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ON THE DOMINATION NUMBER AND THE 2-PACKING NUMBER OF FIBONACCI CUBES AND LUCAS CUBES

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Abstract

Let Γ_n and Λ_n be the n-dimensional Fibonacci cube and Lucas cube, respectively. The domination number γ of Fibonacci cubes and Lucas cubes is studied. In particular it is proved that $\gamma(\Lambda_n)$ is bounded below by $\left\lceil \frac{L_n - 2n}{n-3} \right\rceil$, where L_n is the n-th Lucas number. The 2-packing number ρ of these cubes is also studied. It is proved that $\rho(\Gamma_n)$ is bounded below by $2^{2^{\frac{\lfloor \lg n \rfloor}{2}} - 1}$ and the exact values of $\rho(\Gamma_n)$ and $\rho(\Lambda_n)$ are obtained for $n \leq 10$. It is also shown that $\mathrm{Aut}(\Gamma_n) \simeq \mathbb{Z}_2$.

Key words: Fibonacci cubes; Lucas cubes; domination number; 2-packing number; automorphism group; computer search

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1 Introduction

Fibonacci cubes form a class of graphs introduced because of their properties applicable for interconnection networks [5]. Lucas cubes [10] are subgraphs of Fibonacci cubes in which certain "non-symmetric" vertices are removed. In this way we get graphs with more symmetries, a fact that will be further justified in this paper. Both classes of cubes have been considered from various points of view, see [1, 2, 3, 8, 11, 13].

In this paper we study Fibonacci cubes and Lucas cubes from the viewpoint of domination and packing. While searching for (vertex) subsets of a graph (like dominating sets) it is useful to know symmetries of the graph, hence we first describe automorphism groups of these graphs in Section 2.

In Section 3 we study the domination number of Fibonacci cubes as initiated in [12], and also investigate that of Lucas cubes. We first give some connections between the domination number of Fibonacci cubes and Lucas cubes and construct dominating sets for 9-dimensional cubes. Then we obtain a lower bound on the domination number of Lucas cubes.

A graph invariant closely related to the domination number is the 2-packing number, which is the topic of Section 4. We first obtain an exponential (in terms of the dimension) lower bound on the 2-packing number of the Lucas cubes which is a natural lower bound for the Fibonacci cubes. Combining computer search with some arguments the exact values for the 2-packing number of both classes of cubes up to and including dimension 10 are obtained.

In the rest of this section we define the concepts needed in this paper. For a connected graph G, the distance $d_G(u,v)$ (or $d(u,v)$ for short) between vertices u and v is the usual shortest path distance.

Let $n \geq 1$. A *Fibonacci string* of length n is a binary string $b_1b_2 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words, Fibonacci strings are binary strings that contain no consecutive 1's. The *Fibonacci cube* Γ_n is the subgraph of Q_n induced by the Fibonacci strings of length n. For convenience we also set $\Gamma_0 = K_1$. A Fibonacci string $b_1b_2 \ldots b_n$ is a *Lucas string* if $b_1 \cdot b_n = 0$. The *Lucas cube* Λ_n is the subgraph of Q_n induced by the Lucas strings of length n. We also set $\Lambda_0 = K_1$.

It is well-known (cf. [5]) that $|V(\Gamma_n)| = F_{n+2}$, where F_n are the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Similarly, $|V(\Lambda_n)| = L_n$, see [10], where L_n are the *Lucas numbers*: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

For $0 \leq k \leq n$, let $\Gamma_{n,k}$ be the set of vertices of Γ_n that contain k 1's. Hence $\Gamma_{n,k}$ is the set of vertices of Γ_n at distance k from 0^n. $\Lambda_{n,k}$ is defined analogously. In particular, $\Gamma_{n,0} = \Lambda_{n,0} = \{0^n\}$ and $\Gamma_{n,1} = \Lambda_{n,1} = \{10^{n-1}, 010^{n-2}, \ldots, 0^{n-1}1\}$. If $uv \in E(\Gamma_n)$, where $u \in \Gamma_{n,k}$ and $v \in \Gamma_{n,k-1}$ $(k \geq 1)$, then we say that v is a *down-neighbor* of u and that u is an *up-neighbor* of v. The same terminology again applies to Lucas cubes.

For a binary string $b = b_1b_2 \ldots b_n$, let $\overline{b}$ be the binary complement of b and let $b^R = b_nb_{n-1} \ldots b_1$ be the reverse of b. For binary strings b and c of equal length, let $b + c$ denote their sum computed bitwise modulo 2. For $1 \leq i \leq n$, let e_i be the binary string of length n with 1 in the i-th position and 0 elsewhere. According to this notation, $\Gamma_{n,1} = \Lambda_{n,1} = \{e_1, e_2, \ldots, e_n\}$.

Let G be a graph. Then $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ is adjacent to some vertex from D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $X \subseteq V(G)$ is called a *2-packing* if $d(u,v) > 2$ for any different vertices u and v of X. The *2-packing number* $\rho(G)$ is the maximum cardinality of a 2-packing of G. It is well known that for any graph G, $\gamma(G) \geq \rho(G)$, cf. [6].

Finally, the automorphism group of a graph G is denoted by $\mathrm{Aut}(G)$. For instance, $\mathrm{Aut}(C_n) = D_{2n}$, where C_n is the n-cycle and D_{2n} is the *dihedral group* on n elements. Recall that D_{2n} can be represented as $\langle x, y \mid x^2 = 1, y^n = 1, (xy)^2 = 1 \rangle$.

2 Automorphism groups

In this section we determine the automorphism groups of Fibonacci cubes and Lucas cubes.

Let $n \geq 1$ and define the *reverse map* $r : \Gamma_n \to \Gamma_n$ with:

$$r(b_1 b_2 \ldots b_n) = b^R = b_n b_{n-1} \ldots b_1 \,. \tag{1}$$

It is easy to observe that r is an automorphism of Γ_n. We are going to prove that r is the only nontrivial automorphism of Γ_n. For this sake, the following lemma is useful.

Lemma 2.1 *Let $n \geq 3$ and $k \geq 2$. Then $u, v \in \Gamma_{n,k}$ have different sets of down-neighbors.*

Proof. Since $u, v \in \Gamma_{n,k}$, $d(u,v) \geq 2$. We distinguish two cases.

Suppose first $d(u,v) = 2$ and let u and v differ in positions i and j. Since $u, v \in \Gamma_{n,k}$, we may assume without loss of generality that $u_i = v_j = 1$ and $u_j = v_i = 0$. Moreover, u and v agree in all the other positions. Since $k \geq 2$, there exists an index $\ell \neq i, j$ such that $u_\ell = v_\ell = 1$. Then $u + e_\ell$ is a down-neighbor of u but not a down-neighbor of v.

Assume now $d(u,v) \geq 3$. Let i be an arbitrary index such that $u_i \neq v_i$. We may assume that $u_i = 1$. Then $u + e_i$ is a down-neighbor of u but not of v. $\square$

Theorem 2.2 *For any $n \geq 1$, $\mathrm{Aut}(\Gamma_n) \simeq \mathbb{Z}_2$.*

Proof. The assertion is clear for $n \leq 2$, hence assume in the rest that $n \geq 3$. Let $\alpha \in \mathrm{Aut}(\Gamma_n)$. Since 0^n is the only vertex of degree n, $\alpha(0^n) = 0^n$. Therefore, α maps $\Gamma_{n,1}$ onto $\Gamma_{n,1}$. Let $\Gamma'_{n,1} = \{10^{n-1}, 0^{n-1}1\}$ and $\Gamma''_{n,1} = \Gamma_{n,1} \setminus \Gamma'_{n,1}$. Since 10^{n-1} and $0^{n-1}1$ are the only vertices of degree $n-1$, α maps $\Gamma'_{n,1}$ and $\Gamma''_{n,1}$ onto themselves. We distinguish two cases.

Case 1: $\alpha(10^{n-1}) = 10^{n-1}$.
Then, because α maps $\Gamma'_{n,1}$ onto $\Gamma'_{n,1}$, we have $\alpha(0^{n-1}1) = 0^{n-1}1$. Among the vertices of $\Gamma''_{n,1}$, only 010^{n-2} has no common up-neighbor with 10^{n-1}. Therefore, $\alpha(010^{n-2}) = 010^{n-2}$. In turn, among the remaining vertices of $\Gamma''_{n,1}$, only 0010^{n-3} has no common up-neighbor with 010^{n-2}. Therefore $\alpha(0010^{n-3}) = 0010^{n-3}$. By proceeding with the same argument, α fixes $\Gamma''_{n,1}$ pointwise and hence fixes $\Gamma_{n,1}$ pointwise. Now apply Lemma 2.1 and induction on k to conclude that α fixes $\Gamma_{n,k}$ pointwise for all k. Therefore $\alpha = \mathrm{id}$ in this case.

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Case 2: $\alpha(10^{n-1}) = 0^{n-1}1$.
Now $\alpha(0^{n-1}1) = 10^{n-1}$. Among the vertices of $\Gamma''_{n,1}$, only 010^{n-2} has no common up-neighbor with 10^{n-1}. Thus $\alpha(010^{n-2}) = 0^{n-2}10$, which is the only element of $\Gamma''_{n,1}$ with no common up-neighbor together with $\alpha(10^{n-1}) = 0^{n-1}1$. By proceeding with the same argument, α reverses all the elements of $\Gamma''_{n,1}$, that is, $\alpha_{\Gamma''_{n,1}} = r_{\Gamma''_{n,1}}$ and consecutively $\alpha_{\Gamma_{n,1}} = r_{\Gamma_{n,1}}$. By Lemma 2.1 and induction on k, the same holds for any $\Gamma_{n,k}$, $k \geq 2$. Therefore $\alpha = r$ in this case. □

Let $n \geq 1$. An equivalent way to define Λ_n is that it is the subgraph of Q_n induced on all the binary strings of length n that have no two consecutive 1's in circular manner. This definition is more symmetric than the definition of the Fibonacci strings, so it is reasonable to expect that $\mathrm{Aut}(\Lambda_n)$ is richer than $\mathrm{Aut}(\Gamma_n)$. This is indeed the case. Define $\varphi : \Lambda_n \to \Lambda_n$ by

$$\varphi(b_1 b_2 \ldots b_n) = b_n b_1 \ldots b_{n-1} \, . \tag{2}$$

By the above remark it is clear that $\varphi \in \mathrm{Aut}(\Lambda_n)$. Zagaglia Salvi [14] proved that the automorphism groups of the Lucas semilattices are the dihedral groups. The arguments that determine the automorphism group of the Lucas cubes are in a way parallel to the arguments from [14], hence we next give just a sketch of them.

Note first that Lemma 2.1 with the same proof applies to Lucas cubes as well. Let $\alpha \in \mathrm{Aut}(\Lambda_n)$. Suppose that for some $a, b \in \{0, 1, \ldots, n-1\}$, $\alpha(10^{n-1}) = 0^a 10^{n-a-1}$ and $\alpha(0^{n-1}1) = 0^b 10^{n-b-1}$, where computations are mod n. Then either $b = a-1$ or $b = a+1$ because $\alpha(10^{n-1})$ and $\alpha(0^{n-1}1)$ cannot have a common up-neighbor. When $b = a-1$ we get $\alpha = \varphi^a$ and in the other case $\alpha = \varphi^{a+1} \circ r$. We conclude that $\mathrm{Aut}(\Lambda_n)$ is generated by r and φ^a for $0 \leq a \leq n-1$, and hence:

Theorem 2.3 *For any $n \geq 3$, $\mathrm{Aut}(\Lambda_n) \simeq D_{2n}$.*

3 The domination number

In this section we consider the domination number of Fibonaci and Lucas cubes. We first interrelate their domination numbers. Then we discuss exact domination numbers for small dimensions. The section is conluded by establishing a general lower bound on the domination number of Lucas cubes.

Proposition 3.1 *Let $n \geq 4$, then*

(i) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3})$,
(ii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_n) \leq \gamma(\Lambda_n) + \gamma(\Gamma_{n-4})$.

Proof. (i) $V(\Lambda_n)$ can be partitioned into vertices that start with 0 and vertices that start with 1. The latter vertices are of the form $10\ldots0$ and hance can be dominated by $\gamma(\Gamma_{n-3})$ vertices while the former vertices can be dominated by $\gamma(\Gamma_{n-1})$ vertices.

(ii) Let D be a minimum dominating set of Γ_n and set

$$D' = \{u \mid u \text{ is a Lucas string from } D\} \cup \{0b_2 \ldots b_{n-1}0 \mid 1b_2 \ldots b_{n-1}1 \in D\} \, .$$

A vertex $1b_2 \ldots b_{n-1}1$ dominates two Lucas vertices, namely $0b_2 \ldots b_{n-1}1$ and $1b_2 \ldots b_{n-1}0$. Since these two vertices are dominated by $0b_2 \ldots b_{n-1}0$, we infer that D' is a dominating set of Λ_n. It follows that $\gamma(\Lambda_n) \leq \gamma(\Gamma_n)$.

A dominating set of Λ_n dominates all vertices of Γ_n but the vertices of the form $10b_3 \ldots b_{n-2}01$. These vertices can be dominated by $\gamma(\Gamma_{n-4})$ vertices. □

It can be easily checked that Proposition 3.1 (i) holds for any $n \geq 2$, and that the first inequality of Proposition 3.1 (ii) holds for any $n \geq 0$.

Pike and Zou [12] obtained exact values of $\gamma(\Gamma_n)$ for $n \leq 8$, see Table 2. By computer search they found 509 minimum dominating sets of Γ_8. Following their approach we have computed the domination numbers of Λ_n, $n \leq 8$, see Table 2 again.

Hence the smallest Fibonacci cube and Lucas cube for which the domination numbers are not known are Γ_9 and Λ_9. Since $\gamma(\Gamma_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2})$, it follows that $\gamma(\Gamma_9) \leq 20$, cf. [12, Lemma 3.1]. Since for an exhaustive search too much computer time would be needed, we have used a local search procedure in order to find a smaller dominating set: to get a new dominating set we have replaced one or more vertices with another vertex. In this way we were able to construct a dominating set of Γ_9 of size 17 given on the left-hand side of Table 1. Similarly we have found a dominating set of Λ_9 of order 16 given on the right-hand side of Table 1. Hence:

Proposition 3.2 *$\gamma(\Gamma_9) \leq 17$ and $\gamma(\Lambda_9) \leq 16$.*

We conjecture that $\gamma(\Gamma_9) = 17$ and $\gamma(\Lambda_9) = 16$ hold.

010000000	000000000
100100000	000010000
010100000	000000100
001000100	000100100
000010010	000100010
000001010	000010010
000001001	101000010
101001000	100101000
101000010	010100001
100010100	010001010
100000101	001001001
001010001	101010100
000101001	101001010
000101010	010101001
000100101	010010101
101010001	001010101
010010101	

Table 1: A dominating set of Γ_9 and a dominating set of Λ_9

Pike and Zou [12] also proved that for any $n \geq 4$,

$$\gamma(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 3}{n-2} \right\rceil .$$

We next prove a parallel lower bound for the domination number of Lucas cubes. For this sake we first consider degrees of some specific vertices in Lucas cubes.

Recall that $\Lambda_{n,1}$ is the set of all the vertices with exactly one 1. In addition, set

$$\Lambda'_{n,2} = \{0^a 1010^{n-a-3} \mid 0 \leq a \leq n-1\},$$

where we again compute by modulo n. Hence $\Lambda'_{n,2}$ is the subset of $\Lambda_{n,2}$ consisting of the Lucas strings containing (in circular manner) 101 as a substring.

Lemma 3.3 *Let $n \geq 7$. Then for the Lucas cube Λ_n, the followings hold.*

(i) The vertex 0^n is the only vertex of the maximum degree n.

(ii) The vertices of $\Lambda_{n,1}$ have degree $n-2$.

(iii) Among the vertices with at least two 1's, only the vertices of $\Lambda'_{n,2}$ have degree $n-3$ and all the other vertices have degree at most $n-4$.

Proof. (i) and (ii) are clear.

(iii) Let $u \in \Lambda_{n,k}$ for some $k \geq 2$. Then u has k down-neighbors. The up-neighbors of u are obtained by switching a bit 0 into 1. Let $i_1 < i_2 < \cdots < i_k$ be the positions in which u contains 1. Throughout the proof, the indices of i's will be considered by modulo k and i_j will be considered by modulo n. As no consecutive bits of 1's are allowed, $i_{j+1} - i_j \geq 2$ for all $1 \leq j \leq k$. Let $I_j = \{i_j - 1, i_j + 1\}$ be the set of the positions which are adjacent to i_j for each $1 \leq j \leq k$ and let $I = \bigcup_{1 \leq j \leq k} I_j$. Then any bit which is not in I can be switched to 1 and hence the number of up-neighbors of u is $n - k - |I|$. Therefore, $\deg(u) = n - |I|$. Note that $I_j \cap I_{j'} = \emptyset$ if $|j - j'| \geq 2$, therefore by pigeon-hole principle, $|I| \geq k$. The equality holds if and only if $I_j \bigcap I_{j+1} \neq \emptyset$ for all $1 \leq j \leq k$, which occurs if and only if $i_{j+1} = i_j + 2$ for all $1 \leq j \leq k$, which is in turn true if and only if n is even and $k = \frac{n}{2}$. But in this case, $\deg(u) = \frac{n}{2} \leq n - 4$ as $n \geq 8$. In the other cases, $|I| \geq k+1$ and hence $\deg(u) \leq n - k - 1$. If $k \geq 3$, then $\deg(u) \leq n-4$. Assume $k = 2$. Then $\deg(u) \leq n - 3$, where the equality holds exactly when $|I| = 3$ and $I_1 \bigcap I_2 \neq \emptyset$ which means that $u \in \Lambda'_{n,2}$. □

Lemma 3.4 *Any l vertices from $\Lambda'_{n,2}$ have at least l down-neighbors, that is, at least l neighbors in $\Lambda_{n,1}$.*

Proof. For $1 \leq i \leq l$, let A_i be the set of down-neighbors of $v_i \in \Lambda'_{n,2}$. Then $|A_i| = 2$ for each i. Considering bits by modulo n, each vertex $0^a 10^{n-a-1}$ in $\Lambda_{n,1}$ can be a down-neighbor of at most two vertices $0^a 1010^{n-a-3}$ and $0^{a-2}1010^{n-a-1}$, and hence at most two of $v_1, \ldots, v_l$. By pigeon-hole principle, the assertion is true. □

To establish the announced lower bound, we will apply the natural concept of over-domination, just as it is done in [12]. It is defined as follows. Let D be a dominating set of a graph G. Then the *over-domination* of G with respect to D is:

$$OD_G(D) = \sum_{v \in D} (\deg_G(v) + 1) - |V(G)| \,. \tag{3}$$

Note that $OD_G(D) = 0$ if and only if D is a perfect dominating set [9, 4], that is, a dominating set such that each vertex is dominated exactly once.

Theorem 3.5 *For any $n \geq 7$, $\gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n-3} \right\rceil$.*

Proof. Let D be a minimum dominating set of Λ_n. Set $D_1 = D \cap \Lambda_{n,1}$ and $D_2 = D \cap \Lambda'_{n,2}$, and let $k = |D \cap \Lambda_{n,1}|$ and $l = |D \cap \Lambda'_{n,2}|$. Then clearly $0 \leq k, l \leq n$. Note that the over-domination of G with respect to D can be rewritten as

$$OD(G) = \sum_{u \in V(\Lambda_n)} \left(|\{v \in D \mid d(u,v) \leq 1\}| - 1\right) \,. \tag{4}$$

For a vertex u of Λ_n, set $t(u) = |\{v \in D \mid d(u,v) \leq 1\}| - 1$. As D is a dominating set, $t(u) \geq 0$ for all $u \in V(\Lambda_n)$. We now distinguish two cases.

Case 1: $0^n \in D$.
Combining Lemma 3.3 with Equation (3) we get

$$\begin{aligned} OD(D) &\leq (n+1) + k(n-1) + l(n-2) + (\gamma(\Lambda_n) - k - l - 1)(n-3) - L_n \\ &= \gamma(\Lambda_n)(n-3) + 2k + l + 4 - L_n \,. \end{aligned}$$

Also as $t(u) \geq 0$ for all $u \in V$, Equation (4) implies

$$OD(D) \geq t(0^n) + \sum_{v \in D_1} t(v) \geq 2k \,.$$

Therefore $\gamma(\Lambda_n) \geq \left\lceil \frac{L_n - l - 4}{n-3} \right\rceil \geq \left\lceil \frac{L_n - n - 4}{n-3} \right\rceil$.

Case 2: $0^n \notin D$.
Again, combining Lemma 3.3 with Equation (3) we infer

$$\begin{aligned} OD(D) &\leq k(n-1) + l(n-2) + (\gamma(\Lambda_n) - k - l)(n-3) - L_n \\ &= \gamma(\Lambda_n)(n-3) + 2k + l - L_n \,. \end{aligned}$$

Let A be the set of down-neighbors of D_2. Then for $u \in D_1 \cap A$, $t(u) \geq 1$. By Lemma 3.4, $|A| \geq l$ and hence $|D_1 \bigcap A| \geq k + l - n$. Therefore by Equation (4),

$$OD(D) \geq \sum_{v \in D_1 \bigcap A} t(v) \geq k + l - n \,.$$

Thus $\gamma(\Lambda_n) \geq \left\lceil \frac{L_n - k - n}{n-3} \right\rceil \geq \left\lceil \frac{L_n - 2n}{n-3} \right\rceil$.

By Case 1 and Case 2, $\gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n-3} \right\rceil$. □

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4 The 2-packing number

We now turn to the 2-packing number and first prove the following asymptotical lower bound.

Theorem 4.1 *For any $n \geq 8$, $\rho(\Gamma_n) \geq \rho(\Lambda_n) \geq 2^{2^{\frac{\lfloor \lg n \rfloor}{2}-1}}$.*

Proof. Since for any $n \geq 1$, Λ_n is an isometric subgraph of Γ_n, cf. [7], a 2-packing of Λ_n is also a 2-packing of Γ_n. Therefore $\rho(\Gamma_n) \geq \rho(\Lambda_n)$.

Let $r, s \geq 1$ and let X and Y be maximum 2-packings of Λ_r and Λ_s, respectively. Then $\{x0y \mid x \in X, y \in Y\}$ is a 2-packings of Λ_{r+s+1} of size $\rho(\Lambda_s)\rho(\Lambda_s)$. It follows that

$$\rho(\Lambda_{r+s+1}) \geq \rho(\Lambda_r)\rho(\Lambda_s)\,.$$

Set now $k = \lfloor \lg n \rfloor$. Then $\rho(\Lambda_{2^k}) \geq \rho(\Lambda_{2^{k-1}+1}) \geq \rho(\Lambda_{2^{k-2}})^2$. By repeatedly applying this argument we get

$$\rho(\Lambda_n) \geq \rho(\Lambda_{2^k}) \geq \rho(\Lambda_{2^{k-2l}})^{2^l}\,.$$

When k is even, take $l = \frac{k-2}{2}$ to get $\rho(\Lambda_n) \geq \rho(\Lambda_4)^{2^{\frac{k-2}{2}}} = 2^{2^{\frac{k-2}{2}}}$. When k is odd, take $l = \frac{k-3}{2}$ to get $\rho(\Lambda_n) \geq \rho(\Lambda_8)^{2^{\frac{k-3}{2}}} \geq 8^{2^{\frac{k-3}{2}}} = 2^{3\times 2^{\frac{k-3}{2}}} \geq 2^{2^{\frac{k-2}{2}}}$. □

Using computer we obtained the 2-packing numbers of Γ_n and Λ_n for $n \leq 10$ given in Table 2.

n	0	1	2	3	4	5	6	7	8	9	10
$\gamma(\Gamma_n)$	1	1	1	2	3	4	5	8	12	$\leq$17	-
$\rho(\Gamma_n)$	1	1	1	2	2	3	5	6	9	14	20
$\gamma(\Lambda_n)$	1	1	1	1	3	4	5	7	11	$\leq$16	-
$\rho(\Lambda_n)$	1	1	1	1	2	3	5	6	8	13	18

Table 2: Domination numbers and 2-packing numbers of small cubes

Table 2 needs several comments.

- The computer search found exactly ten 2-packings of size 20 in Γ_{10}. This already implies that $\rho(\Gamma_{10}) = 20$. Indeed, if Γ_{10} would contain a 2-packing of size 21, then it would contain twenty-one 2-packings of size 20.
- By exhaustive search with computer no 2-packing of size 19 in Λ_{10} was found, hence $\rho(\Lambda_{10}) = 18$.
- There is only one (up to isomorphisms of the graphs considered) maximum 2-packing of Λ_5, Λ_6, Λ_7, Λ_9, as well as Γ_6. There are two non-isomorphic 2-packings of maximum cardinality of Γ_9, they are presented in the first two columns of Table 3.

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Since the reverse map given in (1) is an automorphism of Fibonacci cubes, the reverse of a 2-packing is also a 2-packing. Interestingly, the maximum 2-packing of Γ_9 shown on the left-hand side of Table 3, denoted X, is also invariant under the reverse map. That is, $r(X) = X$.

Similarly, the shifts φ^i, where φ is given in (2) and $i \in \{1, \dots, n-1\}$, are automorphisms of Lucas cubes, hence they map 2-packings into 2-packings. Now consider the 2-packing of Λ_9 shown on the right-hand side of Table 3, denote it Y. Then it can be checked that $\varphi^3(Y) = Y$. As a consequence, $\varphi^6(Y) = Y$.

000 001 010	000 001 000	100 100 100
010 100 000	000 100 100	000 010 001
000 100 101	001 000 010	000 101 001
101 001 000	001 010 001	001 000 010
001 000 001	010 000 101	001 000 101
100 000 100	010 010 000	010 001 000
010 001 001	010 100 010	010 010 100
100 100 010	010 101 001	010 100 010
010 010 101	100 010 010	010 100 101
101 010 010	100 010 101	100 010 010
001 010 100	100 100 001	100 101 010
010 010 010	100 101 010	101 001 000
100 010 001	101 000 100	101 010 100
100 101 001	101 001 001	

Table 3: Maximum 2-packings of Γ_9 and of Λ_9

5 Concluding remarks

Based on the data from Table 2 we ask whether some of the followings are true.

Problem 5.1 *Is it true that*

(i) $\gamma(\Gamma_n) - \rho(\Gamma_n) \geq \gamma(\Lambda_n) - \rho(\Lambda_n)$ *for* $n \geq 0$*?*
(ii) $\gamma(\Lambda_n) \geq \rho(\Gamma_n)$ *for* $n \geq 4$*?*
(iii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3}) - 1$ *for* $n \geq 6$*?*

Note that the last question, if it has an affirmative answer, reduces the bound of $\gamma(\Lambda_n)$ in Proposition 3.1 (i) by 1. Moreover, in that case one can also ask whether $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-4})$ holds for $n \geq 6$.

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