$$

$$\left. - 2 \cdot \frac{\partial^2 \bar{\breve{\psi}}_z^{H_1}(\xi,z,\omega)}{\partial z^2} \right\} \cdot J_1(\xi r) \cdot d\xi$$

As an example of a homogeneous layer a homogeneous half space will be considered from here onwards. To satisfy the radiation conditions at $h \to \infty$ we introduce branch cuts in such a way that α and β in Eqs. (77)-(80) are positive for all values of ξ, which further implies that constant C_1, C_3, C_5, and C_7 must be equal to zero:

$$\bar{\breve{\varphi}}^{H_1} = C_2 \cdot e^{-\alpha \cdot z} \tag{92}$$

$$\bar{\breve{\chi}}^{H_0} = C_4 \cdot e^{-\beta \cdot z} \tag{93}$$

$$\bar{\breve{\kappa}}^{H_2} = C_6 \cdot e^{-\beta \cdot z} \tag{94}$$

$$\bar{\breve{\psi}}_z^{H_1} = C_8 \cdot e^{-\beta \cdot z} \tag{95}$$

Boundary conditions on the surface of the half-space can be stated as:

$$\bar{\bar{\sigma}}_{rz}(r,z,\omega)\Big|_{z=0} = -\frac{\bar{Q}(\omega) \cdot \delta(r)}{2 \cdot \pi \cdot r} \tag{96}$$

$$\bar{\bar{\sigma}}_{\vartheta z}(r,z,\omega)\Big|_{z=0} = \frac{\bar{Q}(\omega) \cdot \delta(r)}{2 \cdot \pi \cdot r} \tag{97}$$

$$\bar{\bar{\sigma}}_{zz}(r,z,\omega)\Big|_{z=0} = 0 \tag{98}$$

Introducing expressions (92)-(95) into the boundary conditions (96)-(98) and taking into account Eqs. (89)-(91) yields:

$$\bar{\bar{\sigma}}_{rz}(r,0,\omega) = \frac{\mu}{2} \cdot \left\{ \int_0^\infty \xi \cdot \left\{ -\left[-2 \cdot \xi \cdot \frac{\partial \bar{\breve{\bar{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z} + \frac{2}{\xi} \cdot \frac{\partial^3 \bar{\breve{\bar{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z^3} \right. \right. \right.$$

$$\left. + \frac{\partial^2 \bar{\breve{\bar{\kappa}}}^{H_2}(\xi,0,\omega)}{\partial z^2} + \xi^2 \cdot \bar{\breve{\kappa}}^{H_2}(\xi,0,\omega) \right] \cdot J_0(\xi r) +$$

$$+ \left[-2 \cdot \xi \cdot \frac{\partial \bar{\breve{\bar{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z} + 2 \cdot \xi \cdot \frac{\partial \bar{\breve{\bar{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z} \right. \tag{99}$$

$$\left. \left. \left. + \frac{\partial^2 \bar{\breve{\bar{\kappa}}}^{H_2}(\xi,0,\omega)}{\partial z^2} + \xi^2 \cdot \bar{\breve{\kappa}}^{H_2}(\xi,0,\omega) \right] \cdot J_2(\xi r) \right\} \cdot d\xi \right\} = -\frac{\bar{Q}(\omega) \cdot \delta(r)}{2 \cdot \pi \cdot r}$$

$$\bar{\bar{\sigma}}_{z\vartheta}(r,0,\omega) = \frac{\mu}{2} \cdot \left\{ \int_0^\infty \xi \cdot \left\{ \left[-2 \cdot \xi \cdot \frac{\partial \bar{\breve{\bar{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z} + \xi^2 \cdot \bar{\breve{\kappa}}^{H_2}(\xi,0,\omega) + \right. \right. \right.$$

$$\left. + \frac{2}{\xi} \cdot \frac{\partial^3 \bar{\breve{\bar{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z^3} + \frac{\partial^2 \bar{\breve{\kappa}}^{H_2}(\xi,0,\omega)}{\partial z^2} \right] \cdot J_0(\xi r) +$$

$$+ \left[-2 \cdot \xi \cdot \frac{\partial}{\partial z} \bar{\breve{\varphi}}^{H_1}(\xi,0,\omega) + \frac{\partial^2 \bar{\breve{\bar{\kappa}}}^{H_2}(\xi,0,\omega)}{\partial z^2} + \right.$$

$$\tag{100}$$

$$\left. \left. \left. + 2 \cdot \xi \cdot \frac{\partial \bar{\breve{\bar{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z} + \xi^2 \cdot \bar{\breve{\kappa}}^{H_2}(\xi,0,\omega) \right] \cdot J_2(\xi r) \right\} \cdot d\xi \right\} = \frac{\bar{Q}(\omega) \cdot \delta(r)}{2 \cdot \pi \cdot r}$$

$$\overline{\overline{\sigma}}_{zz}(r,0,\omega) = \mu \cdot \int_0^\infty \xi \cdot \left\{ \left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) \cdot \left[\frac{\partial^2 \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z^2} - \xi^2 \cdot \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega) \right] \right.$$

$$+ 2 \cdot \xi^2 \cdot \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega) - 2 \cdot \xi \cdot \frac{\partial}{\partial z} \breve{\overline{\overline{\kappa}}}^{H_2}(\xi,0,\omega) + \qquad (101)$$

$$\left. - 2 \cdot \frac{\partial^2 \breve{\overline{\overline{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z^2} \right\} \cdot J_1(\xi r) \cdot d\xi = 0$$

It ca be shown that Eqs. (99)-(101) can be written in the following form using just three of four potentials $\breve{\overline{\overline{\varphi}}}^{H_1}$, $\breve{\overline{\overline{\kappa}}}^{H_2}$ and $\breve{\overline{\overline{\psi}}}_z^{H_1}$:

$$-2 \cdot \xi \cdot \frac{\partial \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z} + \frac{2}{\xi} \cdot \frac{\partial^3 \breve{\overline{\overline{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z^3} + \frac{\partial^2 \breve{\overline{\overline{\kappa}}}^{H_2}(\xi,0,\omega)}{\partial z^2} + \xi^2 \cdot \breve{\overline{\overline{\kappa}}}^{H_2}(\xi,0,\omega) = \frac{\overline{Q}(\omega)}{\pi \cdot \mu} \quad (102)$$

$$-2 \cdot \xi \cdot \frac{\partial \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z} + 2 \cdot \xi \cdot \frac{\partial \breve{\overline{\overline{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z} + \frac{\partial^2 \breve{\overline{\overline{\kappa}}}^{H_2}(\xi,0,\omega)}{\partial z^2} + \xi^2 \cdot \breve{\overline{\overline{\kappa}}}^{H_2}(\xi,0,\omega) = 0 \quad (103)$$

$$\frac{1}{2} \cdot \left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) \cdot \left[\frac{\partial^2 \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega)}{\partial z^2} - \xi^2 \cdot \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega) \right] +$$

$$(104)$$

$$+ \xi^2 \cdot \breve{\overline{\overline{\varphi}}}^{H_1}(\xi,0,\omega) - \frac{\partial^2 \breve{\overline{\overline{\psi}}}_z^{H_1}(\xi,0,\omega)}{\partial z^2} - \xi \cdot \frac{\partial}{\partial z} \breve{\overline{\overline{\kappa}}}^{H_2}(\xi,0,\omega) = 0$$

The substitution of general solutions for wave potentials Eqs. (92)-(95) evaluated at z=0 into Eqs. (102)-(104) yields:

$$C_2 \cdot [2 \cdot \xi \cdot \alpha] + C_6 \cdot [\xi^2 + \beta^2] - C_8 \cdot \left[\frac{2}{\xi} \cdot \beta^3 \right] = \frac{\overline{Q}(\omega)}{\pi \cdot \mu} \quad (105)$$

$$C_2 \cdot [2 \cdot \xi \cdot \alpha] + C_6 \cdot [\xi^2 + \beta^2] - C_8 \cdot [2 \cdot \xi \cdot \beta] = 0 \quad (106)$$

$$C_2 \cdot \left[\frac{\alpha^2}{2} \cdot \left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) - \frac{\xi^2}{2} \cdot \left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) + \xi^2 \right] + C_6 \cdot [\beta \cdot \xi] - C_8 \cdot [\beta^2] = 0 \quad (107)$$

The above equations for constants C_2, C_6 and C_8 can be presented in a matrix form as:

$$\begin{bmatrix} 2 \cdot \xi \cdot \alpha & \xi^2 + \beta^2 & -\frac{2}{\xi} \cdot \beta^3 \\ 2 \cdot \xi \cdot \alpha & \xi^2 + \beta^2 & -2 \cdot \xi \cdot \beta \\ \frac{\alpha^2}{2} \cdot \left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) - \frac{\xi^2}{2} \cdot \left(\frac{\lambda + 2 \cdot \mu}{\mu} \right) + \xi^2 & \beta \cdot \xi & -\beta^2 \end{bmatrix} \cdot \begin{Bmatrix} C_2 \\ C_6 \\ C_8 \end{Bmatrix} = \begin{Bmatrix} \frac{\overline{Q}(\omega)}{\pi \cdot \mu} \\ 0 \\ 0 \end{Bmatrix} \quad (108)$$

Upon the solution of system (108) we can write the final form of solution for Eqs. (71)-(74):

$$\breve{\bar{\bar{\varphi}}}^{H_1} = -\frac{\bar{Q}(\omega)}{\pi \cdot \mu} \cdot \frac{\eta \cdot \sqrt{\eta^2 - 1}}{k_T^2 \cdot F(\eta)} \cdot e^{-\alpha \cdot z} \qquad (109)$$

$$\breve{\bar{\bar{\chi}}}^{H_0} = 0 \cdot e^{-\beta \cdot z} = 0 \qquad (110)$$

$$\breve{\bar{\bar{\kappa}}}^{H_2} = \frac{\bar{Q}(\omega)}{\pi \cdot \mu} \cdot \frac{E(\eta)}{k_T^2 \cdot F(\eta)} \cdot e^{-\beta \cdot z} \qquad (111)$$

$$\breve{\bar{\bar{\psi}}}_z^{H_1} = \frac{\bar{Q}(\omega)}{\pi \cdot \mu} \cdot \frac{\eta}{2 \cdot k_T^2 \cdot \sqrt{(\eta^2 - 1)}} \cdot e^{-\beta \cdot z} \quad , \quad (112)$$

where:

$$F(\eta) = \left(2 \cdot \eta^2 - 1\right)^2 - 4 \cdot \eta^2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2} \qquad (113)$$

$$E(\eta) = \eta^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right) - 2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right] \quad . \quad (114)$$

Displacements on the surface can be derived from Eqs. (86)-(88) in the following form:

$$\bar{\bar{u}}_r(r,0,\omega) = \bar{\bar{u}}_{r1}(r,0,\omega) + \bar{\bar{u}}_{r2}(r,0,\omega) \qquad (115)$$

$$\bar{\bar{u}}_\vartheta(r,0,\omega) = \bar{\bar{u}}_{\vartheta 1}(r,0,\omega) + \bar{\bar{u}}_{\vartheta 2}(r,0,\omega) \qquad (116)$$

$$\bar{\bar{u}}_z(r,0,\omega) = \frac{\bar{Q}(\omega)}{\pi \cdot \mu} \cdot \int_0^\infty \xi \cdot \left[\left(\frac{\eta \cdot \sqrt{\eta^2 - 1}}{k_T^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right)^2 - 4 \cdot \eta^2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right]} \cdot \alpha \right) + \right.$$

$$+ \left(\frac{\eta}{2 \cdot k_T^2 \cdot \sqrt{(\eta^2 - 1)}} \cdot \beta \right) + \left[\eta^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right) - 2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right] \middle/ \right. \qquad (117)$$

$$\left. \left. k_T^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right)^2 - 4 \cdot \eta^2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right] \right] \cdot \xi \right] \right\} J_1(\xi r) \cdot d\xi$$

where:

$$\bar{\bar{u}}_{r1}(r,0,\omega) = \frac{1}{2} \cdot \frac{\bar{Q}(\omega)}{\pi \cdot \mu} \cdot \int_0^\infty \xi \cdot \left[\left(-\frac{\eta \cdot \sqrt{\eta^2 - 1}}{k_T^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right)^2 - 4 \cdot \eta^2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right]} \cdot \xi \right) + \right.$$

$$+ \left(\frac{\eta}{2 \cdot k_T^2 \cdot \sqrt{(\eta^2 - 1)}} \cdot \xi \right) - \left(\frac{\eta}{2 \cdot k_T^2 \cdot \sqrt{(\eta^2 - 1)}} \cdot \frac{2}{\xi} \cdot \beta^2 \right) +$$

$$+ \left[\eta^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right) - 2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right] \middle/ \right. \qquad (118)$$

$$\left. \left. k_T^2 \cdot \left[\left(2 \cdot \eta^2 - 1\right)^2 - 4 \cdot \eta^2 \cdot \sqrt{\eta^2 - 1} \cdot \sqrt{\eta^2 - \gamma^2}\right] \right] \cdot \beta \right] \right\} J_0(\xi) \cdot d\xi$$

$$\bar{\bar{u}}_{r2}(r,0,\omega) = \frac{1}{2}\cdot\frac{\bar{Q}(\omega)}{\pi\cdot\mu}\cdot\int_0^\infty \xi\cdot\left\{\left(\frac{\eta\cdot\sqrt{\eta^2-1}}{k_T^2\cdot\left[\left(2\cdot\eta^2-1\right)^2-4\cdot\eta^2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]}\cdot\xi\right)+\right.$$

$$+\left(\frac{\eta}{2\cdot k_T^2\cdot\sqrt{\left(\eta^2-1\right)}}\cdot\xi\right)+\left[\left[\eta^2\cdot\left[\left(2\cdot\eta^2-1\right)-2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]\right/\right. \tag{119}$$

$$\left.\left.\left. k_T^2\cdot\left[\left(2\cdot\eta^2-1\right)^2-4\cdot\eta^2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]\right]\cdot\beta\right]\right\}J_2(\xi r)\cdot d\xi$$

$$\bar{\bar{u}}_{\vartheta1}(r,0,\omega) = \frac{1}{2}\cdot\frac{\bar{Q}(\omega)}{\pi\cdot\mu}\cdot\int_0^\infty \xi\cdot\left\{-\left(-\frac{\eta\cdot\sqrt{\eta^2-1}}{k_T^2\cdot\left[\left(2\cdot\eta^2-1\right)^2-4\cdot\eta^2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]}\cdot\xi\right)+\right.$$

$$+\left(\frac{\eta}{2\cdot k_T^2\cdot\sqrt{\left(\eta^2-1\right)}}\cdot\xi\right)-\left(\frac{\eta}{2\cdot k_T^2\cdot\sqrt{\left(\eta^2-1\right)}}\cdot\frac{2}{\xi}\cdot\beta^2\right)+$$

$$+\left[\left[\eta^2\cdot\left[\left(2\cdot\eta^2-1\right)-2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]\right/\right. \tag{120}$$

$$\left.\left.\left. k_T^2\cdot\left[\left(2\cdot\eta^2-1\right)^2-4\cdot\eta^2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]\right]\cdot\beta\right]\right\}J_0(\xi)\cdot d\xi$$

$$\bar{\bar{u}}_{\vartheta2}(r,0,\omega) = \frac{1}{2}\cdot\frac{\bar{Q}(\omega)}{\pi\cdot\mu}\cdot\int_0^\infty \xi\cdot\left\{\left(\frac{\eta\cdot\sqrt{\eta^2-1}}{k_T^2\cdot\left[\left(2\cdot\eta^2-1\right)^2-4\cdot\eta^2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]}\cdot\xi\right)+\right.$$

$$+\left(\frac{\eta}{2\cdot k_T^2\cdot\sqrt{\left(\eta^2-1\right)}}\cdot\xi\right)+\left[\left[\eta^2\cdot\left[\left(2\cdot\eta^2-1\right)-2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]\right/\right. \tag{121}$$

$$\left.\left.\left. k_T^2\cdot\left[\left(2\cdot\eta^2-1\right)^2-4\cdot\eta^2\cdot\sqrt{\eta^2-1}\cdot\sqrt{\eta^2-\gamma^2}\right]\right]\cdot\beta\right]\right\}J_2(\xi r)\cdot d\xi$$

Transformation of our results from cylindrical to Cartesian coordinates reconfirms Kobayashi's [4] findings.

Finally, evaluating the integrals in Eqs. (115)-(117) one can obtain the solution surface Green's function in the frequency domain. The strait forward evaluation of In evaluating the inverse Hankel transform integrals in the above mentioned equations can be very problematic. Due to their complex structure they cannot be evaluated analytically. Their numeric evaluation through a FFT like scheme, as proposed for a similar problem by Vostrukov

[2], is some how problematic. At a numerical evaluation one has to reduce the semi-infinite integration range to a finite one, therefore neglecting the contributions to the integral value coming at large values of integration variable and one is faced with the problems coming from integration through or nearby a singularity of integrand. The closer inspection of Eqs. (115)-(117) shows that the inversion integrals in these equations have the same basic mathematical structure as the semi-infinite integrals Kobayashi [4] succeeded to transform to the ones with the finite range of integration. For the evaluation of

integrals in Eqs. (115)-(117), despite the differences in detail, we adopt the concept, which has been forwarded by Kobayashi showing hereby that his approach to deal with homogeneous half-space can be generalized and extended to a homogeneous layer.

3 NUMERICAL EXAMPLE

A numerical example is given for homogeneous half-space as a special example of a layer, subjected to a general point load.

3.1 INTEGRAND

The surfaces shown in Figs. (3)-(10) represent integrands drawn for $\nu = 1/3$. In Fig. (3)-(6) the discontinuities at $\eta = \gamma$ can be clearly seen. Borders between meshed and unmeshed areas show curves for a=10. As shapes of integrands are very smooth, a numerical integration can be performed without difficulties.
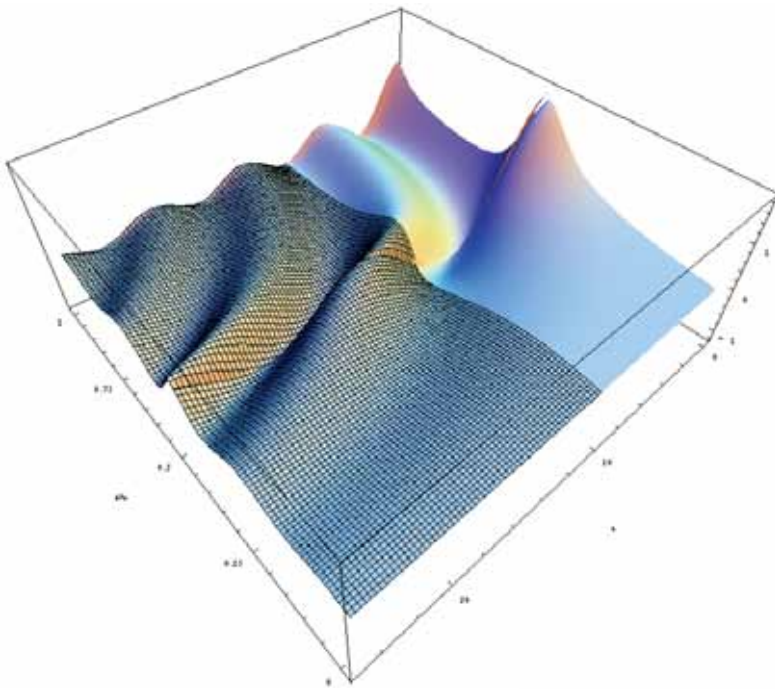
Figure 3. Real part of the integrand of the horizontal fundamental function of axi-symmetric componet
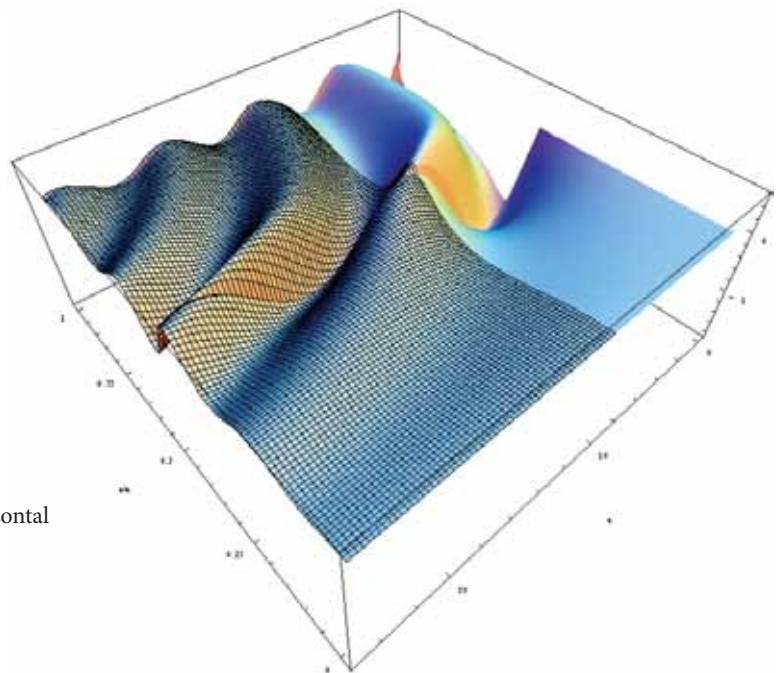
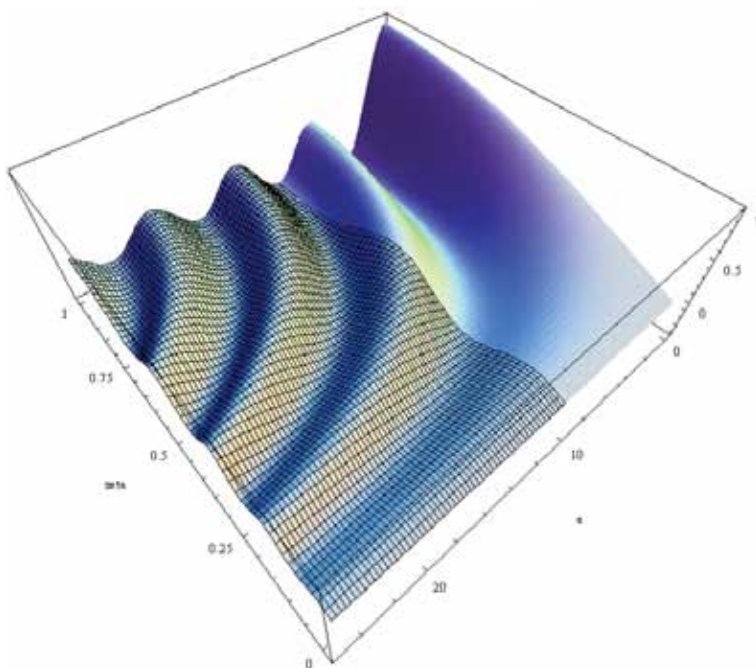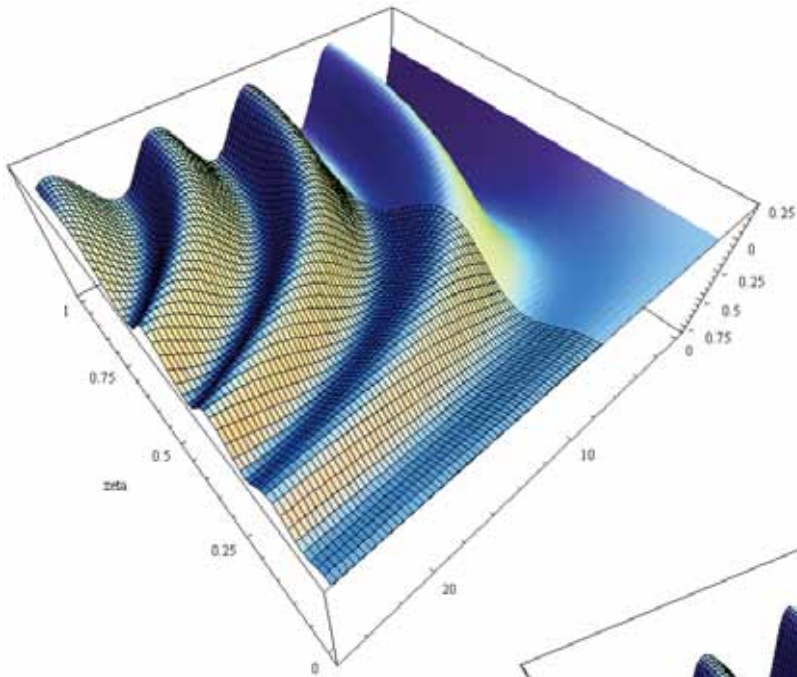Figure 4. Imaginary part of the integrand of the horizontal fundamental function of axi-symmetric component
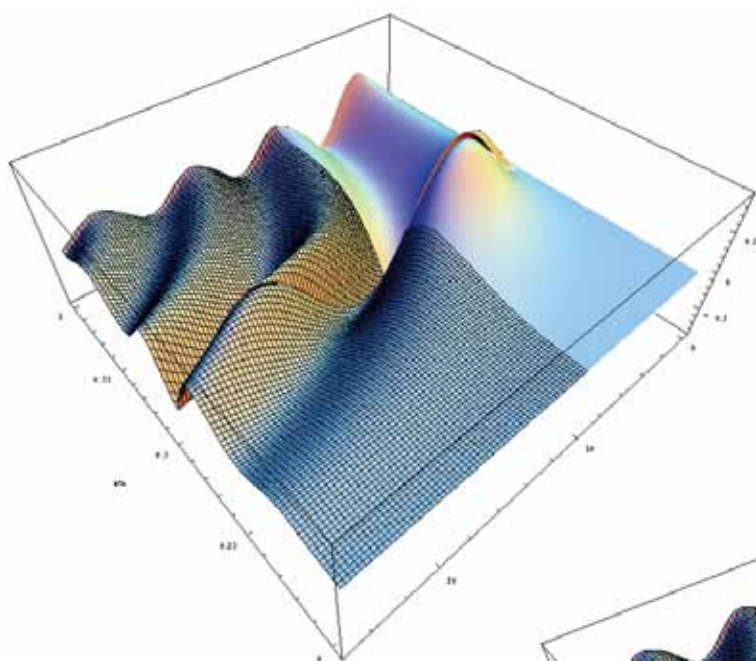
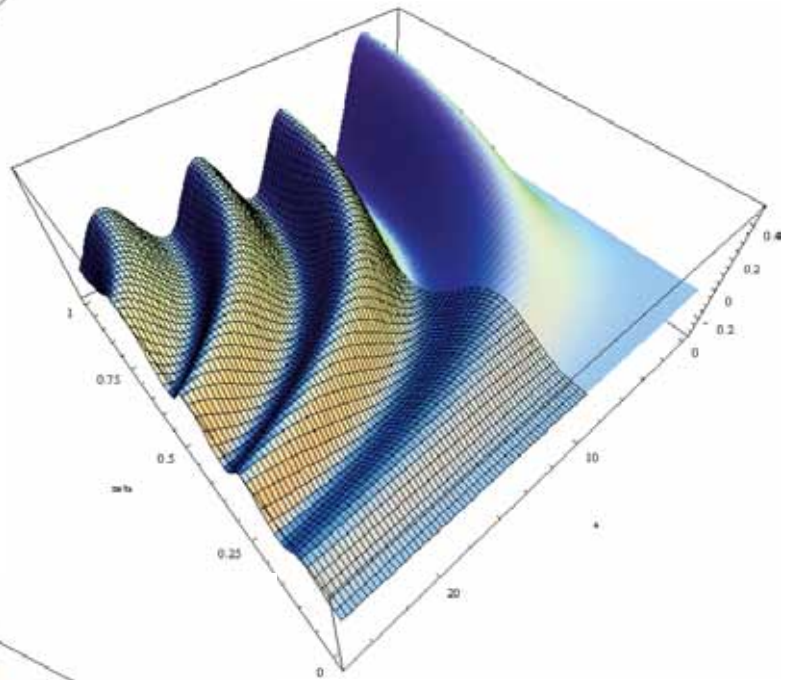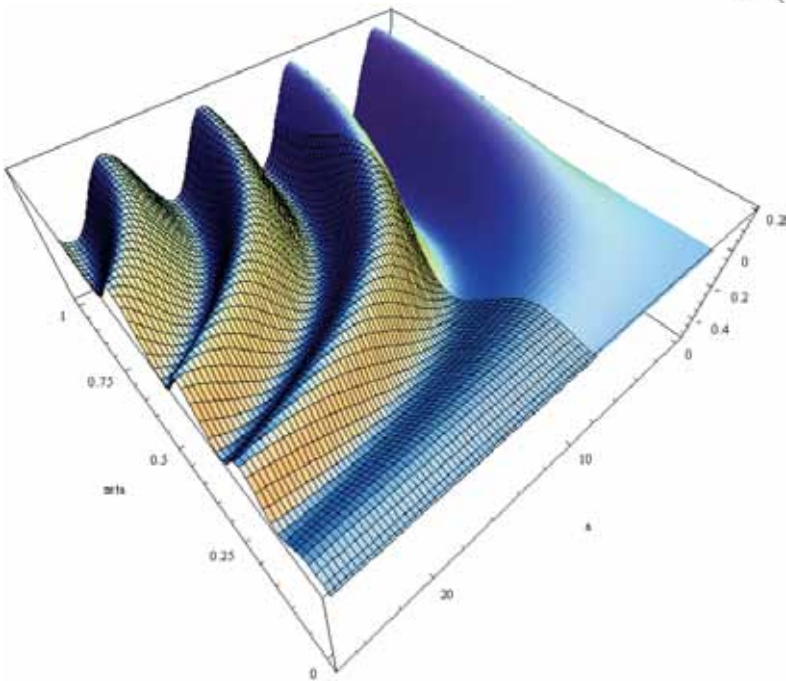**Figure 5**. Real part of the integrand of the horizontal fundamental function of anti-symmetric component
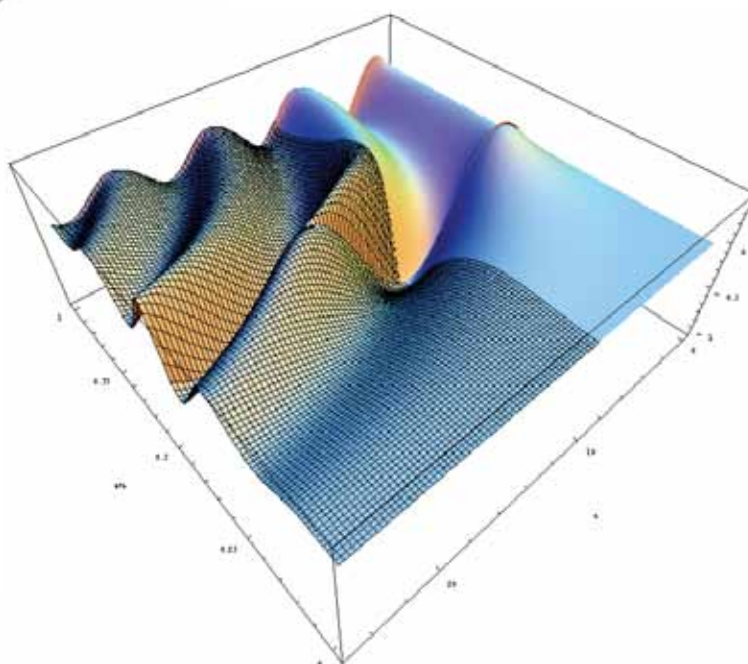
Figure 6. Imaginary part of the integrand of the horizontalfundamental function of anti-symmetric component

Figure 7. Real part of the integrand of the horizontal fundamental function of axi-symmetric component

Figure 8. Imaginary part of the integrand of the horizontal fundamental func·· of axi-symmetric compor

Figure 9. Real part of the integrand of the horizontal fundamental function of anti-symmetric component

Figure 10. Imaginary part of the integrand of the horizontal fundamental function of anti-symmetric componen

3.2 FUNDAMENTAL FUNCTION

The horizontal Green's function is represented as:

$$u_x(r,0,\omega) = \frac{\overline{Q}(\omega)}{4 \cdot \pi \cdot \mu} \cdot \frac{1}{r} \cdot \left\{ \ f_{H_0}^1 + i \cdot f_{H_0}^2 + \left[f_{H_2}^3 + i \cdot f_{H_2}^4 \right] \cdot \cos(2\vartheta) \ \right\}$$
(129)

where, $f_{H_0}^1, f_{H_0}^2$ are the horizontal fundamental function

of the axi-symmetric component, and $f_{H_2}^3, f_{H_2}^4$ are the horizontal fundamental function of the anti-symmetric component. These fundamental functions are defined by two variables, i.e. Poisson's ratio ν and non-dimensional frequency a. The blue curve in Figs. (11)-(12) shows the real part and the red one the imaginary part of the horizontal fundamental function of axi and anti-symmetric component.
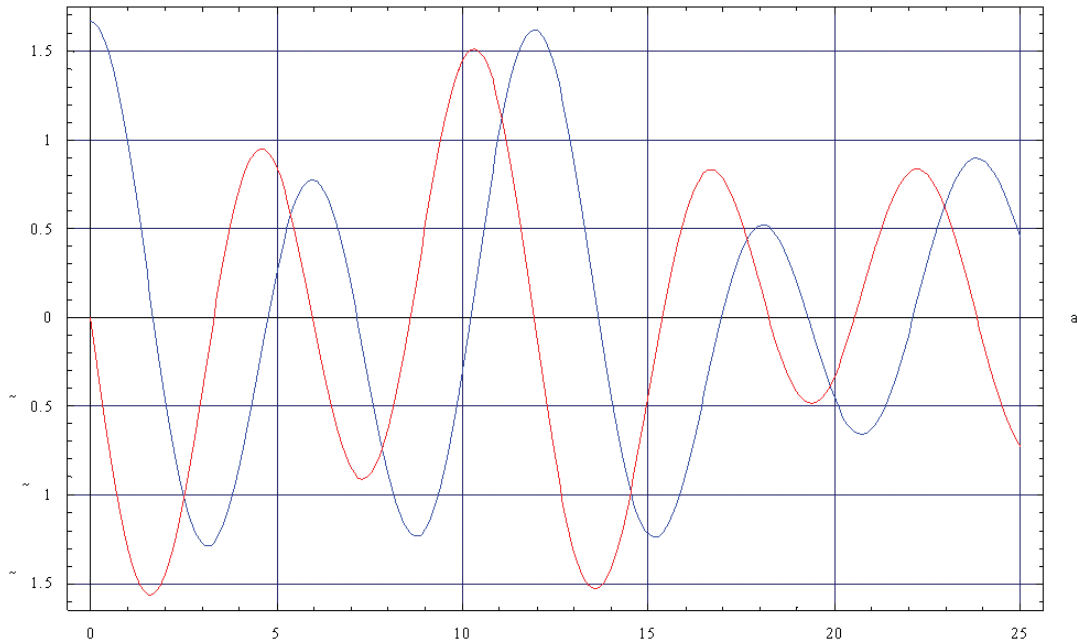

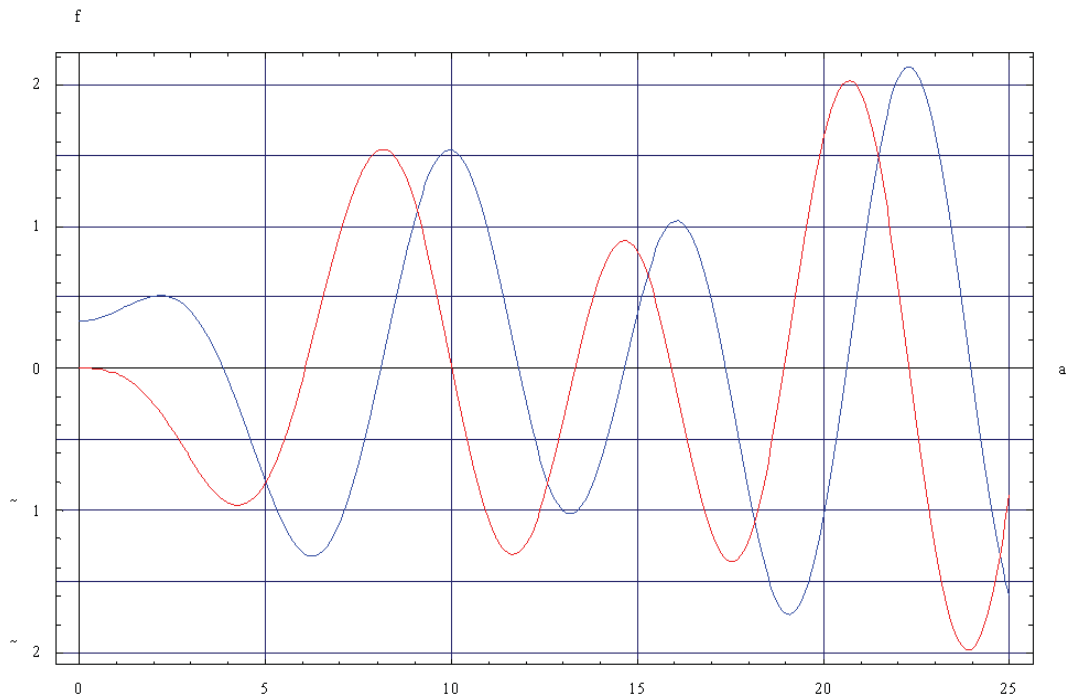**Figure 11**. Horizontal fundamental function of axi-symmetric component

Figure 12. Horizontal fundamental function of anti-symmetric component

The numerical solution for vertical Green's function is obtained through the similar process and is shown in Fig. (13). The blue curve in Fig. (13) shows the real part and the red one the imaginary part of the vertical Green's function.
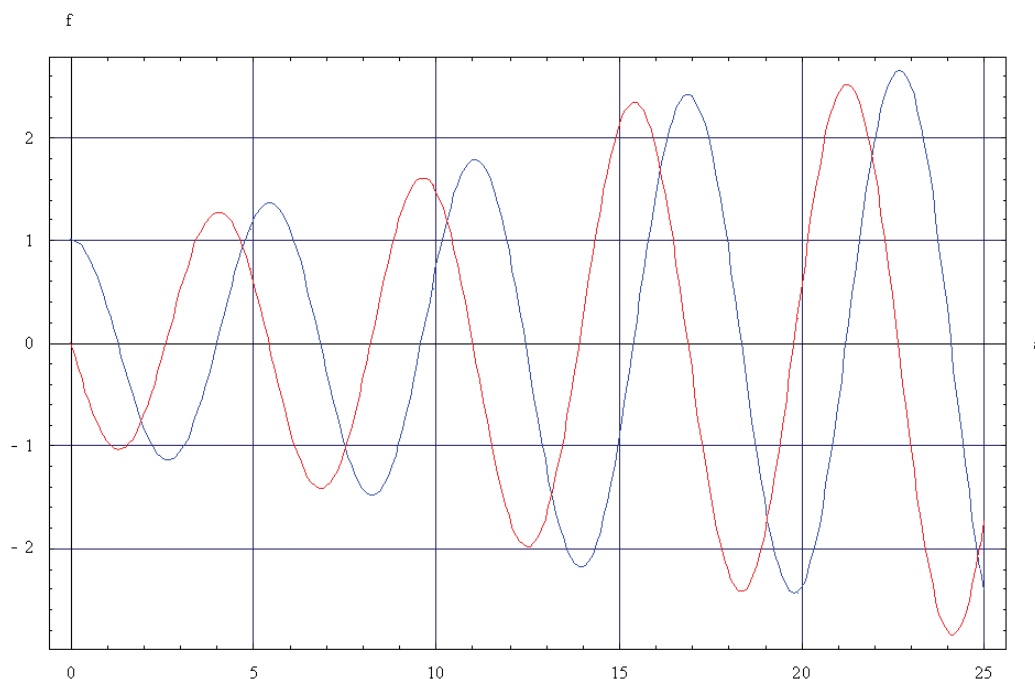
Figure 13. The vertical Green's function

4 CONCLUSIONS

The integral representations of the surface displacements for an elastic layer due to a harmonic point load are presented. The form of these expressions permits us to consider the homogeneous, elastic half-space as a layer of infinite depth. They are expressed by integrals of semi-infinite range. These integrals are usually evaluated by numerical integration. It has been demonstrated in this paper that they can be reduced to the integrals of finite range along the branch cuts of their integrands and the residues at the poles of the integrands. This method yields accurate expressions for singularities of the Green's functions and considerably better results for distances where contributions at large values of integration variable are significant. Numerical results for a homogeneous half-space are presented, which are in agreement with the findings of other researchers derived by a different approach.

Our results for a homogeneous layer, which are believed to be original, allows us to extend our method through superposition of layers to the case of a layered half-space. Our findings in dealing with this problem will be reported in the forthcoming papers.

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