

Proceedings to the 20<sup>th</sup> Workshop  
**What Comes Beyond the  
Standard Models**

Bled, July 9–17, 2017

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## Workshops organized at Bled

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## Preface

The series of annual workshops on "What Comes Beyond the Standard Models?" started in 1998 with the idea of Norma and Holger for organizing a real workshop, in which participants would spend most of the time in discussions, confronting different approaches and ideas. Workshops take place in the picturesque town of Bled by the lake of the same name, surrounded by beautiful mountains and offering pleasant walks and mountaineering.

This year was the 20th anniversary workshop. We celebrated this by offering a talk to the general audience of Bled with the title "How far do we understand the Universe in this moment?", given by Holger Bech Frits Nielsen in the lecture hall of the Bled School of Management. The lecture hall was kindly offered by the founder of the school Danica Purg.

In our very open minded, friendly, cooperative, long, tough and demanding discussions several physicists and even some mathematicians have contributed. Most of topics presented and discussed in our Bled workshops concern the proposals how to explain physics beyond the so far accepted and experimentally confirmed both standard models — in elementary particle physics and cosmology — in order to understand the origin of assumptions of both standard models and be consequently able to make predictions for future experiments. Although most of participants are theoretical physicists, many of them with their own suggestions how to make the next step beyond the accepted models and theories, and several knowing running experiments in details, the participants from the experimental laboratories were very appreciated, helping a lot to understand what do measurements really tell and which kinds of predictions can best be tested.

The (long) presentations (with breaks and continuations over several days), followed by very detailed discussions, have been extremely useful, at least for the organizers. We hope and believe, however, that this is the case also for most of participants, including students. Many a time, namely, talks turned into very pedagogical presentations in order to clarify the assumptions and the detailed steps, analyzing the ideas, statements, proofs of statements and possible predictions, confronting participants' proposals with the proposals in the literature or with proposals of the other participants, so that all possible weak points of the proposals showed up very clearly. The ideas therefore seem to develop in these years considerably faster than they would without our workshops.

This year the gravitational waves were again confirmed, this time from two merging neutron stars — the predicted possible source of heavy elements in the universe — measured also with the for a few second delayed electromagnetic signal. Such events offer new opportunity to be explained by theories, proposed and discussed in our workshops, showing the way beyond the standard models.

This year particle physics experiments have not brought much new, although a lot of work and effort has been put in, but the news will hopefully come when further

analyses of the data gathered with 13 TeV on the LHC will be done. The analyses might show whether there are the new family to the observed three and the new scalar fields, which determine the higgs and the Yukawa couplings, as well as the heavy fifth family explaining the dark matter content, all these predicted by the spin-charge-family theory and discussed in this proceedings. Such analysis might provide a test also of the hypothesis that dark atoms, composed of new stable double charged particles, can explain the puzzling excess of slow positrons, annihilating in the center of Galaxy, as well as the excess of high energy cosmic positrons.

The new data might answer the question, whether laws of nature are elegant (as predicted by the spin-charge-family theory and also — up to the families — other Kaluza-Klein-like theories and the string theories) or “she is just using gauge groups when needed” (what many models assume, some of them with additional discrete symmetries, as in several in this proceedings).

Shall the study of Grassmann space in confrontation with Clifford space for the description of the internal degrees of freedom for fermions, discussed in this proceedings in the first and second quantization of fields, help to better understand the “elegance of the laws of nature” and consequently the laws of nature? Will the complex action including future and past, also studied in this proceedings, help? Both studies have for the working hypotheses that “all the mathematics is a part of nature”. Will the assumption that “nature started” with bosons (as commuting fields) only, fermionizing bosons to obtain anti commuting fermion fields, as discussed in this proceedings, help? Might the extension of the Dirac sea to bosons (which are their own antiparticles), also presented in this proceedings, help as well to understand better the elegance of nature?

Although the supersymmetry might not be confirmed in the low energy regime, yet the regularization by higher derivatives in  $N = 1$  supersymmetric gauge theories, in some cases to all the orders, might speak for the “elegance of the nature”.

The fact that the spin-charge-family theory offers the explanation for all the assumptions of the standard model, predicting the symmetry  $\widetilde{\text{SU}}(2) \times \widetilde{\text{SU}}(2) \times \text{U}(1)$  of mass matrices for four rather than three observed families, explaining also other phenomenas, like the dark matter existence and the matter/antimatter asymmetry (even “miraculous” cancellation of the triangle anomaly in the standard model seems natural in the spin-charge-family theory), it might very well be that there is the fourth family. New data on mixing matrices of quarks and leptons, when accurate enough, will help to determine in which interval can masses of the fourth family members be expected. There are several papers in this proceedings manifesting that the more work is put into the spin-charge-family theory the more explanations for the observed phenomena and the better theoretical grounds for this theory offers.

There are attempts in this proceedings to recognize the origin of families by guessing symmetries of the  $3 \times 3$  mass matrices (this would hardly work if the  $3 \times 3$  mass matrices are indeed the submatrices of the  $4 \times 4$  mass matrices). There are also attempts in this proceedings to understand the appearance of families by guessing new degrees of freedom at higher energies.



The idea of compositeness of quarks and leptons are again coming back in a new context, presented in this proceedings, opening again the question whether the compositeness exists at all — could such clusters be at all massless — and how far can one continue with compositeness.

As every year also this year there has been not enough time to mature the very discerning and innovative discussions, for which we have spent a lot of time, into the written contributions, although some of the ideas started in previous workshops and continued through several years. Since the time to prepare the proceedings is indeed very short, less than two months, authors did not have a time to polish their contributions carefully enough, but this is compensated by the fresh content of the contributions.

Questions and answers as well as lectures enabled by M.Yu. Khlopov via Virtual Institute of Astroparticle Physics ([viavca.in2p3.fr/site.html](http://viavca.in2p3.fr/site.html)) of APC have in ample discussions helped to resolve many dilemmas. Google Analytics, showing more than 226 thousand visits to this site from 152 countries, indicates world wide interest to the problems of physics beyond the Standard models, discussed at Bled Workshop.

The reader can find the records of all the talks delivered by cosmopia since Bled 2009 on [viavca.in2p3.fr/site.html](http://viavca.in2p3.fr/site.html) in Previous - Conferences. The three talks delivered by: Norma Mankoč Borštnik (Spin-charge-family theory explains all the assumptions of the standard model, offers explanation for the dark matter, for the matter/antimatter asymmetry, explains miraculous triangle anomaly cancellation,...making several predictions), Abdelhak Djouadi (A deeper probe of new physics scenarii at the LHC) and M. Yu. Khlopov and Yu. S. Smirnov (Search for double charged particles as direct test for Dark Atom Constituents), can be accessed directly at

[http://viavca.in2p3.fr/what\\_comes\\_beyond\\_the\\_standard\\_model\\_2017.html](http://viavca.in2p3.fr/what_comes_beyond_the_standard_model_2017.html)

Most of the talks can be found on the workshop homepage

<http://bsm.fmf.uni-lj.si/>.

Bled Workshops owe their success to participants who have at Bled in the heart of Slovene Julian Alps enabled friendly and active sharing of information and ideas, yet their success was boosted by videoconferences.

Let us conclude this preface by thanking cordially and warmly to all the participants, present personally or through the teleconferences at the Bled workshop, for their excellent presentations and in particular for really fruitful discussions and the good and friendly working atmosphere.

*Norma Mankoč Borštnik, Holger Bech Nielsen, Maxim Y. Khlopov,  
(the Organizing committee)*

*Norma Mankoč Borštnik, Holger Bech Nielsen, Dragan Lukman,  
(the Editors)*

*Ljubljana, December 2017*

# 1 Predgovor (Preface in Slovenian Language)

Serija delavnic „Kako preseči oba standardna modela, kozmološkega in elektrošibkega“ (“What Comes Beyond the Standard Models?”) se je začela leta 1998 z idejo Norme in Holgerja, da bi organizirali delavnice, v katerih bi udeleženci v izčrpnih diskusijah kritično soočili različne ideje in teorije. Delavnicapoteka na Bledu ob slikovitem jezeru, kjer prijetni sprehodi in pohodi na čudovite gore, ki kipijo nad mestom, ponujajo priložnosti in vzpodbudo za diskusije.

To leto smo imeli jubilejno 20. delavnico. To smo proslavili s predavanjem za splošno občinstvo na Bledu z naslovom “Kako dobro razumemo naše Vesolje v tem trenutku?”, ki ga je imel Holger Bech Frits Nielsen v predavalnici IEDC (Blejska šola za management). Predavalnico nam je prijazno odstopila ustanoviteljica te šole, gospa Danica Purg.

K našim zelo odprtim, prijateljskim, dolgim in zahtevnim diskusijam, polnim iskrivega sodelovanja, je prispevalo veliko fizikov in celo nekaj matematikov. Večina predlogov teorij in modelov, predstavljenih in diskutiranih na naših Blejskih delavnicah, išče odgovore na vprašanja, ki jih v fizikalni skupnosti sprejeta in s številnimi poskusi potrjena standardni model osnovnih fermionskih in bozonskih polj ter kozmološki standardni model puščata odprta. Čeprav je večina udeležencev teoretičnih fizikov, mnogi z lastnimi idejami kako narediti naslednji korak onkraj sprejetih modelov in teorij, in tudi taki, ki poznajo zelo dobro potek poskusov, so še posebej dobrodošli predstavniki eksperimentalnih laboratorijev, ki nam pomagajo v odprtih diskusijah razjasniti resnično sporočilo meritev in ugotoviti, kakšne napovedi so potrebne, da jih lahko s poskusi dovolj zanesljivo preverijo.

Organizatorji moramo priznati, da smo se na blejskih delavnicah v (dolgih) predstavitev (z odmori in nadaljevanji čez več dni), ki so jim sledile zelo podrobne diskusije, naučili veliko, morda več kot večina udeležencev. Upamo in verjamemo, da so veliko odnesli tudi študentje in večina udeležencev. Velikokrat so se predavanja spremenila v zelo pedagoške predstavitve, ki so pojasnile predpostavke in podrobne korake, soočile predstavljene predloge s predlogi v literaturi ali s predlogi ostalih udeležencev ter jasno pokazale, kje utegnejo tičati šibke točke predlogov. Zdi se, da so se ideje v teh letih razvijale bistveno hitreje, zahvaljujoč prav tem delavnicam.

To leto so ponovno zaznali gravitacijske valove, tokrat iz zlitja dveh nevtronskih zved — verjame se, da se pri takih pojavih tvori večina zelo težkih elementov, ki so prisotni v vesolju — kar je omogočilo spremljanje posledic zlitja tudi z elektromagnetnimi valovi. Takšni pojavi ponujajo nove možnosti za razlago s teorijami, ki jih predstavljamo in o katerih razpravljamo na naših delavnicah in kažejo pot onkraj standardnih modelov.

To leto poskusi niso prinesli veliko novega, četudi je bilo v eksperimente vloženega ogromno dela, idej in truda. Nove rezultate in z njimi nova spoznanja je pričakovati

še, ko bodo narejene podrobnejše analize podatkov, pridobljenih na posodobljenem trkalniku (the Large Hadron Collider) pri 13 TeV. Tedaj bomo morda izvedeli ali obstajajo nova družina in nova skalarna polja, ki dolov cajo Higgsove in Yukawine sklopitve, pa tudi težka peta družina, ki razlaga temno snov (kar napoveduje teorija spinov-nabojev-družin obravnavana v več prispevkih in diskusijah v tem zborniku). Take analize bi lahko omogočile preveritev hipoteze, da obstoj temnih atomov, ki jih sestavljajo novi nabiti delci z dvojnimi nabojem, lahko pojasni presežek počasnih pozitronov, ki se anihilirajo v centru Rimske Ceste in presežek kozmičnih pozitronov visokih energij.

Novi podatki bodo morda dali odgovor na vprašanje, ali so zakoni narave preprosti (kot napove teorija spinov-nabojev-družin kakor tudi — razen družin — ostale teorije Kaluza-Kleinovega tipa, pa tudi teorije strun), ali pa narava preprosto "uporabi umeritvene grupe, kadar jih potrebuje" (kar počne veliko modelov, nekateri z dodatnimi diskretnimi simetrijami, kot v tem zborniku).

Bo študij uporabe Grassmannovega prostora v namesto Cliffordovega prostora za opis vseh notranjih prostostnih stopenj fermionov ter prva in druga kvantizacija polj v vsakem od obeh prostorov, kar obravnavamo v tem zborniku, pripomogla k boljšemu razumevanju "elegance naravnih zakonov" ter posledično zakonov? Bo pripomoglo k ugotovitvi, kakši so zakoni narave, proučevanje enačb gibanja, ki sledijo iz kompleksne akcije, ki vključuje preteklost in prihodnost, kar je prav tako predstavljeno v tem zborniku? Oba pristopa privzameta kot delovno hipotezo, da je "vsa matematika del narave". Ali bo pomagala predpostavka, da je "narava začela" samo z bozoni (ki so komutirajoča polja), nato fermionizirala bozone, kar je dalo antikomutirajoča fermionska polja (prav tako predstavljeno v zborniku)? Lahko razširitev Diracovega morja na bozone (ki so sami sebi antidelci), tudi predstavljena v zborniku, pomaga bolje razumeti eleganco narave?

Čeprav supersimetrije pri nizkih energijah morda ne bo opazili, lahko regularizacijo supersimetričnih umeritvenih teorij za  $N = 1$ , v nekaterih primerih v vseh redih, razumemo kot argument za "eleganco narave".

Dejstvo, da teorija spinov-nabojev-družin ponuja razlago predpostavk standardnega modela, napove simetrijo  $SU(2) \times SU(2) \times U(1)$  masnih matrik za štiri družine, namesto opaženih treh družin, ter pojasni še druge pojave, kot je obstoj temne snovi in asimetrija snovi/antisnovi (celo "čudežno" odpravo trikotniške anomalije v standardnem modelu), je argument za možen obstoj četrte družine. Novi podatki o mešalnih matrikah kvarkov in leptonov bodo, če bodo dovolj natančni, pomagali določiti interval pričakovanih mas za člane četrte družine. V tem zborniku je nekaj prispevkov, ki kažejo, da z več vloženim delu ter napoveduje nove pojave.

V zborniku predstavimo tudi pristope, v katerih poskušajo pojasniti izvor družin z ugibanjem simetrij masnih matrik za tri družine (kar v primeru, da so te matrike v resnici podmatrike  $3 \times 3$  matrik  $4 \times 4$  ne bo dosti pomagalo). V zborniku so predstavljeni tudi poskusi, da bi razumeli pojav družin z ugibanjem novih prostostnih stopenj pri visokih energijah.

Poskusi, da bi lastnosti leptonov in kvarkov razložili kot gručo delcev, se znova pojavljajo, tokrat v zborniku v novem kontekstu, ki znova odpira vprašanje, ali so lahko takšne gručice sploh lahko (skoraj) brez mase in kako daleč lahko s podstruk-turo strukture smiselno nadaljujemo.

Kot vsako leto nam tudi letos ni uspelo predstaviti v zborniku kar nekaj zelo obetavnih diskusij, ki so tekle na delavnici in za katere smo porabili veliko časa. Premalo je bilo časa do zaključka redakcije, manj kot dva meseca, zato avtorji niso mogli povsem izpiliti prispevkov, vendar upamo, da to nadomesti svežina prispevkov.

Četudi so k uspehu „Blejskih delavnic“ največ prispevali udeleženci, ki so na Bledu omogočili prijateljsko in aktivno izmenjavo mnenj v osrčju slovenskih Julijcev, so k uspehu prispevale tudi videokonference, ki so povezale delavnice z laboratoriji po svetu. Vprašanja in odgovori ter tudi predavanja, ki jih je v zadnjih letih omogočil M.Yu. Khlopov preko Virtual Institute of Astroparticle Physics ([viavca.in2p3.fr/site.html](http://viavca.in2p3.fr/site.html), APC, Pariz), so v izčrpnih diskusijah pomagali razčistiti marsikatero dilemo. Storitev Google Analytics pokaže več kot 226 tisoč obiskov te spletne strani iz več kot 152 držav sveta, kar kaže na širok interes v svetu za probleme fizike onkraj standardnih modelov, ki jih obravnavamo na blejskih delavnicah.

Bralec najde zapise vseh predavanj, objavljenih preko „cosmovia“ od leta 2009, na [viavca.in2p3.fr/site.html](http://viavca.in2p3.fr/site.html) v povezavi Previous - Conferences. Troje letošnjih predavanj,

Norma Mankoč Borštnik (Spin-charge-family theory explains all the assumptions of the standard model, offers explanation for the dark matter, for the matter/antimatter asymmetry, explains miraculous triangle anomaly cancellation, ... making several predictions), Abdelhak Djouadi (A deeper probe of new physics scenarii at the LHC) in M. Yu. Khlopov ter Yu. S. Smirnov (Search for double charged particles as direct test for Dark Atom Constituents), je dostopnih na [http://viavca.in2p3.fr/what\\_comes\\_beyond\\_the\\_standard\\_model\\_2017.html](http://viavca.in2p3.fr/what_comes_beyond_the_standard_model_2017.html)

Večino predavanj najde bralec na spletni strani delavnice na <http://bsm.fmf.uni-lj.si/>.

Naj zaključimo ta predgovor s pristrčno in toplo zahvalo vsem udeležencem, prisotnim na Bledu osebno ali preko videokonferenc, za njihova predavanja in še posebno za zelo plodne diskusije in odlično vzdušje.

*Norma Mankoč Borštnik, Holger Bech Nielsen, Maxim Y. Khlopov,*  
(Organizacijski odbor)

*Norma Mankoč Borštnik, Holger Bech Nielsen, Dragan Lukman,*  
(uredniki)

*Ljubljana, grudna (decembra) 2017*

## **Talk Section**

All talk contributions are arranged alphabetically with respect to the authors' names.





# 1 Texture Zero Mass Matrices and Their Implications

G. Ahuja \*

Department of Physics, Panjab University, Chandigarh, India

**Abstract.** We have made an attempt to briefly address the issue of texture zero fermion mass matrices from the ‘bottom-up’ perspective. Essentials pertaining to texture zero mass matrices have been summarized and using the facility of Weak Basis transformations, the implications of the texture zero mass matrices so obtained have been examined for the quark as well as the lepton sector.

**Povzetek.** Avtorica obravnava masne matrike za kvarke in leptone, ki imajo ničelne elemente razporejene po določenih vzorcih. Povzame bistvene značilnosti takih masnih matrik, ki jih transformira v šibko bazo ter določi proste parametre iz eksperimentalnih podatkov.

**Keywords:** Texture zero mass matrices, Weak Basis transformations, Quark mass matrices, Lepton mass matrices

## 1.1 Introduction

Understanding fermion masses and mixings is of paramount importance in the field of High Energy Physics. Regarding the quark case, at present one has a fairly good idea of the masses as well as the mixing angles [1]. In particular, one finds that both the quark masses as well as the mixing angles exhibit a clear cut hierarchy. For the case of neutrinos, although, recently refinements of the reactor mixing angle  $s_{13}$  [2,3], the solar mixing angle  $s_{12}$  and the atmospheric mixing angle  $s_{23}$  have been carried out, however, regarding the neutrino masses, in the absence of their absolute measurements, one has their interpretation only in terms of the neutrino mass-squared differences [4].

In order to understand the underlying pattern of fermion masses and flavor mixings, experimental efforts in the form of continuous refinements of the fermion mixing data are being carried out regularly. Along with these attempts, large amounts of efforts at the phenomenological end are also being made. In the present context, we have followed the “bottom-up” approach which involves phenomenological formulation of mass matrices which may eventually provide clues for the efforts carried out through the “top-down” approach. In this context, an interesting idea being investigated in the quark as well as leptonic sector is that of the texture zero mass matrices [5]-[8]. In the present paper, after presenting a brief outline of the essentials pertaining to the texture zero mass matrices in

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Section 2, the details of the analyses corresponding to the quark and leptonic sectors have been presented in Sections 3.1 and 3.2 respectively. Finally, Section 4, summarizes our conclusions.

## 1.2 Essentials pertaining to texture zero mass matrices

Fermion masses, along with fermion mixings, provide a good opportunity to hunt for physics beyond the SM. In view of the relationship of fermion mixing phenomenon with that of the fermion mass matrices, understanding flavor physics essentially implies formulating fermion mass matrices. The lack of a viable approach from the top-down perspective brings up the need for formulating fermion mass matrices from a bottom-up approach. In this context, initially, incorporating the texture zero approach, several ansätze were suggested for quark mass matrices.

### 1.2.1 Quark mass matrices

In the Standard Model (SM), the fermion mass matrices, having their origin in the Higgs fermion couplings, are completely arbitrary, therefore, the number of free parameters available with a general mass matrix is larger than the physical observables. For example, if no restrictions are imposed, there are 36 real free parameters in the two  $3 \times 3$  general complex mass matrices,  $M_U$  and  $M_D$ , which in the quark sector need to describe 10 physical observables, i.e., 6 quark masses, 3 mixing angles and 1 CP violating phase. Similarly, in the leptonic sector, physical observables described by lepton mass matrices are 6 lepton masses, 3 mixing angles and 1 CP violating phase for Dirac neutrinos (2 additional phases in case neutrinos are Majorana particles). Therefore, to develop viable phenomenological fermion mass matrices, as a first step, one needs to constrain the number of free parameters associated with the mass matrices so as to obtain valuable clues for developing an understanding of fermion mixing phenomenology.

In the SM and its extensions in which righthanded quarks are singlets, the above mentioned task is accomplished by considering the fermion mass matrices to be Hermitian. This brings down the number of real free parameters from 36 to 18, which however, is still a large number compared to the number of observables. To this end, Weinberg implicitly and Fritzsch [9,10] explicitly proposed the idea of texture zero mass matrices which imparted considerable predictability to the fermion mass matrices. This approach involves assuming certain elements of the Hermitian quark mass matrices to be zero, e.g., the typical Fritzsch texture zero Hermitian quark mass matrices are given by

$$M_U = \begin{pmatrix} 0 & A_U & 0 \\ A_U^* & 0 & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & A_D & 0 \\ A_D^* & 0 & B_D \\ 0 & B_D^* & C_D \end{pmatrix}, \quad (1.1)$$

where  $M_U$  and  $M_D$  refer to the mass matrices in the up and down sector respectively. Such matrices were named as texture zero mass matrices with a particular matrix defined as texture 'n' zero if the sum of the number of diagonal zeros and



half the number of the symmetrically placed off diagonal zeros is 'n'. Each of the above matrix is texture three zero type, together these are known as texture six zero Fritzsch mass matrices. On lines of these ansatz, by considering lesser number of texture zeros, several possible Fritzsch like texture zero mass matrices can be formulated. Also, one can get non Fritzsch like mass matrices by shifting the position of  $C_i$  ( $i = U, D$ ) on the diagonal as well as by shifting the position of zeros among the non diagonal elements. One can thus obtain a very large number of possible texture zero mass matrices.

An analysis of these mass matrices involves firstly diagonalizing them using bi-unitary orthogonal transformations and then obtaining the fermion mixing matrix using the relationship between the mass matrices and the mixing matrices. The corresponding mixing matrix is compared with the experimentally available mixing matrix which then determines the viability of a given texture zero mass matrix. As an example, we present here essentials pertaining to the diagonalization of texture 4 zero mass matrices. A general Fritzsch-like texture 2 zero mass matrix can be expressed as

$$M_k = \begin{pmatrix} 0 & A_k & 0 \\ A_k^* & D_k & B_k \\ 0 & B_k^* & C_k \end{pmatrix}, \quad (1.2)$$

where  $k = l, \nu D$ , for neutrino case and  $k = U, D$ , for quark case. Considering both the matrices of either the up and the down sector for quarks or the charged lepton or neutrino sector for leptons to be the texture 2 zero type, one essentially obtains the case of texture 4 zero mass matrices. Texture 6 zero mass matrices can be obtained from the above mentioned matrices by taking both  $D_k$  to be zero in both sets of mass matrices. Texture 5 zero matrices can be obtained by taking  $D_k = 0$  in one of the two mass matrices.

To fix the notations and conventions, we detail the formalism connecting the mass matrix to the mixing matrix. The mass matrices, for Hermitian as well as symmetric case, can be exactly diagonalized. To facilitate diagonalization, the mass matrix  $M_k$  can be expressed as

$$M_k = Q_k M_k^r P_k, \quad (1.3)$$

or

$$M_k^r = Q_k^\dagger M_k P_k^\dagger, \quad (1.4)$$

where  $M_k^r$  is a real symmetric matrix with real eigenvalues and  $Q_k$  and  $P_k$  are diagonal phase matrices. For the Hermitian case  $Q_k^\dagger = P_k$ , whereas for the symmetric case under certain conditions  $Q_k = P_k$ . In general, the real matrix  $M_k^r$  is diagonalized by the orthogonal transformation  $O_k$ , e.g.,

$$M_k^{\text{diag}} = O_k^T M_k^r O_k, \quad (1.5)$$

which on using equation (4) can be written as

$$M_k^{\text{diag}} = O_k^T Q_k^\dagger M_k P_k^\dagger O_k. \quad (1.6)$$

Using the method, mentioned above, we reproduce the general diagonalizing transformation  $O_k$ , e.g.,

$$\begin{pmatrix} \pm O_k(11) \pm O_k(12) \pm O_k(13) \\ \pm O_k(21) \mp O_k(22) \pm O_k(23) \\ \mp O_k(31) \pm O_k(32) \pm O_k(33) \end{pmatrix}, \quad (1.7)$$

where

$$\begin{aligned} O_k(11) &= \sqrt{\frac{m_2 m_3 (m_3 - m_2 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 - m_1)(m_1 + m_2)}}, \\ O_k(12) &= \sqrt{\frac{m_1 m_3 (m_1 + m_3 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 + m_2)(m_1 + m_2)}}, \\ O_k(13) &= \sqrt{\frac{m_1 m_2 (m_2 - m_1 + D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 + m_2)(m_3 - m_1)}}, \\ O_k(21) &= \sqrt{\frac{m_1 (m_3 - m_2 - D_k)}{(m_3 - m_1)(m_1 + m_2)}}, \\ O_k(22) &= \sqrt{\frac{m_2 (m_3 + m_1 - D_k)}{(m_2 + m_3)(m_1 + m_2)}}, \\ O_k(23) &= \sqrt{\frac{m_3 (m_2 - m_1 + D_k)}{(m_3 + m_2)(m_1 + m_2)}}, \\ O_k(31) &= \sqrt{\frac{m_1 (m_2 - m_1 + D_k)(m_1 + m_3 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 - m_1)(m_1 + m_2)}}, \\ O_k(32) &= \sqrt{\frac{m_2 (m_2 - m_1 + D_k)(m_3 - m_2 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 + m_2)(m_1 + m_2)}}, \\ O_k(33) &= \sqrt{\frac{m_3 (m_3 - m_2 - D_k)(m_1 + m_3 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 - m_1)(m_3 + m_2)}}, \end{aligned} \quad (1.8)$$

$m_1, -m_2, m_3$  being the eigenvalues of  $M_k$ .

While carrying out the analysis of texture zero mass matrices, the viability of the formulated mass matrices is ensured by checking the compatibility of the mixing matrices so obtained from these with the low energy data. In order to obtain the mixing matrix, we note that in the SM, the quark mass terms for three generations of quarks can be expressed as

$$\bar{q}_{U_L} M_U q_{U_R} + \bar{q}_{D_L} M_D q_{D_R}, \quad (1.9)$$

where  $q_{u_{L(R)}}$  and  $q_{d_{L(R)}}$  are the left (right) handed quark fields for the up sector (u, c, t) and down sector (d, s, b) respectively.  $M_U$  and  $M_D$  are the mass matrices for the up and the down sector of quarks. In order to re-express above equation in terms of the physical quark fields, one can diagonalize the mass matrices by the following bi-unitary transformations

$$V_{u_L}^\dagger M_U V_{u_R} = M_U^{\text{diag}} \equiv \text{Diag}(m_u, m_c, m_t), \quad (1.10)$$

$$V_{d_L}^\dagger M_D V_{d_R} = M_D^{\text{diag}} \equiv \text{Diag}(m_d, m_s, m_b), \quad (1.11)$$

where  $M_{u,D}^{\text{diag}}$  are real and diagonal, while  $V_{u_L}$ ,  $V_{u_R}$  etc. denote the eigenvalues of the mass matrices, i.e., the physical quark masses. Using the above equations, one can rewrite equation (9) as

$$\bar{q}_{u_L} V_{u_L} M_U^{\text{diag}} V_{u_R}^\dagger q_{u_R} + \bar{q}_{d_L} V_{d_L} M_D^{\text{diag}} V_{d_R}^\dagger q_{d_R} \quad (1.12)$$

which can be re-expressed in terms of physical quark fields as

$$\bar{q}_{u_L}^{\text{phys}} M_U^{\text{diag}} q_{u_R}^{\text{phys}} + \bar{q}_{d_L}^{\text{phys}} M_D^{\text{diag}} q_{d_R}^{\text{phys}}, \quad (1.13)$$

where  $\bar{q}_{u_L}^{\text{phys}} = V_{u_L}^\dagger q_{u_L}$  and  $\bar{q}_{d_L}^{\text{phys}} = V_{d_L}^\dagger q_{d_L}$  and so on. The mismatch of diagonalizations of up and down quark mass matrices leads to the quark mixing matrix  $V_{\text{CKM}}$ , referred to as the Cabibbo-Kobayashi-Maskawa (CKM) matrix [11] given as

$$V_{\text{CKM}} = V_{u_L}^\dagger V_{u_R}. \quad (1.14)$$

Over the past few years, both in the quark as well as lepton sector, a large number of analyses [5]-[8] have been carried out which establish the texture zero approach as a viable one for explaining the fermion mixing data. However, as mentioned earlier, since the number of possible texture zero mass matrices is very large, one has to carry out an exhaustive analysis of all possible texture zero mass matrices. To account for this limitation, therefore, Branco *et al.* [12,13] and Fritzsch and Xing [14,15] have proposed the concept of ‘Weak Basis (WB) transformations’.

Within the SM and some of its extensions, one has the facility of making Weak Basis (WB) transformations  $W$  on the quark fields, e.g.,  $q_L \rightarrow W q_L$ ,  $q_R \rightarrow W q_R$ ,  $q'_L \rightarrow W q'_L$ ,  $q'_R \rightarrow W q'_R$ . These are unitary transformations which leave the gauge currents real and diagonal but transform the mass matrices as

$$M_U \rightarrow W^\dagger M_U W, \quad M_D \rightarrow W^\dagger M_D W. \quad (1.15)$$

Without loss of generality, this approach introduces zeros in the quark mass matrices leading to a reduction in the number of parameters defining the mass matrices. Following this, one can arrive at two kinds of structures of the mass matrices, e.g., Branco *et al.* [12,13] give

$$M_q = \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}, \quad M_{q'} = \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad q, q' = U, D, \quad (1.16)$$

whereas Fritzsche and Xing [14,15] give

$$M_q = \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}, \quad q = U, D. \quad (1.17)$$

The mass matrices so obtained can thereafter be considered texture zero mass matrices and same methodology can be used to analyze these. Interestingly, one now has an additional advantage that the large number of possible structures are not all independent. Several of these are related through WB transformations and therefore yield the same structure of the diagonalizing transformations leading to similar mixing matrices, making the number of matrices to be analyzed much less than before. However, there is a limitation too, i.e, this idea does not result in constraining the parameter space of the elements of the mass matrices. To overcome this, one can further impose a condition on the elements of the mass matrices by considering the following hierarchy for these [8]

$$(1, i) \lesssim (2, j) \lesssim (3, 3); \quad i = 1, 2, 3, \quad j = 2, 3. \quad (1.18)$$

### 1.2.2 Lepton mass matrices

Keeping in mind the quark lepton universality [16], similar to the case of texture zero quark mass matrices discussed in the previous section, it becomes desirable to carry out a corresponding analysis in the lepton sector also. In the case of leptons, several attempts have been made to formulate the phenomenological mass matrices considering charged leptons to be diagonal, usually referred to as the flavor basis case [17]. However, in the present work, we have considered the non flavor basis [18], wherein, texture is imposed on both the charged lepton mass matrix as well as on the neutrino mass matrix. The ‘smallness’ of the neutrino masses is best described in terms of ‘seesaw mechanism’ [19] given by

$$M_\nu = -M_{\nu D}^T M_R^{-1} M_{\nu D}, \quad (1.19)$$

with  $M_\nu$ ,  $M_{\nu D}$  and  $M_R$  corresponding to the light Majorana neutrino mass matrix, the Dirac neutrino mass matrix and the heavy right handed Majorana neutrino mass matrix respectively.

The methodology of analyzing the texture zero lepton mass matrices remains essentially the same as that for the case of quarks. One can impose texture on the charged lepton mass matrix  $M_l$  and on the Dirac neutrino mass matrix  $M_{\nu D}$ . Equation (1.19) can then be used to obtain the Majorana neutrino matrix  $M_\nu$  which along with the matrix  $M_l$  allows the construction of the Pontecorvo Maki Nakagawa Sakata (PMNS) matrix [20] for examining the viability of the mass matrices. Using these ideas, in the following we have briefly summarized the results of the analyses in the case of quarks [21] as well as leptons [22].

### 1.3 Results and discussion

#### 1.3.1 Texture zero quark mass matrices

We begin with the the most general Hermitian mass matrices, given by

$$M_q = \begin{pmatrix} E_q & A_q & F_q \\ A_q^* & D_q & B_q \\ F_q^* & B_q^* & C_q \end{pmatrix} \quad (q = U, D). \quad (1.20)$$

Invoking WB transformations, zeros can be introduced in these matrices using a unitary matrix  $W$ , leading to

$$M_U = \begin{pmatrix} E_U & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \quad M_D = \begin{pmatrix} E_D & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix}. \quad (1.21)$$

One may note that these matrices are, in fact, texture one zero each, together these are referred as texture two zero mass matrices.

To check the viability of these mass matrices, one needs to examine the compatibility of the CKM matrix reproduced through these with the recent quark mixing data. Results of a detailed analysis of these matrices, carried out in Ref. [21], reveal that using the following quark masses and the mass ratios at the  $M_Z$  scale as inputs [23]

$$\begin{aligned} m_u &= 1.38_{-0.41}^{+0.42} \text{ MeV}, & m_d &= 2.82 \pm 0.48 \text{ MeV}, & m_s &= 57_{-12}^{+18} \text{ MeV}, \\ m_c &= 0.638_{-0.084}^{+0.043} \text{ GeV}, & m_b &= 2.86_{-0.06}^{+0.16} \text{ GeV}, & m_t &= 172.1 \pm 1.2 \text{ GeV}, \\ m_u/m_d &= 0.553 \pm 0.043, & m_s/m_d &= 18.9 \pm 0.8 \end{aligned} \quad (1.22)$$

and imposing the latest values [1] of the three mixing angles as constraints for the construction of the CKM matrix, one arrives at

$$V_{\text{CKM}} = \begin{pmatrix} 0.9739 - 0.9745 & 0.2246 - 0.2259 & 0.00337 - 0.00365 \\ 0.2224 - 0.2259 & 0.9730 - 0.9990 & 0.0408 - 0.0422 \\ 0.0076 - 0.0101 & 0.0408 - 0.0422 & 0.9990 - 0.9999 \end{pmatrix}, \quad (1.23)$$

this being fully compatible with the one given by Particle Data Group (PDG) [1]. In order to examine whether these mass matrices can accommodate CP violation in the quark sector, in the present work we have made an attempt to reproduce the CP violating Jarlskog's rephasing invariant parameter  $J$ . One obtains a range of  $J = (2.494 - 3.365) \times 10^{-5}$ , this again being compatible with its latest experimental value  $(3.04_{-0.20}^{+0.21}) \times 10^{-5}$  [1].

#### 1.3.2 Texture zero lepton mass matrices

Similar to the quark case, using the facility of WB transformations, wherein it is possible to make a unitary transformation, one can reduce the general lepton mass matrices to

$$M_l = \begin{pmatrix} E_l & A_l & 0 \\ A_l^* & D_l & B_l \\ 0 & B_l^* & C_l \end{pmatrix}, \quad M_{\nu D} = \begin{pmatrix} E_{\nu D} & A_{\nu D} & 0 \\ A_{\nu D}^* & D_{\nu D} & B_{\nu D} \\ 0 & B_{\nu D}^* & C_{\nu D} \end{pmatrix}. \quad (1.24)$$

A detailed analysis of these mass matrices has been carried out in Ref. [22]. In the present work, for the normal and inverted ordering of neutrino masses, we have first examined the viability of these mass matrices and then we have investigated their implications for CP violation in the leptonic sector.

The latest situation regarding neutrinos masses and mixing angles at  $3\sigma$  C.L. is summarized as follows [24]

$$\Delta m_{21}^2 = (7.02 - 8.09) \times 10^{-5} \text{eV}^2; \quad \Delta m_{23}^2 = (2.325 - 2.599) \times 10^{-3} \text{eV}^2; \quad (1.25)$$

$$\sin^2 \theta_{12} = 0.270 - 0.344; \quad \sin^2 \theta_{23} = 0.385 - 0.644; \quad \sin^2 \theta_{13} = 0.0188 - 0.0251. \quad (1.26)$$

The  $3\sigma$  C.L. ranges of the PMNS matrix elements recently constructed by Garcia *et al.*[24] are as follows

$$U_{\text{PMNS}} = \begin{pmatrix} 0.801 - 0.845 & 0.514 - 0.580 & 0.137 - 0.158 \\ 0.225 - 0.517 & 0.441 - 0.699 & 0.164 - 0.793 \\ 0.246 - 0.529 & 0.464 - 0.713 & 0.590 - 0.776 \end{pmatrix}. \quad (1.27)$$

For the inverted and normal neutrino mass orderings, the mass matrices mentioned in equation (1.24) yield the following magnitudes of the corresponding PMNS matrix elements [22] respectively

$$U_{\text{PMNS}}^{\text{IO}} = \begin{pmatrix} 0.034 - 0.859 & 0.0867 - 0.593 & 0.135 - 0.996 \\ 0.250 - 0.971 & 0.068 - 0.812 & 0.043 - 0.808 \\ 0.103 - 0.621 & 0.395 - 0.822 & 0.088 - 0.810 \end{pmatrix}. \quad (1.28)$$

$$U_{\text{PMNS}}^{\text{NO}} = \begin{pmatrix} 0.444 - 0.993 & 0.123 - 0.837 & 0.004 - 0.288 \\ 0.061 - 0.816 & 0.410 - 0.941 & 0.047 - 0.872 \\ 0.012 - 0.848 & 0.049 - 0.779 & 0.460 - 0.992 \end{pmatrix}. \quad (1.29)$$

For both the mass orderings, one finds that the  $3\sigma$  C.L. ranges of the PMNS matrix elements given by Garcia *et al.* are inclusive in the ranges of the PMNS matrix elements found here, thereby ensuring the viability of texture two zero mass matrices considered here. Further, analogous to the case of quarks, we have made an attempt to find constraints for the CP violating Jarlskog's rephasing invariant parameter in the leptonic sector also. For the inverted mass ordering, one obtains a range of  $J$  from  $-0.05 - 0.05$ , whereas, for the normal mass ordering the, parameter  $J$  is obtained in the range  $-0.03 - 0.03$ . These observations, therefore, lead one to conclude that the texture two zero leptonic mass matrices are not only compatible with the recent leptonic mixing data but also provide interesting bounds for the Jarlskog's rephasing invariant parameter.

## 1.4 Summary and Conclusions

To summarize, in the present work, we have made an attempt to provide an overview of texture zero fermion mass matrices. For the case of both quarks and leptons, incorporating the texture zero approach as well as using the WB transformations, analyses of the "general" fermion mass matrices have been discussed.

After examining the viability of these mass matrices, we have obtained interesting bounds on the Jarlskog's rephasing invariant parameter in the quark and leptonic sector.

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### References

1. C. Patrignani *et al.*, Particle Data Group, Chin. Phys. **C 40**, 100001 (2016).
2. F. P. An *et al.*, DAYA-BAY Collaboration, Phys. Rev. Lett. **108**, 171803 (2012).
3. J. K. Ahn *et al.*, RENO Collaboration, Phys. Rev. Lett. **108**, 191802 (2012).
4. F. Capozzi *et al.*, Nucl. Phys. **B 908**, 218 (2016).
5. H. Fritzsch, Z. Z. Xing, Nucl. Phys. **B 556**, 49 (1999) and references therein.
6. Z. Z. Xing, H. Zhang, J. Phys. **G 30**, 129 (2004) and references therein.
7. N. G. Deshpande, M. Gupta, P. B. Pal, Phys. Rev. **D 45**, 953 (1992); P. S. Gill, M. Gupta, Pramana **45**, 333 (1995); M. Gupta, G. Ahuja, Int. J. Mod. Phys. **A 26**, 2973 (2011).
8. M. Gupta, G. Ahuja, Int. J. Mod. Phys. **A 27**, 1230033 (2012) and references therein.
9. H. Fritzsch, Phys. Lett. **B 70**, 436 (1977).
10. H. Fritzsch, Phys. Lett. **B 73**, 317 (1978).
11. N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi, T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
12. G. C. Branco, D. Emmanuel-Costa, R. G. Felipe, Phys. Lett. **B 477**, 147 (2000).
13. G. C. Branco, D. Emmanuel-Costa, R. G. Felipe, H. Serodio, Phys. Lett. **B 670**, 340 (2009).
14. H. Fritzsch, Z. Z. Xing, Phys. Lett. **B 413**, 396 (1997) and references therein.
15. H. Fritzsch, Z. Z. Xing, Nucl. Phys. **B 556**, 49 (1999) and references therein.
16. M. A. Schmidt, A. Yu. Smirnov, Phys. Rev. **D 74**, 113003 (2006).
17. P. H. Frampton, S. L. Glashow, D. Marfatia, Phys. Lett. **B 536**, 79 (2002); Z. Z. Xing, Phys. Lett. **B 530**, 159 (2002); B. R. Desai, D. P. Roy, A. R. Vaucher, Mod. Phys. Lett. **A 18**, 1355 (2003); Z. Z. Xing, Int. J. Mod. Phys. **A 19**, 1 (2004); A. Merle, W. Rodejohann, Phys. Rev. **D 73**, 073012 (2006); S. Dev, S. Kumar, S. Verma, S. Gupta, R. R. Gautam, Phys. Rev. **D 81**, 053010 (2010) and references therein.
18. M. Fukugita, M. Tanimoto, T. Yanagida, Prog. Theor. Phys. **89**, 263 (1993); *ibid.* Phys. Lett. **B 562**, 273 (2003), hep-ph/0303177; M. Randhawa, G. Ahuja, M. Gupta, Phys. Rev. **D 65**, 093016 (2002); G. Ahuja, S. Kumar, M. Randhawa, M. Gupta, S. Dev, Phys. Rev. **D 76**, 013006 (2007); G. Ahuja, M. Gupta, M. Randhawa, R. Verma, Phys. Rev. **D 79**, 093006 (2009); M. Fukugita, *et al.*, Phys. Lett. **B 716**, 294 (2012), arXiv:1204.2389; P. Fakay, S. Sharma, R. Verma, G. Ahuja, M. Gupta, Phys. Lett. **B 720**, 366 (2013).
19. H. Fritzsch, M. Gell-Mann, P. Minkowski, Phys. Lett. **B 59**, 256 (1975); P. Minkowski, Phys. Lett. **B 67**, 421 (1977); T. Yanagida, *proceedings of the Workshop on Unified Theories and Baryon Number in the Universe, Tsukuba, 1979*, eds. A. Sawada, A. Sugamoto, KEK Report No. 79-18, Tsukuba; M. Gell-Mann, P. Ramond, R. Slansky, in *Supergravity*, edited by F. van Nieuwenhuizen, D. Freedman (North Holland, Amsterdam, 1979) 315; S. L. Glashow, in *Quarks and Leptons*, edited by M. Lévy *et al.* (Plenum, New York, 1980) 707; R. N. Mohapatra, G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980); J. Schechter, J. W. F. Valle, Phys. Rev. **D 22**, 2227 (1980).

20. B. Pontecorvo, Zh. Eksp. Theor. Fiz. (JETP) **33**, 549 (1957); *ibid.* **34**, 247 (1958); *ibid.* **53**, 1771 (1967); Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
21. S. Sharma, P. Fakay, G. Ahuja, M. Gupta, Phys. Rev. **D 91**, 053004 (2015).
22. S. Sharma, G. Ahuja, M. Gupta, Phys. Rev. **D 94**, 113004 (2016).
23. Z. Z. Xing, H. Zhang, S. Zhou, Phys. Rev. **D 86**, 013013 (2012).
24. M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz, Nucl. Phys. **B 908**, 199 (2016).





## 2 Search for Double Charged Particles as Direct Test for Dark Atom Constituents

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**Abstract.** The nonbaryonic dark matter of the Universe is assumed to consist of new stable particles. Stable particle candidates for cosmological dark matter are usually considered as neutral and weakly interacting. However stable charged leptons and quarks can also exist hidden in elusive dark atoms and can play a role of dark matter. Such possibility is strongly restricted by the constraints on anomalous isotopes of light elements that form positively charged heavy species with ordinary electrons. This problem might be avoided, if stable particles with charge -2 exist and there are no stable particles with charges +1 and -1. These conditions cannot be realized in supersymmetric models, but can be satisfied in several alternative scenarios, which are discussed in this paper. The excessive -2 charged particles are bound with primordial helium in O-helium atoms, maintaining specific nuclear-interacting form of the dark matter. O-helium dark matter can provide solution for the puzzles of dark matter searches. The successful development of composite dark matter scenarios appeals to experimental search for doubly charged constituents of dark atoms. Estimates of production cross section of such particles at LHC are presented and discussed. Signatures of double charged particles in the ATLAS experiment are outlined.

**Povzetek.** Avtorji predpostavijo, da temno snov, ki ni iz barionov poznanih treh družin, sestavljajo novi stabilni delci. Običajno za njih predlagajo nevtralne delce s šibko interakcijo, pa tudi stabilne nabite leptone in kvarke, vezane v “temne atome”. To zadnjo možnost so poskusi močno omejili. Tem omejitvam se lahko izognemo, če privzamemo, da obstajajo stabilni (temni) delci z nabojem  $-2$ , ni pa stabilnih (temnih) delcev z naboji  $+1$  in  $-1$ . Tega privzetka ne moremo narediti v modelih s supersimetrijo, lahko pa ga naredimo v alternativnih modelih, ki jih avtorji obravnavajo. Višek delcev z nabojem  $-2$  se veže s prvotnim helijem v “atome” O-helija, katerega interakcija je posledično običajna jedrska. Taka temna snov pojasni rezultate tistih poskusov iskanja temne snovi, ki je še niso zaznali. Obravnavajo preseke za te delce pri poskusih na LHC in predvidijo rezultate meritev na ATLASu.

\* Presented first part of talk at Bled

\*\* Presented second part of talk by VIA teleconference

Keywords: Elementary particles, Dark matter, Dark atoms, Stable double charged particles

## 2.1 Introduction

The observation of exotic stable multi-charge objects would represent striking evidence for physics beyond the Standard Model. Cosmological arguments put severe constraints on possible properties of such objects. Such particles (see e.g. Ref. [1] for review and reference) should be stable, provide the measured dark matter density and be decoupled from plasma and radiation at least before the beginning of matter dominated stage. The easiest way to satisfy these conditions is to involve neutral elementary weakly interacting massive particles (WIMPs). SUSY Models provide a list of possible WIMP candidates: neutralino, axino, gravitino etc., However it may not be the only particle physics solution for the dark matter problem.

One of such alternative solutions is based on the existence of heavy stable charged particles bound in neutral dark atoms. Dark atoms offer an interesting possibility to solve the puzzles of dark matter searches. It turns out that even stable electrically charged particles can exist hidden in such atoms, bound by ordinary Coulomb interactions (see [1–3] and references therein). Stable particles with charge -1 are excluded due to overproduction of anomalous isotopes. However, there doesn't appear such an evident contradiction for negatively doubly charged particles.

There exist several types of particle models where heavy stable -2 charged species,  $O^{--}$ , are predicted:

- (a) AC-leptons, predicted as an extension of the Standard Model, based on the approach of almost-commutative geometry [4–7].
- (b) Technileptons and anti-technibaryons in the framework of Walking Technicolor (WTC) [8–14].
- (c) stable "heavy quark clusters"  $\bar{U}\bar{U}\bar{U}$  formed by anti-U quark of 4th generation [4,15–19]
- (d) and, finally, stable charged clusters  $\bar{u}_5\bar{u}_5\bar{u}_5$  of (anti)quarks  $\bar{u}_5$  of 5th family can follow from the approach, unifying spins and charges[20].

All these models also predict corresponding +2 charge particles. If these positively charged particles remain free in the early Universe, they can recombine with ordinary electrons in anomalous helium, which is strongly constrained in terrestrial matter. Therefore a cosmological scenario should provide a mechanism which suppresses anomalous helium. There are two possible mechanisms than can provide a suppression:

- (i) The abundance of anomalous helium in the Galaxy may be significant, but in terrestrial matter a recombination mechanism could suppress this abundance below experimental upper limits [4,6]. The existence of a new U(1) gauge symmetry, causing new Coulomb-like long range interactions between charged dark matter particles, is crucial for this mechanism. This leads inevitably to the existence of dark radiation in the form of hidden photons.

- (ii) Free positively charged particles are already suppressed in the early Universe and the abundance of anomalous helium in the Galaxy is negligible [2,16].

These two possibilities correspond to two different cosmological scenarios of dark atoms. The first one is realized in the scenario with AC leptons, forming neutral AC atoms [6]. The second assumes a charge asymmetry of the  $O^{--}$  which forms the atom-like states with primordial helium [2,16].

If new stable species belong to non-trivial representations of the  $SU(2)$  electroweak group, sphaleron transitions at high temperatures can provide the relation between baryon asymmetry and excess of -2 charge stable species, as it was demonstrated in the case of WTC [8,21–23].

After it is formed in the Standard Big Bang Nucleosynthesis (BBN),  ${}^4\text{He}$  screens the  $O^{--}$  charged particles in composite ( ${}^4\text{He}^{++}O^{--}$ ) OHe “atoms” [16]. In all the models of OHe,  $O^{--}$  behaves either as a lepton or as a specific “heavy quark cluster” with strongly suppressed hadronic interactions. The cosmological scenario of the OHe Universe involves only one parameter of new physics – the mass of  $O^{--}$ . Such a scenario is insensitive to the properties of  $O^{--}$  (except for its mass), since the main features of the OHe dark atoms are determined by their nuclear interacting helium shell. In terrestrial matter such dark matter species are slowed down and cannot cause significant nuclear recoil in the underground detectors, making them elusive in direct WIMP search experiments (where detection is based on nuclear recoil) such as CDMS, XENON100 and LUX. The positive results of DAMA experiments (see [24] for review and references) can find in this scenario a nontrivial explanation due to a low energy radiative capture of OHe by intermediate mass nuclei [2,1,3]. This explains the negative results of the XENON100 and LUX experiments. The rate of this capture is proportional to the temperature: this leads to a suppression of this effect in cryogenic detectors, such as CDMS.

OHe collisions in the central part of the Galaxy lead to OHe excitations, and de-excitations with pair production in E0 transitions can explain the excess of the positron-annihilation line, observed by INTEGRAL in the galactic bulge [1,3,21,27].

One should note that the nuclear physics of OHe is in the course of development and its basic element for a successful and self-consistent OHe dark matter scenario is related to the existence of a dipole Coulomb barrier, arising in the process of OHe-nucleus interaction and providing the dominance of elastic collisions of OHe with nuclei. This problem is the main open question of composite dark matter, which implies correct quantum mechanical solution [28]. The lack of such a barrier and essential contribution of inelastic OHe-nucleus processes seem to lead to inevitable overproduction of anomalous isotopes [29].

Production of pairs of elementary stable doubly charged heavy leptons is characterized by a number of distinct experimental signatures that would provide effective search for them at the experiments at the LHC and test the atom-like structure of the cosmological dark matter. Moreover, astrophysical consequences of composite dark matter model can reproduce experimentally detected excess in positron annihilation line and high energy positron fraction in cosmic rays only if the mass of double charged X particles is in the 1 TeV range (Section 2). We discuss confrontation of these predictions and experimental data in Section 3. The

current status and expected progress in experimental searches for stable double charged particles as constituents of composite dark matter are summarized in the concluding Section 4.

## 2.2 Indirect effects of composite dark matter

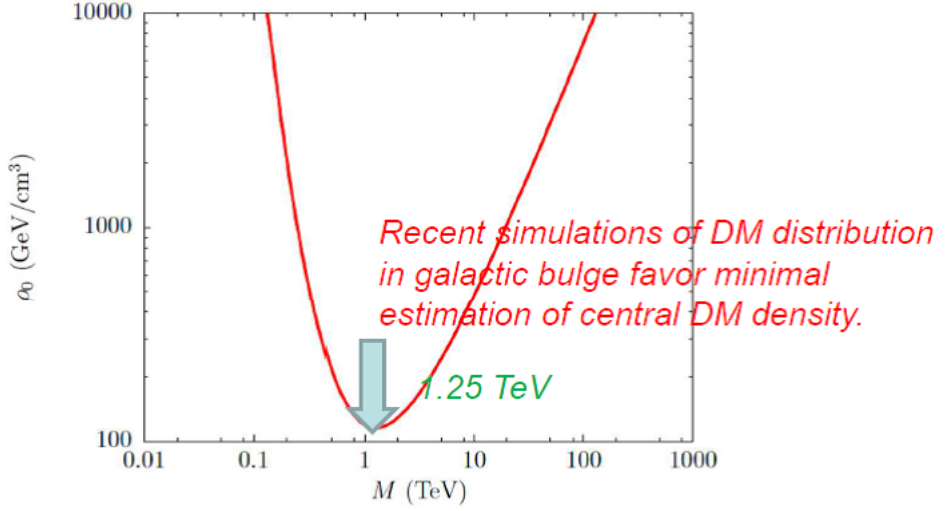
The existence of such form of matter as O-helium should lead to a number of astrophysical signatures, which can constrain or prove this hypothesis. One of the signatures of O-helium can be a presence of an anomalous low  $Z/A$  component in the cosmic ray flux. O-helium atoms that are present in the Galaxy in the form of the dark matter can be destroyed in astrophysical processes and free  $X$  can be accelerated as ordinary charged particles. O-helium atoms can be ionized due to nuclear interaction with cosmic rays or in the front of a shock wave in the Supernova explosions, where they were effectively accumulated during star evolution [16]. If the mechanisms of  $X$  acceleration are effective, the low  $Z/A$  component with charge 2 should be present in cosmic rays at the level of  $F_X/F_p \sim 10^{-9} m_o^{-1}$  [21], and might be observed by PAMELA and AMS02 cosmic ray experiments. Here  $m_o$  is the mass of O-helium in TeV,  $F_X$  and  $F_p$  are the fluxes of  $X$  and protons, respectively.

### 2.2.1 Excess of positron annihilation line in the galactic bulge

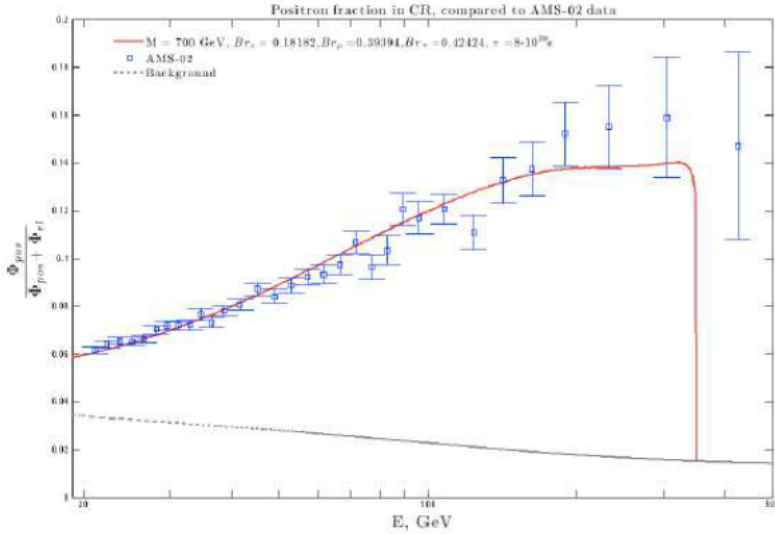
Another signature of O-helium in the Galaxy is the excess of the positron annihilation line in cosmic gamma radiation due to de-excitation of the O-helium after its interaction in the interstellar space. If 2S level of O-helium is excited, its direct one-photon transition to the 1S ground state is forbidden and the de-excitation mainly goes through direct pair production. In principle this mechanism of positron production can explain the excess in positron annihilation line from the galactic bulge, measured by the INTEGRAL experiment. Due to the large uncertainty of DM distribution in the galactic bulge this interpretation of the INTEGRAL data is possible in a wide range of masses of O-helium with the minimal required central density of O-helium dark matter at  $m_o = 1.25$  TeV [25,26]. For the smaller values of  $m_o$  one needs larger central density to provide effective excitation of O-helium in collisions. Current analysis favors lowest values of central dark matter density, making possible O-helium explanation for this excess only for a narrow window around this minimal value (see Fig. 2.1)

### 2.2.2 Composite dark matter solution for high energy positron excess

In a two-component dark atom model, based on Walking Technicolor, a sparse WIMP-like component of atom-like state, made of positive and negative doubly charged techniparticles, is present together with the dominant OHe dark atom and the decays of doubly positive charged techniparticles to pairs of same-sign leptons can explain the excess of high-energy cosmic-ray positrons, found in PAMELA and AMS02 experiments [17]. This explanation is possible for the mass of decaying



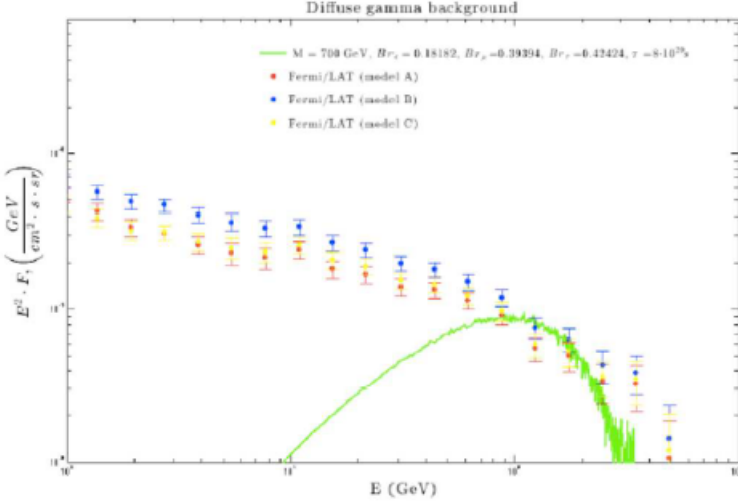
**Fig.2.1.** Dark matter is subdominant in the central part of Galaxy. It strongly suppresses its dynamical effect and causes large uncertainty in dark matter density and velocity distribution. At this uncertainty one can explain the positron line excess, observed by INTERGRAL, for a wide range of  $m_\phi$  given by the curve with minimum at  $m_\phi = 1.25$  TeV. However, recent analysis of possible dark matter distribution in the galactic bulge favors minimal value of its central density.



**Fig.2.2.** Best fit high energy positron fluxes from decaying composite dark matter in confrontation with the results of AMS02 experiment.

+2 charged particle below 1 TeV and depends on the branching ratios of leptonic channels (See Fig. 2.2).

Since even pure lepton decay channels are inevitably accompanied by gamma radiation the important constraint on this model follows from the measurement of cosmic gamma ray background in FERMI/LAT experiment. The multi-parameter



**Fig. 2.3.** Gamma ray flux accompanying the best fit high energy positron fluxes from decaying composite dark matter reproducing the results of AMS02 experiment, in confrontation with FERMI/LAT measurement of gamma ray background.

analysis of decaying dark atom constituent model determines the maximal model independent value of the mass of decaying +2 charge particle, at which this explanation is possible

$$m_o < 1\text{TeV}.$$

One should take into account that according to even in this range hypothesis on decaying composite dark matter, distributed in the galactic halo, can lead to gamma ray flux exceeding the measured by FERMI/LAT.

### 2.2.3 Sensitivity of indirect effect of composite dark matter to the mass of their double charged constituents

We see that indirect effects of composite dark matter strongly depend on the mass of its double charged constituents.

To explain the excess of positron annihilation line in the galactic bulge mass of double charged constituent of O-helium should be in a narrow window around

$$m_o = 1.25\text{TeV}.$$

To explain the excess of high energy cosmic ray positrons by decays of constituents of composite dark matter with charge +2 and to avoid overproduction of gamma background, accompanying such decays, the mass of such constituent should be in the range

$$m_o < 1 \text{ TeV}.$$

These predictions should be confronted with the experimental data on the accelerator search for stable double charged particles.

### 2.3 Searches for stable multi-charged particles

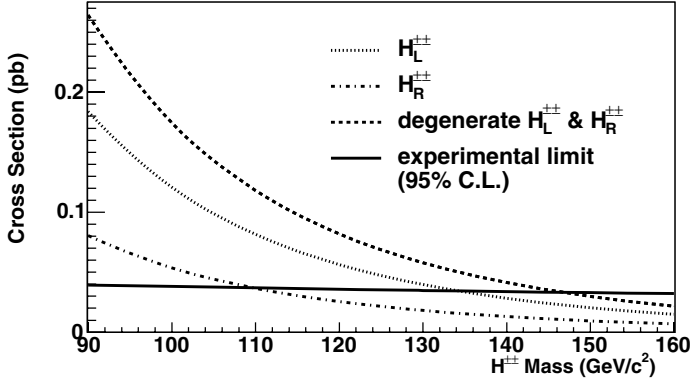
A new charged massive particle with electric charge  $\neq 1e$  would represent a dramatic deviation from the predictions of the Standard Model, and such a spectacular discovery would lead to fundamental insights and critical theoretical developments. Searches for such kind of particles were carried out in many cosmic ray and collider experiments (see for instance review in [38]). Experimental search for double charged particles is of a special interest because of important cosmological sequences discussed in previous sections. In a tree approximation, such particles cannot decay to pair of quarks due to electric charge conservation and only decays to the same sign leptons are possible. The latter implies lepton number nonconservation, being a profound signature of new physics. In general, it makes such states sufficiently long-living in a cosmological scale.

Obviously, such experiments are limited to look only for the “long-lived” particles, which do not decay within a detector, as opposed to truly stable particles, which do not decay at all. Since the kinematics and cross sections for double charged particle production processes cannot be reliably predicted, search results at collider experiments are usually quoted as cross section limits for a range of charges and masses for well-defined kinematic models. In these experiments, the mass limit was set at the level of up to 100 GeV (see e.g. for review [38]).

The CDF experiment collaboration performed a search for long-lived double charged Higgs bosons ( $H^{++}$ ,  $H^{--}$ ) with  $292 \text{ pb}^{-1}$  of data collected in  $p\bar{p}$  collisions at  $\sqrt{s} = 1.96 \text{ TeV}$  [39]. The dominant production mode considered was  $p\bar{p} \rightarrow \gamma^*/Z + X \rightarrow H^{++}H^{--} + X$ .

Background sources include jets fragmenting to high- $p_T$  tracks,  $Z \rightarrow ee$ ,  $Z \rightarrow \mu\mu$ , and  $Z \rightarrow \tau\tau$ , where at least one  $\tau$  decays hadronically. Number of events expected from these backgrounds in the signal region was estimated to be  $< 10^{-5}$  at 68% confidence level (CL).

Not a single event with a  $H^{++}$  or  $H^{--}$  was found in experimental data. This allows to set cross-section limit shown in Fig. 2.4 as a horizontal solid line. Next-to-leading order theoretical calculations of the cross-section for pair production of  $H^{\pm\pm}$  bosons for left-handed and right-handed couplings are also shown in this figure. Comparison of expected and observed cross-section limits gives the following mass constraints: 133 and 109 GeV on the masses of long-lived  $H_L^{\pm\pm}$  and  $H_R^{\pm\pm}$ , respectively, at 95% CL as shown in Fig. 2.4. In case of degenerate  $H_L^{\pm\pm}$  and  $H_R^{\pm\pm}$  bosons, the mass limit was set to 146 GeV.



**Fig. 2.4.** The comparison of the experimental cross section upper limit with the theoretical next-to-leading order cross section for pair production of  $H^{\pm\pm}$  bosons. The theoretical cross sections are computed separately for bosons with left-handed ( $H_L^{\pm\pm}$ ) and right-handed ( $H_R^{\pm\pm}$ ) couplings, and summed for the case that their masses are degenerate, [39].

### 2.3.1 Searches at Large Hadron Collider

Significant increase of beam energy at the Large Hadron Collider (LHC) opens a new era in the high energy physics and allows accessing uncharted territories of particle masses. In this section the results of searches for the multi-charged particles, performed by the ATLAS and the CMS collaborations at LHC, will be described.

Calculations of the cross sections assume that these particles are generated as new massive spin-1/2 ones, neutral under  $SU(3)_C$  and  $SU(2)_L$ .

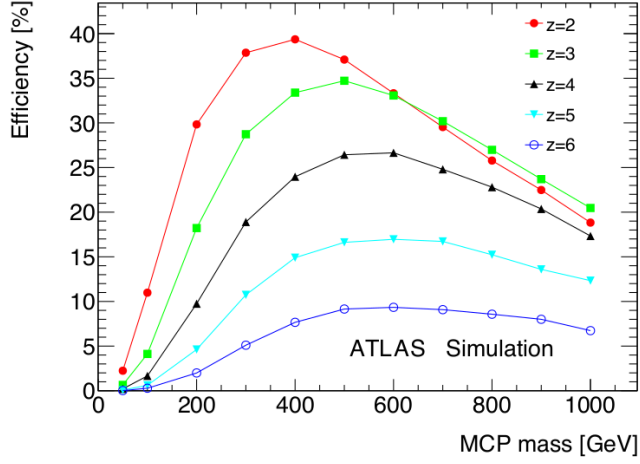
**ATLAS experiment at LHC** In Run 1 (2010–2012), the ATLAS [40] collaboration at LHC performed two searches for long-lived multi-charged particles, including the double charged particles: one search with  $4.4 \text{ fb}^{-1}$  of data collected in pp collisions at  $\sqrt{s} = 7 \text{ TeV}$  [41], and another one with  $20.3 \text{ fb}^{-1}$  collected at  $\sqrt{s} = 8 \text{ TeV}$  [42].

Both these searches feature particles with large transverse momentum values, traversing the entire ATLAS detector. An energy loss of a double charged particle is by a factor of  $q^2 = 4$  higher than that of single charged particle. Such particles will leave a very characteristic signature of high ionization in the detector. More specifically, the searches look for particles with anomalously high ionization on their tracks in three independent detector subsystems: silicon pixel detector (Pixel) and transition radiation tracker (TRT) in the ATLAS inner detector, and monitoring drift tubes (MDT) in the muon system.

The estimate of the expected background originating from the SM processes and providing input into the signal region D was calculated to be  $0.013 \pm 0.002(\text{stat.}) \pm 0.003(\text{syst.})$  events.

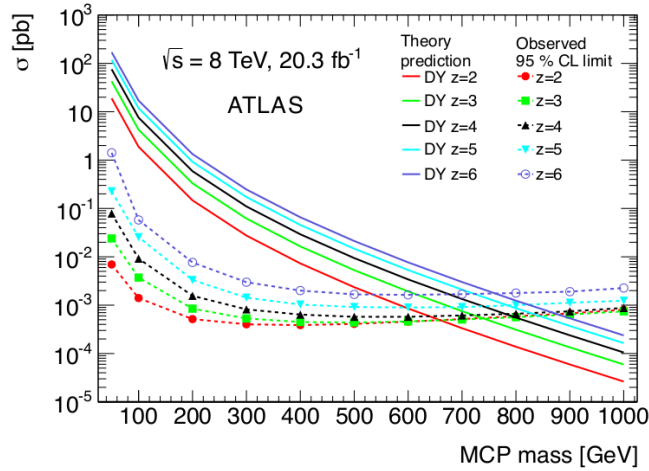
In order to define cross section, a reconstruction efficiency of signal particles has to be known. This value is defined as a fraction of simulated events with at





**Fig.2.5.** The signal efficiencies for different masses and charges of the multi-charged particles for the DY production model. Double charged particles are denoted as “ $z = 2$ ” (red points and line). The picture is taken from [42].

least one multi-charged particle satisfying all of the aforementioned criteria over the number of all generated events. In other words, it is a search sensitivity to find a hypothetical particle with the ATLAS experiment. These values are shown in Fig. 2.5 for each considered signal sample containing particles with charges from 2 to 6.



**Fig.2.6.** Observed 95% CL cross-section upper limits and theoretical cross-sections as functions of the multi-charged particles mass. Again, the double charged particles are denoted as “ $z = 2$ ” (red points and lines). The picture is taken from [42].

No events with double charged particles were found in neither 2011 or 2012 experimental data sets, setting the lower mass limits to 430 and 660 GeV, respectively, at 95% CL. The comparison of observed cross-section upper limits and theoretically predicted cross-sections is shown in Fig. 2.6.

**CMS experiment at LHC** In parallel to the ATLAS experiment, the CMS [43] collaboration at LHC performed a search for double charged particles, using  $5.0 \text{ fb}^{-1}$  of data collected in pp collisions at  $\sqrt{s} = 7 \text{ TeV}$  and  $18.8 \text{ fb}^{-1}$  collected at  $\sqrt{s} = 8 \text{ TeV}$  [44].

The search features particles with high ionization along their tracks in the inner silicon pixel and strip tracker. Tracks with specific ionization  $I_h > 3 \text{ MeV/cm}$  were selected. The muon system was used to measure the time-of-flight values. Tracks with  $1/\beta > 1.2$  were considered.

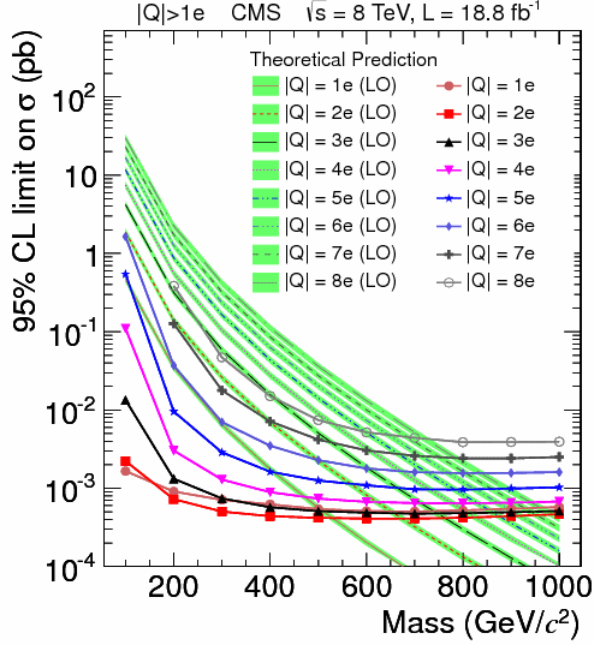
For the part of the search based on the  $\sqrt{s} = 7 \text{ TeV}$  data, the number of events in the signal region, expected from SM processes, was estimated to be  $0.15 \pm 0.04$ , whereas for the  $\sqrt{s} = 8 \text{ TeV}$  part it was  $0.52 \pm 0.11$  events. The uncertainties include both statistical and systematical contributions. 0 and 1 events were observed in the signal regions for the 7 and 8 TeV analyses, respectively, which is consistent with the predicted event rate.

Comparison between observed upper cross section limits and theoretically predicted cross section values for the 8 TeV is shown in Fig. 2.7.

For the 8 TeV search, the mass limit of 665 GeV was obtained. This result (within uncertainties) is very close to the ATLAS limit of 660 GeV for the 8 TeV data set. Also, CMS treated the results obtained at 7 and 8 TeV as combined. This allowed to push the lower mass limit to 685 GeV at 95% CL. A combination of the results of two experiments for 8 TeV would mean an increase of statistics by a factor of 2. Having said that, one can conclude that the mass limit based on the results of both experiment for double charged particles can be set at the level of about 730 GeV.

**What one expects from LHC Run 2** Considering a recent CMS Physics Analysis Summary [45] and an ATLAS paper in preparation, both on the searches for double charged particles in data delivered by LHC to these experiments in 2015–2016, we may conclude that each of these two experiments will be able to set a lower mass limit on the double charged particles at  $m = 1000 \pm 50 \text{ GeV}$ . Going further and considering the data taking periods of ATLAS and CMS until the end of Run 2 (end of 2018), we can estimate a low limit on the double charged particles mass corresponding to the Run 2 data set. Several assumptions are made in these speculations:

- by the end of 2018, ATLAS and CMS will each record about  $120 \text{ fb}^{-1}$  of  $\sqrt{s} = 13 \text{ TeV}$  data;
- signal efficiency will remain at a present level in both experiments, without being compromised by high detector occupancy or any other effects;
- no double charged particle candidates will be observed in the first place.



**Fig.2.7.** Observed 95% CL cross-section upper limits and theoretical cross-sections as functions of the multi-charged particles mass in CMS search at the  $\sqrt{s} = 8 \text{ TeV}$ . The double charged particles are denoted as “ $|Q| = 2e$ ”. The picture is taken from [44].

Considering all of the above, the ATLAS and CMS collaborations may each be expected to set the lower mass limits at the level of 1.2 TeV based on their analyses of the entire 13 TeV data set. If these two experiments combined their independently gathered statistics together for this kind of search, the limits would go as high as up to 1.3 TeV.

## 2.4 Conclusions

The existence of heavy stable neutral particles is one of the popular solutions for the dark matter problem. However, DM considered to be electrically neutral, potentially can be formed by stable heavy charged particles bound in neutral atom-like states by Coulomb attraction. Analysis of the cosmological data and atomic composition of the Universe gives the constraints on the particle charge showing that only -2 charged constituents, being trapped by primordial helium in neutral O-helium states, can avoid the problem of overproduction of the anomalous isotopes of chemical elements, which are severely constrained by observations. Cosmological model of O-helium dark matter can even explain puzzles of direct dark matter searches.

Stable charge -2 states ( $X^{--}$ ) can be elementary like AC-leptons or technileptons, or look like technibaryons. The latter, composed of techniquarks, reveal their

structure at much higher energy scale and should be produced at colliders and accelerators as elementary species. They can also be composite like “heavy quark clusters”  $\bar{U}\bar{U}\bar{U}$  formed by anti-U quark in one of the models of fourth generation [16] or  $\bar{u}_5\bar{u}_5\bar{u}_5$  of (anti)quarks  $\bar{u}_5$  of stable 5th family in the approach [20].

In the context of composite dark matter scenario accelerator search for stable doubly charged leptons acquires the meaning of direct critical test for existence of charged constituents of cosmological dark matter.

The signature for AC leptons and techniparticles is unique and distinctive what allows to separate them from other hypothetical exotic particles. Composite dark matter models can explain observed excess of high energy positrons and gamma radiation in positron annihilation line at the masses of  $X^{--}$  in the range of 1 TeV, it makes search for double charged particles in this range direct experimental test for these predictions of composite dark matter models.

Test for composite  $X^{--}$  can be only indirect: through the search for heavy hadrons, composed of single U or  $\bar{U}$  and light quarks (similar to R-hadrons).

The ATLAS and CMS collaborations at the Large Hadron Collider are searching for the double charged particles since 2011. The most stringent results achieved so far exclude the existence of such particles up to their mass of 680 GeV. This value was obtained by both ATLAS and CMS collaborations independently. It is expected that if these two collaborations combine their independently gathered statistics of LHC Run 2 (2015–2018), the lower mass limit of double charged particles could reach the level of about 1.3 TeV.

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## References

1. M. Yu. Khlopov, Fundamental Particle Structure in the Cosmological Dark Matter Int. J. Mod. Phys. A **28** (2013) 1330042 .
2. M. Y. Khlopov, Physics of Dark Matter in the Light of Dark Atoms Mod. Phys. Lett. A **26** (2011) 2823 [arXiv:1111.2838 [astro-ph.CO]].
3. M. Yu. Khlopov, Dark Atoms and Puzzles of Dark Matter Searches Int. J. Mod. Phys. A **29** (2014) 1443002 .
4. M. Y. Khlopov, “New symmetries in microphysics, new stable forms of matter around us”, arXiv:astro-ph/0607048.
5. C. A. Stephan, Almost-commutative geometries beyond the standard model, arXiv:hep-th/0509213.
6. D. Fargion, M. Khlopov and C. A. Stephan, Cold dark matter by heavy double charged leptons?, Class. Quantum Grav. **23** (2006) 7305 .
7. A. Connes *Noncommutative Geometry* (Academic Press, London and San Diego, 1994).
8. M. Y. Khlopov and C. Kouvaris, Strong Interactive Massive Particles from a Strong Coupled Theory, Phys. Rev. D **77** (2008) 065002 .

9. F. Sannino and K. Tuominen, Orientifold Theory Dynamics and Symmetry Breaking, *Phys. Rev. D* **71** (2005) 051901 .
10. D. K. Hong, S. D. Hsu and F. Sannino, Composite Higgs from higher representations, *Phys. Lett. B* **597** (2004) 89 .
11. D. D. Dietrich, F. Sannino and K. Tuominen, Light composite Higgs from higher representations versus electroweak precision measurements: Predictions for LHC, *Phys. Rev. D* **72** (2005) 055001 .
12. D. D. Dietrich, F. Sannino and K. Tuominen, Light composite Higgs and precision electroweak measurements on the Z resonance: An update, *Phys. Rev. D* **73** (2006) 037701 .
13. S. B. Gudnason, C. Kouvaris and F. Sannino, Towards working technicolor: Effective theories and dark matter, *Phys. Rev. D* **73** (2006) 115003 .
14. S. B. Gudnason, C. Kouvaris and F. Sannino, Dark matter from new technicolor theories, *Phys. Rev. D* **74** (2006) 095008 .
15. K.M.Belotsky *et al*, Heavy hadrons of 4th family hidden in our Universe and close to detection? *Gravitation and Cosmology* **11** (2005) 3 .
16. M.Yu. Khlopov, Composite dark matter from 4th generation *JETP Lett.* **83** (2006) 1 .
17. K. Belotsky, M. Khlopov and K. Shibaev, "Stable matter of 4th generation: Hidden in the Universe and close to detection?", arXiv:astro-ph/0602261.
18. K. Belotsky, M. Khlopov and K. Shibaev, Composite dark matter and its charged constituents, *Gravitation and Cosmology* **12** (2006) 1 .
19. K. Belotsky, M.Yu.Khlopov,K.I.Shibaev, Stable quarks of the 4th family? in Eds. N. L. Watson and T. M. Grant: "The Physics of Quarks: New Research." ( Horizons in World Physics, V.265), NOVA Publishers, Hauppauge NY, 2009, PP.19-47; arXiv:0806.1067 [astro-ph].
20. N. Mankoc-Borstnik, Unification of spins and charges in Grassmann space?, *Mod. Phys. Lett.A* **10**, 587 (1995).
21. M. Y. Khlopov and C. Kouvaris, Composite dark matter from a model with composite Higgs boson, *Phys. Rev. D* **78** (2008) 065040
22. M. Y. Khlopov, The puzzles of dark matter searches, *AIP Conf. Proc.* **1241** (2010) 388
23. M. Y. Khlopov, A. G. Mayorov and E. Y. Soldatov, Composite Dark Matter and Puzzles of Dark Matter Searches, *Int. J. Mod. Phys. D* **19** (2010) 1385.
24. R. Bernabei, *et al*. Dark Matter investigation by DAMA at Gran Sasso *Int. J. Mod. Phys. A* **2013**, 28, 1330022 [71 pages]
25. D. P. Finkbeiner and N. Weiner, Exciting Dark Matter and the INTEGRAL/SPI 511 keV signal, *Phys. Rev. D* **76** 083519 (2007).
26. B. J. Teegarden et al., INTEGRAL/SPI Limits on Electron-Positron Annihilation Radiation from the Galactic Plane, *Astrophys. J.* **621** 296300 (2005).
27. J.-R. Cudell, M.Yu.Khlopov and Q.Wallemacq Dark atoms and the positron-annihilation-line excess in the galactic bulge. *Advances in High Energy Physics*, vol. 2014, Article ID 869425, 5 pages, 2014; e-Print: arXiv: 1401.5228.
28. J.-R. Cudell, M.Yu.Khlopov and Q.Wallemacq, The nuclear physics of OHe. *Bled Workshops in Physics* **13** (2012) 10
29. J.-R. Cudell, M.Yu.Khlopov and Q.Wallemacq, Some Potential Problems of OHe Composite Dark Matter, *Bled Workshops in Physics* **15** (2014) 66
30. K. Belotsky, M. Khlopov, C. Kouvaris, M. Laletin, Decaying Dark Atom constituents and cosmic positron excess, *Advances in High Energy Physics*, vol. 2014, Article ID 214258, 10 p., arXiv:1403.1212.
31. PAMELA Collaboration, *Phys. Rev. Lett.* **111** (2013) 081102, arXiv:1308.0133.
32. AMS collaboration, *Phys. Rev. Lett.* **110** (2013), 141102.
33. M. Aguilar et al., *Phys. Rev. Lett.* **113**, 121102 (2014).

- 34. L. Accardo et al., Phys. Rev. Lett. **113**, 121101 (2014).
- 35. M. Ackermann et al. (Fermi/LAT Collaboration), Phys.Rev. D **86** (2012), arXiv:1205.2739v1
- 36. T. Sjostrand, S. Mrenna and P. Skands, JHEP 0605, 026 (2006)
- 37. E. A. Baltz and J. Edsjo, Phys.Rev. D **59**, 023511 (1999)
- 38. M. Fairbairn, et al., Stable Massive Particles at Colliders, Phys. Rep. **438** (2007) 1.
- 39. CDF Collaboration, D. Acosta et al., Search for long-lived doubly-charged Higgs bosons in  $p\bar{p}$  collisions at  $\sqrt{s} = 1.96$  TeV, Phys. Rev. Lett. **95** 071801 (2005).
- 40. ATLAS Collaboration, G. Aad et al., The ATLAS Experiment at the CERN Large Hadron Collider, JINST **3** S08003 (2008).
- 41. ATLAS Collaboration, G. Aad et al., Search for long-lived, multi-charged particles in pp collisions at  $\sqrt{s} = 7$  TeV using the ATLAS detector, Phys. Lett. B **722** 305 (2013).
- 42. ATLAS Collaboration, G. Aad et al., Search for heavy long-lived multi-charged particles in pp collisions at  $\sqrt{s} = 8$  TeV using the ATLAS detector, Eur. Phys. J. C **75** 362 (2015).
- 43. CMS Collaboration, S. Chatrchyan et al., The CMS Experiment at the CERN LHC, JINST **3** S08004 (2008).
- 44. CMS Collaboration, S. Chatrchyan et al., Searches for long-lived charged particles in pp collisions at  $\sqrt{s} = 7$  and 8 TeV, JHEP **1307** 122 (2013).
- 45. CMS Collaboration, Search for heavy stable charged particles with  $12.9 \text{ fb}^{-1}$  of 2016 data, CMS-PAS-EXO-16-036.



### 3 Eightfold Way for Composite Quarks and Leptons

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*To memory of V.N. Gribov (1930–1997)*

**Abstract.** The L-R symmetric composite model for quarks and leptons where constituent preons are bound by the  $SO(n)_L \times SO(n)_R$  gauge forces is reconsidered. We find that just the eight left-handed and right-handed preons, their local metaflavor symmetry  $SU(8)_{MF}$  and accompanying global chiral symmetry  $SU(8)_L \times SU(8)_R$  may determine physical world at small distances. This result for an admissible number of preons filling the fundamental multiplet of some  $SU(N)_{MF}$  symmetry group appears as a solution to the 't Hooft's anomaly matching condition precisely for  $N = 8$ , provided that this condition is satisfied separately for the L-preon and R-preon composites which fill individually a single multiplet of the  $SU(N)$  rather than a set of its multiplets. We next show that an appropriate L-R symmetry violation reduces an initially emerged vectorlike  $SU(8)$  theory down to the conventional  $SU(5)$  GUT with an extra local family symmetry  $SU(3)_F$  and three standard generations of quarks and leptons. Though the tiny radius of compositeness for universal preons composing both quarks and leptons makes it impossible to immediately confirm their composite nature, theory predicts the several extra heavy  $SU(5) \times SU(3)_F$  multiplets located at the scales from  $O(1)$  TeV up to the Planck mass scale that may appear of actual experimental interest.

**Povzetek.** Avtor obravnava model sestavljenih kvarkov in leptonov s simetrijo L-R, v katerem sestavne delce (preone) veže sila umeritve  $SO(n)_L \times SO(n)_R$ . Fiziko na majhnih razdaljah v tem primeru določa samo osem levoročnih in desnoročnih preonov z njihovo lokalno meta-okusno simetrijo  $SU(N)_{MF}$  in ustrezno globalno kiralno simetrijo  $SU(N)_L \times SU(N)_R$ . Ta rezultat za  $N = 8$  zadosti 't Hooftovemu pogoju za ujemanje anomalij, če velja posebej za gruč levoročnih in posebej za gruč desnoročnih preonov, če vsako od gruč določa bodisi levoročni bodisi desnočni multiplet. Avtor pokaže, da približna zlomitev simetrije L – R vodi do GUT  $SU(5)$ , ki pa ima dodatno lokalno simetrijo  $SU(3)_F$ , s katero opiše tri družine družine kvarkov in leptonov. Teorija napove več zelo težkih multipletov  $SU(5) \times SU(3)_F$ , ki imajo mase od  $O(1)$  TeV do Planckove skale, kar bi lahko opazili pri poskusih.

**Keywords:** Composite, Preon, Family, Grand unified theory

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### 3.1 Preamble

It has long recognized that there is no meaningful internal symmetry scheme beyond the known Grand Unified Theories like as the  $SU(5)$ ,  $SO(10)$ , or  $E(6)$  GUTs which could be well suited for classification of all observed quarks and leptons. Any attempt to describe all three quark-lepton families in the grand unified framework leads to higher symmetries with enormously extended representations which also contain lots of exotic states that never been detected in an experiment. This may motivate us to continue seeking a solution in some subparticle or preon models for quark and leptons just like as in the nineteen-sixties the spectroscopy of hadrons had required to seek a solution in the quark model for hadrons in the framework of the so-called Eightfold Way. This term was coined by Murray Gell-Mann in 1961 to describe a classification scheme for hadrons, that he had devised, according to which the known baryons and mesons are grouped into the eight-member families of some global hadron flavor symmetry  $SU(3)$  [1]. This concept had finally led to the hypothesis of quarks locating in the fundamental triplet of this symmetry, and consequently to a compositeness of baryons and mesons observed. We try to show now that the Eightfold Way idea looks much more adequate when it is applied to a next level of the matter elementarity, namely, to elementary preons and composite quarks and leptons. Remarkably, just the eight preons and their generic  $SU(8)$  symmetry seem to determine in a somewhat special way the fundamental entities of the physical world and its total internal symmetry. Interestingly, not only the number "eight" for preons but also its breakdown into some special subdivisions corresponds to the spirit of the Eightfold Way that will be seen from a brief sketch given below.

In more detail, the Eightfold Way or Noble Eightfold Path [2] is a summary of the path of Buddhist practices leading, as supposed, to a true liberation. Keeping in mind the particle physics we propose that the eight spoke Dharma wheel which symbolizes the Noble Eightfold Path could be associated with eight preon fields (or superfields, in general)  $P_i$  ( $i = 1, \dots, 8$ ) being the fundamental octet of the basic flavor symmetry  $SU(8)$ . They may carry out the eight fundamental quantum numbers which has been detected so far. These numbers are related to the weak isospin, color and families of quarks and leptons. Accordingly, we will refer to these preons as a collection of "isons"  $P_w$  ( $w = 1, 2$ ), "chromons"  $P_c$  ( $c = 1, 2, 3$ ) and "famons"  $P_f$  ( $f = 1, 2, 3$ ). Surprisingly, the Noble Eightfold Path is also originally divided into three similar basic divisions. They are:

- (1) The Insight consisting of the Right view and the Right resolve,
- (2) The Moral virtue consisting of the Right speech, the Right action and the Right livelihood,
- (3) The Meditation consisting of the Right effort, the Right mindfulness and the Right Concentration.

This analogy with a similar decomposition of the sacred number eight,  $8 = 2 + 3 + 3$ , which appears in the expected breakdown of the generic preon  $SU(8)$  symmetry

$$SU(8) \rightarrow SU(2)_W \times SU(3)_C \times SU(3)_F, \quad (3.1)$$

looks indeed rather impressive.



In principle, it is not necessary to generically relate the Eightfold Way concept to preons and composite quarks and leptons. First of all, it is related to the eight fundamental quantum charges of particle physics presently observed. They correspond in fact to the two weak isospin orientations, the three types of colors and the three species of quark-lepton families, all of which may be accommodated in the unified  $SU(8)$  theory. Their carriers could be or could not be the elementary preons, though the preon model composing the observed quark and leptons at appropriate distances seems to reflect this concept in the most transparent way.

We find, resurrecting to an extent the old Eightfold Way idea in an initially L-R symmetric and  $SU(N)$  invariant physical world, that just the eight left-handed and right-handed preons and their basic flavor symmetry  $SU(8)$  appear as a solution to the 't Hooft's anomaly matching condition [3] providing the chiral symmetry preservation at all distances involved and, therefore, masslessness of emerged composite fermions. We show that this happens if (1) this condition is satisfied separately for the L-preon and R-preon composites and (2) each of these two series of composites fill only one irreducible representation of the starting  $SU(N)$  symmetry group rather than a set of its representations. We next show that an appropriate L-R symmetry violation reduces an emerged vectorlike  $SU(8)$  theory down to one of its chiral remnants being of significant physical interest. Particularly, this violation implies that, while there still remains the starting chiral symmetry for the left-handed preons and their composites, for the right-handed states we only have the broken chiral symmetry  $[SU(5) \times SU(3)]_R$ . Therefore, whereas nothing changes for the left-handed preon composites still filling the total multiplet of the  $SU(8)$ , the right-handed preon composites will fill only some particular submultiplets in it. As a result, we eventually come to the conventional  $SU(5)$  GUT with an extra local family symmetry  $SU(3)_F$  and three standard generations of quarks and leptons. Moreover, the theory has the universal gauge coupling constant running down from the  $SU(8)$  unification scale, and also predicts some extra heavy  $SU(5) \times SU(3)_F$  multiplets located at the scales from  $O(1)$  TeV up to the Planck mass that may appear of actual experimental interest. For simplicity, we largely work in an ordinary spacetime framework, though extension to the conventional  $N = 1$  supersymmetry with preons and composites treated as standard scalar superfields could generally be made. All these issues are successively considered in the subsequent sections 3.2–3.7, and in the final section 3.8 we present our conclusion.

Some attempt to classify quark-lepton families in the framework of the  $SU(8)$  GUT with composite quarks and leptons had been made quite a long ago [4], though with some special requirements which presently seem not necessary or could be in principle derived rather than postulated. Since then also many other things became better understood, especially the fact that the chiral family symmetry subgroup  $SU(3)_F$  of the  $SU(8)$ , taken by its own, was turned out to be rather successful in description of quark-lepton generations. At the same time, there have not yet appeared, as mentioned above, any other meaningful internal symmetry for an appropriate classification of all the observed quarks and leptons. All that motivates us to address this essential problem once again.

### 3.2 Preons - metaflavors and metacolors

We start formulating a few key elements of preon models (for some significant references, see [5,6]), partially refining some issues given in our old paper [4].

- We propose that there is an exact L-R symmetry at small distances where  $N$  elementary massless left-handed and right-handed preons,  $P_{iL}$  and  $P_{iR}$  ( $i = 1, \dots, N$ ), possess a local metaflavor symmetry  $SU(N)_{MF}$  including the known physical charges, such as weak isospin, color, and family number. The preons, both  $P_{iL}$  and  $P_{iR}$ , are located in its fundamental representation.
- The preons also possess a local metacolor symmetry  $G_{MC} = G_{MC}^L \times G_{MC}^R$  with  $n$  metacolors ( $n$  is odd) which bind preons into composites - quarks, leptons and other states. In contrast to their common metaflavors, preons have different metacolors for the left-right and left-handed components,  $P_{iL}^\alpha$  and  $P_{iR}^{\alpha'}$ , where  $\alpha$  and  $\alpha'$  are indices of the corresponding metacolor subgroups  $G_{MC}^L$  ( $\alpha = 1, \dots, n$ ) and  $G_{MC}^R$  ( $\alpha' = 1, \dots, n$ ), respectively. As a consequence, there are two types of composites at large distances being composed from them separately with a radius of compositeness,  $R_{MC} \sim 1/\Lambda_{MC}$ , where  $\Lambda_{MC}$  corresponds to the scale of the preon confinement for the asymptotically free (or infrared divergent)  $G_{MC}^{L,R}$  symmetries. Obviously, the preon condensate  $\langle \bar{P}_L P_R \rangle$  which could cause the  $\Lambda_{MC}$  order masses for composites is principally impossible. This is in sharp contrast to an ordinary QCD case where the left-handed and right-handed quarks forms the  $\langle \bar{q}_L q_R \rangle$  condensate thus leading to the  $\Lambda_C$  order masses ( $\Lambda_C \sim (0.1 \div 1) \text{ GeV}$ ) for mesons and baryons. Due to the L-R symmetry, the metacolor symmetry groups  $G_{MC}^L$  and  $G_{MC}^R$  are taken identical with a similar confinement for both of sets of preons. If one also proposes that the preon metacolor symmetry  $G_{MC}$  is generically anomaly-free for any matter content involved, then for independent left-handed and right-handed preons one comes to an input chiral orthogonal symmetry of the type

$$G_{MC} = SO(n)_{MC}^L \times SO(n)_{MC}^R, \quad n = 3, 5, \dots \quad (3.2)$$

for the  $n$ -preon configurations of composites. For reasons of economy, it is usually proposed that fermion composites have the minimal 3-preon configuration.

- Apart from the local symmetries, metacolors and metaflavors, the preons  $P_{iL}^\alpha$  and  $P_{iR}^{\alpha'}$  possess an accompanying chiral global symmetry

$$K(N) = SU(N)_L \times SU(N)_R \quad (3.3)$$

being unbroken at the small distances. We omitted above the Abelian chiral  $U(1)_{L,R}$  symmetries in  $K(N)$  since the corresponding currents have Adler-Bell-Jackiw anomalies in the triangle graph where they couple to two metagluons [7]. In fact, their divergences for massless preons are given by

$$\partial_\mu J_{L,R}^\mu = n \frac{g_{L,R}^2}{16\pi^2} G_{L,R}^{\mu\nu} G_{L,R}^{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \quad (3.4)$$

where  $G_{L,R}^{\mu\nu}$  are the metagluon field strengths for the  $SO(n)_{MC}^{L,R}$  metacolors, respectively, while  $g_{L,R}$  are the appropriate gauge coupling constants. Thereby,

the chiral symmetries  $U(1)_{L,R}$ , which would present the conserved chiral hypercharges in the classical Lagrangian with massless preons, appear broken by the quantum corrections (3.4) that make us to leave only the non-Abelian chiral symmetry (3.3) in the theory. Nevertheless, one could presumably still use these symmetries at the small distances,  $r \ll R_{MC}$ , where the corrections (3.4) may become unessential due to asymptotic freedom in the metacolor theory considered. We will refer to this regime as the valent preon approximation in which one may individually recognize each preon no matter it is free or bound in a composite fermion. Therefore, the chiral preon numbers or hypercharges  $Y_{L,R}$  related to the symmetries  $U(1)_{L,R}$  may be considered in this approximation as the almost conserved classical charges according to which the preon and composite states are allowed to be classified.

- The fact that the left-handed and right-handed preons do not form the  $\langle \bar{P}_L P_R \rangle$  condensate may be generally considered as a necessary but not yet a sufficient condition for masslessness of composites. The genuine massless fermion composites are presumably only those which preserve chiral symmetry of preons (3.3) at large distances that is controlled by the 't Hooft's anomaly matching (AM) condition [3]. For reasons of simplicity, we do not consider below boson composites, the effective scalar or vector fields. Generally, they being no protected by any symmetry will become very heavy (with masses of the order of the compositeness scale  $\Lambda_{MC}$ ) and decouple from a low-lying particle spectrum.

### 3.3 AM conditions for N metaflavors

The AM condition [3] states in general that triangle anomalies related to N massless elementary preons, both left-handed and right-handed, have to match those for massless fermions (including quarks and leptons) being composed by the metacolor forces arranged by the proposed local symmetry  $SO(n)_{MC}^L \times SO(n)_{MC}^R$ . Based on the starting L-R symmetry in our model we will require, in some contrast to the original AM condition [3], that fermions composed from the left-handed and right-handed preons with their own metacolors,  $SO(n)_{MC}^L$  and  $SO(n)_{MC}^R$ , have to satisfy the AM condition separately, whereas the metaflavor triangle anomalies of the L-preon and R-preon composites may in general compensate each other. Therefore, in our L-R symmetric preon model one does not need to specially introduce elementary metacolor singlet fermions, called the "spectator fermions" [3], to cancel these anomalies both at the small and large distances.

The AM condition puts in general powerful constraints on the classification of massless composite fermions with respect to the underlying local metaflavor symmetry  $SU(N)_{MF}$  or some of its subgroups, depending on the extent to which the accompanying global chiral symmetry (3.3) of preons remains at large distances. In one way or another, the AM condition

$$\sum_r i_r a(r) = na(N) \quad (3.5)$$

for preons (the right side) and composite fermions (the left side) should be satisfied. Here  $a(N)$  and  $a(r)$  are the group coefficients of triangle anomalies related to the

groups  $SU(N)_L$  or  $SU(N)_R$  in (3.3) whose coefficients are calculated in an ordinary way,

$$a(r)d^{ABC} = \text{Tr}(\{T^A T^B\}T^C)_r, \quad \text{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB} \quad (3.6)$$

where  $T^A$  ( $A, B, C = 1, \dots, N^2 - 1$ ) are the  $SU(N)$  generators taken in the corresponding representation  $r$ . The  $a(N)$  corresponds a fundamental representation and is trivially equal to  $\pm 1$  (for left-handed and right-handed preons, respectively), while  $a(r)$  is related to a representation  $r$  for massless composite fermions. The values of the factors  $i_r$  give a number of times the representation  $r$  appears in a spectrum of composite fermions and are taken positive for the left-handed states and negative for the right-handed ones.

The anomaly coefficients for composites  $a(r)$  contain an explicit dependence on the number of preons  $N$ , due to which one could try to find this number from the AM condition. However, in general, there are too many solutions to the condition (3.5) for any value of  $N$ . Nevertheless, for some special, though natural, requirements an actual solution may only appear for  $N = 8$ , as we will see below.

Indeed, to strengthen the AM condition one could think that it would more appropriate to have all composite quarks and leptons in a single representation of the unified symmetry group rather than in some set of its representations. This, though would not largely influence the gauge sector of the unified theory, could make its Yukawa sector much less arbitrary. Apart from that, the composites belonging to different representations would have in general different preon numbers that could look rather unnatural. Let us propose for the moment that we only have the minimal three-preon fermion composites formed by the metacolor forces which correspond to the  $SO(3)_{MC}^L \times SO(3)_{MC}^R$  symmetry case in (3.2). We will, therefore, require that only some single representation  $r_0$  for massless three-preon states has to satisfy the AM condition that simply gives in (3.5)

$$a(r_0) = 3 \quad (3.7)$$

individually for L-preon and R-preon composites.

Now, calculating the anomaly coefficients for all possible three L-preon and three R-preon composites one respectively has (see also [4,8])

$$\begin{aligned} &\Psi_{\{ijk\}L,R}, N^2/2 + 9N/2 + 9, \\ &\Psi_{[ijk]L,R}, N^2/2 - 9N/2 + 9, \\ &\Psi_{\{[ij]k\}L,R}, N^2 - 9, \\ &\Psi_{\{jk\}L,R}^i, N^2/2 + 7N/2 - 1, \\ &\Psi_{[jk]L,R}^i, N^2/2 - 7N/2 - 1 \end{aligned} \quad (3.8)$$

with all appropriate  $SU(N)_{L,R}$  representations listed (anomaly coefficients for right-handed composites have to be taken with an opposite sign). Putting then each of the above anomaly coefficients in (3.8) into the AM condition (3.7) one can readily find that there is a solution with an integer  $N$  only for the last tensors  $\Psi_{[jk]L,R}^i$ , and this is in fact the unique "eightfold" solution

$$N^2/2 - 7N/2 - 1 = 3, \quad N = 8. \quad (3.9)$$

Remarkably, the same solution  $N = 8$  independently appears if one requires that the  $SU(N)$  symmetry has to possess the right  $SU(5)$  GUT assignment [5] for the observed quark-lepton families in order to be in fact in accordance with observations. This means that one of its specified representations has to contain an equal numbers of the  $SU(5)$  anti-quintets  $\bar{5}^k$  and decuplets  $10_{[k,l]}$  ( $k, l$  are the  $SU(5)$  indices). Indeed, decomposing  $SU(N)$  into  $SU(5) \times SU(N-5)$  one find that this equality does not exist for any 3-index representation (3.8) but the last one  $\Psi_{[jk]}^i$ , for which it reads as

$$(N-5)(N-6)/2 = N-5, \quad N=8 \quad (3.10)$$

thus leading again to the "eightfold"  $SU(8)$  metaflavor symmetry.

Let us note that, apart the minimal 3-preon states, there are also possible some alternative higher preon configurations for composite quarks and leptons. Moreover, the  $SO(3)_{MC}^{L,R}$  metacolors providing the three-preon structure of composite quarks and leptons may appear insufficient for the preon confinement, unless one invokes some special strong coupling regime [9]. For the asymptotically free  $SO(n)$  metacolor, one must generally require  $n > 2 + 2N/11$ , due to which the composite quarks and leptons have to appear at least as five-preon states. Checking generally all possible  $n$ -index representations of  $SU(N)$  we find that the AM condition only works for some combination of its "traced" tensors  $\Psi_{[jk]L,R}^i$  and  $\Psi_{iL,R}$  obtained after taking traces out of the proper  $n$ -index tensors  $\Psi_{[jk\dots]L,R}^{i\dots}$ . This eventually leads to the equation generalizing the above anomaly matching condition (3.9)

$$N^2/2 - 7N/2 - 1 + p = n \quad (3.11)$$

where  $p$  is a number of the traced fundamental multiplets  $\Psi_{iL,R}$  for composites. One can see that there appear some reasonable solutions only for  $n - p = 3$  and, therefore, one has again solutions for the "eightfold" metaflavor symmetry  $SU(8)$ .

Apart from the AM condition (3.5) there would be in general another kind of constraint on composite models which has been also proposed in [3]. This constraint requires the anomaly matching for preons and composites, even if some of introduced  $N$  preons become successively heavier than the scale of compositeness and consequently decouple from the entire theory. As a result, the AM condition should work for any number of preons remained massless (thus basically being independent of  $N$ ) that could make generally classification of composite fermions quite arbitrary. Fortunately, such an extra constraint is not applicable to our L-R symmetric model where the Dirac masses for preons are not possible by definition, whereas the Majorana masses would mean breaking of the input local metaflavor  $SU(N)_{MF}$  symmetry<sup>1</sup>.

Most importantly, the orthogonal symmetry for metacolor (3.2) allows to consider more possible composite configurations than it is in the case of an unitary metacolor symmetry, as in the conventional  $SU(3)$  color for QCD. The above strengthening of the AM condition, according to which composites only fill a

<sup>1</sup> Apart from that, it has been generally argued [5] that the nonperturbative effects may not be analytic in the preon mass so that for the large and small preon masses the theories may be quite different, thus avoiding this additional constraint.

single multiplet of the metaflavor  $SU(N)_{MF}$  symmetry group, has unambiguously led us to the composite multiplets  $\Psi_{[jk]L,R}^i$  having the same classical  $U(1)_{L,R}$  fermion numbers or hypercharges  $Y_{L,R}$  as the preons themselves. We argued in the previous section that these hypercharges may be considered in the valent preon approximation as the almost conserved classical charges according to which the preon and composite states could be classified. With all that in mind, one could assume that there may work some extra selection rule according to which only composites satisfying the condition

$$Y_{L,R}(\text{preons}) = Y_{L,R}(\text{composites}) \quad (3.12)$$

appear in physical spectrum in the orthogonal left-right metacolor case.

We can directly see that the condition (3.12) trivially works for the simplest composite states which could be constructed out of a single preon  $P_{iL}^\alpha$  or  $P_{iR}^{\alpha'}$ , whose metacolor charge is screened by the metagluon fields  $A_{L\mu}^\alpha$  and  $A_{R\mu}^{\alpha'}$  of  $SO(n)_{MC}^L$  and  $SO(n)_{MC}^R$ , respectively. These composites will also satisfy the general AM condition (3.5) provided that one admits the  $n$  left-handed and right-handed fundamental multiplets of the  $SU(N)_{MF}$  to participate ( $i_N = n$ ). In our L-R symmetric model, however, such massless composites will necessarily pair up, thus becoming very massive and decoupling from the low-lying particle spectrum, no matter the starting L-R symmetry becomes later broken or not. This in sharp contrast to the models [8] with the orthogonal metacolor group  $SO(n)$  for the single chirality preons, where such massless composite generally appear to be in contradiction with observations. Moreover, in this case the composite multi-preon states for quarks and leptons seem hardly to be stable, since they could freely dissociate into three screened preon states.

One could wonder why the condition (3.12) does not work in the familiar QCD case with elementary quarks and composite baryons. The point is that, despite some conceptual similarity, QCD is the principally different theory. The first and immediate is that the unitary color  $SU(3)_C$ , in contrast to the orthogonal ones, allows by definition no other quark number for baryons but  $Y_B = 3Y_q$ . The most important aspect of this difference is, however, that the color symmetry  $SU(3)_C$  is vectorlike due to which chiral symmetry in QCD is broken by quark-antiquark condensates with the corresponding zero-mass Goldstone bosons (pions, kaons etc.) providing the singularity of the three-point function. As a consequence, the AM condition implies in this case that dynamics requires spontaneous breakdown of chiral symmetry rather than an existence of massless composite fermions, as happens in the orthogonal metacolor case discussed above.

We find below in section 3.5 that, though the proposed condition (3.12) looks rather trivial in the L-R symmetry phase of the theory, it may become rather significant when this symmetry becomes spontaneously broken.

### 3.4 Composites - the L-R symmetry phase

So, we have at small distances the preons given by the Weil fields

$$P_{iL}^\alpha, P_{iR}^{\alpha'} \quad (i = 1, \dots, 8; \alpha = 1, 2, 3; \alpha' = 1, 2, 3) \quad (3.13)$$

belonging to the fundamental octet of the local metaflavor symmetry  $SU(8)_{MF}$  and to triplets of the metacolor symmetry  $SO(3)_{MC}^L \times SO(3)_{MC}^R$  which are local, and there is also the accompanying global chiral symmetry

$$K(8) = SU(8)_L \times SU(8)_R \quad (3.14)$$

of the eight preon species (3.13). At large distances, on the other hand, we have composites located, respectively, in the left-handed and right-handed multiplets of the  $SU(8)_{MF}$

$$\Psi_{[jkl]L}^i(216), \Psi_{[jkl]R}^i(216), \quad (3.15)$$

where their dimensions are explicitly indicated. The chiral symmetry (3.14), according to the AM condition taken, remains at large distances. Due to a total L-R symmetry of preons and composites the triangle anomalies both at small and large distances appears automatically compensated. Decomposing the  $SU(8)_{MF}$  composite multiplets (3.15) into the  $SU(5) \times SU(3)_F$  components one has

$$216_{L,R} = [(\bar{5} + 10, \bar{3}) + (45, 1) + (5, 8) + (24, 3) + (1, 3) + (1, \bar{6})]_{L,R} \quad (3.16)$$

where the first term for the left-handed composites,  $(\bar{5} + 10, \bar{3})_L$ , could be associated with the standard  $SU(5)$  GUT assignment for quarks and leptons [5] extended by some family symmetry  $SU(3)_F$ , while other multiplets are somewhat exotic and, hopefully, could be made heavy to decouple them from an observed low-lying particle spectrum.

The determination of the explicit form of the wave function for composite states (3.15) is a complicated dynamical problem related to the yet unknown dynamics of the preon confinement. We propose that some basic feature of these composites are simply given by an expression

$$\Psi_{[jkl]L}^i(x) \propto \epsilon_{\alpha\beta\gamma} \left( \bar{P}_L^{\alpha i} \gamma_\mu P_{L[j}^\beta \right) \gamma^\mu P_{kl]L}^\gamma(x) \quad (3.17)$$

where indices  $\alpha, \beta, \gamma$  belong to the metacolor symmetry  $SO(3)_{MC}^L$ . In the valent preon approximation, the preon current (3.17) corresponds to a bound state of three left-handed preons with zero mass ( $p^2 = (p_1 + p_2 + p_3)^2 = 0$ ) being formed by massless preons ( $p_1^2 = p_2^2 = p_3^2 = 0$ ) which are moving in a common direction. It is then clear that a state with a spin of  $1/2$  (and a helicity  $-1/2$ ) can be only obtained by assembling two preons and one antipreon into a quark or lepton. In a similar way one can construct the preon current  $\Psi_{[jkl]R}^i$  which corresponds to a multiplet of states again with a spin of  $1/2$  (but a helicity  $+1/2$ ) composed from right-handed preons. This is simply achieved by making the proper replacements in (3.17) leading to the composite states

$$\Psi_{[jkl]R}^i(x) \propto \epsilon_{\alpha'\beta'\gamma'} \left( \bar{P}_R^{\alpha' i} \gamma_\mu P_{R[j}^{\beta'} \right) \gamma^\mu P_{kl]R}^{\gamma'}(x) \quad (3.18)$$

where indices  $\alpha', \beta', \gamma'$  belong now to the metacolor symmetry  $SO(3)_{MC}^R$ . For the simplest composite states which can be constructed out of a single preon  $P_{iL}^\alpha$  or  $P_{iR}^{\alpha'}$ , whose metacolor charge is screened by the metagluon fields  $A_{L\mu}^\alpha$  and  $A_{R\mu}^{\alpha'}$  of  $SO(3)_{MC}^L$  and  $SO(3)_{MC}^R$ , the wave functions may be written as

$$\Psi_{iL}(x) \propto A_{L\mu}^\alpha \gamma^\mu P_{iL}^\alpha(x), \quad \Psi_{iR}(x) \propto A_{R\mu}^{\alpha'} \gamma^\mu P_{iR}^{\alpha'}(x), \quad (3.19)$$

respectively.

Let us remark in conclusion that the whole theory so far considered is certainly vectorlike with respect to preons (3.13), as well as composites (3.15). This means that, while preons are left massless being protected by their metacolors, all the L-preon and R-preon composites being metacolor singlets will pair up due to some quantum gravitational transitions and, therefore, acquire some Dirac masses. We find below that due to closeness of the compositeness scale  $\Lambda_{MC}$  to the Planck scale  $M_{Pl}$  the masses of all composites appear very heavy that has nothing in common with reality. It is rather clear that such a theory is meaningless unless the L-R symmetry is somehow broken that seems to be in essence a basic point in our model. One could expect that such breaking may somehow exclude the right-handed submultiplet  $(\bar{5} + 10, \bar{3})_R$  in the composite spectrum (3.16), while leaving there its left-handed counterpart,  $(\bar{5} + 10, 3)_L$ , which can be then uniquely associated with the observed three families of ordinary quarks and leptons.

### 3.5 Composites - partially broken L-R symmetry

We propose that there a partial breaking of the chiral symmetry (3.14) in the right-handed preon sector of the type

$$K(8) \rightarrow SU(8)_L \times [SU(5) \times SU(3)]_R \quad (3.20)$$

For convenience, we consider a supersymmetric model where this breaking may be caused presumably due to the asymmetric preon condensation

$$\epsilon_{\alpha\beta\gamma} \langle P_{iL}^\alpha P_{jL}^\beta P_{kL}^\gamma \rangle = 0, \quad \epsilon_{\alpha'\beta'\gamma'} \langle P_{iR}^{\alpha'} P_{jR}^{\beta'} P_{kR}^{\gamma'} \rangle = \delta_i^a \delta_j^b \delta_k^c \epsilon_{abc} \Lambda_{MC}^4 \quad (3.21)$$

emerging for preon superfields with their fermion and scalar field components involved. Here antisymmetric third-rank tensors  $\epsilon_{\alpha\beta\gamma}$  and  $\epsilon_{\alpha'\beta'\gamma'}$  belong to the metacolor symmetries  $SO(3)_{MC}^L$  and  $SO(3)_{MC}^R$ , respectively, while  $\epsilon_{abc}$  ( $a, b, c = 1, 2, 3$ ) to the symmetry  $SU(3)_R$ . Remarkably, the breaking (3.20) is only possible when the number of metacolors  $n$  is equal 3 or 5, as is actually implied in our model. For the minimal case,  $n = 3$ , the vacuum configurations (3.21) could spontaneously appear in some L-R symmetric model with the properly arranged high-dimensional preon interactions<sup>2</sup>

$$\begin{aligned} & \sum_{n=1}^{\infty} \{ G_n^{LL} [(\bar{P}_L \bar{P}_L \bar{P}_L) (P_L P_L P_L)]^n + G_n^{RR} [(\bar{P}_R \bar{P}_R \bar{P}_R) (P_R P_R P_R)]^n \\ & + G_n^{LR} [(\bar{P}_L \bar{P}_L \bar{P}_L) (P_R P_R P_R)]^n + G_n^{RL} [(\bar{P}_R \bar{P}_R \bar{P}_R) (P_L P_L P_L)]^n \} \end{aligned} \quad (3.22)$$

with coupling constants satisfying the conditions  $G_n^{LL} = G_n^{RR}$  and  $G_n^{LR} = G_n^{RL}$ . This model is evidently non-renormalizable and can be only considered as an effective theory valid at sufficiently low energies. The dimensionful couplings  $G_n$

<sup>2</sup> This L-R symmetry breaking model looks somewhat similar to the well-known multi-fermion interaction schemes used in the other contexts for chiral symmetry breaking [10] or spontaneous Lorentz violation [11].



are proportional to appropriate powers of some UV cutoff  $\Lambda$  which in our case can be ultimately related to the preon confinement energy scale  $\Lambda_{MC}$ ,  $G_n \sim \Lambda_{MC}^{4-8n}$ . For some natural choice of these coupling constants one may come to the asymmetric solution (3.21).

A more conventional way of getting the L-R asymmetry may follow from the symmetric scalar field potential [5]

$$U = M^2(\Phi_L^2 + \Phi_R^2) + h(\Phi_L^2 + \Phi_R^2)^2 + h'\Phi_L^2\Phi_R^2 + P(\Phi_L, \Phi_R) \quad (3.23)$$

containing two elementary third-rank antisymmetric scalar fields,  $\Phi_L^{[ijk]}$  and  $\Phi_R^{[ijk]}$ , interacting with L- and R-preons, respectively. For some natural area of the parameters in the potential,  $M^2 < 0$  and  $h, h' > 0$ , and properly chosen couplings for scalars  $\Phi_L^{[ijk]}$  and  $\Phi_R^{[ijk]}$  in the polynomial  $P(\Phi_L, \Phi_R)$  they may readily develop the totally asymmetric VEV configuration

$$\langle \Phi_L^{[ijk]} \rangle = 0, \quad \langle \Phi_R^{[ijk]} \rangle = \delta_a^i \delta_b^j \delta_c^k \epsilon^{abc} M_{LR} \quad (a, b, c = 1, 2, 3) \quad (3.24)$$

where the mass  $M_{LR}$  corresponds to the L-R symmetry breaking scale and indices  $a, b, c$  belong to the  $SU(3)_R$ . Due to these VEVs, the higher dimension terms in the effective superpotential induced generally by gravity

$$\frac{G_L}{M_{Pl}} (P_{iL} P_{jL} P_{kL}) \Phi_L^{[ijk]} + \frac{G_R}{M_{Pl}} (P_{iR} P_{jR} P_{kR}) \Phi_R^{[ijk]} \quad (3.25)$$

( $G_{L,R}$  are dimensionless coupling constants) will change the AM conditions for right-handed states leaving those for the left-handed ones intact. This modification is related to an appearance of a new Yukawa interaction for preons

$$G'_R \epsilon^{abc} (P_{aR}^{(f)} C P_{bR}^{(f)}) P_{cR}^{(s)}, \quad G'_R = G_R \frac{M_{LR}}{M_{Pl}} \quad (3.26)$$

where  $P_{iR}^{(f,s)}$  are, respectively, the fermion and scalar field components of the right-handed preon superfield  $P_{iR}$ . This interaction will give some extra radiative corrections to the triangle graphs with circulating "family" preons  $P_{aR}^{(f)}$  ( $a = 1, 2, 3$ ) and their composites. As a result, the triangle anomalies corresponding to all generators of the  $SU(8)_R$ , besides those of the  $[SU(5) \times SU(3)]_R$ , are left uncompensated, that causes the proper decreasing of the chiral symmetry, just as is indicated in (3.20).

Eventually, while there still remains the starting chiral symmetry for the left-handed preons and their composites, for the right-handed states we only have the broken symmetry given in (3.20). Therefore, whereas nothing changes for the L-preon composites filling the total multiplet  $216_L$  in (3.16), the R-preons will only compose some particular submultiplets in  $216_R$  (3.16) which in general may not include the three right-handed quark-lepton families  $(\bar{5} + 10, 3)_R$ . We can simply postulate it as some possible ansatz being allowed by the different chiral symmetries in the L-preon and R-preon sectors in the L-R symmetry broken phase. Nonetheless, it would be interesting to argue it using the preon number matching

condition (3.12) which we discussed in section 3.3. Note first that the  $U(1)_R$  symmetry in the right-handed sector reduces after the L-R symmetry breaking (3.21, 3.24) to

$$U(1)_R \rightarrow U(1)_R^{(5)} \times Z(3)_R^{(3)} \quad (3.27)$$

while the  $U(1)_L$  symmetry is left intact. Here,  $U(1)_R^{(5)}$  and  $Z(3)_R^{(3)}$  stand for the survived continuous and discrete symmetries of quintet preons  $P_{sR}$  ( $s = 1, \dots, 5$ ) of  $SU(5)_R$  and triplet preons  $P_{aR}$  ( $a = 1, 2, 3$ ) of  $SU(3)_R$ , respectively, which are thereby separated. Namely, the R-preon hypercharge group in the broken L-R symmetry phase is given by the product (3.27) rather than the universal  $U(1)_R$  for all eight preons, as was in its unbroken phase. Now, if we require the preon number matching for preons and composites the states collected in  $(\bar{5} + 10, \bar{3})_R$  will never appear in physical spectrum. Indeed, as one can easily check, both the  $U(1)_R^{(5)}$  hypercharge and discrete  $Z(3)_R^{(3)}$  symmetry values for these states are quite different from those for the preons  $P_{sR}$  and  $P_{aR}$ , respectively. At the same time, all other composite submultiplets in  $216_R$  (3.16) readily match the both symmetry values for preons.

One way or another, the simplest combination of the  $216_R$  submultiplets which may simultaneously satisfy the AM conditions for the  $[SU(5) \times SU(3)]_R$  symmetry, as well as the above preon number matching condition is in fact given by the collection

$$(45, 1)_R + (5, 8 + 1)_R + 3(1, 3)_R \quad (3.28)$$

where the submultiplet  $(1, 3)_R$  has to appear three times in (3.28) in order to appropriately restore the anomaly coefficient balance for the R-preon composites. Of course, this collection of states can also appear by its own without any reference to the preon number matching condition that we have used above as some merely heuristic argument.

### 3.6 Physical sector - quarks and leptons

We can see that after chiral symmetry breaking in the sector of the right-handed preon composites the starting metaflavor symmetry  $SU(8)_{MF}$  at large distances is reduced to the product of the standard  $SU(5)$  GUT and chiral family symmetry  $SU(3)_F$

$$SU(8)_{MF} \rightarrow SU(5) \times SU(3)_F \quad (3.29)$$

presumably with the equal gauge coupling constants  $g_5$  and  $g_{3F}$  at the grand unification scale. This is in essence the chiral remnant of the initially emerged vectorlike  $SU(8)_{MF}$  symmetry. The massless composite fermions, due to pairing up of the similar L-preon and R-preon composites and decoupling them from a low-energy spectrum, are given now by the collection of the  $SU(5) \times SU(3)_F$  multiplets

$$(\bar{5} + 10, \bar{3})_L + (24, 3)_L + 2(1, 3)_R + (1, \bar{6})_R \quad (3.30)$$

which automatically appear free from both the  $SU(5)$  and  $SU(3)_F$  anomalies. They contain just three conventional families of quarks and leptons plus massive multiplets located on the family symmetry scale  $M_F$ . In order to sufficiently suppress

all flavor-changing transitions, which would induce the family gauge boson exchanges, this scale should be at least of the order  $10^{5\div 6}$  GeV, though in principle it could be as large as the SU(5) GUT scale. In the latter case, some of the heavy states in (3.30) could be considered as candidates for the superheavy right-handed neutrinos. One can argue that the physical composite multiplets (3.30) appear not only for the triple metacolor,  $n = 3$ , but in general case as well. Indeed, using the remark concerning the generalized AM condition (3.11) and properly extending the left-handed multiplets in (3.16) and the right-handed multiplets in (3.28) by the new  $n - 3$  fundamental composite octets  $[(5, 1) + (1, 3)]_{L,R}$  to have anomaly matching for any number  $n$  of metacolors, one comes after pairing of the identical multiplets to the same physical remnant (3.30) as in the triple metacolor case.

It is important to note that the tiny radius of compositeness for universal preons composing both quarks and leptons makes it impossible to directly observe their composite nature [12]. Indeed, one can readily see that the quark pair  $u + d$  contains the same preons as the antiquark-antilepton pair  $\bar{u} + e^+$  that will lead to the process

$$u + d \rightarrow \bar{u} + e^+ \quad (3.31)$$

and consequently to the proton decay  $p \rightarrow \pi^0 + e^+$  just due to a simple rearrangement of preons in a proton. To prevent this one should take the compositeness scale  $\Lambda_{MC}$  of the order of the scale of the SU(5) GUT or even larger,  $\Lambda_{MC} \gtrsim M_{GUT} \approx 2 \cdot 10^{16}$  GeV, and, respectively,  $R_{MC} \leq 5 \cdot 10^{-31}$  sm.

This limit on the radius of compositeness may in turn cause limits on the composite fermions masses appearing as a result of the quantum gravitational transitions of the identical states in the left-handed multiplets (3.15) and right-handed multiplets (3.28),

$$(45, 1)_{L,R} + (5, 8 + 1)_{L,R} + (1, 3)_{L,R}, \quad (3.32)$$

into each other. From dimensional arguments related to a general structure of the composites proposed above (3.17), these masses could be of the order

$$(\Lambda_{MC}/M_{Pl})^5 \Lambda_{MC}$$

(that corresponds in fact to the 6-fermion interaction of the left-handed and right-handed preons) and, in fact, are very sensitive to the confinement scale  $\Lambda_{MC}$ . Actually, for the metacolor scales,  $M_{GUT} \leq \Lambda_{MC} \leq M_{Pl}$ , the heavy fermion masses may be located at the scales from O(1) TeV up to the Planck mass scale. Therefore, the heavy composite states may be of direct observation interest if they are located near the low limit, or otherwise they will populate the SU(5) GUT desert. Interestingly, the screened preon states (3.19)

$$(5, 1)_{L,R} + (1, 3)_{L,R} \quad (3.33)$$

acquire much heavier masses when being pairing with each other. Again, from the dimensional arguments one may conclude that these masses has a natural order  $(\Lambda_{MC}/M_{Pl})\Lambda_{MC}$  that is significantly larger than masses of the 3-preon states (3.17, 3.18).

Note that some of the heavy states (3.32) can mix with ordinary quarks and leptons given by the multiplet  $(\bar{5} + 10, \bar{3})_L$  in (3.30). Particularly, there could be the large mixing term of the part  $(\bar{5}, \bar{3})_L$  containing the lepton doublet and down antiquarks with the multiplet  $(5, 8 + 1)_R$  in (3.32). This term has a form

$$(\bar{5}, \bar{3})_L (5, 8 + 1)_R (1, 3) \quad (3.34)$$

where  $(1, 3)$  stands for some pure "horizontal" scalar field being a triplet of the family symmetry  $SU(3)_F$ . Actually, this mixing is related again to the 6-fermion gravitational interaction of the left-handed and right-handed preons, thus leading to the nondiagonal masses of the order  $(\Lambda_{MC}/M_{Pl})^5 M_F$ . Thereby, in order not to significantly disturb the masses of quarks and leptons in (3.30) one has to generally propose  $M_F \ll \Lambda_{MC}$ . This in fact is readily satisfied even for high family scales, namely, in the case when the scale  $M_F$  is taken near the grand unification scale  $M_{GUT}$ , while the scale  $\Lambda_{MC}$  near the Planck scale  $M_{Pl}$ . The more liberal limitations appears when that part  $(\bar{5}, \bar{3})_L$  mixes with the screen preon states  $(5, 1)_R$  in (3.33) due to the same scalar triplet  $(1, 3)$  of the  $SU(3)_F$ . Now, this mixing caused by the 4-fermion interaction leads to the nondiagonal mass of the order  $(\Lambda_{MC}/M_{Pl})^2 M_F$  that may be naturally much lesser than diagonal mass  $(\Lambda_{MC}/M_{Pl})\Lambda_{MC}$  derived above for the screened preon state. Nevertheless, depending on real values of the scales  $\Lambda_{MC}$  and  $M_F$  there could be expected some violation of unitarity in the conventional  $3 \times 3$  mass matrices of leptons and down quarks which may be of a special interest for observations. Other mixings of quarks and leptons with heavy states (3.32) and (3.33) will necessarily include an ordinary Higgs quintet of the grand unified  $SU(5)$  (or a doublet of the SM) and, therefore, are negligibly small.

To conclude, our preon model predicts three types of states which are (1) three families of ordinary quarks and leptons  $(\bar{5} + 10, \bar{3})_L$  in (3.30) with masses at the electroweak scale, (2) the heavy chiral multiplets  $(24, 3)_L + 2(1, 3)_R + (1, \bar{6})_R$  (3.30) with the Majorana type masses at the family scale  $M_F = 10^{6 \div 16}$  GeV and (3) the heavy paired multiplets (3.32) with masses in the interval  $10^{3 \div 19}$  GeV which are related to the gravitational transition amplitudes of the L-preon composites into the R-preon ones. However, the most important prediction of the left-right preon model considered here is, indeed, an existence of the local chiral family (or horizontal) symmetry  $SU(3)_F$  for quark-lepton generations which is briefly presented below.

### 3.7 The chiral family symmetry $SU(3)_F$

The flavor mixing of quarks and leptons is certainly one of the major problems that presently confront particle physics. Many attempts have been made to interpret the pattern of this mixing in terms of various family symmetries - discrete or continuous, global or local. Among them, the chiral family symmetry  $SU(3)_F$  derived first in the similar preon framework [4] and developed then by its own by many authors [13-22] seems most promising. As was shown, the spontaneous breaking of this symmetry gives some guidance to the observed hierarchy between elements of the quark-lepton mass matrices, on the one hand, and to presence

of texture zeros in them, on the other, that leads to relationships between the mass and mixing parameters. In the framework of the supersymmetric Standard Model, it leads, at the same time, to an almost uniform mass spectrum for the superpartners, with a high degree of flavor conservation, that makes its existence even more significant in the SUSY case.

Generically, the chiral family symmetry  $SU(3)_F$  possesses four basically attractive features:

- (i) It provides a natural explanation of the number three of observed quark-lepton families, correlated with three species of massless or light ( $m_\nu < M_Z/2$ ) neutrinos contributing to the invisible Z boson partial decay width;
- (ii) Its local nature conforms with the other local symmetries of the Standard Model, such as the weak isospin symmetry  $SU(2)_w$  or color symmetry  $SU(3)_c$ , thus leading to the family-unified SM with a total symmetry  $SM \times SU(3)_F$ ;
- (iii) Its chiral nature, according to which both left-handed and right-handed fermions are proposed to be fundamental triplets of the  $SU(3)_F$ , provides the hierarchical mass spectrum of quark-lepton families as a result of a spontaneous symmetry breaking at some high scale  $M_F$  which could in principle be located in the area from  $10^{5 \div 6}$  GeV (to properly suppress the flavor-changing processes) up to the grand unification scale  $M_{GUT}$  and even higher. Actually, any family symmetry should be completely broken in order to conform with reality at lower energies. This symmetry should be chiral, rather than a vectorlike, since a vectorlike symmetry would not forbid the invariant mass, thus leading in general to degenerate rather than hierarchical mass spectra. Interestingly, both known examples of local vectorlike symmetries, electromagnetic  $U(1)_{EM}$  and color  $SU(3)_C$ , appear to be exact symmetries, while all chiral symmetries including conventional grand unifications [5]  $SU(5)$ ,  $SO(10)$  and  $E(6)$  (where fermions and antifermions lie in the same irreducible representations) appear broken;
- (iv) Thereby, due to its chiral structure, the  $SU(3)_F$  admits a natural unification with all known GUTs in a direct product form, both in an ordinary and supersymmetric framework, thus leading to the family-unified GUTs,  $GUT \times SU(3)_F$ , beyond the Standard Model.

So, if one takes these naturality criteria seriously, all the candidates for flavor symmetry can be excluded except for local chiral  $SU(3)_F$  symmetry. Indeed, the  $U(1)$  family symmetry does not satisfy the criterion (i) and is in fact applicable to any number of quark-lepton families. Also, the  $SU(2)$  family symmetry can contain, besides two light families treated as its doublets, any number of additional (singlets or new doublets of  $SU(2)$ ) families. All the global non-Abelian symmetries are excluded by criterion (ii), while the vectorlike symmetries are excluded by the last criteria (iii) and (iv).

Among applications of the  $SU(3)_F$  symmetry, the most interesting ones are the description of the quark and lepton masses and mixings in the Standard Model and GUTs [13], neutrino masses and oscillations [15] and rare processes [16] including their astrophysical consequences [22]. Remarkably, the  $SU(3)_F$  invariant Yukawa coupling are always accompanied by an accidental global chiral  $U(1)$  symmetry, which can be identified with the Peccei-Quinn symmetry [18] provided it is not explicitly broken in the Higgs sector, thus giving a solution to

the strong CP problem [17]. In the SUSY context [19], the  $SU(3)_F$  model leads to a special relation between (s)fermion masses and the soft SUSY breaking terms at the GUT scale in a way that all the dangerous flavor-changing processes are naturally suppressed. The special sector of applications is related to a new type of topological defects - flavored cosmic strings and monopoles appearing during the spontaneous violation of the  $SU(3)_F$  which may be considered as possible candidates for the cold dark matter in the Universe [20].

Let us note in conclusion that if the family symmetry  $SU(3)_F$  arises from the preon model proposed above one can expect that in the emerged  $SU(5) \times SU(3)_F$  GUT the gauge coupling constants  $g_5$  and  $g_{3F}$  should be equal at the  $SU(8)_{MF}$  unification scale. The study of flavor changing processes  $\mu \rightarrow e + \gamma$ ,  $D^0 - \bar{D}^0$ ,  $B^0 - \bar{B}^0$  and others caused by the  $SU(3)_F$  gauge boson exchanges could in principle show whether the family symmetry has an origin in the preon model or it is, rather, independently postulated. However, the most crucial difference between these two cases is related to the existence in the preon model of some heavy  $SU(5) \times SU(3)_F$  multiplets located at scales from  $O(1)$  TeV up to the Planck mass. If they are relatively light, they may be of direct observational interest by them own. If they are heavy, they still strongly affect the quark-lepton mass matrices due to their large mixings with the down quarks and leptons, as was shown in (3.34). Remarkably, even if the family symmetry  $SU(3)_F$  is taken at the GUT scale the difference between these cases is still left. Indeed, now all flavor-changing transitions due to the family gauge boson exchange will be extremely suppressed, while for the independently introduced family symmetry these transitions may significantly contribute into rates of the nondiagonal processes. Moreover, for the high scale family symmetry one has some natural candidates for massive right-handed neutrinos in terms of the extra heavy states given in (3.30).

### 3.8 Conclusion and outlook

We have shown that, apart from somewhat inspirational religious and philosophical aspects ensured by the Eightfold Way, the  $SU(8)$  symmetry as a basic internal symmetry of the physical world is indeed advocated by preon model for composite quarks and leptons.

In fact, many preon models have been discussed and considered in the past (some significant references can be found in [5,6]), though they were not turned out to be too successful and attractive, especially compared with other theory developments, like as supersymmetry and supergravity, appeared at almost the same time. However, there is still left a serious problem in particle physics with classification of all observed quark-lepton families. As in the hadron spectroscopy case, this may motivate us to continue seeking a solution in some subparticle or preon models for quarks and leptons, rather than in the less definitive extra dimension or superstring theories.

Let us briefly outline the main results presented here. We have started with the L-R symmetric preon model and found that an admissible metaflavor symmetry  $SU(8)_{MF}$  appears as a solution to the 't Hooft's anomaly matching condition

providing preservation of the accompanying chiral symmetry  $SU(8)_L \times SU(8)_R$  at all scales involved. In contrast to a common point of view, we require that states composed from the left-handed and right-handed preons with their own metacolors,  $SO(3)_{MC}^L \times SO(3)_{MC}^R$ , have to satisfy AM condition separately, though their triangle anomalies may compensate each other. The point is, however, this  $SU(8)_{MF}$  theory emerges as the vectorlike theory with respect both to preons and composites. As a consequence, while preons are left massless being protected by their metacolors, all L-preon and R-preon composites being metacolor singlets will pair up and, therefore, acquire superheavy Dirac masses. It is rather clear that such a theory is meaningless unless the L-R symmetry is partially broken that seems to be a crucial point in our model. In this connection, the natural mechanisms for spontaneous L-R symmetry breaking have been proposed according to which some R-preons, in contrast to L-preons, may be condensed or such asymmetry may be caused by the properly arranged scalar field potential. As result, an initially emerged vectorlike  $SU(8)$  theory reduces down to the conventional  $SU(5)$  GUT with an extra local family symmetry  $SU(3)_F$  and three standard generations of quarks and leptons. Though the tiny radius of compositeness for universal preons composing both quarks and leptons makes it impossible to immediately confirm their composite nature, the theory necessarily predicts a few special  $SU(5) \times SU(3)_F$  multiplets of composite fermions located at the scales from  $O(1)$  TeV up to the Planck mass scale that may appear of actual experimental interest. Some of them may be directly observed, the others populate the  $SU(5)$  GUT desert. Due to their mixing with ordinary quark-lepton families there may be expected some violation of unitarity in the mass matrices for leptons and down quarks depending on the interplay between the compositeness scale  $\Lambda_{MC}$  and scale of the family symmetry  $SU(3)_F$ .

For the reasons of simplicity, we have not considered here boson composites which could appear as the effective scalar or vector fields in the theory. Generally, they will become very heavy (with masses of the order of the compositeness scale  $\Lambda_{MC}$ ) unless their masses are specially protected by the low-scale supersymmetry. The point is, however, that some massless composite vector fields could nonetheless appear in a theory as the Goldstone bosons related to spontaneous violation of Lorentz invariance through the multi-preon interactions similar to those given in the section 3.5 (3.22). In principle, one could start with a global metaflavor symmetry  $SU(N)_{MF}$  which is then converted into the local one through the contact multi-preon interactions [11] or some nonlinear constraint put on the preon currents (see in this connection [23] and the later works [24]). If so, the quarks and leptons, on the one hand, and the gauge fields (photons, weak bosons, gluons etc.), on the other, could be composed at the same order distances determined by the preon confinement scale  $\Lambda_{MC}$ . In other words, there may be a lower limit to the division of matter beyond which one can not go. Indeed, a conventional division of matter from atoms to quarks is naturally related to the fact that matter is successively divided, whereas the mediator gauge fields (photons, gluons, gravitons etc.) are left intact. However, situation may be drastically changed if these spontaneously emerging gauge fields become composite as well. We will address this and other related questions elsewhere.

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## References

1. M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W.A. Benjamin, New York, 1964).
2. [https://en.wikipedia.org/wiki/Noble\\_Eightfold\\_Path](https://en.wikipedia.org/wiki/Noble_Eightfold_Path).
3. G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G.'t Hooft et al (Plenum, New-York, 1980).
4. J.L. Chkareuli, JETP Lett. 32 (1980) 671;  
J.L. Chkareuli, *Composite Quarks And Leptons: From  $Su(5)$  To  $Su(8)$  Symmetry*, In Proc. of the Int. Seminar QUARKS-82, pp 149-156, edited by A.L. Kataev (INR, Moscow, 1982).
5. R.N. Mohapatra, *Unification and Supersymmetry* (Springer-Verlag, New-York, 2003).
6. I.A. D'Souza and C.S. Kalman, *Preons: Models of Leptons, Quarks and Gauge Bosons as Composite Objects* (World Scientific, Singapore, 1992);  
H. Terazawa, Y. Chikashige and K. Akama, Phys. Rev. D 15 (1977) 480 ; H. Terazawa, ibid. 22 (1980) 184.
7. G. 't Hooft, Phys. Rev. Letters 37 (1976) 8.
8. R. Barbieri, L. Maiani and R. Petronzio, Phys. Lett. B 96 (1980) 63.
9. K.G. Wilson, Phys. Rev. 10 (1974) 2245;  
M. Creutz, Phys. Rev. Lett. 43 (1979) 553.
10. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.
11. J.D. Bjorken, Ann. Phys. (N.Y.) 24 (1963) 174;  
Per Kraus and E.T. Tomboulis, Phys. Rev. D 66 (2002) 045015.
12. A.A. Anselm, Sov. Phys. JETP 53 (1981) 23.
13. Z.G. Berezhiani and J.L. Chkareuli, Sov. J. Nucl. Phys. 37 (1983) 618;  
Z.G. Berezhiani, Phys. Lett. B129 (1983) 99; ibid B150 (1985) 177;  
Z. Berezhiani and M. Yu.Khlopov, Sov. J. Nucl. Phys. 51(1990) 935;  
T. Appelquist, Y. Bai and M. Piai, Phys. Lett. B 637 (2006) 245 .
14. F. Wilczek, AIP Conf. Proc. 96 (1983) 313.
15. Z. G. Berezhiani, J. L. Chkareuli, JETP Lett. 37 (1983) 338;  
Z.G. Berezhiani and J.L. Chkareuli, Sov. Phys. Usp. 28 (1985) 104;  
T. Appelquist, Y. Bai and M. Piai, Phys. Rev. D 74 (2006) 076001 .
16. J.L. Chkareuli, Phys. Lett. B246 (1990) 498; ibid B 300 (1993) 361.
17. Z.G. Berezhiani and J.L. Chkareuli, *Horizontal Unification Of Quarks And Leptons*, In Proc. of the Int. Seminar QUARKS-84, vol. 1, pp 110-121, edited by A.N. Tavkhelidze et al (INR, Moscow, 1984);  
Z. Berezhiani and M. Yu.Khlopov, Sov. J. Nucl. Phys. 51(1990) 739;  
T. Appelquist, Y. Bai and M. Piai, Rev. D 75(2007) 073005.
18. R.D. Peccei and H.R. Quinn, Phys. Rev. D16 (1977) 1891.



19. Z.G. Berezhiani, J.L. Chkareuli, G.R. Dvali and M.R. Jibuti, *Supersymmetry and generations of quarks and leptons*, In Proc. of the Int. Seminar *QUARKS-86*, pp 209-223, edited by A.N. Tavkhelidze et al (INR, Moscow, 1986);  
 Z. Berezhiani, Phys. Lett. B 417 (1998) 287;  
 S.F. King and G.G. Ross, Phys. Lett. B 520 (2001) 243;  
 J.L. Chkareuli, C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B 626 (2002) 307.
20. J.L. Chkareuli, Phys. Lett. B272 (1991) 207;  
 G. Dvali and G. Senjanovic, Phys. Rev. Lett. 72 (1994) 9;  
 D. Spergel and U.-Li Pen, Astrophys. J. 491 (1997) L67;  
 S.M. Carroll and M. Trodden, Phys. Rev. D57 (1998) 5189.
21. P. Ramond, hep-ph/9809459.
22. M. Yu. Khlopov, *Cosmoparticle Physics* (World Scientific, Singapore, 1999).
23. Y. Nambu, Progr. Theor. Phys. Suppl. E 68 (1968) 190.
24. J.L. Chkareuli, C.D. Froggatt, J.G. Jejelava and H.B. Nielsen, Nucl. Phys. B 796 (2008) 211;  
 J.L. Chkareuli, C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B 848 (2011) 498.



## 4 A Deeper Probe of New Physics Scenarii at the LHC

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**Abstract.** The implications of the discovery of a Higgs boson at the LHC with a mass of 125 GeV are summarised in the context of the Standard Model of particle physics and in new physics scenarios beyond it, taking the example of the minimal supersymmetric Standard Model extension, the MSSM. The perspectives for Higgs and new physics searches at the next LHC upgrades as well as at future hadron and lepton colliders are then briefly summarized.

**Povzetek.** Avtor povzame posledice odkritja higgsovega bozona z maso 125 GeV na pospeševalniku LHC v kontekstu standardnega modela osnovnih delcev in scenarijev možne nove fizike onkraj tega modela. Kot primer vzame minimalno supersimetrično razširitev standardnega modela znano kot MSSM. Pregleda obete za iskanje znakov nove fizike v naslednji nadgradnji LHC in na bodočih leptonskih in hadronskih pospeševalnikih.

Keywords: Higgs boson, new physics scenarios, supersymmetry, MSSM

### 4.1 Introduction

The ATLAS and CMS historical discovery of a particle with a mass of 125 GeV [1] and properties that are compatible with those of a scalar Higgs boson [2,3] has far reaching consequences not only for the Standard Model (SM) but also for new physics models beyond it. In the SM, electroweak symmetry breaking is achieved spontaneously via the Brout–Englert–Higgs mechanism [2], wherein the neutral component of an isodoublet scalar field acquires a non-zero vacuum expectation value  $v$ . This gives rise to nonzero masses for the fermions and the electroweak gauge bosons while preserving the  $SU(2) \times U(1)$  gauge symmetry. One of the four degrees of freedom of the original isodoublet field, corresponds to a physical particle [3]: a scalar boson with  $J^{PC} = 0^{++}$  quantum numbers under parity and charge conjugation. The couplings of the Higgs boson to the fermions and gauge bosons are related to the masses of these particles and are thus decided by the symmetry breaking mechanism. In contrast, the Higgs mass itself  $M_H$ , although expected to be in the vicinity of the weak scale  $v \approx 250$  GeV, is undetermined. Let

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us summarise the known information on this parameter before the start of the LHC.

A direct information was the lower limit  $M_H \gtrsim 114$  GeV at 95% confidence level (CL) established at LEP2 [4]. Furthermore, a global fit of the electroweak precision data to which the Higgs boson contributes, yields the value  $M_H = 92^{+34}_{-26}$  GeV, corresponding to a 95% CL upper limit of  $M_H \lesssim 160$  GeV [4]. From the theoretical side, the presence of this new weakly coupled degree of freedom is a crucial ingredient for a unitary electroweak theory. Indeed, the SM without the Higgs particle leads to scattering amplitudes of the W/Z bosons that grow with the square of the center of mass energy and perturbative unitarity would be lost at energies above the TeV scale. In fact, even in the presence of a Higgs boson, the W/Z bosons could interact very strongly with each other and, imposing the unitarity requirement leads to the important mass bound  $M_H \lesssim 700$  GeV [5], implying that the particle is kinematically accessible at the LHC.

Another theoretical constraint emerges from the fact that the Higgs self-coupling,  $\lambda \propto M_H^2$ , evolves with energy and at some stage, becomes very large and even infinite and the theory completely loses its predictability. If the energy scale up to which the couplings remains finite is of the order of  $M_H$  itself, one should have  $M_H \lesssim 650$  GeV [6]. On the other hand, for small values of  $\lambda$  and hence  $M_H$ , the quantum corrections tend to drive the self-coupling to negative values and completely destabilize the scalar Higgs potential to the point where the minimum is not stable anymore [6]. Requiring  $\lambda \geq 0$ , up to the TeV scale implies that  $M_H \gtrsim 70$  GeV. If the SM is to be extended to the Planck scale  $M_P \sim 10^{18}$  GeV, the requirements on  $\lambda$  from finiteness and positivity constrain the Higgs mass to lie in the range  $130 \text{ GeV} \lesssim M_H \lesssim 180 \text{ GeV}$  [6]. This narrow margin is close to the one obtained from the direct and indirect experimental constraints.

The discovery of the Higgs particle with a mass of 125 GeV, a value that makes the SM perturbative, unitary and extrapolable to the highest possible scales, is therefore a consecration of the model and crowns its past success in describing all experimental data available. In particular, the average mass value measured by the ATLAS and CMS teams,  $M_H = 125.1 \pm 0.24$  GeV [7], is remarkably close to the best-fit of the precision data which should be considered as a great achievement and a triumph for the SM. In addition, a recent analysis that includes the state-of-the-art quantum corrections [8] gives for the condition of absolute stability of the electroweak vacuum,  $\lambda(M_P) \geq 0$ , the bound  $M_H \gtrsim 129$  GeV for the present value of the top quark mass and the strong coupling constant,  $m_t^{\text{exp}} = 173.2 \pm 0.9$  GeV and  $\alpha_s(M_Z) = 0.1184 \pm 0.0007$  [4]. Allowing for a  $2\sigma$  variation of  $m_t^{\text{exp}}$ , one obtains  $M_H \geq 125.6$  GeV that is close to the measured  $M_H$  value [7]. In fact, for an unambiguous and well-defined determination of the top mass, one should rather use the total cross section for top pair production at hadron colliders which can unambiguously be defined theoretically; this mass has a larger error,  $\Delta m_t \approx 3$  GeV, which allows more easily absolute stability of the SM vacuum up to  $M_P$  [9].

Nevertheless, the SM is far from being perfect in many respects. It does not explain the proliferation of fermions and the large hierarchy in their mass spectra and does not say much about the small neutrino masses. The SM does not unify in a satisfactory way the electromagnetic, weak and strong forces, as one has

three different symmetry groups with three coupling constants which shortly fail to meet at a common value during their evolution with the energy scale; it also ignores the fourth force, gravitation. Furthermore, it does not contain a particle that could account for the cosmological dark matter and fails to explain the baryon asymmetry in the Universe.

However, the main problem that calls for beyond the SM is related to the special status of the Higgs boson which, contrary to fermions and gauge bosons has a mass that cannot be protected against quantum corrections. Indeed, these are quadratic in the new physics scale which serves as a cut-off and hence, tend to drive  $M_H$  to very large values, ultimately to  $M_P$ , while we need  $M_H = \mathcal{O}(100 \text{ GeV})$ . Thus, the SM cannot be extrapolated beyond  $\mathcal{O}(1 \text{ TeV})$  where some new physics should emerge. This is the reason why we expect something new to manifest itself at the LHC.

There are three avenues for the many new physics scenarios beyond the SM. There are first theories with extra space-time dimensions that emerge at the TeV scale (the cut-off is then not so high) and, second, composite models inspired from strong interactions also at the TeV scale (and thus the Higgs is not a fundamental spin-zero particle). Some versions of these scenarios do not incorporate any Higgs particle in their spectrum and are thus ruled out by the Higgs discovery. However, the option that emerges in the most natural way is Supersymmetry (SUSY) [10] as it solves most of the SM problems discussed above. In particular, SUSY protects  $M_H$  as the quadratically divergent radiative corrections from standard particles are exactly compensated by the contributions of their supersymmetric partners. These new particles should not be much heavier than 1 TeV not to spoil this compensation [11] and, thus, they should be produced at the LHC.

The Higgs discovery is very important for SUSY and, in particular, for its simplest low energy manifestation, the minimal supersymmetric SM (MSSM) that indeed predicts a light Higgs state. In the MSSM, two Higgs doublet fields  $H_u$  and  $H_d$  are required, leading to an extended Higgs consisting of five Higgs bosons, two CP-even  $h$  and  $H$ , a CP-odd  $A$  and two charged  $H^\pm$  states [12]. Nevertheless, only two parameters are needed to describe the Higgs sector at tree-level: one Higgs mass, which is generally taken to be that of the pseudoscalar boson  $M_A$ , and the ratio of vacuum expectation values of the two Higgs fields,  $\tan \beta = v_d/v_u$ , expected to lie in the range  $1 \lesssim \tan \beta \lesssim 60$ . The masses of the CP-even  $h$ ,  $H$  and the charged  $H^\pm$  states, as well as the mixing angle  $\alpha$  in the CP-even sector are uniquely defined in terms of these two inputs at tree-level, but this nice property is spoiled at higher orders [13]. For  $M_A \gg M_Z$ , one is in the so-called decoupling regime in which the  $h$  state is light and has almost exactly the SM-Higgs couplings, while the other CP-even  $H$  and the charged  $H^\pm$  bosons become heavy,  $M_H \approx M_{H^\pm} \approx M_A$ , and decouple from the massive gauge bosons. In this regime, the MSSM Higgs sector thus looks almost exactly as the one of the SM with its unique Higgs boson.

Nevertheless, contrary to the SM Higgs boson, the lightest MSSM CP-even  $h$  mass is bounded from above and, depending on the SUSY parameters that enter the important quantum corrections, is restricted to  $M_h^{\max} \lesssim 130 \text{ GeV}$  [13] if one assumes a SUSY breaking scale that is not too high,  $M_s \lesssim \mathcal{O}(1 \text{ TeV})$ , in order to avoid too much fine-tuning in the model. Hence, the requirement that the MSSM

$h$  boson coincides with the one observed at the LHC, i.e. with  $M_h \approx 125$  GeV and almost SM-like couplings as the LHC data seem to indicate, would place very strong constraints on the MSSM parameters, in particular the SUSY-breaking scale  $M_S$ . This comes in addition to the LHC limits obtained from the search of the heavier Higgs states and the superparticles.

In this talk, the implications of the discovery of the Higgs boson at the LHC and the measurement of its properties will be summarised and the prospects for the searches of new physics, in particular in the SUSY context, in the future will be discussed.

## 4.2 Implications: Standard Model and beyond

In many respects, the Higgs particle was born under a very lucky star as the mass value of  $\approx 125$  GeV allows to produce it at the LHC in many redundant channels and to detect it in a variety of decay modes. This allows detailed studies of the Higgs properties.

### 4.2.1 Higgs production and decay

We start by summarizing the production and decay at the LHC of a light SM-like Higgs particle, which should correspond to the lightest MSSM  $h$  boson in the decoupling regime. First, for  $M_H \approx 125$  GeV, the Higgs mainly decays [14] into  $b\bar{b}$  pairs but the decays into  $WW^*$  and  $ZZ^*$  final states, before allowing the gauge bosons to decay leptonically  $W \rightarrow \ell\nu$  and  $Z \rightarrow \ell\ell$  ( $\ell = e, \mu$ ), are also significant. The  $H \rightarrow \tau^+\tau^-$  channel (as well as the  $gg$  and  $c\bar{c}$  decays that are not detectable at the LHC) is also of significance, while the clean loop induced  $H \rightarrow \gamma\gamma$  mode can be easily detected albeit its small rates. The very rare  $H \rightarrow Z\gamma$  and even  $H \rightarrow \mu^+\mu^-$  channels should be accessible at the LHC but only with a much larger data sample.

On the other hand, many Higgs production processes have significant cross sections [15–17]. While the by far dominant gluon fusion mechanism  $gg \rightarrow H$  (ggF) has extremely large rates ( $\approx 20$  pb at  $\sqrt{s} = 7\text{--}8$  TeV), the subleading channels, i.e. the vector boson fusion (VBF)  $qq \rightarrow Hqq$  and the Higgs-strahlung (HV)  $q\bar{q} \rightarrow HV$  with  $V = W, Z$  mechanisms, have cross sections which should allow for Higgs studies of the already at  $\sqrt{s} \gtrsim 7$  TeV with the  $\approx 25 \text{ fb}^{-1}$  data collected by each experiment. The associated process  $pp \rightarrow t\bar{t}H$  (ttH) would require higher energy and luminosity.

This pattern already allows the ATLAS and CMS experiments to observe the Higgs boson in several channels and to measure some of its couplings in a reasonably accurate way. The channels that have been searched are  $H \rightarrow ZZ^* \rightarrow 4\ell^\pm$ ,  $H \rightarrow WW^* \rightarrow 2\ell 2\nu$ ,  $H \rightarrow \gamma\gamma$  where the Higgs is mainly produced in ggF with subleading contributions from Hjj in the VBF process,  $H \rightarrow \tau\tau$  where the Higgs is produced in association with one (in ggF) and two (in VBF) jets, and finally  $H \rightarrow b\bar{b}$  with the Higgs produced in the HV process. One can ignore for the moment the low sensitivity  $H \rightarrow \mu\mu$  and  $H \rightarrow Z\gamma$  channels.

A convenient way to scrutinize the couplings of the produced  $H$  boson is to look at their deviation from the SM expectation. One then considers for a given

search channel the signal strength modifier  $\mu$  which for the  $H \rightarrow XX$  decay mode measures the deviation compared to the SM expectation of the Higgs production cross section times decay branching fraction  $\mu_{XX}$ . ATLAS and CMS have provided the signal strengths for the various final states with a luminosity of  $\approx 5 \text{ fb}^{-1}$  for the 2011 run at  $\sqrt{s} = 7 \text{ TeV}$  and  $\approx 20 \text{ fb}^{-1}$  for the 2012 run at  $\sqrt{s} = 8 \text{ TeV}$ . The constraints given by the two collaborations, when combined, lead to a global signal strength  $\mu_{\text{tot}}^{\text{ATLAS}} = 1.18 \pm 0.15$  and  $\mu_{\text{tot}}^{\text{CMS}} = 1.00 \pm 0.14$  [7]. The global value being very close to unity implies that the observed Higgs is SM-like.

Hence, already with the rather limited statistics at hand, the accuracy of the ATLAS and CMS measurements is reaching the 15% level. This is at the same time impressive and worrisome. Indeed, the main Higgs production channel is the top and bottom quark loop mediated gluon fusion mechanism and, at  $\sqrt{s}=7$  or 8 TeV, the three other mechanisms contribute at a total level below 15%. The majority of the signal events observed at LHC, in particular in the search channels  $H \rightarrow \gamma\gamma$ ,  $H \rightarrow ZZ^* \rightarrow 4\ell$ ,  $H \rightarrow WW^* \rightarrow 2\ell 2\nu$  and to some extent  $H \rightarrow \tau\tau$ , thus come from the ggF mechanism which is known to be affected by large theoretical uncertainties.

Indeed, although  $\sigma(gg \rightarrow H)$  is known up next-to-next-to-leading order (NNLO) in perturbative QCD (and at least at NLO for the electroweak interaction) [15,16], there is a significant residual scale dependence which points to the possibility that still higher order contributions cannot be totally excluded. In addition, as the process is of  $\mathcal{O}(\alpha_s^2)$  at LO and is initiated by gluons, there are sizable uncertainties due to the gluon parton distribution function (PDF) and the value of the coupling  $\alpha_s$ . A third source of theoretical uncertainties, the use of an effective field theory (EFT) approach to calculate the radiative corrections beyond NLO should also be considered [15]. In addition, large uncertainties arise when  $\sigma(gg \rightarrow H)$  is broken into the jet categories  $H+0j$ ,  $H+1j$  and  $H+2j$  [18]. In total, the combined theoretical uncertainty is estimated to be  $\Delta^{\text{th}} \approx \pm 15\%$  [16] and would increase to  $\Delta^{\text{th}} \approx \pm 20\%$  if the EFT uncertainty is also included. The a priori cleaner VBF process will be contaminated by the  $gg \rightarrow H+2j$  mode making the total error in the  $H+jj$  “VBF” sample also rather large [18].

Hence, the theoretical uncertainty is already at the level of the accuracy of the cross section measured by the ATLAS and CMS collaborations. Another drawback of the analyses is that they involve strong theoretical assumptions on the total Higgs width since some contributing decay channels not accessible at the LHC are assumed to be SM-like and possible invisible Higgs decays in scenarios beyond the SM do not occur.

In Ref. [17], following earlier work [19] it has been suggested to consider the ratio  $D_{XX}^p = \sigma^p(pp \rightarrow H \rightarrow XX)/\sigma^p(pp \rightarrow H \rightarrow VV)$  for a specific production process  $p$  and for a given decay channel  $H \rightarrow XX$  when the reference channel  $H \rightarrow VV$  is used. In these ratios, the cross sections and hence, their significant theoretical uncertainties will cancel out, leaving out only the ratio of partial decay widths which are better known. The total decay width which includes contributions from channels not under control such as possible invisible Higgs decays, do not appear in the ratios  $D_{XX}^p$ . Some common experimental systematical uncertainties such as the one from the luminosity measurement and the small uncertainties in the

Higgs decay branching ratios also cancel out. We are thus left with only with the statistical and some (non common) systematical errors [17].

The ratios  $D_{XX}$  involve, up to kinematical factors and known radiative corrections, only the ratios  $|c_X|^2 / |c_V|^2$  of the Higgs reduced couplings to the particles  $X$  and  $V$  compared to the SM expectation,  $c_X \equiv g_{HXX} / g_{HXX}^{\text{SM}}$ . For the time being, three independent ratios can be considered:  $D_{\gamma\gamma}$ ,  $D_{\tau\tau}$  and  $D_{bb}$ . In order to determine these ratios, the theoretical uncertainties have to be treated as a bias (and not as if they were associated with a statistical distribution) and the fit has to be performed for the two  $\mu$  extremal values:  $\mu_i|_{\text{exp}} \pm \delta\mu_i / \mu_i|_{\text{th}}$  with  $\delta\mu_i / \mu_i|_{\text{th}} \approx \pm 20\%$  [20].

A large number of analyses of the Higgs couplings from the LHC data have been performed and in most cases, it is assumed that the couplings of the Higgs boson to the massive  $W$ ,  $Z$  gauge bosons are equal to  $g_{HZZ} = g_{HWW} = c_V$  and the couplings to all fermions are also the same  $g_{Hff} = c_f$ . However, as for instance advocated in Ref. [21] to characterize the Higgs particle at the LHC, at least three independent  $H$  couplings should be considered, namely  $c_t$ ,  $c_b$  and  $c_V$ . While the couplings to  $W$ ,  $Z$ ,  $b$ ,  $\tau$  particles are derived by considering the decays of the Higgs boson to these particles, the  $Ht\bar{t}$  coupling is derived indirectly from  $\sigma(gg \rightarrow H)$  and  $\text{BR}(H \rightarrow \gamma\gamma)$ , two processes that are generated by triangular loops involving the top quarks in the SM. One can assume, in a first approximation, that  $c_c = c_t$  and  $c_\tau = c_b$  and possible invisible Higgs decays are absent. In Ref. [21], a three-dimensional fit of the  $H$  couplings was performed in the space  $[c_t, c_b, c_V]$ , when the theory uncertainty is taken as a bias and not as a nuisance. The best-fit value for the couplings, with the  $\sqrt{s} = 7+8$  TeV ATLAS and CMS data turns out to be  $c_t = 0.89$ ,  $c_b = 1.01$  and  $c_V = 1.02$ , ie very close to the SM values.

## 4.2.2 Implications of the Higgs couplings measurement

The precise measurements of Higgs couplings allow to draw several important conclusions.

*i) A fourth generation fermions is excluded.* Indeed, in addition to the direct LHC searches that exclude heavier quarks  $m_{b'}, m_{t'} \lesssim 600$  GeV [23], strong constraints can be also obtained from the loop induced Higgs–gluon and Higgs–photon vertices in which any heavy particle coupling to the Higgs proportionally to its mass will contribute. For instance the additional 4th generation  $t'$  and  $b'$  contributions increase  $\sigma(gg \rightarrow H)$  by a factor of  $\approx 9$  at LO but large  $\mathcal{O}(G_F m_{t'}^2)$  electroweak corrections should be considered. It has been shown [23] that with a fourth family, the Higgs signal would have not been observable and the obtained Higgs results unambiguously rule out this possibility.

*ii) The invisible Higgs decay width should be small.* Invisible decays would affect the properties of the observed Higgs boson and could be constrained if the total decay width is determined. But for a 125 GeV Higgs,  $\Gamma_H^{\text{tot}} = 4$  MeV, is too small to be resolved experimentally. Nevertheless, in  $pp \rightarrow VV \rightarrow 4f$ , a large fraction of the Higgs cross section lies in the high-mass tail [24] allowing to put loose constraints  $\Gamma_H^{\text{tot}} / \Gamma_H^{\text{SM}} \approx 5\text{--}10$  [25]. The invisible Higgs decay width  $\Gamma_H^{\text{inv}}$  can be better constrained indirectly by a fit of the Higgs couplings and in particular with the

signal strength in the  $H \rightarrow ZZ$  process:  $\mu_{ZZ} \propto \Gamma(H \rightarrow ZZ)/\Gamma_H^{\text{tot}}$  with  $\Gamma_H^{\text{tot}} = \Gamma_H^{\text{inv}} + \Gamma_H^{\text{SM}}$ ; one obtains  $\Gamma_H^{\text{inv}}/\Gamma_H^{\text{SM}} \lesssim 50\%$  at 95% CL with the assumption  $c_f = c_V = 1$  [20].

A more model independent approach would be to perform direct searches for missing transverse energy. These have been conducted in  $pp \rightarrow HV$  with  $V \rightarrow jj, \ell\ell$  and in VBF,  $qq \rightarrow qqE_\tau$  leading to  $\text{BR}_{\text{inv}} \lesssim 50\%$  at 95%CL for SM-like Higgs couplings [7]. A more promising search for invisible decays is the monojet channel  $gg \rightarrow Hj$  which has large rates [26]. While the most recent monojet ATLAS and CMS searches are only sensitive to  $\text{BR}_{\text{inv}} \sim 1$ , more restrictive results can be obtained in the future.

The Higgs invisible rate and the dark matter detection rate in direct astrophysical searches are correlated in Higgs portal models and it turns out that LHC constraints are competitive [27] with those derived from direct dark matter search experiments [28].

iii) *The spin-parity quantum numbers are those of a standard Higgs.* One also needs to establish that the observed Higgs state is indeed a CP even scalar and hence with  $J^{\text{PC}} = 0^{++}$  quantum numbers. For the spin, the observation of the  $H \rightarrow \gamma\gamma$  decay rules out the spin-1 case [29]. The Higgs parity can be probed by studying kinematical distributions in the  $H \rightarrow ZZ^* \rightarrow 4\ell$  decay channel and in the VH and VBF production modes [30] and with the  $25 \text{ fb}^{-1}$  data collected so far, ATLAS and CMS found that the observed Higgs is more compatible with a  $0^+$  state and the  $0^-$  possibility is excluded at the 98%CL [7]. Other useful diagnostics of the Higgs CP nature that also rely on the tensorial structure of the HVV coupling can be made in the VBF process [31]. Nevertheless, there is a caveat in the analyses relying on the HVV couplings: a CP-odd state has no tree-level VV couplings [32]. In fact, a better way to measure the Higgs parity is to study the signal strength in the  $H \rightarrow VV$  channels and in Ref. [20] it was demonstrated that the observed Higgs has indeed a large CP component,  $\gtrsim 50\%$  at the 95%CL. In fact, the less unambiguous way to probe the Higgs CP nature would be to look at final states in which the particle decays hadronically, e.g.  $pp \rightarrow HZ \rightarrow b\bar{b}\ell\ell$  [32]. These processes are nevertheless extremely challenging even at the upgraded LHC.

### 4.2.3 Implications for Supersymmetry

We turn now to the implications of the LHC Higgs results for the MSSM Higgs sector and first make a remark on the Higgs masses and couplings, which at tree-level depend only on  $M_A$  and  $\tan\beta$ , when the important radiative corrections are included. In this case many parameters such as the masses of the third generation squarks  $m_{\tilde{t}_1}, m_{\tilde{b}_1}$  and their trilinear couplings  $A_t, A_b$  enter  $M_h$  and  $M_H$  through quantum corrections. These are introduced by a general  $2 \times 2$  matrix  $\Delta\mathcal{M}_{ij}^2$  but the leading one is controlled by the top Yukawa coupling and is proportional to  $m_t^4, \log M_S$  with  $M_S = \sqrt{m_{\tilde{t}_1} m_{\tilde{t}_2}}$  the SUSY-breaking scale and the stop mixing parameter  $X_t$  [13]. The maximal value  $M_h^{\text{max}}$  is then obtained for a decoupling regime  $M_A \sim \mathcal{O}(\text{TeV})$ , large  $\tan\beta$ , large  $M_S$  that implies heavy stops and maximal mixing  $X_t = \sqrt{6}M_S$  [33]. If the parameters are optimized as above, the maximal  $M_h$  value reaches the level of 130 GeV.

It was pointed out in Refs. [34,35,21] that when the measured value  $M_h = 125$  GeV is taken into account, the MSSM Higgs sector with only the largely dominant



correction discussed above, can be again described with only the two parameters  $\tan \beta$  and  $M_A$ ; in other words, the loop corrections are fixed by the value of  $M_h$ . This observation leads to a rather simple but accurate parametrisation of the MSSM Higgs sector, called hMSSM.

The reduced couplings of the CP-even  $h$  state (as is the case for the heavier  $H$ ) depend in principle only on the angles  $\beta$  and  $\alpha$  (and hence  $\tan \beta$  and  $M_A$ ),  $c_V^0 = \sin(\beta - \alpha)$ ,  $c_t^0 = \cos \alpha / \sin \beta$ ,  $c_b^0 = -\sin \alpha / \cos \beta$ , while the couplings of  $A$  and  $H^\pm$  (as well as  $H$  in the decoupling regime) to gauge boson are zero and those to fermions depend only on  $\beta$ : for  $\tan \beta > 1$ , they are enhanced ( $\propto \tan \beta$ ) for  $b, \tau$  and suppressed ( $\propto 1/\tan \beta$ ) for tops.

*i) Implications from the Higgs mass value:* In the so-called “phenomenological MSSM” (pMSSM) [37] in which the model involves only 22 free parameters, a large scan has been performed [36] using the RGE program *Suspect* [38] that calculates the maximal  $M_h$  value and the result confronted to the measured mass  $M_h \sim 125$  GeV. For  $M_S \lesssim 1$  TeV, only scenarios with  $X_t/M_S$  values close to maximal mixing  $X_t/M_S \approx \sqrt{6}$  survive. The no-mixing scenario  $X_t \approx 0$  is ruled out for  $M_S \lesssim 3$  TeV, while the typical mixing scenario,  $X_t \approx M_S$ , needs large  $M_S$  and moderate to large  $\tan \beta$  values. In constrained MSSM scenarios (cMSSM) such the minimal supergravity (mSUGRA) model and the gauge and anomaly mediated SUSY-breaking scenarios, GMSB and AMSB, only a few basic inputs are needed and the mixing parameter cannot take arbitrary values. A scan in these models with  $M_S \lesssim 3$  TeV not to allow for too much fine-tuning [11] leads  $M_h^{\max} \lesssim 122$  GeV in AMSB and GMSB thus disfavoring these scenarios while one has  $M_h^{\max} = 128$  GeV in mSUGRA. In high-scale SUSY scenarios,  $M_S \gg 1$  TeV, the radiative corrections are very large and need to be resummed [39]. For low  $\tan \beta$  values, large scales, at least  $M_S \gtrsim 10^4$  GeV, are required to obtain  $M_h = 125$  GeV and even higher in most cases

*ii) Implications from the production rates of the observed state.* Besides the corrections to the Higgs masses and couplings discussed above, there are also direct corrections to the Higgs couplings and the most ones are those affecting the  $h b \bar{b}$  vertex [40] and the stop loop contributions to the  $g g \rightarrow h$  production and  $h \rightarrow \gamma \gamma$  decay rates [41]. A fit of the  $c_t$ ,  $c_b$  and  $c_V$  couplings shows that the latter are small [20]. In turn, ignoring the direct corrections and using the input  $M_h \approx 125$  GeV, one can make a fit in the plane  $[\tan \beta, M_A]$ . The best-fit point is  $\tan \beta = 1$  and  $M_A = 550$  GeV which implies a large SUSY scale,  $M_S = \mathcal{O}(100)$  TeV. In all, cases one also has  $M_A \gtrsim 200\text{--}350$  GeV.

*iii) Implications from heavy Higgs boson searches.* At high  $\tan \beta$  values, the strong enhancement of the  $b, \tau$  couplings makes that the  $\Phi = H/A$  states decay dominantly into  $\tau^+ \tau^-$  and  $b \bar{b}$  pairs and are mainly produced in  $g g \rightarrow \Phi$  fusion with the  $b$ -loop included and associated production with  $b$ -quarks,  $g g / q \bar{q} \rightarrow b \bar{b} + \Phi$  [42]. The most powerful LHC search channel is thus  $p p \rightarrow g g + b \bar{b} \rightarrow \Phi \rightarrow \tau^+ \tau^-$ . For the charged Higgs, the dominant mode is  $H^\pm \rightarrow \tau \nu$  with the  $H^\pm$  light enough to be produced in top decays  $t \rightarrow H^+ b \rightarrow \tau \nu b$ . In the low  $\tan \beta$  regime,  $\tan \beta \lesssim 3$ , the phenomenology of the  $A, H, H^\pm$  states is richer [34]. For the production, only  $g g \rightarrow \Phi$  process with the dominant  $t$  and sub-dominant  $b$  contributions provides large rates. The  $H/A/H^\pm$  decay pattern is in turn rather involved. Above the

$t\bar{t}$  (tb) threshold  $H/A \rightarrow t\bar{t}$  and  $H^+ \rightarrow t\bar{b}$  are by far dominant. Below threshold, the  $H \rightarrow WW, ZZ$  decays are significant. For  $2M_h \lesssim M_H \lesssim 2m_t$  ( $M_A \gtrsim M_h + M_Z$ ),  $H \rightarrow hh$  ( $A \rightarrow hZ$ ) is the dominant  $H(A)$  decay mode. But the  $A \rightarrow \tau\tau$  channel is still important with rates  $\gtrsim 5\%$ . In the case of  $H^\pm$ , the channel  $H^+ \rightarrow Wh$  is important for  $M_{H^\pm} \lesssim 250$  GeV, similarly to the  $A \rightarrow hZ$  case.

In Ref. [34] an analysis of these channels has been performed using current information given by ATLAS and CMS in the context of the SM, MSSM [43] or other scenarios. The outcome is impressive. The ATLAS and CMS  $H/A \rightarrow \tau^+\tau^-$  constraint is extremely restrictive and  $M_A \lesssim 250$  GeV, it excludes almost the entire intermediate and high  $\tan\beta$  regimes. The constraint is less effective for a heavier  $A$  but even for  $M_A \approx 400$  GeV the high  $\tan\beta \gtrsim 10$  region is excluded and one is even sensitive to  $M_A \approx 800$  GeV for  $\tan\beta \gtrsim 50$ . For  $H^\pm$ , almost the entire  $M_{H^\pm} \lesssim 160$  GeV region is excluded by the process  $t \rightarrow H^+b$  with the decay  $H^+ \rightarrow \tau\nu$ . The other channels, in particular  $H \rightarrow VV$  and  $H/A \rightarrow t\bar{t}$ , are very constraining as they cover the entire low  $\tan\beta$  area that was previously excluded by the LEP2 bound up to  $M_A \approx 500$  GeV. Even  $A \rightarrow hZ$  and  $H \rightarrow hh$  would be visible at the current LHC in small portions of the parameter space.

### 4.3 Perspectives for Higgs and New Physics

The last few years were extremely rich and exciting for particle physics. With the historical discovery of a Higgs boson by the LHC collaborations ATLAS and CMS, crowned by a Nobel prize in fall 2013, and the first probe of its basic properties, they witnessed a giant step in the unraveling of the mechanism that breaks the electroweak symmetry and generates the fundamental particle masses. They promoted the SM as the appropriate theory, up to at least the Fermi energy scale, to describe three of Nature's interactions, the electromagnetic, weak and strong forces. However, it is clear that these few years have also led to some frustration as no signal of physics beyond the SM has emerged from the LHC data. The hope of observing some signs of the new physics models that were put forward to address the hierarchy problem, that is deeply rooted in the Higgs mechanism, with Supersymmetric theories being the most attractive ones, did not materialize.

The Higgs discovery and the non-observation of new particles has nevertheless far reaching consequences for supersymmetric theories and, in particular, for their simplest low energy formulation, the MSSM. The mass of approximately 125 GeV of the observed Higgs boson implies that the scale of SUSY-breaking is rather high, at least  $\mathcal{O}(\text{TeV})$ . This is backed up by the limits on the masses of strongly interacting SUSY particles set by the ATLAS and CMS searches, which in most cases exceed the TeV range. This implies that if SUSY is indeed behind the stabilization of the Higgs mass against very high scales that enter via quantum corrections, it is either fine-tuned at the permille level at least or its low energy manifestation is more complicated than expected.

The production and decay rates of the observed Higgs particles, as well as its spin and parity quantum numbers, as measured by ATLAS and CMS with the  $\approx 25 \text{ fb}^{-1}$  data collected at  $\sqrt{s}=7+8$  TeV, indicate that its couplings to fermions

and gauge bosons are almost SM-like. In the context of the MSSM, this implies that we are close to the decoupling regime and this particle is the lightest  $h$  boson, while the other  $H/A/H^\pm$  states must be heavier than approximately the Fermi scale. This last feature is also backed up by LHC direct searches of these heavier Higgs states.

This drives up to the question that is now very often asked: what to do next? The answer is, for me, obvious: we are only in the beginning of a new era. Indeed, it was expected since a long time that the probing of the electroweak symmetry breaking mechanism will be at least a two chapters story. The first one is the search and the observation of a Higgs-like particle that will confirm the scenario of the SM and most of its extensions, that is, a spontaneous symmetry breaking by a scalar field that develops a non-zero vev. This long chapter has just been closed by the ATLAS and CMS collaborations with the spectacular observation of a Higgs boson. This observation opens a second and equally important chapter: the precise determination of the Higgs profile and the unraveling of the electroweak symmetry breaking mechanism itself.

A more accurate measurement of the Higgs couplings to fermions and gauge bosons will be mandatory to establish the exact nature of the mechanism and, eventually, to pin down effects of new physics if additional ingredients beyond those of the SM are involved. This is particularly true in weakly interacting theories such as SUSY in which the quantum effects are expected to be small. These measurements could be performed at the upgraded LHC with an energy close to  $\sqrt{s} = 14$  TeV, in particular if a very high luminosity, a few  $\text{ab}^{-1}$ , is achieved [43,44].

At this upgrade, besides improving the measurements performed so far, rare but important channels such as associated Higgs production with top quarks,  $p\bar{p} \rightarrow t\bar{t}H$ , and Higgs decays into  $\mu^+\mu^-$  and  $Z\gamma$  states could be probed. Above all, a determination of the self-Higgs coupling could be made by searching for double Higgs production e.g. in the gluon fusion channel  $gg \rightarrow HH$  [45]; this would be a first step towards the reconstruction of the scalar potential that is responsible of electroweak symmetry breaking. This measurement would be difficult at the LHC even with high-luminosity but a proton collider with  $\sqrt{s} = 30$  to 100 TeV could do the job [44].

In a less near future, a high-energy lepton collider, which is nowadays discussed in various options (ILC, TLEP, CLIC,  $\mu$ -collider) would lead to a more accurate probing of the Higgs properties [46], promoting the scalar sector to the very high-precision level of the gauge and fermion sectors achieved by the LEP and SLC colliders in the 1990s [4]. At electron-positron colliders, the process  $e^+e^- \rightarrow HZ$ , just looking at the recoiling  $Z$  boson allows to measure the Higgs mass, the CP parity and the absolute HZZ coupling, allowing to derive the total decay width  $\Gamma_H^{\text{tot}}$ . One can then measure precisely, already at  $\sqrt{s} \approx 250$  GeV where  $\sigma(e^+e^- \rightarrow HZ)$  is maximal, the absolute Higgs couplings to gauge bosons and light fermions from the decay branching ratios. The important couplings to top quarks and the Higgs self-couplings can be measured at the 10% level in the higher-order processes  $e^+e^- \rightarrow t\bar{t}H$  and  $e^+e^- \rightarrow HHZ$  at energies of at least 500 GeV with a high-luminosity.

Besides the high precision study of the already observed Higgs, one should also continue to search for the heavy states that are predicted by SUSY, not only the superparticles but also the heavier Higgs bosons. The energy upgrade to  $\approx 14$  TeV (and eventually beyond) and the planed order of magnitude (or more) increase in luminosity will allow to probe much higher mass scales than presently. In fact, more generally, one should continue to search for any sign of new physics or new particles, new gauge bosons and fermions, as predicted in most of the SM extensions.

In conclusion, it is not yet time to give up on SUSY and more generally on New Physics but, rather, to work harder to be fully prepared for the more precise and larger data set that will be delivered by the upgraded LHC. It will be soon enough to “philosophize” then as the physics landscape will become more clear.

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### References

1. ATLAS collaboration, Phys. Lett. B716 (2012) 1; CMS collaboration, Phys. Lett. B716 (2012) 30.
2. F. Englert and R. Brout, Phys. Rev. Lett. 13 (1964) 321; P. Higgs, Phys. Rev. Lett. 13 (1964) 508; P. Higgs, Phys. Lett. 12 (1964) 132; G. Guralnik, C. Hagen, T. Kibble, Phys. Rev. Lett. 13 (1964) 585.
3. For a review of SM Higgs, see: A. Djouadi, Phys. Rept. 457 (2008) 1.
4. K. Olive et al., Particle Data Group, Chin. Phys. C38 (2014) 090001.
5. B.W. Lee, C. Quigg and H.B. Thacker, Phys. Rev. D16 (1977) 1519.
6. N. Cabibbo et al., Nucl. Phys. B158 (1979) 295; M. Sher, Phys. Rept. 179 (1989) 273; G. Altarelli and G. Isidori, Phys. Lett. B337 (1994) 141.
7. See the talks given at the summer conferences.
8. G. Degrandi et al., JHEP 1208 (2012) 098; F. Bezrukov et al., JHEP 1210 (2012) 140.
9. S. Alekhin, A. Djouadi and S. Moch, Phys. Lett. B716 (2012) 214.
10. M. Drees, R. Godbole, P. Roy, *Theory and phenomenology of sparticles*, World Sci., 2005.
11. E. Witten, Nucl. Phys. B188 (1981) 513; R. Barbieri and G. Giudice, Nucl. Phys. B306 (1988) 63; M. Papucci, J. Ruderman and A. Weiler, JHEP 1209 (2012) 035.
12. A. Djouadi, Phys. Rept. 459 (2008) 1; Eur. Phys. J. C74 (2014) 2704.
13. S. Heinemeyer, W. Hollik and G. Weiglein, Phys. Rept. 425 (2006) 265; M. Carena and H. Haber, Prog. Part. Nucl. Phys. 50 (2003) 63; B. Allanach et al., JHEP 0409 (2004) 044.
14. A. Djouadi, J. Kalinowski and M. Spira, Comput. Phys. Commun. 108 (1998) 56; A. Djouadi, M. Muhlleitner and M. Spira, Acta. Phys. Polon. B38 (2007) 635.
15. J. Baglio and A. Djouadi, JHEP 1103 (2011) 055; JHEP 1010 (2010) 064; J. Baglio et al., Phys. Lett. B716 (2012) 203.
16. S. Dittmaier et al., the LHC Higgs xsWG, arXiv:1101.0593 [hep-ph].
17. A. Djouadi, Eur. Phys. J. C73 (2013) 2498; A. Djouadi, J. Quevillon and R. Vega-Morales, arXiv:1509.03913 [hep-ph].
18. S. Dittmaier et al., the LHC Higgs xsWG, arXiv:1201.3084 [hep-ph].
19. D. Zeppenfeld et al. Phys. Rev. D62 (2000) 013009; A. Djouadi et al., hep-ph/0002258; M. Dührssen et al., Phys. Rev. D70 (2004) 113009.

20. A. Djouadi and G. Moreau, *Eur. Phys. J. C* **73** (2013) 2512.
21. A. Djouadi, L. Maiani, G. Moreau, A. Polosa, J. Quevillon, V. Riquer, *Eur. Phys. J. C* **73** (2013) 2650; *JHEP* 1506 (2015) 168.
22. A. Djouadi and A. Lenz, *Phys. Lett. B* **715** (2012) 310; E. Kuflik, Y. Nir and T. Volansky, *Phys. Rev. Lett.* **110** (2013) 091801; A. Denner et al., *Eur. Phys. J. C* **72** (2012) 1992; A. Djouadi, P. Gambino, B. Kniehl, *Nucl. Phys. B* **523** (1998) 17; A. Djouadi and P. Gambino, *Phys. Rev. Lett.* **73** (1994) 2528.
23. CMS collaboration, *Phys. Rev. D* **86** (2012) 112003; ATLAS collaboration, *Phys. Rev. Lett.* **109** (2012) 032001.
24. See e.g., N. Kauer and G. Passarino, *JHEP* 1208 (2012) 116.
25. ATLAS collaboration, ATLAS-CONF-2014-042; CMS collaboration, *Phys. Lett. B* **736** (2014) 64.
26. A. Djouadi, A. Falkowski, Y. Mambrini and J. Quevillon, *Eur. Phys. J. C* **73** (2013) 2455.
27. A. Djouadi, O. Lebedev, Y. Mambrini and J. Quevillon, *Phys. Lett. B* **709** (2012) 65.
28. M. Goodman and E. Witten, *Phys. Rev. D* **31** (1985) 3059; M. Drees and M. Nojiri, *Phys. Rev. D* **48** (1993) 3483; A. Djouadi and M. Drees, *Phys. Lett. B* **484** (2000) 183.
29. L. Landau, *Dokl. Akad. Nauk Ser. Fiz.* **60** (1948) 207; C. Yang, *Phys. Rev.* **77** (1950) 242.
30. For a review, see e.g. S. Kraml (ed.) et al., *hep-ph/0608079*.
31. T. Plehn, D. Rainwater and D. Zeppenfeld, *Phys. Rev. Lett.* **88** (2002) 051801; A. Djouadi, R. Godbole, B. Mellado, K. Mohan, *Phys. Lett. B* **723** (2013) 307.
32. V. Barger et al., *Phys. Rev. D* **49** (1994) 79; B. Grzadkowski, J. Gunion and X. He, *Phys. Rev. Lett.* **77** (1996) 5172; J. Gunion and J. Pliszka, *Phys. Lett. B* **444** (1998) 136; P. Bhupal Dev et al., *Phys. Rev. Lett.* **100** (2008) 051801; J. Ellis, V. Sanz, T. You, *Phys. Lett. B* **726** (2013) 244.
33. M. Carena et al., *Eur. J. Phys. C* **26** (2003) 601; *Eur. Phys. J. C* **73** (2013) 2552.
34. A. Djouadi and J. Quevillon, *JHEP* 1310 (2013) 028.
35. L. Maiani, A. Polosa, V. Riquer, *Phys. Lett. B* **718** (2012) 465; *ibid.* **724** (2013) 274.
36. A. Arbey et al., *Phys. Lett. B* **708** (2012) 162; *JHEP* 1209 (2012) 107; *Phys. Lett. B* **720** (2013) 153.
37. A. Djouadi et al. (MSSM working group), *hep-ph/9901246*.
38. A. Djouadi, J.L. Kneur and G. Moultaka, *Comput. Phys. Commun.* **176** (2007) 426; M. Muhlleitner et al., *Comput. Phys. Commun.* **168** (2005) 46.
39. L. Hall and Y. Nomura, *JHEP* 03 (2010) 076; G. Giudice and A. Strumia, *Nucl. Phys. B* **858** (2012) 63; N. Bernal, A. Djouadi, P. Slavich, *JHEP* 0707 (2007) 016.
40. See e.g., M. Carena et al., *Nucl. Phys. B* **577** (2000) 88.
41. A. Djouadi et al., *Phys. Lett. B* **435** (1998) 101; *Eur. Phys. J. C* **1** (1998) 149; *Eur. Phys. J. C* **1** (1998) 163; A. Arvanitaki, G. Villadoro, *JHEP* 02 (2012) 144; A. Delgado et al., *Eur. Phys. J. C* **73** (2013) 2370.
42. M. Spira et al., *Phys. Lett. B* **264** (1991) 440; *Phys. Lett. B* **318** (1993) 347; *Nucl. Phys. B* **453** (1995) 17; R. Harlander and W. Kilgore, *Phys. Rev. D* **68** (2003) 013001; R. Harlander, S. Liebler and H. Mantler, *Comp. Phys. Comm.* **184** (2013) 1605.
43. ATLAS coll., *arXiv:1307.7292*; CMS coll., *arXiv:1307.7135*.
44. SN. Arkani-Hamed et al. *arXiv:1511.06495* [hep-ph]; J. Baglio, A. Djouadi and J. Quevillon, *arXiv:1511.07853* [hep-ph].
45. See e.g., J. Baglio et al., *JHEP* 1304 (2013) 151.
46. M. Bicer et al., *arXiv:1308.6176*; H. Baer et al., *arXiv:1306.6352*; G. Arons et al., *arXiv:0709.1893*; G. Weiglein et al. *Phys. Rept.* **426** (2006) 47; J. Aguilar-Saavedra, *hep-ph/0106315*; E. Accomando et al., *Phys. Rept.* **299** (1998) 1; A. Djouadi, *Int. J. Mod. Phys. A* **10** (1995) 1.



## 5 $\Delta F = 2$ in Neutral Mesons From a Gauged $SU(3)_F$ Family Symmetry

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**Abstract.** Within a broken local gauge vector-like  $SU(3)_F$  family symmetry, we study some  $\Delta F = 2$  processes induced by the tree level exchange of the new massive horizontal gauge bosons, which introduce flavor-changing couplings. We find out that some of the dangerous FCNC processes, like for instance;  $K^0 - \bar{K}^0$ ,  $D^0 - \bar{D}^0$  mixing, may be properly suppressed if the first stage of the Spontaneous Symmetry Breaking (SSB),  $SU(3)_F \rightarrow SU(2)_F$ , occurs at a high scale  $\Lambda \sim 10^{11}$  GeV, with the  $SU(2)_F$  gauge bosons acting on the light families. We provide a parameter space region where this framework can accommodate the hierarchical spectrum of quark masses and mixing and simultaneously suppress properly the contribution to  $K^0 - \bar{K}^0$  mixing as well as the  $\mathcal{O}_{LL}$  and  $\mathcal{O}_{RR}$  effective operators for the  $\Delta C = 2$  processes.

**Povzetek.** Avtor obravnava procese, pri katerih se družinsko kvantno število spremeni za 2. Uporabi model, v katerem opiše družinsko kvantno število kvarkov in leptonov z grupo  $SU(3)$ , lokalna umeritvena polja grupe  $SU(3)$  pa poskrbijo za interakcijo med fermioni, ki nosijo ustrezna kvantna števila. Masivni umeritveni bozoni dopuščajo sicer nevtralne prehode (FCNC) med fermioni iste družine, vendar so taki prehodi, kot primer navaja mešanje  $K^0 - \bar{K}^0$  ter  $D^0 - \bar{D}^0$ , dovolj malo verjetni, če le pride do spontane zlomitve družinske simetrije  $SU(3)_F \rightarrow SU(2)_F$  pri energiji  $\Lambda \sim 10^{11}$  GeV. Poišče območje parametrov, v katerem imajo kvarki opazljive lastnosti.

Keywords: Quark and lepton masses and mixing, Flavor symmetry,  $\Delta F = 2$  Processes.

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### 5.1 Introduction

Flavor physics and rare processes play an important role to test any Beyond Standard Model(BSM) physics proposal, and hence, it is crucial to explore the possibility to suppress properly these type of flavor violating processes.

Within the framework of a vector-like gauged  $SU(3)_F$  family symmetry model[1,2], we study the contribution to  $\Delta F = 2$  processes[3]-[6] in neutral mesons

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at tree level exchange diagrams mediated by the gauge bosons with masses of the order of some TeV's, corresponding to the lower scale of the  $SU(3)_F$  family symmetry breaking.

The reported analysis is performed in a scenario where light fermions obtain masses from radiative corrections mediated by the massive bosons associated to the broken  $SU(3)_F$  family symmetry, while the heavy fermions; top and bottom quarks and tau lepton become massive from tree level See-saw mechanisms. Previous theories addressing the problem of quark and lepton masses and mixing with spontaneously broken  $SU(3)$  gauge symmetry of generations include the ones with chiral local  $SU(3)_H$  family symmetry as well as other  $SU(3)$  family symmetries. See for instance [7]-[20] and references therein.

## 5.2 $SU(3)_F$ flavor symmetry model

The model is based on the gauge symmetry

$$G \equiv SU(3)_F \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \quad (5.1)$$

where  $SU(3)_F$  is a completely vector-like and universal gauged family symmetry. That is, the corresponding gauge bosons couple equally to Left and Right Handed ordinary Quarks and Leptons, with  $g_H$ ,  $g_s$ ,  $g$  and  $g'$  the corresponding coupling constants. The content of fermions assumes the standard model quarks and leptons:

$$\Psi_q^o = (3, 3, 2, \frac{1}{3})_L \quad , \quad \Psi_l^o = (3, 1, 2, -1)_L$$

$$\Psi_u^o = (3, 3, 1, \frac{4}{3})_R \quad , \quad \Psi_d^o = (3, 3, 1, -\frac{2}{3})_R \quad , \quad \Psi_e^o = (3, 1, 1, -2)_R$$

where the last entry is the hypercharge  $Y$ , with the electric charge defined by  $Q = T_{3L} + \frac{1}{2}Y$ .

The model includes two types of extra fermions:

- Right Handed Neutrinos:  $\Psi_{\nu_R}^o = (3, 1, 1, 0)_R$  , introduced to cancel anomalies [21],
- and a new family of  $SU(2)_L$  weak singlet vector-like fermions:

$$U_L^o, U_R^o = (1, 3, 1, \frac{4}{3}) \quad , \quad D_L^o, D_R^o = (1, 3, 1, -\frac{2}{3}) \quad (5.2)$$

Vector Like electrons:  $E_L^o, E_R^o = (1, 1, 1, -2)$

and

New Sterile Neutrinos:  $N_L^o, N_R^o = (1, 1, 1, 0)$  ,

The particle content and gauge symmetry assignments are summarized in Table 5.1. Notice that all  $SU(3)_F$  non-singlet fields transform as the fundamental representation under the  $SU(3)_F$  symmetry.

	$SU(3)_F$	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$\psi_q^o$	3	3	2	$\frac{1}{3}$
$\psi_{uR}^o$	3	3	1	$\frac{4}{3}$
$\psi_{dR}^o$	3	3	1	$-\frac{2}{3}$
$\psi_l^o$	3	1	2	-1
$\psi_{eR}^o$	3	1	1	-2
$\psi_{\nu R}^o$	3	1	1	0
$\Phi^u$	3	1	2	-1
$\Phi^d$	3	1	2	+1
$\eta_i$	3	1	1	0
$U_{L,R}^o$	1	3	1	$\frac{4}{3}$
$D_{L,R}^o$	1	3	1	$-\frac{2}{3}$
$E_{L,R}^o$	1	1	1	-2
$N_{L,R}^o$	1	1	1	0

**Table 5.1.** Particle content and charges under the gauge symmetry

### 5.3 $SU(3)_F$ family symmetry breaking

To implement the SSB of  $SU(3)_F$ , we introduce the flavon scalar fields:  $\eta_i$ ,  $i = 2, 3$ ,

$$\eta_i = (3, 1, 1, 0) = \begin{pmatrix} \eta_{i1}^o \\ \eta_{i2}^o \\ \eta_{i3}^o \end{pmatrix}, \quad i = 2, 3$$

with the “Vacuum ExpectationValues” (VEV’s):

$$\langle \eta_2 \rangle^T = (0, \Lambda_2, 0) \quad , \quad \langle \eta_3 \rangle^T = (0, 0, \Lambda_3) . \quad (5.3)$$

It is worth to mention that these two scalars in the fundamental representation is the minimal set of scalars to break down completely the  $SU(3)_F$  family symmetry. The interaction Lagrangian of the  $SU(3)_F$  gauge bosons to the SM massless fermions is

$$i\mathcal{L}_{\text{int}, SU(3)_F} = g_H \begin{pmatrix} \bar{f}_1^o & \bar{f}_2^o & \bar{f}_3^o \end{pmatrix} \gamma_\mu \begin{pmatrix} \frac{Z_1^\mu}{2} + \frac{Z_2^\mu}{2\sqrt{3}} & \frac{Y_1^{+\mu}}{\sqrt{2}} & \frac{Y_2^{+\mu}}{\sqrt{2}} \\ \frac{Y_1^{-\mu}}{\sqrt{2}} & -\frac{Z_1^\mu}{2} + \frac{Z_2^\mu}{2\sqrt{3}} & \frac{Y_3^{+\mu}}{\sqrt{2}} \\ \frac{Y_2^{-\mu}}{\sqrt{2}} & \frac{Y_3^{-\mu}}{\sqrt{2}} & -\frac{Z_2^\mu}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} f_1^o \\ f_2^o \\ f_3^o \end{pmatrix}$$



where  $g_H$  is the  $SU(3)_F$  coupling constant,  $Z_1, Z_2$  and  $Y_j^\pm = \frac{Y_j^1 \mp i Y_j^2}{\sqrt{2}}$ ,  $j = 1, 2, 3$  are the eight gauge bosons.

Thus, the contribution to the horizontal gauge boson masses from the VEV's in Eq.(5.3) read

- $\langle \eta_2 \rangle$  :  $\frac{g_{H_2}^2 \Lambda_2^2}{2} (Y_1^+ Y_1^- + Y_3^+ Y_3^-) + \frac{g_{H_2}^2 \Lambda_2^2}{4} (Z_1^2 + \frac{Z_2^2}{3} - 2Z_1 \frac{Z_2}{\sqrt{3}})$
- $\langle \eta_3 \rangle$  :  $\frac{g_{H_3}^2 \Lambda_3^2}{2} (Y_2^+ Y_2^- + Y_3^+ Y_3^-) + g_{H_3}^2 \Lambda_3^2 \frac{Z_2^2}{3}$

The "Spontaneous Symmetry Breaking" (SSB) of  $SU(3)_F$  occurs in two stages

$$SU(3)_F \times G_{SM} \xrightarrow{\langle \eta_3 \rangle} SU(2)_F \times G_{SM} \xrightarrow{\langle \eta_2 \rangle} G_{SM}$$

**FCNC ?**

$\Lambda_3$ : 5 very heavy boson masses ( $\geq 100 \text{ TeV's}$ )

$\Lambda_2$ : 3 heavy boson masses (may be a few  $\text{TeV's}$ ).

Notice that the hierarchy of scales  $\Lambda_3 \gg \Lambda_2$  define an "approximate  $SU(2)$  global symmetry" in the spectrum of  $SU(2)_F$  gauge boson masses. To suppress properly the FCNC like, for instance:  $\mu \rightarrow e\gamma$ ,  $\mu \rightarrow e e e$ ,  $K^0 - \bar{K}^0$ , and  $D^0 - \bar{D}^0$ , it is crucial to choose properly the  $SU(2)_F$  symmetry at the lower scale.

Therefore, neglecting tiny contributions from electroweak symmetry breaking, we obtain the gauge boson mass terms.

$$M_2^2 Y_1^+ Y_1^- + M_3^2 Y_2^+ Y_2^- + (M_2^2 + M_3^2) Y_3^+ Y_3^- + \frac{1}{2} M_2^2 Z_1^2 + \frac{1}{2} \frac{M_2^2 + 4M_3^2}{3} Z_2^2 - \frac{1}{2} (M_2^2) \frac{2}{\sqrt{3}} Z_1 Z_2 \quad (5.4)$$

$$M_2^2 = \frac{g_{H_2}^2 \Lambda_2^2}{2}, \quad M_3^2 = \frac{g_{H_3}^2 \Lambda_3^2}{2}, \quad y \equiv \frac{M_3}{M_2} = \frac{\Lambda_3}{\Lambda_2} \quad (5.5)$$

	$Z_1$	$Z_2$
$Z_1$	$M_2^2$	$-\frac{M_2^2}{\sqrt{3}}$
$Z_2$	$-\frac{M_2^2}{\sqrt{3}}$	$\frac{M_2^2 + 4M_3^2}{3}$

**Table 5.2.**  $Z_1 - Z_2$  mixing mass matrix

Diagonalization of the  $Z_1 - Z_2$  squared mass matrix yield the eigenvalues

$$M_-^2 = \frac{2}{3} \left( M_2^2 + M_3^2 - \sqrt{(M_3^2 - M_2^2)^2 + M_2^2 M_3^2} \right) \quad (5.6)$$

$$M_+^2 = \frac{2}{3} \left( M_2^2 + M_3^2 + \sqrt{(M_3^2 - M_2^2)^2 + M_2^2 M_3^2} \right) \quad (5.7)$$

and finally

$$M_2^2 Y_1^+ Y_1^- + M_3^2 Y_2^+ Y_2^- + (M_2^2 + M_3^2) Y_3^+ Y_3^- + M_-^2 \frac{Z_-^2}{2} + M_+^2 \frac{Z_+^2}{2}, \quad (5.8)$$

where

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} Z_- \\ Z_+ \end{pmatrix} \quad (5.9)$$

$$\cos \phi \sin \phi = \frac{\sqrt{3}}{4} \frac{M_2^2}{\sqrt{M_2^4 + M_3^2(M_3^2 - M_2^2)}}$$

$$Z_1 = \cos \phi Z_- - \sin \phi Z_+ \quad , \quad Z_2 = \sin \phi Z_- + \cos \phi Z_+ \quad (5.10)$$

## 5.4 Electroweak symmetry breaking

For electroweak symmetry breaking we introduction two triplets of  $SU(2)_L$  Higgs doublets, namely;

$$\Phi^u = (3, 1, 2, -1) \quad , \quad \Phi^d = (3, 1, 2, +1),$$

and the VEV's

$$\langle \Phi^u \rangle = \begin{pmatrix} \langle \Phi_1^u \rangle \\ \langle \Phi_2^u \rangle \\ \langle \Phi_3^u \rangle \end{pmatrix} \quad , \quad \langle \Phi^d \rangle = \begin{pmatrix} \langle \Phi_1^d \rangle \\ \langle \Phi_2^d \rangle \\ \langle \Phi_3^d \rangle \end{pmatrix} ,$$

where

$$\langle \Phi_i^u \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_{ui} \\ 0 \end{pmatrix} \quad , \quad \langle \Phi_i^d \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_{di} \end{pmatrix} .$$

The contributions from  $\langle \Phi^u \rangle$  and  $\langle \Phi^d \rangle$  yield the  $W$  and  $Z_o$  gauge boson masses and mixing with the  $SU(3)_F$  gauge bosons

$$\begin{aligned} & \frac{g^2}{4} (v_u^2 + v_d^2) W^+ W^- + \frac{(g^2 + g'^2)}{8} (v_u^2 + v_d^2) Z_o^2 \\ & + \text{tiny contribution to the } SU(3)_F \text{ gauge boson masses and mixing with} \\ & \text{the gauge boson } Z_o , \end{aligned}$$

$v_u^2 = v_{1u}^2 + v_{2u}^2 + v_{3u}^2$ ,  $v_d^2 = v_{1d}^2 + v_{2d}^2 + v_{3d}^2$ . So, if we define  $M_W = \frac{1}{2}gv$ , we may write  $v = \sqrt{v_u^2 + v_d^2} \approx 246$  GeV.

## 5.5 Fermion masses

### 5.5.1 Dirac See-saw mechanisms

The scalars and fermion content allow the gauge invariant Yukawa couplings

$$H_u \overline{\psi}_q^o \Phi^u U_R^o + h_{iu} \overline{\psi}_{uR}^o \eta_i U_L^o + M_u \overline{U}_L^o U_R^o + h.c \quad (5.11)$$

$$H_d \overline{\psi}_q^o \Phi^d D_R^o + h_{id} \overline{\psi}_{dR}^o \eta_i D_L^o + M_D \overline{D}_L^o D_R^o + h.c \quad (5.12)$$

$$H_e \overline{\psi}_l^o \Phi^d E_R^o + h_{ie} \overline{\psi}_{eR}^o \eta_i E_L^o + M_E \overline{E}_L^o E_R^o + h.c \quad (5.13)$$

$$H_\nu \overline{\psi}_l^o \Phi^u N_R^o + h_{i\nu} \overline{\psi}_{\nu R}^o \eta_i N_L^o + M_{N_D} \overline{N}_L^o N_R^o + h.c \quad (5.14)$$

$$h_L \overline{\psi}_l^o \Phi^u (N_L^o)^c + m_L \overline{N}_L^o (N_L^o)^c + h.c \quad (5.15)$$

$$h_{iR} \overline{\psi}_{\nu R}^o \eta_i (N_R^o)^c + m_R \overline{N}_R^o (N_R^o)^c + h.c \quad (5.16)$$

$M_u, M_D, M_E, M_{N_D}, m_L, m_R$  are free mass parameters and  $H_u, H_d, H_e, H_\nu, h_{iu}, h_{id}, h_{ie}, h_{i\nu}, h_L, h_{iR}$  are Yukawa coupling constants. When the involved scalar fields acquire VEV's, we get in the gauge basis  $\psi_{L,R}^{oT} = (e^o, \mu^o, \tau^o, E^o)_{L,R}$ , the mass terms  $\overline{\psi}_L^o \mathcal{M}^o \psi_R^o + h.c$ , where

$$\mathcal{M}^o = \begin{pmatrix} 0 & 0 & 0 & h v_1 \\ 0 & 0 & 0 & h v_2 \\ 0 & 0 & 0 & h v_3 \\ 0 & h_2 \Lambda_2 & h_3 \Lambda_3 & M \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & b_2 & b_3 & M \end{pmatrix}. \quad (5.17)$$

$\mathcal{M}^o$  is diagonalized by applying a biunitary transformation  $\psi_{L,R}^o = V_{L,R}^o \chi_{L,R}$ . Using the possible parametrizations for the orthogonal matrices  $V_L^o$  and  $V_R^o$  are written explicitly in the Appendix A, Using one obtains

$$V_L^{oT} \mathcal{M}^o V_R^o = \text{Diag}(0, 0, -\lambda_3, \lambda_4) \quad (5.18)$$

$$V_L^{oT} \mathcal{M}^o \mathcal{M}^{oT} V_L^o = V_R^{oT} \mathcal{M}^{oT} \mathcal{M}^o V_R^o = \text{Diag}(0, 0, \lambda_3^2, \lambda_4^2). \quad (5.19)$$

where  $\lambda_3$  and  $\lambda_4$  are the nonzero eigenvalues defined in Eqs.(5.56-5.58),  $\lambda_4$  being the fourth heavy fermion mass, and  $\lambda_3$  of the order of the top, bottom and tau

mass for  $u$ ,  $d$  and  $e$  fermions, respectively. We see from Eqs.(5.18,5.19) that from tree level the See-saw mechanism yields two massless eigenvalues associated to the light fermions:

It is worth to mention that the Yukawa couplings in Eqs.5.11–5.16 are invariant under the global symmetry  $U(1)_B \times U(1)_Y \times U(1)_\alpha \times U(1)_\beta$ , where  $B$  is the baryon number,  $Y$  is the hypercharge, and  $U(1)_\alpha$ ,  $U(1)_\beta$  are two additional symmetries, and one of them could play the role of a Peccei-Quinn symmetry to address the strong CP problem[22].

## 5.6 One loop contribution to fermion masses

After tree level contributions the first two generations remain massless. Therefore, in this scenario light fermion masses, including neutrinos, may get small masses from radiative corrections mediated by the  $SU(3)_F$  heavy gauge bosons.

The one loop diagram of Fig. 1 gives the generic contribution to the mass term  $m_{ij} \bar{e}_{iL}^o e_{jR}^o$ , where

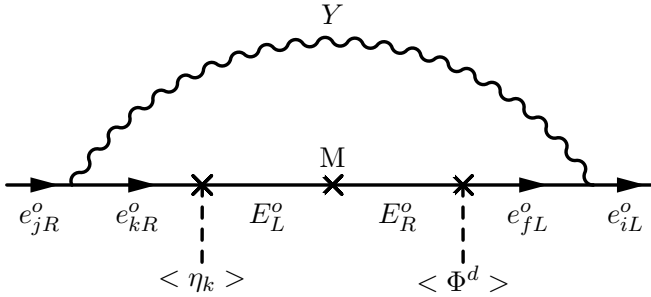


Fig. 5.1. Generic one loop diagram contribution to the mass term  $m_{ij} \bar{e}_{iL}^o e_{jR}^o$

$$m_{ij} = c_Y \frac{\alpha_H}{\pi} \sum_{k=3,4} m_k^o (V_L^o)_{ik} (V_R^o)_{jk} f(M_Y, m_k^o) \quad , \quad \alpha_H \equiv \frac{g_H^2}{4\pi} \quad , \quad (5.20)$$

$M_Y$  being the mass of the gauge boson,  $c_Y$  is a factor coupling constant, Eq.(5.3),  $m_3^o = -\lambda_3$  and  $m_4^o = \lambda_4$  are the See-saw mass eigenvalues, Eq.(5.18),  $f(x, y) = \frac{x^2}{x^2 - y^2} \ln \frac{x^2}{y^2}$ , and

$$\sum_{k=3,4} m_k^o (V_L^o)_{ik} (V_R^o)_{jk} f(M_Y, m_k^o) = \frac{a_i b_j M}{\lambda_4^2 - \lambda_3^2} F(M_Y) \quad , \quad (5.21)$$

$i = 1, 2, 3$ ,  $j = 2, 3$ , and  $F(M_Y) \equiv \frac{M_Y^2}{M_Y^2 - \lambda_4^2} \ln \frac{M_Y^2}{\lambda_4^2} - \frac{M_Y^2}{M_Y^2 - \lambda_3^2} \ln \frac{M_Y^2}{\lambda_3^2}$ . Adding up all possible the one loop contributions, we get the mass terms  $\bar{\psi}_L^o \mathcal{M}_1^o \psi_R^o + \text{h.c.}$ ,

$$\mathcal{M}_1^o = \begin{pmatrix} D_{11} & D_{12} & D_{13} & 0 \\ 0 & D_{22} & D_{23} & 0 \\ 0 & D_{32} & D_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\alpha_H}{\pi}, \quad (5.22)$$

$$D_{11} = \frac{1}{2}(\mu_{22}F_1 + \mu_{33}F_2), \quad D_{12} = \mu_{12}\left(-\frac{F_{Z_1}}{4} + \frac{F_{Z_2}}{12}\right), \quad D_{13} = -\mu_{13}\left(\frac{F_{Z_2}}{6} + F_m\right),$$

$$D_{22} = \mu_{22}\left(\frac{F_{Z_1}}{4} + \frac{F_{Z_2}}{12} - F_m\right) + \frac{1}{2}\mu_{33}F_3, \quad D_{23} = -\mu_{23}\left(\frac{F_{Z_2}}{6} - F_m\right),$$

$$D_{32} = -\mu_{32}\left(\frac{F_{Z_2}}{6} - F_m\right), \quad D_{33} = \mu_{33}\frac{F_{Z_2}}{3} + \frac{1}{2}\mu_{22}F_3,$$

$$\alpha_H = \frac{g_H^2}{4\pi}, \quad F_1 \equiv F(M_{Y_1}), \quad F_2 \equiv F(M_{Y_2}), \quad F_3 \equiv F(M_{Y_3})$$

$$F_{Z_1} = \cos^2 \phi F(M_-) + \sin^2 \phi F(M_+), \quad F_{Z_2} = \sin^2 \phi F(M_-) + \cos^2 \phi F(M_+)$$

$$F_m = \frac{\cos \phi \sin \phi}{2\sqrt{3}} [F(M_-) - F(M_+)].$$

$F_{Z_1}, F_{Z_2}$  are the contributions from the diagrams mediated by the  $Z_1, Z_2$  gauge bosons,  $F_m$  comes from the  $Z_1 - Z_2$  mixing diagrams, with  $M_2, M_3, M_-, M_+$  the horizontal boson masses, Eqs.(5.5-5.7),

$$\mu_{ij} = \frac{a_i b_j M}{\lambda_4^2 - \lambda_3^2} = \frac{a_i b_j}{a b} \lambda_3 c_\alpha c_\beta, \quad (5.23)$$

with  $c_\alpha = \cos \alpha, c_\beta = \cos \beta, s_\alpha = \sin \alpha, s_\beta = \sin \beta$  the mixing angles coming from the diagonalization of  $\mathcal{M}^o$ . Therefore, up to one loop corrections the fermion masses are

$$\bar{\psi}_L^o \mathcal{M}^o \psi_R^o + \bar{\psi}_L^o \mathcal{M}_1^o \psi_R^o = \bar{\chi}_L \mathcal{M} \chi_R, \quad (5.24)$$

where  $\psi_{L,R}^o = V_{L,R}^o \chi_{L,R}$ , and  $\mathcal{M} \equiv [\text{Diag}(0, 0, -\lambda_3, \lambda_4) + V_L^{oT} \mathcal{M}_1^o V_R^o]$  can be written as:

$$\mathcal{M} = \begin{pmatrix} m_{11} & m_{12} & c_\beta m_{13} & s_\beta m_{13} \\ m_{21} & m_{22} & c_\beta m_{23} & s_\beta m_{23} \\ c_\alpha m_{31} & c_\alpha m_{32} & (-\lambda_3 + c_\alpha c_\beta m_{33}) & c_\alpha s_\beta m_{33} \\ s_\alpha m_{31} & s_\alpha m_{32} & s_\alpha c_\beta m_{33} & (\lambda_4 + s_\alpha s_\beta m_{33}) \end{pmatrix}, \quad (5.25)$$

The explicit expression for the  $m_{ij}$  mass terms depends on the used parametrization for  $V_L^o, V_R^o$ .

The diagonalization of  $\mathcal{M}$ , Eq.(5.25) gives the physical masses for u and d quarks, e charged leptons and  $\nu$  Dirac neutrino masses.

Using a new biunitary transformation

$$\chi_{L,R} = V_{L,R}^{(1)} \Psi_{L,R}; \quad \bar{\chi}_L \mathcal{M} \chi_R = \bar{\Psi}_L V_L^{(1)\dagger} \mathcal{M} V_R^{(1)} \Psi_R,$$

with  $\Psi_{L,R}^T = (f_1, f_2, f_3, F)_{L,R}$  the mass eigenfields, that is

$$V_L^{(1)\dagger} \mathcal{M} \mathcal{M}^T V_L^{(1)} = V_R^{(1)\dagger} \mathcal{M}^T \mathcal{M} V_R^{(1)} = \text{Diag}(m_1^2, m_2^2, m_3^2, M_F^2), \quad (5.26)$$

$m_1^2 = m_e^2, m_2^2 = m_\mu^2, m_3^2 = m_\tau^2$  and  $M_F^2 = M_E^2$  for charged leptons. So, the rotations from massless to mass fermions eigenfields in this scenario reads

$$\psi_L^o = V_L^o V_L^{(1)} \Psi_L \quad \text{and} \quad \psi_R^o = V_R^o V_R^{(1)} \Psi_R \quad (5.27)$$

### 5.6.1 Quark Mixing Matrix $V_{CKM}$

We recall that vector like quarks, Eq.(5.2), are  $SU(2)_L$  weak singlets, and they do not couple to  $W$  boson in the interaction basis. In this way, the interaction of L-handed up and down quarks;  $f_{uL}^o{}^T = (u^o, c^o, t^o)_L$  and  $f_{dL}^o{}^T = (d^o, s^o, b^o)_L$ , to the  $W$  charged gauge boson is

$$\frac{g}{\sqrt{2}} \bar{f}_{uL}^o \gamma_\mu f_{dL}^o W^{+\mu} = \frac{g}{\sqrt{2}} \bar{\Psi}_{uL} [ (V_{uL}^o V_{uL}^{(1)})_{3 \times 4} ]^T (V_{dL}^o V_{dL}^{(1)})_{3 \times 4} \gamma_\mu \Psi_{dL} W^{+\mu}, \quad (5.28)$$

Hence, in this scenario the non-unitary  $V_{CKM}$  of dimension  $4 \times 4$  is identified as

$$(V_{CKM})_{4 \times 4} = [ (V_{uL}^o V_{uL}^{(1)})_{3 \times 4} ]^T (V_{dL}^o V_{dL}^{(1)})_{3 \times 4} \quad (5.29)$$

## 5.7 Numerical results for quark masses and mixing

As an example of the possible spectrum of quark masses and mixing from this scenario, we consider the following set of parameters at the  $M_Z$  scale [23]

Using the input values for the horizontal boson masses, Eq.(5.5), and the coupling constant of the  $SU(3)_F$  symmetry:

$$M_2 = 6.0 \text{ TeV} \quad , \quad M_3 = 1.5 \times 10^8 \text{ TeV} \quad , \quad \frac{\alpha_H}{\pi} = 0.2, \quad (5.30)$$

we show in the interaction basis the following tree level  $\mathcal{M}_q^o$ , and one loop  $\mathcal{M}_q^o$  quark mass matrices, and the corresponding mass eigenvalues and mixing:

**u-quarks:**

Tree level see-saw mass matrix:

$$\mathcal{M}_u^o = \begin{pmatrix} 0 & 0 & 0 & 5573.43 \\ 0 & 0 & 0 & 23883.8 \\ 0 & 0 & 0 & 397346. \\ 0 & -1.931 \times 10^8 & 5.193 \times 10^6 & 2.470 \times 10^8 \end{pmatrix} \text{ MeV}, \quad (5.31)$$

the mass matrix up to one loop corrections:

$$\mathcal{M}_{u1}^o = \begin{pmatrix} 1.42 & -220.786 & 34.6742 & 0 \\ 0 & -944.713 & 148.589 & 0 \\ 0 & -91921.3 & 11631.6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ MeV} \quad (5.32)$$

the u-quark mass eigenvalues

$$(m_u, m_c, m_t, M_U) = (1.382, 633.289, 172968, 313.606 \times 10^6) \text{ MeV} \quad (5.33)$$

and the mixing matrices:

$$V_{uL} = V_{uL}^o V_{uL}^{(1)}:$$

$$\begin{pmatrix} 0.973838 & -0.226464 & -0.0188217 & 0.0000144353 \\ -0.227244 & -0.970491 & -0.080663 & 0.0000618569 \\ 9.34208 \times 10^{-7} & 0.0828299 & -0.996563 & 0.0011792 \\ -2.14837 \times 10^{-9} & -0.0000343726 & 0.00118041 & 0.999999 \end{pmatrix} \quad (5.34)$$

$$V_{uR} = V_{uR}^o V_{uR}^{(1)}:$$

$$\begin{pmatrix} 1. & 0.000507791 & 1.54519 \times 10^{-7} & 0 \\ -7.6088 \times 10^{-6} & 0.0147444 & 0.787788 & -0.61577 \\ 0.000507462 & -0.999362 & 0.0316481 & 0.0165598 \\ -0.0000166153 & 0.0325336 & 0.615132 & 0.787752 \end{pmatrix} \quad (5.35)$$

**d-quarks:**

$$\mathcal{M}_d^o = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3102.75 \\ 0 & 0 & 0 & 61977.5 \\ 0 & -9.805 \times 10^7 & 2.837 \times 10^6 & 6.046 \times 10^8 \end{pmatrix} \text{ MeV} \quad (5.36)$$

$$\mathcal{M}_{d1}^o = \begin{pmatrix} 2.82 & 0 & 0 & 0 \\ 0 & -130.851 & 10.43 & 0 \\ 0 & -7200.83 & 664.801 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ MeV} \quad (5.37)$$

the d-quark mass eigenvalues

$$(m_d, m_s, m_b, M_D) = (2.82, 52.087, 2861.96, 612.541 \times 10^6) \text{ MeV} \quad (5.38)$$

the mixing matrices:

$$V_{dL} = V_{dL}^0 V_{dL}^{(1)}:$$

$$\begin{pmatrix} 1. & 0 & 0 & 0 \\ 0 & 0.991883 & -0.127155 & 5.03428 \times 10^{-6} \\ 0 & -0.127155 & -0.991883 & 0.000101762 \\ 0 & 7.94612 \times 10^{-6} & 0.000101576 & 1. \end{pmatrix} \quad (5.39)$$

$$V_{dR} = V_{dR}^0 V_{dR}^{(1)}:$$

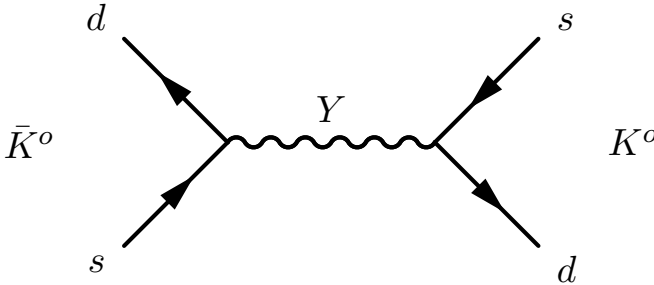
$$\begin{pmatrix} 1. & 0 & 0 & 0 \\ 0 & -0.127762 & 0.9788 & -0.160083 \\ 0 & 0.99148 & 0.130175 & 0.00463162 \\ 0 & -0.0253722 & 0.158128 & 0.987093 \end{pmatrix} \quad (5.40)$$

and the quark mixing matrix

$$V_{CKM} = \begin{pmatrix} 0.97383 & 0.2254 & 0.02889 & -1.14 \times 10^{-6} \\ -0.22646 & 0.97314 & 0.04124 & 3.54 \times 10^{-6} \\ -0.01882 & -0.04670 & 0.99873 & -0.00010 \\ 1.44 \times 10^{-5} & 8.85 \times 10^{-5} & -0.00117 & 1.20 \times 10^{-7} \end{pmatrix} \quad (5.41)$$

## 5.8 $\Delta F = 2$ Processes in Neutral Mesons

Here we study the tree level FCNC interactions that contribute to  $K^0 - \bar{K}^0$ ,  $D^0 - \bar{D}^0$  mixing via  $Z_1$ ,  $Y_1^\pm$  exchange from the depicted diagram in Fig. 2.



**Fig. 5.2.** Generic tree level exchange contribution to  $K^0 - \bar{K}^0$  from the SU(3) horizontal gauge bosons.

The  $Z_1$ ,  $Y_1^\pm$  gauge bosons have flavor changing couplings in both left- and right-handed fermions, and then contribute the  $\Delta S = 2$  effective operators



$$\mathcal{O}_{LL} = (\bar{d}_L \gamma_\mu s_L)(\bar{d}_L \gamma^\mu s_L) \quad , \quad \mathcal{O}_{RR} = (\bar{d}_R \gamma_\mu s_R)(\bar{d}_R \gamma^\mu s_R) \quad (5.42)$$

$$\mathcal{O}_{LR} = (\bar{d}_L \gamma_\mu s_L)(\bar{d}_R \gamma^\mu s_R) \quad (5.43)$$

The  $SU(3)_F$  couplings to fermions, Eq.5.3, when written in the mass basis yield the gauge couplings

$$\mathcal{L}_{\text{int}, Z_1} = \frac{g_H}{2} (C_{L Z_1} \bar{d}_L \gamma_\mu s_L + C_{R Z_1} \bar{d}_R \gamma_\mu s_R) Z_1^\mu \quad (5.44)$$

$$\mathcal{L}_{\text{int}, Y_1^1} = \frac{g_H}{2} (C_{L Y_1^1} \bar{d}_L \gamma_\mu s_L + C_{R Y_1^1} \bar{d}_R \gamma_\mu s_R) Y_1^{1\mu} \quad (5.45)$$

$$\mathcal{L}_{\text{int}, Y_1^2} = \frac{g_H}{2} (C_{L Y_1^2} \bar{d}_L \gamma_\mu s_L + C_{R Y_1^2} \bar{d}_R \gamma_\mu s_R) Y_1^{2\mu} \quad (5.46)$$

with the coefficients

$$C_{L Z_1} = L_{11} L_{12} - L_{21} L_{22} \quad , \quad C_{R Z_1} = R_{11} R_{12} - R_{21} R_{22}$$

$$C_{L Y_1^1} = L_{12} L_{21} + L_{11} L_{22} \quad , \quad C_{R Y_1^1} = R_{12} R_{21} + R_{11} R_{22} \quad (5.47)$$

$$C_{L Y_1^2} = (L_{12} L_{21} - L_{11} L_{22}) \quad , \quad C_{R Y_1^2} = (R_{12} R_{21} - R_{11} R_{22})$$

where  $L_{ij} = V_{L ij}$  and  $R_{ij} = V_{R ij}$ . For each gauge boson, the effective four-fermion hamiltonian at the scale of the gauge boson mass is

$$\mathcal{H}_{Z_1} = \frac{g_H^2}{4M_{Z_1}^2} (C_{L Z_1}^2 \mathcal{O}_{LL} + 2 C_{L Z_1} C_{R Z_1} \mathcal{O}_{LR} + C_{R Z_1}^2 \mathcal{O}_{RR}) \quad (5.48)$$

$$\mathcal{H}_{Y_1^1} = \frac{g_H^2}{4M_{Y_1^1}^2} (C_{L Y_1^1}^2 \mathcal{O}_{LL} + 2 C_{L Y_1^1} C_{R Y_1^1} \mathcal{O}_{LR} + C_{R Y_1^1}^2 \mathcal{O}_{RR}) \quad (5.49)$$

$$\mathcal{H}_{Y_1^2} = -\frac{g_H^2}{4M_{Y_1^2}^2} (C_{L Y_1^2}^2 \mathcal{O}_{LL} + 2 C_{L Y_1^2} C_{R Y_1^2} \mathcal{O}_{LR} + C_{R Y_1^2}^2 \mathcal{O}_{RR}) \quad (5.50)$$

with  $M_{Y_1} = M_{Y_2} = M_2$ . Therefore, the total four-fermion hamiltonian  $\mathcal{H}_{SU(2)} = \mathcal{H}_{Z_1} + \mathcal{H}_{Y_1^1} + \mathcal{H}_{Y_1^2}$  can be written as

$$\begin{aligned} \mathcal{H}_{SU(2)} = & \frac{g_H^2}{4M_2^2} \left[ (C_{L Z_1}^2 + C_{L Y_1^1}^2 - C_{L Y_1^2}^2) \mathcal{O}_{LL} + (C_{R Z_1}^2 + C_{R Y_1^1}^2 + C_{R Y_1^2}^2) \mathcal{O}_{RR} \right. \\ & \left. + 2(C_{L Z_1} C_{R Z_1} + C_{L Y_1^1} C_{R Y_1^1} - C_{L Y_1^2} C_{R Y_1^2}) \mathcal{O}_{LR} \right] \\ & + \frac{g_H^2}{4} \left( \frac{1}{M_{Z_1}^2} - \frac{1}{M_2^2} \right) [C_{L Z_1}^2 \mathcal{O}_{LL} + C_{R Z_1}^2 \mathcal{O}_{RR} + 2 C_{L Z_1} C_{R Z_1} \mathcal{O}_{LR}] \quad (5.51) \end{aligned}$$

From the coefficients in Eq.5.47 we get:

$$C_{LZ_1}^2 + C_{LY_1^1}^2 - C_{LY_1^2}^2 = \delta_L^2 \quad , \quad C_{RZ_1}^2 + C_{RY_1^1}^2 - C_{RY_1^2}^2 = \delta_R^2 \quad , \quad ,$$

$$C_{L,Z_1} C_{R,Z_1} + C_{L,Y_1^1} C_{R,Y_1^1} - C_{L,Y_1^2} C_{R,Y_1^2} = \delta_L \delta_R \\ + 2(L_{11} R_{21} - L_{21} R_{11})(L_{22} R_{12} - L_{12} R_{22}) \quad ,$$

and finally we can write

$$\mathcal{H}_{SU(2)} = \frac{g_H^2}{4M_1^2} [\delta_L^2 \mathcal{O}_{LL} + \delta_R^2 \mathcal{O}_{RR} + \delta_{LR}^2 \mathcal{O}_{LR}] \quad (5.52)$$

$$+ \frac{g_H^2}{4} \left( \frac{1}{M_{Z_1}^2} - \frac{1}{M_1^2} \right) [(L_{11}L_{12} - L_{21}L_{22})^2 \mathcal{O}_{LL} + (R_{11}R_{12} - R_{21}R_{22})^2 \mathcal{O}_{RR} \\ + 2(L_{11}L_{12} - L_{21}L_{22})(R_{11}R_{12} - R_{21}R_{22}) \mathcal{O}_{LR}] \quad (5.53)$$

with

$$\delta_L = L_{11} L_{12} + L_{21} L_{22} \quad , \quad \delta_R = R_{11} R_{12} + R_{21} R_{22}$$

$$\delta_{LR} = \sqrt{2(\delta_L \delta_R + 2(L_{11} R_{21} - L_{21} R_{11})(L_{22} R_{12} - L_{12} R_{22}))}$$

### 5.8.1 $K^0 - \bar{K}^0$ meson mixing

The numerical fit of parameters provided in section 7 yield the mixing angles  $V_{d12} = V_{d21} = 0$  for left- and right-handed d-quarks, and then all the contributions to the effective operators, Eqs.5.42–5.43, for  $\Delta S = 2$  vanish.

### 5.8.2 $D^0 - \bar{D}^0$ meson mixing

The reported parameter space region in section 7 generate  $M_{Z_1} = M_2$  with very good approximation, and then only the four-fermion Hamiltonian in Eq.5.52 contribute. For this case we compute the numerical values

$$\delta_L = -7.73804 \times 10^{-8} \quad , \quad \frac{M_2}{\frac{g_H}{2} \delta_L} = -5.51894 \times 10^7 \text{ TeV} \\ \delta_R = 5.07679 \times 10^{-4} \quad , \quad \frac{M_2}{\frac{g_H}{2} \delta_R} = 8411.97 \text{ TeV} \quad (5.54) \\ \delta_{LR} = 0.0508636 \quad , \quad \frac{M_2}{\frac{g_H}{2} \delta_{LR}} = 83.9614 \text{ TeV}$$

Accordingly to the Review "The CKM quark - mixing matrix" in PDG2016[24], the  $\Delta C = 2$  effective operators  $\mathcal{O}_{LL}$  and  $\mathcal{O}_{RR}$  are within suppression limits.

## 5.9 Conclusions

Horizontal gauge bosons from the local  $SU(3)_F$  introduce flavor changing couplings, and in particular mediate  $\Delta F = 2$  processes at tree level. We reported the analytic contribution to  $K^0 - \bar{K}^0$  and  $D^0 - \bar{D}^0$  meson mixing from tree level exchange diagrams mediated by the  $SU(2)_F$  gauge bosons  $Z_1, Y_1^\pm$  with masses in the TeV region. We provide a particular parameter space region in section 7 where this scenario can accommodate the hierarchy spectrum of quark masses and simultaneously suppress properly the  $K^0 - \bar{K}^0$  meson mixing, and the effective operators  $\mathcal{O}_{LL}$  and  $\mathcal{O}_{RR}$  for the  $\Delta C = 2$  processes.

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## 5.10 APPENDIX: Diagonalization of the generic Dirac See-saw mass matrix

$$\mathcal{M}^o = \begin{pmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & b_2 & b_3 & c \end{pmatrix} \quad (5.55)$$

The tree level  $\mathcal{M}^o$   $4 \times 4$  See-saw mass matrix is diagonalized by a biunitary transformation  $\psi_L^o = V_L^o \chi_L$  and  $\psi_R^o = V_R^o \chi_R$ . The diagonalization of  $\mathcal{M}^o \mathcal{M}^{o\top}$  ( $\mathcal{M}^{o\top} \mathcal{M}^o$ ) yield the nonzero eigenvalues

$$\lambda_3^2 = \frac{1}{2} \left( B - \sqrt{B^2 - 4D} \right) \quad , \quad \lambda_4^2 = \frac{1}{2} \left( B + \sqrt{B^2 - 4D} \right) \quad (5.56)$$

and rotation mixing angles

$$\begin{aligned} \cos \alpha &= \sqrt{\frac{\lambda_4^2 - a^2}{\lambda_4^2 - \lambda_3^2}} \quad , \quad \sin \alpha = \sqrt{\frac{a^2 - \lambda_3^2}{\lambda_4^2 - \lambda_3^2}} \quad , \\ \cos \beta &= \sqrt{\frac{\lambda_4^2 - b^2}{\lambda_4^2 - \lambda_3^2}} \quad , \quad \sin \beta = \sqrt{\frac{b^2 - \lambda_3^2}{\lambda_4^2 - \lambda_3^2}} \quad . \end{aligned} \quad (5.57)$$

$$B = a^2 + b^2 + c^2 = \lambda_3^2 + \lambda_4^2, \quad D = a^2 b^2 = \lambda_3^2 \lambda_4^2, \quad (5.58)$$

$$a^2 = a_1^2 + a_2^2 + a_3^2, \quad b^2 = b_1^2 + b_2^2 + b_3^2$$

The rotation matrices  $V_L^o, V_R^o$  admit several parametrizations related to the two zero mass eigenstates.

### 5.10.1 Parametrization P12

$$V_L^o = \begin{pmatrix} c_1 & c_2 s_1 & s_1 s_2 c_\alpha & s_1 s_2 s_\alpha \\ -s_1 & c_1 c_2 & c_1 s_2 c_\alpha & c_1 s_2 s_\alpha \\ 0 & -s_2 & c_2 c_\alpha & c_2 s_\alpha \\ 0 & 0 & -s_\alpha & c_\alpha \end{pmatrix}, \quad V_R^o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_r & s_r c_\beta & s_r s_\beta \\ 0 & -s_r & c_r c_\beta & c_r s_\beta \\ 0 & 0 & -s_\beta & c_\beta \end{pmatrix}$$

$$a_p = \sqrt{a_1^2 + a_2^2}, \quad b_p = \sqrt{b_1^2 + b_2^2}, \quad a = \sqrt{a_p^2 + a_3^2}, \quad b = \sqrt{b_p^2 + b_3^2},$$

$$s_1 = \frac{a_1}{a_p}, \quad c_1 = \frac{a_2}{a_p}, \quad s_2 = \frac{a_p}{a}, \quad c_2 = \frac{a_3}{a}, \quad s_r = \frac{b_2}{b}, \quad c_r = \frac{b_3}{b}$$

### 5.10.2 Parametrization P13

$$V_L^o = \begin{pmatrix} c_1 & -s_1 s_2 & s_1 c_2 c_\alpha & s_1 c_2 s_\alpha \\ 0 & c_2 & s_2 c_\alpha & s_2 s_\alpha \\ -s_1 & -c_1 s_2 & c_1 c_2 c_\alpha & c_1 c_2 s_\alpha \\ 0 & 0 & -s_\alpha & c_\alpha \end{pmatrix}, \quad V_R^o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_r & s_r c_\beta & s_r s_\beta \\ 0 & -s_r & c_r c_\beta & c_r s_\beta \\ 0 & 0 & -s_\beta & c_\beta \end{pmatrix}$$

$$a_n = \sqrt{a_1^2 + a_3^2}, \quad b_n = \sqrt{b_1^2 + b_3^2}, \quad a = \sqrt{a_n^2 + a_2^2}, \quad b = \sqrt{b_n^2 + b_2^2},$$

$$s_1 = \frac{a_1}{a_n}, \quad c_1 = \frac{a_3}{a_n}, \quad s_2 = \frac{a_2}{a}, \quad c_2 = \frac{a_n}{a}, \quad s_r = \frac{b_2}{b}, \quad c_r = \frac{b_3}{b} \quad (5.59)$$

### 5.10.3 Parametrization P23

$$V_L^o = \begin{pmatrix} c_1 & 0 & s_1 c_\alpha & s_1 s_\alpha \\ -s_1 s_2 & c_2 & c_1 s_2 c_\alpha & c_1 s_2 s_\alpha \\ -s_1 c_2 & -s_2 & c_1 c_2 c_\alpha & c_1 c_2 s_\alpha \\ 0 & 0 & -s_\alpha & c_\alpha \end{pmatrix}, \quad V_R^o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_r & s_r c_\beta & s_r s_\beta \\ 0 & -s_r & c_r c_\beta & c_r s_\beta \\ 0 & 0 & -s_\beta & c_\beta \end{pmatrix}$$

$$a_n = \sqrt{a_2^2 + a_3^2}, \quad b_n = \sqrt{b_2^2 + b_3^2}, \quad a = \sqrt{a_n^2 + a_1^2}, \quad b = \sqrt{b_n^2 + b_1^2},$$

$$s_1 = \frac{a_1}{a}, \quad c_1 = \frac{a_n}{a}, \quad s_2 = \frac{a_2}{a_n}, \quad c_2 = \frac{a_3}{a_n}, \quad s_r = \frac{b_2}{b}, \quad c_r = \frac{b_3}{b} \quad (5.60)$$

## References

1. A. Hernandez-Galeana, Rev. Mex. Fis. **Vol. 50(5)**, (2004) 522. hep-ph/0406315.
2. A. Hernandez-Galeana, Bled Workshops in Physics, (ISSN:1580-4992), **Vol. 17, No. 2**, (2016) Pag. 36; arXiv:1612.07388[hep-ph]; **Vol. 16, No. 2**, (2015) Pag. 47; arXiv:1602.08212[hep-ph]; **Vol. 15, No. 2**, (2014) Pag. 93; arXiv:1412.6708[hep-ph]; **Vol. 14, No. 2**, (2013) Pag. 82; arXiv:1312.3403[hep-ph]; **Vol. 13, No. 2**, (2012) Pag. 28; arXiv:1212.4571[hep-ph]; **Vol. 12, No. 2**, (2011) Pag. 41; arXiv:1111.7286[hep-ph]; **Vol. 11, No. 2**, (2010) Pag. 60; arXiv:1012.0224[hep-ph]; Bled Workshops in Physics, **Vol. 10, No. 2**, (2009) Pag. 67; arXiv:0912.4532[hep-ph];
3. E. Golowich, J. Hewett, S. Pakvasa, and A. Petrov, Phys. Rev. D **76**, 095009 (2007).
4. E. Golowich, J. Hewett, S. Pakvasa, and A. Petrov, Phys. Rev. D **79**, 114030 (2009).
5. M. Kirk, A. Lenz, and T. Rauh, arXiv:1711.02100[hep-ph]; T. Jubb, M. Kirk, A. Lenz, and G. Tetlalmatzi-Xolocotzi, arXiv:1603.07770[hep-ph];
6. C. Bobeth, A. J. Buras, A. Celis, and M. Junk, arXiv:1703.04753[hep-ph]; A. J. Buras, arXiv:1611.06206[hep-ph]; arXiv:1609.05711[hep-ph];
7. Z. Berezhiani and M. Yu. Khlopov: Theory of broken gauge symmetry of families, Sov.J.Nucl.Phys. **51**, 739 (1990).
8. A. S. Sakharov and M. Yu. Khlopov: Horizontal unification as the phenomenology of the theory of everything, Phys.Atom.Nucl. **57**, 651 (1994).
9. M. Yu. Khlopov: *Cosmoparticle physics*, World Scientific, New York -London-Hong Kong - Singapore, 1999.
10. Maxim Khlopov, *Fundamentals of Cosmic Particle physics*, CISP-SPRINGER, Cambridge, 2012.
11. M. Yu. Khlopov: Fundamental Particle Structure in the Cosmological Dark Matter, J. of Mod. Phys. A **28** 1330042 (60 pages) (2013).
12. Z. Berezhiani and M. Yu. Khlopov: Physical and astrophysical consequences of breaking of the symmetry of families, Sov.J.Nucl.Phys. **51**, 935 (1990).
13. Z. Berezhiani, M. Yu. Khlopov and R. R. Khomeriki: On the possible test of quantum flavor dynamics in the searches for rare decays of heavy particles, Sov.J.Nucl.Phys. **52**, 344 (1990).
14. T. Appelquist, Y. Bai and M. Piai:  $SU(3)$  Family Gauge Symmetry and the Axion, Phys. Rev. D **75**, 073005 (2007).
15. T. Appelquist, Y. Bai and M. Piai: Neutrinos and  $SU(3)$  family gauge symmetry, Phys. Rev. D **74**, 076001 (2006).
16. T. Appelquist, Y. Bai and M. Piai: Quark mass ratios and mixing angles from  $SU(3)$  family gauge symmetry, Phys. Lett. B **637**, 245 (2006).
17. J. L. Chkareuli: Quark-Lepton families: From  $SU(5)$  to  $SU(8)$  symmetry, JETP Lett. **32**, 671 (1980).
18. Z. G. Berezhiani, J. L. Chkareuli: Mass of the T quark and the number of quark lepton generations, JETP Lett. **35**, 612 1982.
19. Z. G. Berezhiani, J. L. Chkareuli: Neutrino oscillations in grand unified models with a horizontal symmetry, JETP Lett. **37**, 338 1983.
20. Z.G. Berezhiani: The weak mixing angles in gauge models with horizontal symmetry: A new approach to quark and lepton masses, Phys. Lett. B **129**, 99 (1983).
21. T. Yanagida, Phys. Rev. D **20**, 2986 (1979).
22. A. Hernandez-Galeana and M. Yu. Khlopov, work in progress.
23. Zhi-zhong Xing, He Zhang and Shun Zhou, Phys. Rev. D **86**, 013013 (2012).
24. The Review of Particle Physics (2016), C. Patrignani et al. (Particle Data Group), Chin. Phys. C, **40**, 100001 (2016).
25. See PDG review: The CKM quark-mixing matrix, by A. Ceccucci (CERN), Z. Ligeti (LBNL), and Y. Sakai (KEK).



## 6 Phenomenological Mass Matrices With a Democratic Texture

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**Abstract.** Taking into account all available data on the mass sector, we obtain unitary rotation matrices that diagonalize the quark matrices by using a specific parametrization of the Cabibbo-Kobayashi-Maskawa mixing matrix. The form of the resulting mass matrices is consistent with a democratic scheme with a well-defined, stepwise breaking of the initial flavour symmetry.

**Povzetek.** Avtorica izbere parametrizacijo mešalne matrike Cabibba, Kobayashija in Maskawe, poišče zanjo unitarne rotacijske matrike, ki pri tej parametrizaciji diagonalizirajo masne matrike kvarkov. Izmerjene mase kvarkov zavrti v startni masni matriki, ki sta skladni z demokratično shemo matrik z dobro definirano in postopno zlomljeno začetno simetrije.

Keywords: Mass matrices, CKM matrix, Democratic texture

### 6.1 Mass states and flavour states

In this work, we take a very phenomenological approach on the fermion mass matrices, by assuming that the quark mass matrices can be derived from a (naive) factorization of the Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix  $V$  [1], which appears in the charged current Lagrangian

$$\mathcal{L}_{cc} = -\frac{g}{2\sqrt{2}} \bar{\varphi}_L \gamma^\mu V \varphi'_L W_\mu + \text{h.c.} \quad (6.1)$$

where  $\varphi$  and  $\varphi'$  are quark fields with charges  $Q$  and  $Q - 1$ , correspondingly.

From the perspective of weak interactions,  $\mathcal{L}_{cc}$  describes an interaction between left-handed flavour states. From the point of view of all other interactions, the interaction takes place between mixed physical particle states - where “physical particles” refer to mass eigenstates of the mass matrices  $M$  and  $M'$  appearing in the mass Lagrangian

$$\mathcal{L}_{mass} = \bar{f} M f + \bar{f}' M' f'$$

where  $f, f'$  are fermion flavour states of charge  $2/3$  and  $-1/3$ , respectively, with the corresponding mass matrices denoted as  $M = M(2/3)$  and  $M' = M'(-1/3)$ . Our

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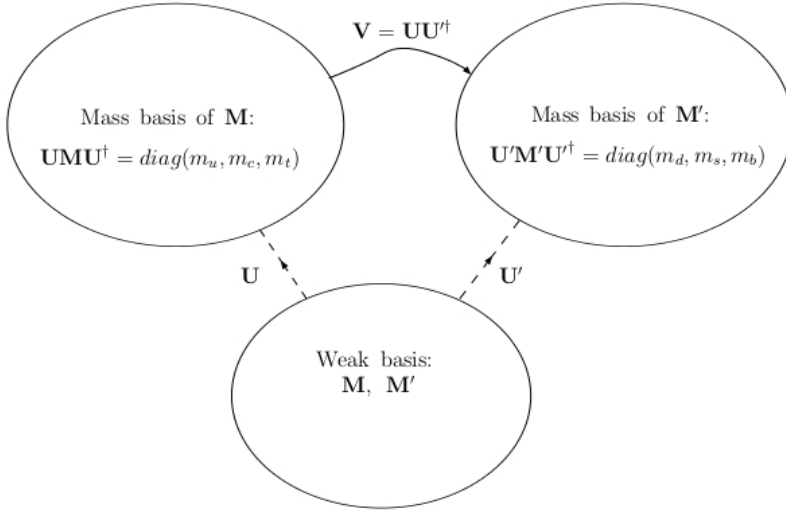
dilemma is in a way how to understand the relation between physical particles and flavour states.

We imagine that all the flavour states live in the same “weak basis” in flavour space, while the mass states of the 2/3-sector and the -1/3-sector live in their separate “mass bases”. We go between the weak basis and the mass bases of the two charge sectors by rotating with the unitary matrices  $U$  and  $U'$ , which are factors of the CKM-matrix,  $V = UU'^\dagger$ .

$$M \rightarrow UMU^\dagger = D = \text{diag}(m_u, m_c, m_t) \quad (6.2)$$

$$M' \rightarrow U'M'U'^\dagger = D' = \text{diag}(m_d, m_s, m_b)$$

Since  $V \neq 1$ , the up-sector mass basis is different from the down-sector mass basis, the CKM matrix thus bridges the two mass bases.



The mass Lagrangian reads

$$\mathcal{L}_{\text{mass}} = \bar{f}Mf + \bar{f}'M'f' = \bar{\psi}D\psi + \bar{\psi}'D'\psi' \quad (6.3)$$

where  $f, f'$  are the flavour states and  $\psi, \psi'$  are the mass states. We of course know the diagonal mass matrices  $D(2/3)$  and  $D'(-1/3)$ , it is  $M(2/3)$  and  $M'(-1/3)$  that we are looking for, in the hope that their form can shed light on (the mechanism behind) the mysterious, hierarchical fermion mass spectra.

Whereas the quark mass eigenstates are perceived as “physical”, and the weakly interacting flavour states are perceived as mixings of physical particles, in the lepton sector the situation is somewhat different, due to the fact that neutrino mass eigenstates don’t ever appear in interactions - they merely propagate in free space. In the realm of neutral leptons it is actually the flavour states  $\nu_e, \nu_\mu, \nu_\tau$

that we perceive as “physical”, since they are the only neutrinos that we “see”, as they appear together with the charged leptons. As the charged leptons  $e, \mu, \tau$  are assumed to be both weak eigenstates and mass eigenstates, the only mixing matrix that appears in the lepton sector is the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix  $U$  which only operates on neutrino states,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{(PMNS)} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

where  $(\nu_1, \nu_2, \nu_3)$  are mass eigenstates, and  $(\nu_e, \nu_\mu, \nu_\tau)$  are the weakly interacting “flavour states”. In the lepton sector, the charged currents are thus interpreted as charged lepton flavours  $(e, \mu, \tau)$  interacting with the neutrino flavour states  $(\nu_e, \nu_\mu, \nu_\tau)$ .

## 6.2 Factorizing the weak mixing matrix

The usual procedure in establishing an ansatz for the quark mass matrices is based on some argument or model. Here we follow a rather phenomenological approach, looking for a factorization of the Cabbibo-Kobayashi-Maskawa mixing matrix, which would give the ‘right’ mass matrices. The CKM matrix can of course be parametrized and factorized in many different ways, and different factorizations correspond to different rotation matrices  $U$  and  $U'$ , and correspondingly to different mass matrices  $M$  and  $M'$ .

We choose what we perceive as the most obvious and “symmetric” factorization of the CKM mixing matrix is, following the standard parametrization [2] with three Euler angles  $\alpha, \beta, 2\theta$ ,

$$V = \begin{pmatrix} c_\beta c_{2\theta} & s_\beta c_{2\theta} & s_{2\theta} e^{-i\delta} \\ -c_\beta s_\alpha s_{2\theta} e^{i\delta} - s_\beta c_\alpha & -s_\beta s_\alpha s_{2\theta} e^{i\delta} + c_\beta c_\alpha & s_\alpha c_{2\theta} \\ -c_\beta c_\alpha s_{2\theta} e^{i\delta} + s_\beta s_\alpha & -s_\beta c_\alpha s_{2\theta} e^{i\delta} - c_\beta s_\alpha & c_\alpha c_{2\theta} \end{pmatrix} = U U'^\dagger \quad (6.4)$$

with the diagonalizing rotation matrices for the up- and down-sectors

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} W(\rho) = \\ &= \begin{pmatrix} c_\theta e^{-i\gamma} & 0 & s_\theta e^{-i\gamma} \\ -s_\alpha s_\theta e^{i\gamma} & c_\alpha & s_\alpha c_\theta e^{i\gamma} \\ -c_\alpha s_\theta e^{i\gamma} & -s_\alpha & c_\alpha c_\theta e^{i\gamma} \end{pmatrix} W(\rho) \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} U' &= \begin{pmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} W(\rho) = \\ &= \begin{pmatrix} c_\beta c_\theta e^{-i\gamma} & -s_\beta & -c_\beta s_\theta e^{-i\gamma} \\ s_\beta c_\theta e^{-i\gamma} & c_\beta & -s_\beta s_\theta e^{-i\gamma} \\ s_\theta e^{i\gamma} & 0 & c_\theta e^{i\gamma} \end{pmatrix} W(\rho) \end{aligned} \quad (6.6)$$



respectively, where  $W(\rho)$  is a unitary matrix which is chosen in such a way that  $\gamma$  is the only phase in either of the mass matrices,

$$\begin{pmatrix} 0 & \cos \rho & \pm \sin \rho \\ 1 & 0 & 0 \\ 0 & \mp \sin \rho & \cos \rho \end{pmatrix}, \quad \begin{pmatrix} \cos \rho & 0 & \pm \sin \rho \\ 0 & 1 & 0 \\ \mp \sin \rho & 0 & \cos \rho \end{pmatrix}, \quad \begin{pmatrix} \cos \rho & \pm \sin \rho & 0 \\ 0 & 0 & 1 \\ \mp \sin \rho & \cos \rho & 0 \end{pmatrix}$$

Here  $\rho$  is unknown, whereas  $\alpha$ ,  $\beta$ ,  $\theta$  and  $\gamma$  correspond to the parameters in the standard parametrization, with  $\gamma = \delta/2$ ,  $\delta = 1.2 \pm 0.08$  rad, and  $2\theta = 0.201 \pm 0.011^\circ$ , while  $\alpha = 2.38 \pm 0.06^\circ$  and  $\beta = 13.04 \pm 0.05^\circ$ . In this factorization scheme,  $\alpha$  and  $\beta$  are rotation angles operating in the up-sector and the down-sector, respectively.

With the rotation matrices  $U(\alpha, \theta, \gamma, \rho)$  and  $U'(\beta, \theta, \gamma, \rho)$ , we obtain the up- and down-sector mass matrices

$$M = U^\dagger \text{diag}(m_u, m_c, m_t)U \quad \text{and} \quad M' = U'^\dagger \text{diag}(m_d, m_s, m_b)U',$$

such that

$$\begin{aligned} M &= \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \\ &= W^\dagger(\rho) \begin{pmatrix} Xc_\theta^2 + Ys_\theta^2 & Zs_\theta e^{-i\gamma} & (X-Y)c_\theta s_\theta \\ Zs_\theta e^{i\gamma} & Y - 2Z \cot 2\alpha & -Zc_\theta e^{i\gamma} \\ (X-Y)c_\theta s_\theta & -Zc_\theta e^{-i\gamma} & Xs_\theta^2 + Yc_\theta^2 \end{pmatrix} W(\rho) \end{aligned} \quad (6.7)$$

where  $X = m_u$ ,  $Z = (m_t - m_c) \sin \alpha \cos \alpha$  and  $Y = m_c \sin^2 \alpha + m_t \cos^2 \alpha$ ; and

$$\begin{aligned} M' &= \begin{pmatrix} M'_{11} & M'_{12} & M'_{13} \\ M'_{21} & M'_{22} & M'_{23} \\ M'_{31} & M'_{32} & M'_{33} \end{pmatrix} = \\ &= W^\dagger(\rho) \begin{pmatrix} X's_\theta^2 + Y'c_\theta^2 & Z'c_\theta e^{i\gamma} & (X'-Y')c_\theta s_\theta \\ Z'c_\theta e^{-i\gamma} & Y' + 2Z' \cot 2\beta & -Z's_\theta e^{-i\gamma} \\ (X'-Y')c_\theta s_\theta & -Z's_\theta e^{i\gamma} & X'c_\theta^2 + Y's_\theta^2 \end{pmatrix} W(\rho) \end{aligned} \quad (6.8)$$

where  $X' = m_b$ ,  $Z' = (m_s - m_d) \sin \beta \cos \beta$  and  $Y' = m_d \cos^2 \beta + m_s \sin^2 \beta$ . The two mass matrices thus have similar textures.

From  $Y = m_c \sin^2 \alpha + m_t \cos^2 \alpha$ ,  $Z = (m_t - m_c) \sin \alpha \cos \alpha$ ,  $Y' = m_d \cos^2 \beta + m_s \sin^2 \beta$  and  $Z' = (m_s - m_d) \sin \beta \cos \beta$ , we moreover have

$$\begin{aligned} m_u &= X, & m_c &= Y - Z \cot \alpha, & m_t &= Y + Z \tan \alpha \\ m_d &= Y' - Z' \tan \beta, & m_s &= Y' + Z' \cot \beta, & m_b &= X' \end{aligned} \quad (6.9)$$

### 6.3 The matrix $W$

We choose the matrix  $W(\rho)$  as

$$W(\rho) = \begin{pmatrix} \cos \rho & -\sin \rho & 0 \\ 0 & 0 & 1 \\ \sin \rho & \cos \rho & 0 \end{pmatrix}, \quad (6.10)$$

which gives the up-sector mass matrix

$$\begin{aligned}
 M &= W^\dagger \begin{pmatrix} Xc_\theta^2 + Ys_\theta^2 & Zs_\theta e^{-i\gamma} & (X-Y)c_\theta s_\theta \\ Zs_\theta e^{i\gamma} & Y - 2Z \cot 2\alpha & -Zc_\theta e^{i\gamma} \\ (X-Y)c_\theta s_\theta & -Zc_\theta e^{-i\gamma} & Xs_\theta^2 + Yc_\theta^2 \end{pmatrix} W = \\
 &= W^\dagger \begin{pmatrix} A & Zs_\theta e^{-i\gamma} & H \\ Zs_\theta e^{i\gamma} & F & -Zc_\theta e^{i\gamma} \\ H & -Zc_\theta e^{-i\gamma} & K \end{pmatrix} W = \\
 &= \begin{pmatrix} Ac_\rho^2 + Ks_\rho^2 + H \sin 2\rho & \frac{1}{2}(K-A) \sin 2\rho + H \cos 2\rho & -Ze^{-i\gamma} \sin(\rho - \theta) \\ \frac{1}{2}(K-A) \sin 2\rho + H \cos 2\rho & As_\rho^2 + Kc_\rho^2 - H \sin 2\rho & -Ze^{-i\gamma} \cos(\rho - \theta) \\ -Ze^{i\gamma} \sin(\rho - \theta) & -Ze^{i\gamma} \cos(\rho - \theta) & F \end{pmatrix}, \tag{6.11}
 \end{aligned}$$

With

$$A = Xc_\theta^2 + Ys_\theta^2, H = (X-Y)c_\theta s_\theta \text{ and } K = Xs_\theta^2 + Yc_\theta^2,$$

we get

$$M = \begin{pmatrix} X \cos^2 \mu + Y \sin^2 \mu & (Y-X) \sin \mu \cos \mu & -Z \sin \mu e^{-i\gamma} \\ (Y-X) \sin \mu \cos \mu & X \sin^2 \mu + Y \cos^2 \mu & -Z \cos \mu e^{-i\gamma} \\ -Z \sin \mu e^{i\gamma} & -Z \cos \mu e^{i\gamma} & F \end{pmatrix} \tag{6.12}$$

where  $\mu = \rho - \theta$ , and as before,  $X = m_u$ ,  $Z = (m_t - m_c) \sin \alpha \cos \alpha$ ,  $Y = m_c \sin^2 \alpha + m_t \cos^2 \alpha$ , and  $F = Xs_\theta^2 + Yc_\theta^2 = Y - 2Z \cot 2\alpha = \text{trace}(M) - X - Y$ .

Now, depending on the value of  $\mu = \rho - \theta$ , we get different matrix textures, e.g.

$\mu = \rho - \theta$	0 or $\pi$	$\pi/4$	$\pi/2$
$M_{11} = Xc_\mu^2 + Ys_\mu^2$	X	$(X+Y)/2$	Y
$M_{12} = \frac{1}{2}(Y-X)s_{2\mu}$	0	$(Y-X)/2$	0
$M_{13} = -Zs_\mu e^{-i\gamma}$	0	$-Ze^{-i\gamma}/\sqrt{2}$	$-Ze^{-i\gamma}$
$M_{22} = Xs_\mu^2 + Yc_\mu^2$	Y	$(X+Y)/2$	X
$M_{23} = -Zc_\mu e^{-i\gamma}$	$-Ze^{-i\gamma}$	$-Ze^{-i\gamma}/\sqrt{2}$	0
$M_{33} = F$	$Y - 2Z \cot 2\alpha$	$Y - 2Z \cot 2\alpha$	$Y - 2Z \cot 2\alpha$

So for  $\rho - \theta = 0$  or  $\pi$ , we get the simple form

$$M(0, \pi) = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & -Ze^{-i\gamma} \\ 0 & -Ze^{i\gamma} & F \end{pmatrix}, \tag{6.13}$$

and for  $\rho - \theta = \pi/2$ , equally simple

$$M(\pi/2) = \begin{pmatrix} Y & 0 & -Ze^{-i\gamma} \\ 0 & X & 0 \\ -Ze^{i\gamma} & 0 & F \end{pmatrix} \tag{6.14}$$

Applying the same procedure on the down-sector, we get the down-sector mass matrix

$$\begin{aligned}
 M' &= W(\rho)^\dagger \begin{pmatrix} X's_\theta^2 + Y'c_\theta^2 & Z'c_\theta e^{i\gamma} & (X' - Y')c_\theta s_\theta \\ Z'c_\theta e^{-i\gamma} & Y' + 2Z'\cot 2\beta & -Z's_\theta e^{-i\gamma} \\ (X' - Y')c_\theta s_\theta & -Z's_\theta e^{i\gamma} & X'c_\theta^2 + Y's_\theta^2 \end{pmatrix} W(\rho) = \\
 &= \begin{pmatrix} X'\sin^2 \mu' + Y'\cos^2 \mu' & (X' - Y')\sin \mu' \cos \mu' & Z'\cos \mu' e^{i\gamma} \\ (X' - Y')\sin \mu' \cos \mu' & X'\cos^2 \mu' + Y'\sin^2 \mu' & -Z'\sin \mu' e^{i\gamma} \\ Z'\cos \mu' e^{-i\gamma} & -Z'\sin \mu' e^{-i\gamma} & F' \end{pmatrix} \quad (6.15)
 \end{aligned}$$

where  $\mu' = \rho + \theta$ , and as before,  $X' = m_b$ ,  $Z' = (m_s - m_d) \sin \beta \cos \beta$ ,  $Y' = m_d \cos^2 \beta + m_s \sin^2 \beta$ , and  $F' = Y' + 2Z'\cot 2\beta = \text{trace}(M') - X' - Y'$ .

Depending on the value of  $\mu' = \rho + \theta$ , we get different matrix textures.

$\mu' = \rho + \theta$	0 or $\pi$	$\pi/4$	$\pi/2$
$M'_{11} = X's_{\mu'}^2 + Y'c_{\mu'}^2$	$Y'$	$(X' + Y')/2$	$X'$
$M'_{12} = \frac{1}{2}(X' - Y')s_{2\mu'}$	0	$(X' - Y')/2$	0
$M'_{13} = Z'c_{\mu'} e^{i\gamma}$	$Z'e^{i\gamma}$	$Z'e^{i\gamma}/\sqrt{2}$	0
$M'_{22} = X'c_{\mu'}^2 + Y's_{\mu'}^2$	$X'$	$(X' + Y')/2$	$Y'$
$M'_{23} = -Z's_{\mu'} e^{i\gamma}$	0	$-Z'e^{i\gamma}/\sqrt{2}$	$-Z'e^{i\gamma}$
$M'_{33} = F'$	$Y' + 2Z'\cot 2\beta$	$Y' + 2Z'\cot 2\beta$	$Y' + 2Z'\cot 2\beta$

So for  $\mu' = \rho + \theta = 0$  or  $\pi$ , we get

$$M'(0, \pi) = \begin{pmatrix} Y' & 0 & Z'e^{i\gamma} \\ 0 & X' & 0 \\ Z'e^{-i\gamma} & 0 & F' \end{pmatrix} \quad (6.16)$$

and for  $\mu' = \rho + \theta = \pi/2$ , we get

$$M'(\pi/2) = \begin{pmatrix} X' & 0 & 0 \\ 0 & Y' & -Z'e^{i\gamma} \\ 0 & -Z'e^{-i\gamma} & F' \end{pmatrix} \quad (6.17)$$

## 6.4 Texture Zero Mass Matrices

The textures (6.13) and (6.14), as well as (6.16) and (6.17), make us wonder if our scheme implies quark mass matrices of texture zero.

Texture zero matrices can be said to have come about because of the need to reduce the number of free parameters, since the fermion mass matrices are 3x3 complex matrices, which without any constraints contain 36 real free parameters. It is however always possible to perform a unitary transformation that renders an arbitrary mass matrix Hermitian [5], so there is no loss of generality to assume that the mass matrices be Hermitian, reducing the number of free parameters to 18. This is still a very big number, which in the end of the 1970-ies prompted Fritzsch

[4], [6] to introduce “texture zero matrices”, mass matrices where a certain number of the entries are zero.

Since then, a huge amount of articles have appeared, with analyses of the very large number of (different types of) texture zero matrices and their phenomenology. In the course of this work, a number of texture zero matrices has been ruled out, singling out a smaller subset of matrices as viable [7]. Among the texture 4 zero matrices the only matrices that are found to be viable are:

$$\begin{pmatrix} A & B & 0 \\ B^* & D & C \\ 0 & C^* & 0 \end{pmatrix}, \begin{pmatrix} A & B & C \\ B^* & D & 0 \\ C^* & 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 & B \\ 0 & 0 & C \\ B^* & C^* & D \end{pmatrix}, \begin{pmatrix} 0 & C & 0 \\ C^* & A & B \\ 0 & B^* & D \end{pmatrix}, \begin{pmatrix} 0 & 0 & C \\ 0 & A & B \\ C^* & B^* & D \end{pmatrix}, \begin{pmatrix} D & C & B \\ C^* & 0 & 0 \\ B^* & 0 & A \end{pmatrix}$$

while

$$\begin{pmatrix} A & 0 & 0 \\ 0 & C & B \\ 0 & B^* & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B^* & 0 & D \end{pmatrix}$$

are among the matrices that are ruled out. In our scheme this precisely corresponds to the matrices (6.13), (6.14), (6.16) and (6.17), which means that our mass matrices  $M$  and  $M'$  are not of texture zero. This can be expressed as a constraint on the values of the angle  $\rho$ ,

$$\rho \neq \frac{1}{2}N\pi \pm \theta \quad (6.18)$$

where  $N \in \mathbb{Z}$ , ruling out the matrices  $M(\frac{1}{2}N\pi - \theta)$  and  $M'(\frac{1}{2}N\pi + \theta)$ , so our mass matrices  $M$  and  $M'$  are not of texture zero. Instead, they display a democratic texture.

## 6.5 Democratic mass matrices

Initially, we were looking for mass matrices with a democratic structure [3], where the assumption is that both the up- and down-sector mass matrices start out from a form of the type  $M_0 = k\mathbf{N}$  and  $M'_0 = k'\mathbf{N}$  where

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The underlying philosophy is that in the Standard Model, where the fermions get their masses from the Yukawa couplings by the Higgs mechanism, there is no reason why there should be a different Yukawa coupling for each fermion. The couplings to the gauge bosons of the strong, weak and electromagnetic interactions are identical for all the fermions in a given charge sector, it thus seems like a natural assumption that they should also have identical Yukawa couplings. The difference is that the weak interactions take place in a specific flavour space basis, while the other interactions are flavour independent.

A matrix of the form  $M = k\mathbf{N}$  moreover has the mass spectrum  $(0, 0, 3k)$ , reflecting the phenomenology of the fermion mass spectra with one very big, and two much smaller mass values. In the weak basis  $M = k\mathbf{N}$  is however totally

flavour symmetric, which means that the (weak) flavours  $f_i$  are indistinguishable (“absolute democracy”).

In the assumed initial stage, since the up-sector mass matrix and the down sector mass matrix are identical except for the dimensional coefficients  $k$  and  $k'$ , the mixing matrix is equal to unity, so there is no CP-violation. In order to obtain the final mass spectra with the three hierarchical non-zero values, the initial flavour symmetry displayed by the matrices  $M_0$  and  $M'_0$  must be broken, in such a way that the mixing matrix becomes the observed CKM matrix (with a CP-violating phase).

An “ansatz” within the democratic scenario then consists of a specific choice of a flavour symmetry breaking scheme. And it is precisely what we are looking for: a credible flavour symmetry breaking scheme that gives the observed mass spectra.

Our initial assumption is that the rotation matrices (6.5), (6.6) which diagonalize the up-sector and down-sector mass matrices, are given by the factorization of the Cabibbi-Kobayashi-Maskawa matrix (6.4), with well-known angles. The only “steering-parameter parameter” is then  $\rho$ , in the sense that different values of  $\rho$  correspond to mass matrices of different form.

### 6.5.1 A democratic substructure

We now reparametrize the mass matrices (6.12) and (6.15),

$$M = \begin{pmatrix} Xc_\mu^2 + Ys_\mu^2 & (Y - X)s_\mu c_\mu & -Zs_\mu e^{-i\gamma} \\ (Y - X)s_\mu c_\mu & Xs_\mu^2 + Yc_\mu^2 & -Zc_\mu e^{-i\gamma} \\ -Zs_\mu e^{i\gamma} & -Zc_\mu e^{i\gamma} & F \end{pmatrix}$$

and

$$M' = \begin{pmatrix} X's_{\mu'}^2 + Y'c_{\mu'}^2 & (X' - Y')s_{\mu'}c_{\mu'} & Z'c_{\mu'} e^{i\gamma} \\ (X' - Y')s_{\mu'}c_{\mu'} & X'c_{\mu'}^2 + Y's_{\mu'}^2 & -Z's_{\mu'} e^{i\gamma} \\ Z'c_{\mu'} e^{-i\gamma} & -Z's_{\mu'} e^{-i\gamma} & F' \end{pmatrix},$$

in a way that reveals their “democratic substructure”:

$$M = \begin{pmatrix} P & & \\ & R & \\ & & Se^{i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P & & \\ & R & \\ & & Se^{-i\gamma} \end{pmatrix} + \begin{pmatrix} X & & \\ & X & \\ & & Q \end{pmatrix} \quad (6.19)$$

and

$$M' = \begin{pmatrix} P' & & \\ & R' & \\ & & S'e^{-i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P' & & \\ & R' & \\ & & S'e^{i\gamma} \end{pmatrix} + \begin{pmatrix} X' & & \\ & X' & \\ & & Q' \end{pmatrix} \quad (6.20)$$

where

$$P = \sqrt{|Y - X|} \sin(\rho - \theta), \quad R = \sqrt{|Y - X|} \cos(\rho - \theta), \quad S = \frac{-Z}{\sqrt{|Y - X|}}, \quad Q = F - S^2,$$

and

$$P' = \sqrt{|Y' - X'|} \cos(\rho + \theta), \quad R' = -\sqrt{|Y' - X'|} \sin(\rho + \theta), \quad S' = \frac{Z'}{\sqrt{|Y' - X'|}}, \quad Q' = F' - S'^2.$$

These matrices can in their turn be rewritten as

$$M = B \begin{pmatrix} \sin \mu & & \\ & \cos \mu & \\ & & Ge^{i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sin \mu & & \\ & \cos \mu & \\ & & Ge^{-i\gamma} \end{pmatrix} + \begin{pmatrix} X & & \\ & X & \\ & & Q \end{pmatrix} \quad (6.21)$$

where

$$\mu = \rho - \theta, \quad B = Y - X, \quad G = -Z/(Y - X), \quad Q = F - BG^2.$$

Likewise,

$$M' = B' \begin{pmatrix} \cos \mu' & & \\ & -\sin \mu' & \\ & & G'e^{-i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \cos \mu' & & \\ & -\sin \mu' & \\ & & G'e^{i\gamma} \end{pmatrix} + \begin{pmatrix} X' & & \\ & X' & \\ & & Q' \end{pmatrix} \quad (6.22)$$

where

$$\mu' = \rho + \theta, \quad B' = Y' - X', \quad G' = Z'/(Y' - X'), \quad Q' = F' - B'G'^2.$$

So without any assumptions about an initial democratic texture, we get a mass matrix structure that can be interpreted as originating from a democratic mass matrix, where the flavour symmetry has subsequently been broken in a very specific manner.

## 6.6 Flavour symmetry breaking mechanisms

The goal of our investigation is to get some hint about the form that the quark mass matrices take in the weak basis - and the hint we get from the matrices (6.21) and (6.22) is that the mass matrices come about from a kind of democratic scenario where the initial flavour symmetry is broken in a stepwise fashion.

Flavour symmetries relate the different flavours  $f_j$ , and in the democratic scenario, where the initial form of the mass matrices is taken to be

$$M_0 = k\mathbf{N} = k \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (6.23)$$

the mass Lagrangian reads

$$\mathcal{L}_{\text{mass}} = k\bar{\mathbf{f}}\mathbf{N}\mathbf{f} = \sum_{i=1, j=1}^3 k \bar{f}_i f_j$$

This means that in the democratic scheme, all the flavours  $f_j$  are initially indistinguishable, with the same Yukawa coupling for all the flavours: a totally flavour symmetric situation.

Following the hint given by our approach, we now postulate that the mass matrices originate from a democratic form (6.23), and that the initial overall flavour symmetries have subsequently undergone a stepwise breaking. To show how this works, we start with a generic matrix  $M_0$ , and take the first symmetry breaking step to be

$$M_0 = k\mathbf{N} \rightarrow M_1 = \begin{pmatrix} E & & \\ & E & \\ & & J \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} E & & \\ & E & \\ & & J \end{pmatrix} \quad (6.24)$$

Here the mass spectrum is basically unchanged even though the flavour symmetry is partially broken, with the mass Lagrangian

$$\mathcal{L}_{\text{mass}} = k\bar{f}M_1f = E^2\bar{\chi}\chi + EJ(\bar{\chi}f_3 + \bar{f}_3\chi) + J^2\bar{f}_3f_3$$

where  $\chi = f_1 + f_2$ ; thus the flavour symmetry  $f_1 \Leftrightarrow f_2$  is still unbroken. In the next step, we lift the remaining flavour symmetry by rotating the two equal terms,

$$(E, E) \rightarrow (L \sin \eta, L \cos \eta),$$

which gives

$$M_1 = k\mathbf{N} \rightarrow M_2 = L^2 \begin{pmatrix} \sin \eta & & \\ & \cos \eta & \\ & & T \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sin \eta & & \\ & \cos \eta & \\ & & T \end{pmatrix}, \quad (6.25)$$

where  $L^2$  is the only dimensional parameter, and  $T = J/L$ . In order to account for CP-violation, we moreover introduce a phase  $\gamma$ , in a way that reflects that CP-violation is connected to the presence of three families (with only two families there is no CP-violation):

$$M_2 \rightarrow M_3 = L^2 \begin{pmatrix} \sin \eta & & \\ & \cos \eta & \\ & & Te^{i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sin \eta & & \\ & \cos \eta & \\ & & Te^{-i\gamma} \end{pmatrix}, \quad (6.26)$$

where the CP-breaking phase is connected to the third family, as it should. We know nothing about the values of  $L$ ,  $\eta$ ,  $T$ , but by the assumption that the trace of the mass matrix is constant through all the flavour symmetry breaking steps, we get

$$L^2 + T^2 = 3k$$

But the matrix  $M_3$  still has determinant zero, and a mass spectrum with two vanishing and one non-zero mass value. We therefore add an extra term to  $M_3$ , which like in (6.21) and (6.22), is of diagonal form. This gives us the final mass matrix

$$M_3 \rightarrow M_4 = L^2 \begin{pmatrix} \sin \eta & & \\ & \cos \eta & \\ & & Te^{i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sin \eta & & \\ & \cos \eta & \\ & & Te^{-i\gamma} \end{pmatrix} + \Lambda \quad (6.27)$$

where  $\Lambda$  is a diagonal matrix.

Our scheme thus reads:

- We start with the democratic matrix  $M_0 = k\mathbf{N}$ ,  $k = \text{Trace}(M)/3$ , with total flavour symmetry  $f_1 \Leftrightarrow f_2 \Leftrightarrow f_3$  in the weak basis.
- Assumption: the trace of the matrix  $M$  is constant throughout every flavour symmetry breaking step.
- First flavour breaking step (6.24):  $M_0 \rightarrow M_1$ . The flavour symmetry  $f_1 \Leftrightarrow f_2$  still remains, and there is still only one non-zero mass value, but  $f_3$  is singled out.
- Next flavour breaking step (6.25):  $M_1 \rightarrow M_2$ , lifting the flavour symmetry  $f_1 \Leftrightarrow f_2$ .
- Introducing a CP-violating phase (6.26):  $M_2 \rightarrow M_3$ .
- Last step (6.27): adding a diagonal matrix to  $M_3$ ,  $M_3 \rightarrow M_4 = M_3 + \Lambda$ , whereby we get the three observed non-zero mass values.

## 6.7 Conclusion

Without introducing any new assumptions, by just factorizing the “standard parametrization” of the CKM weak mixing matrix in a specific way, we obtain mass matrices with a specific type of democratic texture and a well-defined scheme for breaking the initial flavour symmetry. Our approach thus hints at a democratic scenario, which comes from the formalism without any other assumptions than a very natural and straightforward way of factorizing the weak mixing matrix.

## References

1. M. Kobayashi, T. Maskawa; Maskawa (1973), “CP-Violation in the Renormalizable Theory of Weak Interaction”, *Progress of Theoretical Physics* 49 (2): 652657.
2. L.L. Chau and W.-Y. Keung (1984), “Comments on the Parametrization of the Kobayashi-Maskawa Matrix”, *Phys. Rev. Letters* 53 (19): 1802.
3. A. Kleppe, “A democratic suggestion”, hep-ph/1608.08988.
4. H. Fritzsch, *Phys. Lett. B* 70, 436 (1977), *Phys. Lett. B* 73, 317 (1978).
5. D. Emmanuel-Costa and C. Simoes, *Phys. Rev. D* 79, 073006 (2009).
6. H. Fritzsch, “Texture Zero Mass Matrices and Flavor Mixing of Quarks and Leptons”, hep-ph/1503.07927v1.
7. S. Sharma, P. Fakay, G. Ahuja, M. Gupta, *Phys. Rev. D* 91, 053004 (2015).





## 7 Fermions and Bosons in the Expanding Universe by the Spin-charge-family theory <sup>\*</sup>

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**Abstract.** The *spin-charge-family* theory, which is a kind of the Kaluza-Klein theories in  $d = (13 + 1)$  — but with the two kinds of the spin connection fields, the gauge fields of the two Clifford algebra objects,  $S^{ab}$  and  $\tilde{S}^{ab}$  — explains all the assumptions of the *standard model*: The origin of the charges of fermions appearing in one family, the origin and properties of the vector gauge fields of these charges, the origin and properties of the families of fermions, the origin of the scalar fields observed as the Higgs's scalar and the Yukawa couplings. The theory explains several other phenomena like: The origin of the dark matter, of the matter-antimatter asymmetry, the "miraculous" triangle anomaly cancellation in the *standard model* and others. Since the theory starts at  $d = (13 + 1)$  the question arises how and at which  $d$  had our universe started and how it came down to  $d = (13 + 1)$  and further to  $d = (3 + 1)$ . In this short contribution some answers to these questions are presented.

**Povzetek.** Avtorica obravnava teorijo *spinov-nabojev-družin*, ki sodi v družino Kaluza-Kleinovih teorij v  $d = (13 + 1)$  — vendar z dvema vrstama polj spinskih povezav, ki so umeritvena polja dveh vrst objektov v Cliffordovih algebrah — in pojasni predpostavke *standardnega modela*: izvor nabojev fermionov v posamezni družini, izvor in lastnosti vektorskih umeritvenih polj teh nabojev, izvor in lastnosti družin fermionov in izvor skalarnih polj, ki se kažejo kot Higgsov skalar in Yukavine sklopitve. Teorija pojasni tudi druge pojave: izvor temne snovi, izvor asimetrije snov-antisnov, "čudežno" izginotje trikotniške anomalije v *standardnem modelu*. Ker teorija izhaja iz  $d = (13 + 1)$ , se pojavi vprašanje, kao in pri katerem  $d$  se je vesolje začelo in kako je prišlo so  $d = (13 + 1)$  in nato še naprej do  $d = (3 + 1)$ . Avtorica predlaga nekaj odgovorov na ta vprašanja.

**Keywords:** Unifying theories; Beyond the standard model; Kaluza-Klein-like theories; Vector and scalar gauge fields and their origin; Fermions, their families in their properties in the expanding universe.

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## 7.1 Introduction

Both standard models, the *standard model* of elementary fermion and boson fields and the *standard cosmological model*, have quite a lot of assumptions, guessed from the properties of observables. Although in the history physics was and still is (in particular when many degrees of freedom are concerned) relying on small theoretical steps, confirmed by experiments, there are also a few decisive steps, without which no real further progress would be possible. Among such steps there are certainly the *general theory of relativity* and the *standard model* of elementary fermion and boson fields. Both theories enabled much better understanding of our universe and its elementary fields — fermions and bosons.

With more and more accurate experiments is becoming increasingly clear that a new decisive step is again needed in the theory of elementary fields as well as in cosmology.

Both theories rely on observed facts built into innovative mathematical models. However, the assumptions remain unexplained.

Among the non understood assumptions of the *standard model* of the elementary fields of fermions and bosons are: i. The origin of massless family members with their charges related to spins. ii. The origin of families of fermions. iii. The origin of the massless vector gauge fields of the observed charges. iv. The origin of masses of family members and heavy bosons. v. The origin of the Higgs's scalar and the Yukawa couplings. vi. The origin of matter-antimatter asymmetry. vii. The origin of the dark matter. viii. The origin of the electroweak phase transition scale. ix. The origin of the colour phase transition scale. And others.

Among the non understood assumptions of the *cosmological model* are: a. The differences in the origin of the gravity, of the vector gauge fields and the (Higgs's) scalars. b. The origin of the dark matter, of the matter-antimatter asymmetry of the (ordinary) matter. c. The appearance of fermions. d. The origin of the inflation of the universe. e. While it is known how to quantize vector gauge fields, the quantization of gravity is still an open problem.

The L(arge) H(adron) C(ollider) and other accelerators and measuring apparatus produce a huge amount of data, the analyzes of which should help to explain the assumptions of both standard models. But it looks like so far that the proposed models, relying more or less on small extensions of the standard models, can not offer much help. The situation in elementary particle physics is reminiscent of the situation in the nuclear physics before the *standard model* of the elementary fields was proposed, opening new insight into physics of elementary fermion and boson fields.

The deeper into the history of our universe we are succeeding to look by the observations and experiments the more both standard models are becoming entangled, dependent on each other, calling for the next step which would offer the explanation for most of the above mentioned non understood assumptions of both standard models.

The *spin-charge-family* theory [1,2,4,3,5–8] does answer open questions of the *standard model* of the elementary fields and also several of cosmology.

The *spin-charge-family* theory [1,2,4,3] is promising to be the right next step beyond the *standard model* of elementary fermion and boson fields by offering the explanation for all the assumptions of this model. By offering the explanation also for the dark matter and matter-antimatter asymmetry the theory makes a new step also in cosmology, in particular since it starts at  $d \geq 5$  with spinors and gravitational fields only — like the Kaluza-Klein theories (but with the two kinds of the spin connection fields, which are the gauge fields of the two kinds of the Clifford algebra objects). Although there are still several open problems waiting to be solved, common to most of proposed theories — like how do the boundary conditions influence the breaking of the starting symmetry of space-time and how to quantize gravity in any  $d$ , while we know how to quantize at least the vector gauge fields in  $d = (3 + 1)$  — the *spin-charge-family* theory is making several predictions (not just stimulated by the current experiments what most of predictions do).

The *spin-charge-family* theory (Refs. [1,2,4,3,5–11,13,15,12] and the references therein) starts in  $d = (13 + 1)$ : **i.** with the simple action for spinors, Eq. (7.1), which carry two kinds of spins, **i.a.** the Dirac one described by  $\gamma^a$  and manifesting at low energies in  $d = (3 + 1)$  as spins and all the charges of the observed fermions of one family, Table 7.1, **i.b.** the second one named [15] (by the author of this paper)  $\tilde{\gamma}^a$  ( $\{\tilde{\gamma}^a, \gamma^b\}_+ = 0$ , Eq. (7.2)), and manifesting at low energies the family quantum numbers of the observed fermions. **ii.** Spinors interact in  $d = (13 + 1)$  with the gravitational field only, **ii.a.** the vielbeins and **ii.b.** the two kinds of the spin connection fields (Refs. [1,4] and the references therein). Spin connection fields —  $\omega_{stm}$  ( $(s, t) \geq 5$ ,  $m = (0, 1, 2, 3, 4)$ ), Eq. (7.1) — are the gauge fields of  $S^{st}$ , Eq. (7.7), and manifest at low energies in  $d = (3 + 1)$  as the vector gauge fields (the colour, weak and hyper vector gauge fields are directly or indirectly observed vector gauge fields). Spin connections  $\omega_{sts'}$  ( $(s, t) \geq 5$ ,  $s' = (7, 8)$ ) manifest as scalar gauge fields, contributing to the Higgs's scalar and the Yukawa couplings together with the scalar spin connection gauge fields —  $\tilde{\omega}_{abs'}$  ( $(a, b) = (m, s, t)$ ,  $s' = (7, 8)$ ), Eq. (7.1) — which are the gauge fields of  $\tilde{S}^{ab}$ , Eq. (7.7) [4,3,1,2]. Correspondingly these (several) scalar gauge fields determine after the electroweak break masses of the families of all the family members and of the heavy bosons (Refs. [2,4,3,1], and the references therein).

Scalar fields  $\omega_{sts'}$  ( $(s, t) \geq 5$ ,  $s' = (9, \dots, 14)$ ), Ref. [4] (and the references therein), cause transitions from anti-leptons to quarks and anti-quarks into quarks and back. In the presence of the condensate of two right handed neutrinos [2,4] the matter-antimatter symmetry breaks.

## 7.2 Short presentation of the *spin-charge-family* theory

The *spin-charge-family* theory [3,2,4,7–10] assumes a simple action, Eq. (7.1), in an even dimensional space ( $d = 2n$ ,  $d > 5$ ).  $d$  is chosen to be  $(13 + 1)$ , what makes the simple starting action in  $d$  to manifest in  $d = (3 + 1)$  in the low energy regime all the observed degrees of freedom, explaining all the assumptions of the *standard model* as well as other observed phenomena. Fermions interact with the vielbeins  $f^\alpha_a$  and the two kinds of the spin-connection fields —  $\omega_{ab\alpha}$  and  $\tilde{\omega}_{ab\alpha}$  — the

gauge fields of  $S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$  and  $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$ , respectively, where:

$$\mathcal{A} = \int d^d x \, E \, \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + \text{h.c.} + \int d^d x \, E \, (\alpha R + \tilde{\alpha} \tilde{R}), \quad (7.1)$$

here  $p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\}_-$ ,  $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ <sup>1</sup>,

$$R = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\omega_{ab\alpha, \beta} - \omega_{c a \alpha} \omega^c_{b \beta})\} + \text{h.c.},$$

$$\tilde{R} = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\tilde{\omega}_{ab\alpha, \beta} - \tilde{\omega}_{c a \alpha} \tilde{\omega}^c_{b \beta})\} + \text{h.c.}.$$

The action introduces two kinds of the Clifford algebra objects,  $\gamma^a$  and  $\tilde{\gamma}^a$ ,

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+. \quad (7.2)$$

$f^\alpha_a$  are vielbeins inverted to  $e^a_\alpha$ , Latin letters (a, b, ...) denote flat indices, Greek letters ( $\alpha, \beta, \dots$ ) are Einstein indices, (m, n, ...) and ( $\mu, \nu, \dots$ ) denote the corresponding indices in (0, 1, 2, 3), (s, t, ...) and ( $\sigma, \tau, \dots$ ) denote the corresponding indices in  $d \geq 5$ :

$$e^a_\alpha f^\alpha_b = \delta^a_b, \quad e^a_\alpha f^\alpha_b = \delta^a_b, \quad (7.3)$$

$$E = \det(e^a_\alpha)^2.$$

The action  $\mathcal{A}$  offers the explanation for the origin and all the properties of the observed fermions (of the family members and families), of the observed vector gauge fields, of the Higgs's scalar and of the Yukawa couplings, explaining the origin of the matter-antimatter asymmetry, the appearance of the dark matter and predicts the new scalars and the new (fourth) family coupled to the observed three to be measured at the LHC ([2,4] and the references therein).

The *standard model* groups of spins and charges are the subgroups of the  $SO(13, 1)$  group with the generator of the infinitesimal transformations expressible with  $S^{ab}$  — for spins

$$\vec{N}_\pm (= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \quad (7.4)$$

— for the weak charge,  $SU(2)_I$ , and the second  $SU(2)_{II}$ , these two groups are the invariant subgroups of  $SO(4)$

$$\vec{\tau}^1 := \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}),$$

$$\vec{\tau}^2 := \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}), \quad (7.5)$$

<sup>1</sup> Whenever two indexes are equal the summation over these two is meant.

<sup>2</sup> This definition of the vielbein and the inverted vielbein is general, no specification about the curled space is assumed yet, but is valid also in the low energies regions, when the starting symmetry is broken [1].

— for the colour charge  $SU(3)$  and for the "fermion charge"  $U(1)_{II}$ , these two groups are subgroups of  $SO(6)$

$$\begin{aligned}\tau^3 &:= \frac{1}{2} \{S^{9\ 12} - S^{10\ 11}, S^{9\ 11} + S^{10\ 12}, S^{9\ 10} - S^{11\ 12}, \\ &\quad S^{9\ 14} - S^{10\ 13}, S^{9\ 13} + S^{10\ 14}, S^{11\ 14} - S^{12\ 13}, \\ &\quad S^{11\ 13} + S^{12\ 14}, \frac{1}{\sqrt{3}}(S^{9\ 10} + S^{11\ 12} - 2S^{13\ 14})\}, \\ \tau^4 &:= -\frac{1}{3}(S^{9\ 10} + S^{11\ 12} + S^{13\ 14}),\end{aligned}\quad (7.6)$$

— while the hyper charge  $Y$  is  $Y = \tau^{23} + \tau^4$ . The breaks of the symmetries, manifesting in Eqs. (7.4, 7.5, 7.6), are in the *spin-charge-family* theory caused by the condensate and the constant values of the scalar fields carrying the space index (7, 8) (Refs. [3,4] and the references therein). The space breaks first to  $SO(7, 1) \times SU(3) \times U(1)_{II}$  and then further to  $SO(3, 1) \times SU(2)_I \times U(1)_I \times SU(3)$ , what explains the connections between the weak and the hyper charges and the handedness of spinors.

The equivalent expressions for the family charges, expressed by  $\tilde{S}^{ab}$ , follow if in Eqs. (7.4 - 7.6)  $S^{ab}$  are replaced by  $\tilde{S}^{ab}$ .

### 7.2.1 A short inside into the spinor states of the *spin-charge-family* theory

I demonstrate in this subsection on two examples how transparently can properties of spinor and anti-spinor states be read from these states [13,15,3], when the states are expressed with  $\frac{d}{2}$  nilpotents and projectors, formed as odd and even objects of  $\gamma^a$ 's (Eq. (7.10)) and chosen to be the eigenstates of the Cartan subalgebra (Eq. (7.8)) of the algebra of the two groups, as in Table 7.1.

Recognizing that the two Clifford algebra objects ( $S^{ab}, S^{cd}$ ), or ( $\tilde{S}^{ab}, \tilde{S}^{cd}$ ), fulfilling the algebra,

$$\begin{aligned}\{S^{ab}, S^{cd}\}_- &= i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac}), \\ \{\tilde{S}^{ab}, \tilde{S}^{cd}\}_- &= i(\eta^{ad}\tilde{S}^{bc} + \eta^{bc}\tilde{S}^{ad} - \eta^{ac}\tilde{S}^{bd} - \eta^{bd}\tilde{S}^{ac}), \\ \{S^{ab}, \tilde{S}^{cd}\}_- &= 0,\end{aligned}\quad (7.7)$$

commute, if all the indexes (a, b, c, d) are different, the Cartan subalgebra is in  $d = 2n$  selected as follows

$$\begin{aligned}S^{03}, S^{12}, S^{56}, \dots, S^{d-1\ d}, &\quad \text{if } d = 2n \geq 4, \\ \tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1\ d}, &\quad \text{if } d = 2n \geq 4.\end{aligned}\quad (7.8)$$

Let us define as well one of the Casimirs of the Lorentz group — the handedness  $\Gamma$  ( $\{\Gamma, S^{ab}\}_- = 0$ ) in  $d = 2n$ <sup>3</sup>

$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{a\bar{a}}} \gamma^a), \quad \text{if } d = 2n, \quad (7.9)$$

<sup>3</sup> The reader can find the definition of handedness for  $d$  odd in Refs. [13,4] and the references therein.

which can be written also as  $\Gamma^{(d)} = i^{d-1} \cdot 2^{\frac{d}{2}} S^{03} \cdot S^{12} \dots S^{(d-1)d}$ . The product of  $\gamma^a$ 's must be taken in the ascending order with respect to the index  $a$ :  $\gamma^0 \gamma^1 \dots \gamma^d$ . It follows from the Hermiticity properties of  $\gamma^a$  for any choice of the signature  $\eta^{aa}$  that  $\Gamma^{(d)\dagger} = \Gamma^{(d)}$ ,  $(\Gamma^{(d)})^2 = I$ . One proceeds equivalently for  $\tilde{\Gamma}^{(d)}$ , substituting  $\gamma^a$ 's by  $\tilde{\gamma}^a$ 's. We also find that for  $d$  even the handedness anticommutes with the Clifford algebra objects  $\gamma^a$  ( $\{\gamma^a, \Gamma\}_+ = 0$ ).

Spinor states can be, as in Table 7.1, represented as products of nilpotents and projectors defined by  $\gamma^a$ 's

$${}^{ab}(\mathbf{k}) := \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \quad {}^{ab}[\mathbf{k}] := \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \quad (7.10)$$

where  $k^2 = \eta^{aa}\eta^{bb}$ .

It is easy to check that the nilpotent  ${}^{ab}(\mathbf{k})$  and the projector  ${}^{ab}[\mathbf{k}]$  are "eigenstates" of  $S^{ab}$  and  $\tilde{S}^{ab}$

$$\begin{aligned} S^{ab} {}^{ab}(\mathbf{k}) &= \frac{1}{2} k {}^{ab}(\mathbf{k}), & S^{ab} {}^{ab}[\mathbf{k}] &= \frac{1}{2} k {}^{ab}[\mathbf{k}], \\ \tilde{S}^{ab} {}^{ab}(\mathbf{k}) &= \frac{1}{2} k {}^{ab}(\mathbf{k}), & \tilde{S}^{ab} {}^{ab}[\mathbf{k}] &= -\frac{1}{2} k {}^{ab}[\mathbf{k}], \end{aligned} \quad (7.11)$$

where in Eq. (7.11) the vacuum state  $|\psi_0\rangle$  is meant to stay on the right hand sides of projectors and nilpotents. This means that one gets when multiplying nilpotents  ${}^{ab}(\mathbf{k})$  and projectors  ${}^{ab}[\mathbf{k}]$  by  $S^{ab}$  the same objects back multiplied by the constant  $\frac{1}{2}k$ , while  $\tilde{S}^{ab}$  multiply  ${}^{ab}(\mathbf{k})$  by  $k$  and  ${}^{ab}[\mathbf{k}]$  by  $(-k)$  rather than  $k$ .

One can namely see, taking into account Eq. (7.2), that

$$\begin{aligned} \gamma^a {}^{ab}(\mathbf{k}) &= \eta^{aa} {}^{ab}[-\mathbf{k}], \quad \gamma^b {}^{ab}(\mathbf{k}) = -ik {}^{ab}[-\mathbf{k}], \quad \gamma^a {}^{ab}[\mathbf{k}] = (-k) {}^{ab}[\mathbf{k}], \quad \gamma^b {}^{ab}[\mathbf{k}] = -ik\eta^{aa} {}^{ab}(-\mathbf{k}), \\ \tilde{\gamma}^a {}^{ab}(\mathbf{k}) &= -i\eta^{aa} {}^{ab}[\mathbf{k}], \quad \tilde{\gamma}^b {}^{ab}(\mathbf{k}) = -k {}^{ab}[\mathbf{k}], \quad \tilde{\gamma}^a {}^{ab}[\mathbf{k}] = i {}^{ab}(\mathbf{k}), \quad \tilde{\gamma}^b {}^{ab}[\mathbf{k}] = -k\eta^{aa} {}^{ab}(\mathbf{k}) \end{aligned} \quad (7.12)$$

One recognizes also that  $\gamma^a$  transform  ${}^{ab}(\mathbf{k})$  into  ${}^{ab}[-\mathbf{k}]$ , never to  ${}^{ab}[\mathbf{k}]$ , while  $\tilde{\gamma}^a$  transform  ${}^{ab}(\mathbf{k})$  into  ${}^{ab}[\mathbf{k}]$ , never to  ${}^{ab}[-\mathbf{k}]$ .

In Table 7.1 [2,5,3] the left handed ( $\Gamma^{(13,1)} = -1$ , Eq. (7.9)) massless multiplet of one family (Table 7.3) of spinors — the members of the fundamental representation of the  $SO(13,1)$  group — is presented as products of nilpotents and projectors, Eq. (7.10). All these states are eigenstates of the Cartan sub-algebra (Eq. (7.8)). Table 7.1 manifests the subgroup  $SO(7,1)$  of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons [13,15]. The multiplet contains the left handed ( $\Gamma^{(3,1)} = -1$ ) weak  $(SU(2)_I)$  charged ( $\tau^{13} = \pm\frac{1}{2}$ , Eq. (7.5)), and  $SU(2)_{II}$  chargeless ( $\tau^{23} = 0$ , Eq. (7.5)) quarks and leptons and the right handed ( $\Gamma^{(3,1)} = 1$ ) weak  $(SU(2)_I)$  chargeless and  $SU(2)_{II}$  charged ( $\tau^{23} = \pm\frac{1}{2}$ ) quarks and leptons, both with the spin  $S^{12}$  up and down ( $\pm\frac{1}{2}$ , respectively). Quarks and leptons (and separately anti-quarks and anti-leptons) have the same  $SO(7,1)$  part. They distinguish only in the  $SU(3) \times U(1)$  part: Quarks are triplets of three colours ( $\mathbf{c}^i = (\tau^{33}, \tau^{38}) = [\frac{1}{2}, \frac{1}{2\sqrt{3}}], (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$ , Eq. (7.6)) carrying the "fermion

charge" ( $\tau^4 = \frac{1}{6}$ , Eq. (7.6)). The colourless leptons carry the "fermion charge" ( $\tau^4 = -\frac{1}{2}$ ).

The same multiplet contains also the left handed weak ( $SU(2)_I$ ) chargeless and  $SU(2)_{II}$  charged anti-quarks and anti-leptons and the right handed weak ( $SU(2)_I$ ) charged and  $SU(2)_{II}$  chargeless anti-quarks and anti-leptons. Anti-quarks are anti-triplets, carrying the "fermion charge" ( $\tau^4 = -\frac{1}{6}$ ). The anti-colourless anti-leptons carry the "fermion charge" ( $\tau^4 = \frac{1}{2}$ ).  $S^{12}$  defines the ordinary spin  $\pm\frac{1}{2}$ .  $Y = (\tau^{23} + \tau^4)$  is the hyper charge, the electromagnetic charge is  $Q = (\tau^{13} + Y)$ . The vacuum state, on which the nilpotents and projectors operate, is not shown.

All these properties of states can be read directly from the table. *Example 1. and 2.* demonstrate how this can be done.

The states of opposite charges (anti-particle states) are reachable from the particle states (besides by  $S^{ab}$ ) also by the application of the discrete symmetry operator  $\mathcal{C}_{\mathcal{N}} \mathcal{P}_{\mathcal{N}}$ , presented in Refs. [12] and in the footnote of this subsection.

In Table 7.1 the starting state is chosen to be  $(+i) \begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 \\ (+) & (+) & || & (+) & (-) \end{smallmatrix}$   $\begin{smallmatrix} 13 & 14 \\ (-) \end{smallmatrix}$ . We could make any other choice of products of nilpotents and projectors, let say the state  $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) & | & (+) & [-] & || & (+) & (-) & (-) \end{smallmatrix}$ , which is the state in the seventh line of Table 7.1. All the states of one representation can be obtained from the starting state by applying on the starting state the generators  $S^{ab}$ . From the first state, for example, we obtain the seventh one by the application of  $S^{07}$  (or of  $S^{08}$ ,  $S^{37}$ ,  $S^{38}$ ).

Let us make a few examples to get inside how can one read the quantum numbers of states from 7 products of nilpotents and projectors. All nilpotents and projectors are eigen states, Eq. (7.11), of Cartan sub-algebra, Eq. (7.8).

*Example 1.:* Let us calculate properties of the two states: The first state —  $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & | & (+) & (+) & || & (+) & (-) & (-) \end{smallmatrix} |\psi_0\rangle$  — and the seventh state —  $\begin{smallmatrix} 03 & 12 & 56 & 78 \\ [-i] & (+) & | & (+) & [-] \end{smallmatrix}$   $\begin{smallmatrix} 9 & 10 & 11 & 12 & 13 & 14 \\ || & (+) & (-) & (-) & |\psi_0\rangle \end{smallmatrix}$  — of Table 7.1.

The handedness of the whole one Weyl representation (64 states) follows from Eqs. (7.9,7.8):  $\Gamma^{(14)} = i^{13} 2^7 S^{03} S^{12} \dots S^{13 14}$ . This operator gives, when applied on the first state of Table 7.1, the eigenvalue  $= i^{13} 2^7 \frac{i}{2} (\frac{1}{2})^4 (-\frac{1}{2})^2 = -1$  (since the operator  $S^{03}$  applied on the nilpotent  $\begin{smallmatrix} 03 \\ (+i) \end{smallmatrix}$  gives the eigenvalue  $\frac{k}{2} = \frac{i}{2}$ , the rest four operators have the eigenvalues  $\frac{1}{2}$ , and the last two  $-\frac{1}{2}$ , Eq. (7.11)).

In an equivalent way we calculate the handedness  $\Gamma^{(3,1)}$  of these two states in  $d = (3 + 1)$ : The operator  $\Gamma^{(3,1)} = i^3 2^2 S^{03} S^{12}$ , applied on the first state, gives 1 — the right handedness, while  $\Gamma^{(3,1)}$  is for the seventh state  $-1$  — the left handedness.

The weak charge operator  $\tau^{13} (= \frac{1}{2} (S^{56} - S^{78}))$ , Eq. (7.5), applied on the first state, gives the eigenvalue 0:  $\frac{1}{2} (\frac{1}{2} - \frac{1}{2})$ . The eigenvalue of  $\tau^{13}$  is for the seventh state  $\frac{1}{2}$ :  $\frac{1}{2} (\frac{1}{2} - (-\frac{1}{2}))$ ,  $\tau^{23} (= \frac{1}{2} (S^{56} + S^{78}))$ , applied on the first state, gives as its eigenvalue  $\frac{1}{2}$ , while when  $\tau^{23}$  applies on the seventh state gives 0. The "fermion charge" operator  $\tau^4 (= -\frac{1}{3} (S^{9 10} + S^{11 12} + S^{13 14}))$ , Eq. (7.6), gives when applied on any of these two states, the eigenvalues  $-\frac{1}{3} (\frac{1}{2} - \frac{1}{2} - \frac{1}{2}) = \frac{1}{6}$ . Correspondingly is

i	$ \alpha \psi_i\rangle, \Gamma^{(7,1)} = (-1)1, \Gamma^{(6)} = (1) - 1$ (Anti)octet of (anti)quarks and (anti)leptons	$\Gamma^{(3,1)} S^{12}$	$\tau^{13} \tau^{23}$	$\tau^{33}$	$\tau^{38}$	$\tau^4$	Y	Q
1	$u_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & (+) & (+) &    & (+) & (-) & (-) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$u_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & (+) & (+) &    & (+) & (-) & (-) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$d_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & [-] & [-] &    & (+) & (-) & (-) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad -\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$d_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & [-] & [-] &    & (+) & (-) & (-) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad -\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$d_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & [-] & (+) &    & (+) & (-) & (-) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$-\frac{1}{2} \quad 0$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$d_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & [-] & (+) &    & (+) & (-) & (-) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$-\frac{1}{2} \quad 0$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$u_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & (+) & (+) &    & (+) & (-) & (-) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$\frac{1}{2} \quad 0$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	$u_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & (+) & [-] &    & (+) & (-) & (-) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$\frac{1}{2} \quad 0$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$u_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & (+) & (+) &    & [-] & (+) & (-) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad \frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$u_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & (+) & (+) &    & [-] & (+) & (-) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad \frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$d_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & [-] & [-] &    & [-] & (+) & (-) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad -\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$d_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & [-] & [-] &    & [-] & (+) & (-) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad -\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$d_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & [-] & (+) &    & [-] & (+) & (-) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$-\frac{1}{2} \quad 0$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$d_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & [-] & (+) &    & [-] & (+) & (-) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$-\frac{1}{2} \quad 0$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$u_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & (+) & [-] &    & [-] & (+) & (-) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$\frac{1}{2} \quad 0$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	$u_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & (+) & [-] &    & [-] & (+) & (-) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$\frac{1}{2} \quad 0$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$u_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & (+) & (+) &    & [-] & (+) & (+) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad \frac{1}{2}$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$u_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & (+) & (+) &    & [-] & (+) & (+) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad \frac{1}{2}$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$d_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & [-] & [-] &    & [-] & (+) & (+) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad -\frac{1}{2}$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$d_R^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & [-] & [-] &    & [-] & (+) & (+) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad -\frac{1}{2}$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	$d_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & [-] & (+) &    & [-] & (+) & (+) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$-\frac{1}{2} \quad 0$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$d_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & [-] & (+) &    & [-] & (+) & (+) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$-\frac{1}{2} \quad 0$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$u_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & (+) & [-] &    & [-] & (+) & (+) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$\frac{1}{2} \quad 0$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$u_L^c$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & (+) & [-] &    & [-] & (+) & (+) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$\frac{1}{2} \quad 0$	$0$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	$\nu_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & (+) & (+) &    & [-] & (+) & (+) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad \frac{1}{2}$	$0$	$0$	$-\frac{1}{2}$	$0$	$0$
26	$\nu_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & (+) & (+) &    & [-] & (+) & (+) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad \frac{1}{2}$	$0$	$0$	$-\frac{1}{2}$	$0$	$0$
27	$e_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & (+) &   & [-] & [-] &    & (+) & (+) & (+) \end{smallmatrix}$	$1 \quad \frac{1}{2}$	$0 \quad -\frac{1}{2}$	$0$	$0$	$-\frac{1}{2}$	$-1$	$-1$
28	$e_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & [-] &   & [-] & [-] &    & (+) & (+) & (+) \end{smallmatrix}$	$1 \quad -\frac{1}{2}$	$0 \quad -\frac{1}{2}$	$0$	$0$	$-\frac{1}{2}$	$-1$	$-1$
29	$e_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & [-] & (+) &    & (+) & (+) & (+) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$-\frac{1}{2} \quad 0$	$0$	$0$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-1$
30	$e_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & [-] & (+) &    & (+) & (+) & (+) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$-\frac{1}{2} \quad 0$	$0$	$0$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-1$
31	$\nu_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ [-i] & (+) &   & (+) & [-] &    & (+) & (+) & (+) \end{smallmatrix}$	$-1 \quad \frac{1}{2}$	$\frac{1}{2} \quad 0$	$0$	$0$	$-\frac{1}{2}$	$-\frac{1}{2}$	$0$
32	$\nu_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 910 & 1112 & 1314 \\ (+i) & [-] &   & (+) & [-] &    & (+) & (+) & (+) \end{smallmatrix}$	$-1 \quad -\frac{1}{2}$	$\frac{1}{2} \quad 0$	$0$	$0$	$-\frac{1}{2}$	$-\frac{1}{2}$	$0$

**Table 7.1.** The left handed ( $\Gamma^{(13,1)} = -1$ , Eq. (7.9)) multiplet of spinors — the members of (one family of) the fundamental representation of the  $SO(13, 1)$  group of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons, with the charges, spin and handedness manifesting in the low energy regime — is presented in the massless basis using the technique [2,5,3], explained in the text and in *Examples 1,2..*



i	$ \alpha \psi_i\rangle, \Gamma(7,1) = (-1)1, \Gamma(6) = (1) - 1$ (Anti)octet of (anti)quarks and (anti)leptons	$\Gamma(3,1) S^{12}$	$\tau^{13} \tau^{23}$	$\tau^{33}$	$\tau^{38}$	$\tau^4$	Y	Q
33	$\bar{d}_L^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & (+) & (+) &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}_L^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & (+) & (+) &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}_L^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & [-] & [-] &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}_L^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & [-] & [-] &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}_R^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & (+) & (+) &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	1 $\frac{1}{2}$	$\frac{1}{2}$ 0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}_R^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & (+) & (+) &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	1 $-\frac{1}{2}$	$\frac{1}{2}$ 0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}_R^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & [-] & (+) &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	1 $\frac{1}{2}$	$-\frac{1}{2}$ 0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}_R^{-1}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & [-] & (+) &    & [-] & [+ ] & [+ ] \end{smallmatrix}$	1 $-\frac{1}{2}$	$-\frac{1}{2}$ 0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}_L^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & (+) & (+) &    & (-) & (-) & [+ ] \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\bar{d}_L^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & (+) & (+) &    & (+) & (-) & [+ ] \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}_L^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & [-] & [-] &    & (+) & (-) & [+ ] \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\bar{u}_L^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & [-] & [-] &    & (+) & (-) & [+ ] \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\bar{d}_R^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & (+) & (+) &    & (+) & (-) & [+ ] \end{smallmatrix}$	1 $\frac{1}{2}$	$\frac{1}{2}$ 0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\bar{d}_R^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & (+) & (+) &    & (+) & (-) & [+ ] \end{smallmatrix}$	1 $-\frac{1}{2}$	$\frac{1}{2}$ 0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\bar{u}_R^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & [-] & (+) &    & (+) & (-) & [+ ] \end{smallmatrix}$	1 $\frac{1}{2}$	$-\frac{1}{2}$ 0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\bar{u}_R^{-2}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & [-] & (+) &    & (+) & (-) & [+ ] \end{smallmatrix}$	1 $-\frac{1}{2}$	$-\frac{1}{2}$ 0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\bar{d}_L^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & (+) & (+) &    & (+) & [+ ] & (-) \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\bar{d}_L^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & (+) & (+) &    & (+) & [+ ] & (-) \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\bar{u}_L^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & [-] & [-] &    & (+) & [+ ] & (-) \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\bar{u}_L^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & [-] & [-] &    & (+) & [+ ] & (-) \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\bar{d}_R^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & (+) & [-] &    & (+) & [+ ] & (-) \end{smallmatrix}$	1 $\frac{1}{2}$	$\frac{1}{2}$ 0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\bar{d}_R^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & (+) & (+) &    & (+) & [+ ] & (-) \end{smallmatrix}$	1 $-\frac{1}{2}$	$\frac{1}{2}$ 0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\bar{u}_R^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & [-] & (+) &    & (+) & [+ ] & (-) \end{smallmatrix}$	1 $\frac{1}{2}$	$-\frac{1}{2}$ 0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\bar{u}_R^{-3}$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & [-] & (+) &    & (+) & [+ ] & (-) \end{smallmatrix}$	1 $-\frac{1}{2}$	$-\frac{1}{2}$ 0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$e_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & (+) & (+) &    & [-] & (-) & (-) \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$e_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & (+) & (+) &    & [-] & (-) & (-) \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\nu_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) &   & [-] & [-] &    & [-] & (-) & (-) \end{smallmatrix}$	-1 $\frac{1}{2}$	0 $-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\nu_L$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & [-] &   & [-] & [-] &    & [-] & (-) & (-) \end{smallmatrix}$	-1 $-\frac{1}{2}$	0 $-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\nu_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & [-] & (+) &    & [-] & (-) & (-) \end{smallmatrix}$	1 $\frac{1}{2}$	$-\frac{1}{2}$ 0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\nu_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & [-] & (+) &    & [-] & (-) & (-) \end{smallmatrix}$	1 $-\frac{1}{2}$	$-\frac{1}{2}$ 0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$e_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) &   & (+) & [-] &    & [-] & (-) & (-) \end{smallmatrix}$	1 $\frac{1}{2}$	$\frac{1}{2}$ 0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$e_R$ $\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & [-] &   & (+) & [-] &    & [-] & (-) & (-) \end{smallmatrix}$	1 $-\frac{1}{2}$	$\frac{1}{2}$ 0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 7.2. Table 7.1 continued.

the hyper charge  $Y (= \tau^{23} + \tau^4)$  of these two states  $Y = (\frac{2}{3}, \frac{1}{6})$ , respectively, what the *standard model* assumes for  $u_R$  and  $u_L$ , respectively.

One finds for the colour charge of these two states,  $(\tau^{33}, \tau^{38}) (= (\frac{1}{2}(S^{910} - S^{1112}), \frac{1}{\sqrt{3}}(S^{910} + S^{1112} - 2S^{1314})))$  the eigenvalues  $(1/2, 1/(2\sqrt{3}))$ .

The first and the seventh states differ in the handedness  $\Gamma(3,1) = (1, -1)$ , in the weak charge  $\tau^{13} = (0, \frac{1}{2})$  and the hyper charge  $Y = (\frac{2}{3}, \frac{1}{6})$ , respectively. All the states of this octet —  $SO(7,1)$  — have the same colour charge and the same

"fermion charge" (the difference in the hyper charge  $Y$  is caused by the difference in  $\tau^{23} = (\frac{1}{2}, 0)$ ).

The states for the  $d_R$ -quark and  $d_L$ -quark of the same octet follow from the state  $u_R$  and  $u_L$ , respectively, by the application of  $S^{57}$  (or  $S^{58}, S^{67}, S^{68}$ ).

All the  $SO(7, 1)$  ( $\Gamma^{(7,1)} = 1$ ) part of the  $SO(13, 1)$  spinor representation are the same for either quarks of all the three colours (quarks states appear in Table 7.1 from the first to the 24<sup>th</sup> line) or for the colourless leptons (leptons appear in Table 7.1 from the 25<sup>th</sup> line to the 32<sup>nd</sup> line).

Leptons distinguish from quarks in the part represented by nilpotents and projectors, which is determined by the eigenstates of the Cartan subalgebra of ( $S^9{}^{10}, S^{11}{}^{12}, S^{13}{}^{14}$ ). Taking into account Eq. (7.11) one calculates that  $(\tau^{33}, \tau^{38})$  is

for the colourless part of the lepton states ( $v_{R,L}, e_{R,L}$ ) —  $(\cdots \parallel \begin{smallmatrix} 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & (+) & (+) & (+) & (+) \end{smallmatrix})$  — equal to  $(0, 0)$ , while the "fermion charge"  $\tau^4$  is for these states equal to  $-\frac{1}{2}$  (just as assumed by the *standard model*).

Let us point out that the octet  $SO(7, 1)$  manifests how the spin and the weak and hyper charges are related.

*Example 2.:* Let us look at the properties of the anti-quark and anti-lepton states of one fundamental representation of the  $SO(13, 1)$  group. These states are presented in Table 7.1 from the 33<sup>rd</sup> line to the 64<sup>th</sup> line, representing anti-quarks (the first three octets) and anti-leptons (the last octet).

Again, all the anti-octets, the  $SO(7, 1)$  ( $\Gamma^{(7,1)} = -1$ ) part of the  $SO(13, 1)$  representation, are the same either for anti-quarks or for anti-leptons. The last three products of nilpotents and projectors (the part appearing in Table 7.1 after "||") determine anti-colours for the anti-quarks states and the anti-colourless state for anti-leptons.

Let us add that all the anti-spinor states are reachable from the spinor states (and opposite) by the application of the operator  $[12] \mathbb{C}_N \mathcal{P}_N$ <sup>4</sup>. The part of this operator, which operates on only the spinor part of the state (presented in Table 7.1), is  $\mathbb{C}_N \mathcal{P}_N|_{\text{spinor}} = \gamma^0 \prod_{\gamma^s, s=5}^d \gamma^s$ . Taking into account Eq. (7.12) and this operator one finds that  $\mathbb{C}_N \mathcal{P}_N|_{\text{spinor}}$  transforms  $u_R^c$  from the first line of Table 7.1 into  $\bar{u}_L^c$  from the 35<sup>th</sup> line of the same table. When the operator  $\mathbb{C}_N \mathcal{P}_N|_{\text{spinor}}$  applies on  $v_R$  (the 25<sup>th</sup> line of the same table, with the colour chargeless part equal to

$\cdots \parallel \begin{smallmatrix} 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & (+) & (+) & (+) & (+) \end{smallmatrix}$ ), transforms  $v_R$  into  $\bar{v}_L$  (the 59<sup>th</sup> line of the table, with the colour anti-chargeless part equal to  $(\cdots \parallel \begin{smallmatrix} 9 & 10 & 11 & 12 & 13 & 14 \\ (-) & (-) & (-) & (-) & (-) & (-) \end{smallmatrix})$ ).

<sup>4</sup> Discrete symmetries in  $d = (3 + 1)$  follow from the corresponding definition of these symmetries in  $d$ -dimensional space [12]. This operator is defined as:  $\mathbb{C}_N \mathcal{P}_N = \gamma^0 \prod_{\gamma^s, s=5}^d \gamma^s I_{\bar{x}_3} I_{x^6, x^8, \dots, x^d}$ , where  $\gamma^0$  and  $\gamma^1$  are real,  $\gamma^2$  imaginary,  $\gamma^3$  real,  $\gamma^5$  imaginary,  $\gamma^6$  real, alternating imaginary and real up to  $\gamma^d$ , which is in even dimensional spaces real.  $\gamma^{a'}$ s appear in the ascending order. Operators  $I$  operate as follows:  $I_{x^0} x^0 = -x^0$ ;  $I_x x^a = -x^a$ ;  $I_{x^0} x^a = (-x^0, \bar{x})$ ;  $I_{\bar{x}} \bar{x} = -\bar{x}$ ;  $I_{\bar{x}_3} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d)$ ;  $I_{x^5, x^7, \dots, x^{d-1}} (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \dots, x^{d-1}, x^d) = (x^0, x^1, x^2, x^3, -x^5, x^6, -x^7, \dots, -x^{d-1}, x^d)$ ;  $I_{x^6, x^8, \dots, x^d} (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \dots, x^{d-1}, x^d) = (x^0, x^1, x^2, x^3, x^5, -x^6, x^7, -x^8, \dots, x^{d-1}, -x^d)$ ,  $d = 2n$ .

### 7.2.2 A short inside into families in the *spin-charge-family* theory

The operators  $\tilde{S}^{ab}$ , commuting with  $S^{ab}$  (Eq. (7.7)), transform any spinor state, presented in Table 7.1, to the same state of another family, orthogonal to the starting state and correspondingly to all the states of the starting family.

Applying the operator  $\tilde{S}^{03} (= \frac{i}{2}\gamma^0\gamma^3)$ , for example, on  $\nu_R$  (the 25<sup>th</sup> line of Table 7.1 and the last line on Table 7.3), one obtains, taking into account Eq. (7.12), the  $\nu_{R7}$  state belonging to another family, presented in the seventh line of Table 7.3.

Operators  $S^{ab}$  transform  $\nu_R$  (the 25<sup>th</sup> line of Table 7.1, presented in Table 7.3 in the eighth line, carrying the name  $\nu_{R8}$ ) into all the rest of the 64 states of this eighth family, presented in Table 7.1. The operator  $S^{11\,13}$ , for example, transforms  $\nu_{R8}$  into  $u_{R8}$  (presented in the first line of Table 7.1), while it transforms  $\nu_{R7}$  into  $u_{R7}$ .

Table 7.3 represents eight families of neutrinos, which distinguish among themselves in the family quantum numbers:  $(\tilde{\tau}^{13}, \tilde{N}_L, \tilde{\tau}^{23}, \tilde{N}_R, \tilde{\tau}^4)$ . These family quantum numbers can be expressed by  $\tilde{S}^{ab}$  as presented in Eqs. (7.4, 7.5, 7.6), if  $S^{ab}$  are replaced by  $\tilde{S}^{ab}$ .

Eight families decouples into two groups of four families, one (II) is a doublet with respect to  $(\tilde{N}_L$  and  $\tilde{\tau}^1)$  and a singlet with respect to  $(\tilde{N}_R$  and  $\tilde{\tau}^2)$ , the other (I) is a singlet with respect to  $(\tilde{N}_L$  and  $\tilde{\tau}^1)$  and a doublet with respect to  $(\tilde{N}_R$  and  $\tilde{\tau}^2)$ .

All the families follow from the starting one by the application of the operators  $(\tilde{N}_{R,L}^\pm, \tilde{\tau}^{(2,1)\pm})$ , Eq. (7.18). The generators  $(N_{R,L}^\pm, \tau^{(2,1)\pm})$ , Eq. (7.18), transform  $\nu_{R1}$  to all the members belonging to the  $SO(7, 1)$  group of one family,  $S^{s,t}$ ,  $(s, t) = (9 \cdots, 14)$  transform quarks of one colour to quarks of other colours or to leptons.

									$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}_L^3$	$\tilde{N}_R^3$	$\tilde{\tau}^4$
I	$\nu_{R1}$	03	12	56	78	9 10	11 12	13 14	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\nu_{R2}$	03	12	56	78	9 10	11 12	13 14	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\nu_{R3}$	03	12	56	78	9 10	11 12	13 14	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\nu_{R4}$	03	12	56	78	9 10	11 12	13 14	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
II	$\nu_{R5}$	03	12	56	78	9 10	11 12	13 14	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\nu_{R6}$	03	12	56	78	9 10	11 12	13 14	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
II	$\nu_{R7}$	03	12	56	78	9 10	11 12	13 14	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\nu_{R8}$	03	12	56	78	9 10	11 12	13 14	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{3}$

**Table 7.3.** Eight families of the right handed neutrino  $\nu_R$  (appearing in the 25<sup>th</sup> line of Table 7.1), with spin  $\frac{1}{2}$ .  $\nu_{Ri}$ ,  $i = (1, \cdots, 8)$ , carries the family quantum numbers  $\tilde{\tau}^{13}$ ,  $\tilde{N}_L^3$ ,  $\tilde{\tau}^{23}$ ,  $\tilde{N}_R^3$  and  $\tilde{\tau}^4$ . Eight families decouple into two groups of four families.

All the families of Table 7.3 and the family members of the eighth family in Table 7.1 are in the massless basis.

The scalar fields, which are the gauge scalar fields of  $\vec{N}_R$  and  $\vec{\tau}^2$ , couple only to the four families which are doublets with respect to these two groups. The scalar fields which are the gauge scalars of  $\vec{N}_L$  and  $\vec{\tau}^1$  couple only to the four families which are doublets with respect to these last two groups.

After the electroweak phase transition, caused by the scalar fields with the space index (7, 8) [5,11,3,4], the two groups of four families become massive. The lowest of the two groups of four families contains the observed three, while the fourth family remains to be measured. The lowest of the upper four families is the candidate to form the dark matter [4,10].

### 7.2.3 Vector gauge fields and scalar gauge fields in the *spin-charge-family* theory

In the *spin-charge-family* theory [4,2,3], like in all the Kaluza-Klein like theories, either vielbeins or spin connections can be used to represent the vector gauge fields in  $d = (3 + 1)$  space, when space with  $d \geq 5$  has large enough symmetry and no strong spinor source is present. This is proven in Ref. [1] and the references therein. There are the superposition of  $\omega_{stm}$ ,  $m = (0, 1, 2, 3)$ ,  $(s, t) \geq 5$ , which are used in the *spin-charge-family* theory to represent vector gauge fields —  $A_m^{Ai} (= \sum_{s,t} c^{Ai}_{st} \omega^{st}_m)$  — in  $d = (3 + 1)$  in the low energy regime. Here  $Ai$  represent the quantum numbers of the corresponding subgroups, expressed by the operators  $S^{st}$  in Eqs. (7.5, 7.6). Coefficients  $c^{Ai}_{st}$  can be read from Eqs. (7.5, 7.6). These vector gauge fields manifest the properties of all the directly and indirectly observed gauge fields<sup>5</sup>.

In the *spin-charge-family* theory also the scalar fields [2,4,3,9,11,1] have the origin in the spin connection field, in  $\omega_{sts'}$  and  $\tilde{\omega}_{sts'}$ ,  $(s, t, s') \geq 5$ . These scalar fields offer the explanation for the Higgs's scalar and the Yukawa couplings of the *standard model* [9,4].

Both, scalar and vector gauge fields, follow from the simple starting action of the *spin-charge-family* presented in Eq. (7.1).

The Lagrange function for the vector gauge fields follows from the action for the curvature  $R$  in Eq. (7.1) and manifests in the case of the flat  $d = (3 + 1)$  space as assumed by the *standard model*:  $L_v = -\frac{1}{4} \sum_{A,i,m,n} F^{Ai}_{mn} F^{Ai}_{mn}$ ,  $F^{Ai}_{mn} = \partial_m A_n^{Ai} - \partial_n A_m^{Ai} - if^{Aijk} A_m^j A_n^k$ , with

$$\begin{aligned} A_m^{Ai} &= \sum_{s,t} c^{Ai}_{st} \omega^{st}_m, \\ \tau^{Ai} &= \sum_{s,t} c^{Aist} M_{st}, \quad M_{st} = S_{st} + L_{st}. \end{aligned} \quad (7.13)$$

<sup>5</sup> In the *spin-charge-family* theory there are, besides the vector gauge fields of  $(\vec{\tau}^1, \vec{\tau}^3)$ , Eqs. (7.5, 7.6), also the vector gauge fields of  $\vec{\tau}^2$ , Eq. (7.5), and  $\tau^4$ , Eq. (7.6). The vector gauge fields of  $\tau^{21}$ ,  $\tau^{22}$  and  $Y' = \tau^{23} - \tan \theta_2$  gain masses when interacting with the condensate [4] (and the references therein) at around  $10^{16}$  GeV, while the vector gauge field of the hyper charge  $Y = \tau^{23} + \tau^4$  remains massless, together with the gauge fields of  $\vec{\tau}^1$  and  $\vec{\tau}^3$ , manifesting at low energies properties, postulated by the *standard model*.

In the low energy regime only  $S_{st}$  manifest. These expressions can be found in Ref. [1], Eq. (25), for example, and the references therein.

From Eq. (7.1) we read the interaction between fermions, presented in Table 7.1, and the corresponding vector gauge fields in flat  $d = (3 + 1)$  space.

$$\mathcal{L}_{fv} = \bar{\Psi}\gamma^m(p_m - \sum_{A,i} \tau^{Ai} A_m^{Ai})\Psi. \quad (7.14)$$

Particular superposition of spin connection fields, either  $\omega_{sts'}$  or  $\tilde{\omega}_{abs'}$ ,  $(s, t, s') \geq 5, (a, b) = (0, \dots, 8)$ , with the scalar space index  $s' = (7, 8)$ , manifest at low energies as the scalar fields, which contribute to the masses of the family members. The superposition of the scalar fields  $\omega_{stt'}$  with the space index  $t' = (9, \dots, 14)$  contribute to the transformation of matter into antimatter and back, causing in the presence of the condensate [2,4] the matter-antimatter asymmetry of our universe. The interactions of all these scalar fields with fermions follow from Eq. (7.1)

$$\mathcal{L}_{fs} = \{ \sum_{s=7,8} \bar{\Psi}\gamma^s p_{0s} \Psi \} + \{ \sum_{t=5,6,9,\dots,14} \bar{\Psi}\gamma^t p_{0t} \Psi \}, \quad (7.15)$$

where  $p_{0s} = p_s - \frac{1}{2}S^{s's''}\omega_{s's''s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$ ,  $p_{0t} = p_t - \frac{1}{2}S^{t't''}\omega_{t't''t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$ , with  $m \in (0, 1, 2, 3)$ ,  $s \in (7, 8)$ ,  $(s', s'') \in (5, 6, 7, 8)$ ,  $(a, b)$  (appearing in  $\tilde{S}^{ab}$ ) run within either  $(0, 1, 2, 3)$  or  $(5, 6, 7, 8)$ ,  $t$  runs  $\in (5, \dots, 14)$ ,  $(t', t'')$  run either  $\in (5, 6, 7, 8)$  or  $\in (9, 10, \dots, 14)$ . The spinor function  $\Psi$  represents all family members of all the  $2^{\frac{7+1}{2}-1} = 8$  families presented in Table 7.3.

There are the superposition of the scalar fields  $\omega_{s's''s} - (A_{\pm}^Q, A_{\pm}^{Q'}, A_{\pm}^{Y'})^6$  — and the superposition of  $\tilde{\omega}_{s's''s} - (\tilde{A}_{\pm}^{\tilde{N}_L}, \tilde{A}_{\pm}^{\tilde{I}}, \tilde{A}_{\pm}^{\tilde{N}_R}, \tilde{A}_{\pm}^{\tilde{Z}})^7$  — which determine mass terms of family members of spinors after the electroweak break. I shall use  $A_{\pm}^{Ai}$  to represent all the scalar fields, which determine masses of family members, the Yukawa couplings and the weak boson vector fields,  $A_{\pm}^{Ai} = (\sum_{A,i,a,b} c^{Aist}(\omega_{st7\pm} \pm i\omega_{st8}))$  as well as  $= \sum_{A,i,a,b} c^{Aist}(\tilde{\omega}_{ab7\pm} \pm i\omega_{ab8})$ .

The part of the second term of Eq. (7.15), in which summation runs over the space index  $s = (7, 8)$  —  $\sum_{s=7,8} \bar{\Psi}\gamma^s p_{0s}\Psi$  — determines after the electroweak break masses of the two groups of four families. The highest of the lower four families is predicted to be observed at the L(arge)H(adron)C(ollider) [11], the lowest of the higher four families is explaining the origin of the dark matter [10].

The scalar fields in the part of the second term of Eq. (7.15), in which summation runs over the space index  $t = (9, \dots, 14)$  —  $\sum_{t=9,\dots,14} \bar{\Psi}\gamma^t p_{0t}\Psi$  — cause

<sup>6</sup>  $Q := \tau^{13} + Y$ ,  $Q' := -Y \tan^2 \vartheta_1 + \tau^{13}$ ,  $Q' := -\tan^2 \vartheta_1 Y + \tau^{13}$ ,  $Y := \tau^4 + \tau^{23}$ ,  $Y' := -\tan^2 \vartheta_2 \tau^4 + \tau^{23}$ ,  $Q := \tau^{13} + Y$ , and correspondingly  $A_s^Q = \sin \vartheta_1 A_s^{13} + \cos \vartheta_1 A_s^Y$ ,  $A_s^{Q'} = \cos \vartheta_1 A_s^{13} - \sin \vartheta_1 A_s^Y$ ,  $A_s^{Y'} = \cos \vartheta_2 A_s^{23} - \sin \vartheta_2 A_s^4$ ,  $A_s^4 = -(\omega_{910s} + \omega_{1112s} + \omega_{1314s})$ ,  $A_s^{13} = (\omega_{56s} - \omega_{78s})$ ,  $A_s^{23} = (\omega_{56s} + \omega_{78s})$ , with  $(s \in (7, 8))$  (Ref. [3], Eq. (A9)).

<sup>7</sup>  $\tilde{A}_s^{\tilde{I}} = (\tilde{\omega}_{58s} - \tilde{\omega}_{67s}, \tilde{\omega}_{57s} + \tilde{\omega}_{68s}, \tilde{\omega}_{56s} - \tilde{\omega}_{78s})$ ,  $\tilde{A}_s^{\tilde{N}_L} = (\tilde{\omega}_{23s} + i\tilde{\omega}_{01s}, \tilde{\omega}_{31s} + i\tilde{\omega}_{02s}, \tilde{\omega}_{12s} + i\tilde{\omega}_{03s})$ ,  $\tilde{A}_s^{\tilde{Z}} = (\tilde{\omega}_{58s} + \tilde{\omega}_{67s}, \tilde{\omega}_{57s} - \tilde{\omega}_{68s}, \tilde{\omega}_{56s} + \tilde{\omega}_{78s})$  and  $\tilde{A}_s^{\tilde{N}_R} = (\tilde{\omega}_{23s} - i\tilde{\omega}_{01s}, \tilde{\omega}_{31s} - i\tilde{\omega}_{02s}, \tilde{\omega}_{12s} - i\tilde{\omega}_{03s})$ , where  $(s \in (7, 8))$  (Ref. [3], Eq. (A8)).

transitions from anti-leptons into quarks and anti-quarks into quarks and back, transforming antimatter into matter and back. In the expanding universe the condensate of two right handed neutrinos breaks this matter-antimatter symmetry, explaining the matter-antimatter asymmetry of our universe [2].

Spin connection fields  $\omega_{sts'}$  and  $\tilde{\omega}_{sts'}$  interact also with vector gauge fields and among themselves [1]. These interactions can be read from Eq. (7.1).

### 7.3 Discussions and open problems

The *spin-charge-family* theory is offering the next step beyond both *standard models*, by explaining:

- i. The origin of charges of the (massless) family members and the relation between their charges and spins. The theory, namely, starts in  $d = (13 + 1)$  with the simple action for spinors, which interact with the gravity only (Eq.7.1) (through the vielbeins and the two kinds of the spin connection fields), while one fundamental representation of  $SO(13, 1)$  contains, if analyzed with respect to the subgroups  $SO(3, 1)$ ,  $SU(3)$ ,  $SU(2)_I$ ,  $SU(2)_{II}$  and  $U(1)_{II}$  of the group  $SO(13, 1)$ , all the quarks and anti-quarks and all the leptons and anti-leptons with the properties assumed by the *standard model*, relating handedness and charges of spinors as well as of anti-spinors (Table 7.1).
- ii. The origin of families of fermions, since spinors carry two kinds of spins (Eq. (7.2)) — the Dirac  $\gamma^a$  and  $\tilde{\gamma}^a$ . In  $d = (3 + 1)$   $\gamma^a$  take care of the observed spins and charges,  $\tilde{\gamma}^a$  take care of families (Table 7.3).
- iii. The origin of the massless vector gauge fields of the observed charges, represented by the superposition of the spin connection fields  $\omega_{stm}, (s, t) \geq 5, m \leq 3$  [1,4,3].
- iv. The origin of masses of family members and of heavy bosons. The superposition of  $\omega_{sts'}, (s, t) \geq 5, s' = (7, 8)$  and the superposition of  $\tilde{\omega}_{abs'}, (a, b) = (0, \dots, 8), s' = (7, 8)$  namely gain at the electroweak break constant values, determining correspondingly masses of the spinors (fermions) and of the heavy bosons, explaining [4,3,11] the origin of the Higgs's scalar and the Yukawa couplings of the *standard model*.
- v. The origin of the matter-antimatter asymmetry [2], since the superposition of  $\omega_{sts'}, s' \geq 9$ , cause transitions from anti-leptons into quarks and anti-quarks into quarks and back, while the appearance of the scalar condensate in the expanding universe breaks the CP symmetry, enabling the existence of matter-antimatter asymmetry.
- vi. The origin of the dark matter, since there are two groups of decoupled four families in the low energy regime. The neutron made of quarks of the stable of the upper four families explains the appearance of the dark matter [10]<sup>8</sup>.
- vii. The origin of the triangle anomaly cancellation in the standard model. All the quarks and anti-quarks and leptons and anti-leptons, left and right handed,

<sup>8</sup> We followed in Ref. [10] freezing out of the fifth family quarks and anti-quarks in the expanding universe to see whether baryons of the fifth family quarks are the candidates for the dark matter.

appear within one fundamental representation of  $SO(13, 1)$  [4,3].

viii. The origin of all the gauge fields. The *spin-charge-family* theory unifies the gravity with all the vector and scalar gauge fields, since in the starting action there is only gravity (Eq. (7.1)), represented by the vielbeins and the two kinds of the spin connection fields, which in the low energy regime manifests in  $d = (3 + 1)$  as the ordinary gravity and all the directly and indirectly observed vector and scalar gauge fields [1]. If there is no spinor condensate present, only one of the three fields is the propagating field (both spin connections are expressible with the vielbeins). In the presence of the spinor fields the two spin connection fields differ among themselves (Ref. [1], Eq. (4), and the references therein).

The more work is done on the *spin-charge-family* theory, the more answers to the open questions of both *standard models* is the theory offering.

There are, of course, still open questions (mostly common to all the models) like:

a. How has our universe really started? The *spin-charge-family* theory assumes  $d = (13 + 1)$ , but how "has the universe decided" to start with  $d = (13 + 1)$ ? If starting at  $d = \infty$ , how can it come to  $(13 + 1)$  with the massless Weyl representation of only one handedness? We have studied in a toy model the break of symmetry from  $d = (5 + 1)$  into  $(3 + 1)$  [14], finding that there is the possibility that spinors of one handedness remain massless after this break. This study gives a hope that breaking the symmetry from  $(d - 1) + 1$ , where  $d$  is even and  $\infty$ , could go, if the jump of  $(d - 1) + 1$  to  $((d - 4) - 1) + 1$  would be repeated as twice the break suggested in Ref. [14]. These jumps should then be repeated all the way from  $d = \infty$  to  $d = (13 + 1)$ .

b. What did "force" the expanding universe to break the symmetry of  $SO(13, 1)$  to  $SO(7, 1) \times SU(3) \times U(1)_{II}$  and then further to  $SO(3, 1) \times SU(2) \times SU(3) \times U(1)_I$  and finally to  $SO(3, 1) \times SU(3) \times U(1)$ ?

From phase transitions of ordinary matter we know that changes of temperature and pressure lead a particular matter into a phase transition, causing that constituents of the matter (nuclei and electrons) rearrange, changing the symmetry of space.

In expanding universe the temperature and pressure change, forcing spinors to make condensates (like it is the condensate of the two right handed neutrinos in the *spin-charge-family* theory [3,2,4], which gives masses to vector gauge fields of  $SU(2)_{II}$ , breaking  $SU(2)_{II} \times U(1)_{II}$  into  $U(1)_I$ ). There might be also vector gauge fields causing a change of the symmetry (like does the colour vector gauge fields, which "dress" quarks and anti-quarks and bind them to massive colourless baryons and mesons of the ordinary, mostly the first family, matter). Also scalar gauge fields might cause the break of the symmetry of the space (as this do the superposition of  $\omega_{s't's}$  and the superposition of  $\tilde{\omega}_{abs}$ ,  $s = (7, 8)$ ,  $(s', t') \geq 5$ ,  $(a, b) = (0, \dots, 8)$  in the *spin-charge-family* theory [4,3] by gaining constant values in  $d = (3 + 1)$  and breaking correspondingly also the symmetry of the coordinate space in  $d \geq 5$ ).

All these remain to be studied.

c. What is the scale of the electroweak phase transition? How higher is this scale in

comparison with the colour phase transition scale? If the colour phase transition scale is at around 1 GeV (since the first family quarks contribute to baryons masses around 1 GeV), is the electroweak scale at around 1 TeV (of the order of the mass of Higgs's scalar) or this scale is much higher, possibly at the unification scale (since the *spin-charge-family* theory predicts two decoupled groups of four families and several scalar fields — twice two triplets and three singlets [3,11,4])?

d. There are several more open questions. Among them are the origin of the dark energy, the appearance of fermions, the origin of inflation of the universe, quantization of gravity, and several others. Can the *spin-charge-family* theory be — while predicting the fourth family to the observed three, several scalar fields, the fifth family as the origin of the dark matter, the scalar fields transforming anti-leptons into quarks and anti-quarks into quarks and back and the condensate which break this symmetry — the first step, which can hopefully show the way to next steps?

## 7.4 APPENDIX: Some useful formulas and relations are presented [4,5]

$$\begin{aligned}
 S^{ac} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \eta^{cc} \begin{smallmatrix} ab & cd \\ [-k]([-k]) \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix} &= \frac{i}{2} \eta^{aa} \eta^{cc} \begin{smallmatrix} ab & cd \\ [k][k] \end{smallmatrix}, \\
 S^{ac} \begin{smallmatrix} ab & cd \\ [k][k] \end{smallmatrix} &= \frac{i}{2} \begin{smallmatrix} ab & cd \\ (-k)(-k) \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ [k][k] \end{smallmatrix} &= -\frac{i}{2} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix}, \\
 S^{ac} \begin{smallmatrix} ab & cd \\ (k)[k] \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \begin{smallmatrix} ab & cd \\ [-k](-k) \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ (k)[k] \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \begin{smallmatrix} ab & cd \\ [k](k) \end{smallmatrix}, \\
 S^{ac} \begin{smallmatrix} ab & cd \\ [k](k) \end{smallmatrix} &= \frac{i}{2} \eta^{cc} \begin{smallmatrix} ab & cd \\ (-k)(-k) \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ [k](k) \end{smallmatrix} &= \frac{i}{2} \eta^{cc} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix}. \quad (7.16)
 \end{aligned}$$

$$\begin{aligned}
 \begin{smallmatrix} ab & ab \\ (k)(k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ (k)(-k) \end{smallmatrix} &= \eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)(k) \end{smallmatrix} &= \eta^{aa} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)(-k) \end{smallmatrix} &= 0, \\
 \begin{smallmatrix} ab & ab \\ [k][k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ [k](-k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k][k] \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k](-k) \end{smallmatrix} &= \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, \\
 \begin{smallmatrix} ab & ab \\ (k)[k] \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [k](k) \end{smallmatrix} &= \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)[k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ (-k) \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)(-k) \end{smallmatrix} &= 0, \\
 \begin{smallmatrix} ab & ab \\ (k)(-k) \end{smallmatrix} &= \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ [k](-k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k](k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k](-k) \end{smallmatrix} &= \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} \quad (7.17)
 \end{aligned}$$

$$\begin{aligned}
 N_{+}^{\pm} &= N_{+}^1 \pm i N_{+}^2 = -(\mp i) \begin{smallmatrix} 03 & 12 \\ (\pm) \end{smallmatrix}, & N_{-}^{\pm} &= N_{-}^1 \pm i N_{-}^2 = (\pm i) \begin{smallmatrix} 03 & 12 \\ (\pm) \end{smallmatrix}, \\
 \tilde{N}_{+}^{\pm} &= -(\mp i) \begin{smallmatrix} 03 & 12 \\ (\pm) \end{smallmatrix}, & \tilde{N}_{-}^{\pm} &= (\pm i) \begin{smallmatrix} 03 & 12 \\ (\pm) \end{smallmatrix}, \\
 \tau^{1\pm} &= (\mp) \begin{smallmatrix} 56 & 78 \\ (\pm)(\mp) \end{smallmatrix}, & \tau^{2\mp} &= (\mp) \begin{smallmatrix} 56 & 78 \\ (\mp)(\mp) \end{smallmatrix}, \\
 \tilde{\tau}^{1\pm} &= (\mp) \begin{smallmatrix} 56 & 78 \\ (\pm)(\mp) \end{smallmatrix}, & \tilde{\tau}^{2\mp} &= (\mp) \begin{smallmatrix} 56 & 78 \\ (\mp)(\mp) \end{smallmatrix}. \quad (7.18)
 \end{aligned}$$



## References

1. D. Lukman, N.S. Mankoč Borštnik, Eur. Phys. J. C (2017) 33, DOI: 10.1140/epjc/s10052-017-4804-y.
2. N.S. Mankoč Borštnik, Phys. Rev. D **91**, 6 (2015) 065004 ID: 0703013; doi:10.1103/[arXiv:1409.7791, arXiv:1502.06786v1].
3. N.S. Mankoč Borštnik, J. of Mod. Phys. **6** (2015) 2244–2274, doi:10.4236/jmp.2015.615230 [arXiv:1409.4981].
4. N.S. Mankoč Borštnik, "Spin-charge-family theory is offering next step in understanding elementary particles and fields and correspondingly universe", to appear in Proceedings to The 10<sup>th</sup> Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields, IARD conference, Ljubljana 6-9 of June 2016 [arXiv:1607.01618v2].
5. A. Borštnik Bračič and N.S. Mankoč Borštnik, Phys. Rev. D **74**, 073013 (2006) [hep-ph/0301029; hep-ph/9905357, p. 52-57; hep-ph/0512062, p.17-31; hep-ph/0401043, p. 31-57].
6. N.S. Mankoč Borštnik, Phys. Lett. B **292**, 25-29 (1992).
7. N.S. Mankoč Borštnik, J. Math. Phys. **34**, 3731-3745 (1993).
8. N.S. Mankoč Borštnik, Int. J. Theor. Phys. **40**, 315-338 (2001).
9. G. Bregar, M. Breskvar, D. Lukman and N.S. Mankoč Borštnik, New J. of Phys. **10**, 093002 (2008), [arXiv:0606159, arXiv:07082846, arXiv:0612250, p.25-50].
10. G. Bregar and N.S. Mankoč Borštnik, Phys. Rev. D **80**, 1 (2009), 083534 (2009).
11. G. Bregar, N.S. Mankoč Borštnik, "The new experimental data for the quarks mixing matrix are in better agreement with the *spin-charge-family* theory predictions", Proceedings to the 17<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 20-28 of July, 2014, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2014, p.20-45 [ arXiv:1502.06786v1, arXiv:1412.5866].
12. N.S. Mankoč Borštnik, H.B. Nielsen, JHEP **04** (2014) 165-174, doi:10.1007/[arXiv:1212.2362v3].
13. N.S. Mankoč Borštnik, H.B. Nielsen, J. of Math. Phys. **43** (2002) 5782 [hep-th/0111257].
14. D. Lukman, N.S. Mankoč Borštnik, H.B. Nielsen, New J. Phys. **13** (2011) 103027 [hep-th/1001.4679v5].
15. N.S. Mankoč Borštnik, H. B. Nielsen, J. of Math. Phys. **44** (2003) 4817, [hep-th/0303224].



## 8 Why Nature Made a Choice of Clifford and not Grassmann Coordinates \*

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**Abstract.** This is a discussion on fields, the internal degrees of freedom of which are expressed by either the Grassmann or the Clifford “coordinates”. Since both “coordinates” fulfill anticommutation relations, both fields can be second quantized so that their creation and annihilation operators fulfill the requirements of the commutation relations for fermion fields. However, while the internal spin, determined by the generators of the Lorentz group of the Clifford objects  $S^{ab}$  and  $\tilde{S}^{ab}$  (in the *spin-charge-family* theory  $S^{ab}$  determine the spin degrees of freedom and  $\tilde{S}^{ab}$  the family degrees of freedom) have half integer spin, have  $S^{ab}$  (expressible with  $S^{ab} + \tilde{S}^{ab}$ ) integer spin. Nature made obviously a choice of the Clifford algebra.

We discuss here the quantization — first and second — of the fields, the internal degrees of freedom of which are functions of the Grassmann coordinates  $\theta$  and their conjugate momentum, as well as of the fields, the internal degrees of freedom of which are functions of the Clifford  $\gamma^a$ . Inspiration comes from the *spin-charge-family* theory [[1,2,9,3], and the references therein], in which the action for fermions in  $d$ -dimensional space is equal to  $\int d^d x \, E \, \frac{1}{2} (\bar{\Psi} \gamma^a p_{0a} \Psi) + \text{h.c.}$ , with  $p_{0a} = f^a_a p_{0a} + \frac{1}{2E} \{p_a, E f^a_a\}$ ,  $p_{0a} = p_a - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ . We write the basic states of the Grassmann fields and the Clifford fields as a function of products of either Grassmann or Clifford objects, trying to understand *the choice of nature*. We look for the action for free fields which are functions of either the Grassmann coordinates or of the Clifford coordinates in order to understand why Clifford algebra “win” in the competition for the physical degrees of freedom (at least in our observable world).

**Povzetek.** Avtorja obravnavata polja, pri katerih so notranje prostostne stopnje izražene ali z Grassmannovimi ali pa s Cliffordovimi “koordinatami”. Ker obe vrsti “koordinat” zadoščata antikomutacijskim relacijam, lahko za obe vrsti polj naredimo drugo kvantizacijo tako, da kreacijski in anihilacijski operatorji zadoščajo komutacijskim relacijam za fermionska polja. Za razliko od internih spinov, ki jih določajo generatorji Lorentzove grupe Cliffordovih objektov  $S^{ab}$  in  $\tilde{S}^{ab}$  (v teoriji *spinov-nabojev-družin*  $S^{ab}$  določajo spinske prostostne stopnje,  $\tilde{S}^{ab}$  pa družinske prostostne stopnje) in imajo polštevilčni spin), imajo  $S^{ab}$  (ki jih lahko izrazimo z  $S^{ab} + \tilde{S}^{ab}$ ) celoštevilski spin. “Narava se je očitno odločila” za Cliffordovo algebro.

\* This is the part of the talk presented by N.S. Mankoč Borštnik at the 20<sup>th</sup> Workshop “What Comes Beyond the Standard Models”, Bled, 09-17 of July, 2017, and published in the Proceedings to this workshop.

Avtorja obravnavata kvantizacijo — prvo in drugo — za polja, pri katerih so notranje prostostne stopnje funkcije Grassmannovih koordinat  $\theta$  in ustreznih konjugiranih momentov, pa tudi za polja, kjer so interne prostostne stopnje funkcije Cliffordovih koordinat  $\gamma^a$ . Navdih najdeta v teoriji *spinov-nabojev-drušin* [[1,2,9,3], in reference v teh člankih], v kateri je akcija za fermione v d razsežnem prostoru enaka  $\int d^d x \ E \ \frac{1}{2} (\bar{\Psi} \gamma^a p_{0a} \Psi) + h.c.$ , with  $p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\} -$ ,  $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ . Da bi razumela “izbiro narave”, zapišeta osnovna stanja Grassmannovih in Cliffordovih polj kot produkte Grassmannovih ali Cliffordovih objektov. Iščeta akcijo za prosta polja, ki so funkcije Grassmannovih ali pa Cliffordovih koordinat, da bi bolje razumela, zakaj Cliffordova algebra “zmaga” v tekmi za fizikalne prostostne stopnje (vsaj v opazljivem svetu).

Keywords: Spinor representations, Kaluza-Klein theories, Discrete symmetries, Higher dimensional spaces, Beyond the standard model  
PACS: 11.30.Er, 11.10.Kk, 12.60.-i, 04.50.-h

## 8.1 Introduction

This paper is to look for the answers to the questions like: Why our universe “uses” the Clifford rather than the Grassmann coordinates, although both lead in the second quantization procedure to the anticommutation relations required for fermion degrees of freedom? Does the answer lay on the fact that the Clifford degrees of freedom offers the appearance of the families, the half integer spin and the charges as observed so far for fermions, while the Grassmann coordinates offer the groups of (isolated) integer spin states and to charges in the adjoint representations? Can this explain why the simple starting action of the *spin-charge-family* theory of one of us (N.S.M.B.) [9,3,5,8,4,6,7] is doing so far extremely well in manifesting the observed properties of the fermion and boson fields in the low energy regime?

The working hypothesis is that “Nature knows” all the mathematics, accordingly therefore “she knows” for the Grassmann and the Clifford coordinates. To understand why Grassmann space “was not chosen” – we see that the use of the Dirac  $\gamma^a$ ’s enabled to understand the fermions in the first and second quantized theory of fields – or better, to understand why the Clifford algebra (in the *spin-charge-family* theory of two kinds –  $\gamma^a$ ’s and  $\tilde{\gamma}^a$ ’s) is successfully applicable at least in the low energy regime, we work in this paper with both types of spaces.

This work is a part of the project of both authors, which includes the *fermionization* procedure of boson fields or the *bosonization* procedure of fermion fields, discussed in Refs. [10] and in this proceedings for any dimension d (by the authors of this contribution, while one of them, H.B.F.N. [11], has succeeded with another author to do the *fermionization* for  $d = (1 + 1)$ ), and which would hopefully help to better understand the content and dynamics of our universe.

In the *spin-charge-family* theory [9,3,5,8,4,6,7] — which offers the explanation of all the assumptions of the *standard model*, with the appearance of families, the scalar higgs and the Yukawa couplings included, offering also the explanation for the matter-antimatter asymmetry in our universe and for the appearance of the dark matter — a very simple starting action for massless fermions and bosons in

$d = (1 + 13)$  is assumed, in which massless fermions interact with only gravity, the vielbeins  $f^\alpha_a$  (the gauge fields of momentums  $p_a$ ) and the two kinds of the spin connections ( $\omega_{ab\alpha}$  and  $\tilde{\omega}_{ab\alpha}$ , the gauge fields of the two kinds of the Clifford algebra objects  $\gamma^a$  and  $\tilde{\gamma}^a$ , respectively).

$$\mathcal{A} = \int d^d x \left( \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + \text{h.c.} + \int d^d x \left( \alpha R + \tilde{\alpha} \tilde{R} \right), \quad (8.1)$$

with  $p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\}_-$ ,  $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$  and  $R = \frac{1}{2} \{f^\alpha_{[a} f^{\beta b]}\} (\omega_{ab\alpha, \beta} - \omega_{ca\alpha} \omega^c_{b\beta}) + \text{h.c.}$ ,  $\tilde{R} = \frac{1}{2} \{f^\alpha_{[a} f^{\beta b]}\} (\tilde{\omega}_{ab\alpha, \beta} - \tilde{\omega}_{ca\alpha} \tilde{\omega}^c_{b\beta}) + \text{h.c.}$ . The two kinds of the Clifford algebra objects,  $\gamma^a$  and  $\tilde{\gamma}^a$ ,

$$\begin{aligned} \{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+, \\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0. \end{aligned} \quad (8.2)$$

anticommute,  $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$  ( $\gamma^a$  and  $\tilde{\gamma}^b$  are connected with the left and the right multiplication of the Clifford objects, there is no third kind of operators). One of the objects, the generators  $S^{ab} = \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a)$ , determine spins and charges of spinors of any families, another,  $\tilde{S}^{ab} = \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$ , determine the family quantum numbers. Here  $^1 f^\alpha_{[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$ . There are correspondingly two kinds of infinitesimal generators of the Lorentz transformations in the internal degrees of freedom -  $S^{ab}$  for  $SO(13, 1)$  and  $\tilde{S}^{ab}$  for  $\tilde{SO}(13, 1)$ , arranging states into representations.

The curvature  $R$  and  $\tilde{R}$  determine dynamics of the gauge fields — the spin connections and the vielbeins, which manifest in  $d = (1 + 3)$  all the known vector gauge fields as well as the scalar fields [5] which explain the appearance of higgs and the Yukawa couplings, provided that the symmetry breaks from the starting one to  $SO(3, 1) \times SU(3) \times U(1)$ .

The infinitesimal generators of the Lorentz transformations for the gauge fields – the two kinds of the Clifford operators and the Grassmann operators – operate as follows

$$\begin{aligned} \{S^{ab}, \gamma^e\}_- &= -i (\eta^{ae} \gamma^b - \eta^{be} \gamma^a), \\ \{\tilde{S}^{ab}, \tilde{\gamma}^e\}_- &= -i (\eta^{ae} \tilde{\gamma}^b - \eta^{be} \tilde{\gamma}^a), \\ \{S^{ab}, \theta^e\}_- &= -i (\eta^{ae} \theta^b - \eta^{be} \theta^a), \\ \{\mathbf{M}^{ab}, A^{d\dots e\dots g}\}_- &= -i (\eta^{ae} A^{d\dots b\dots g} - \eta^{be} A^{d\dots a\dots g}), \end{aligned} \quad (8.3)$$

where  $\mathbf{M}^{ab}$  are defined by a sum of  $L^{ab}$  plus any of  $S^{ab}$  or  $\tilde{S}^{ab}$ , in the Grassmann case  $\mathbf{M}^{ab}$  is  $L^{ab} + S^{ab}$ , which appear to be  $\mathbf{M}^{ab} = L^{ab} + S^{ab} + \tilde{S}^{ab}$ , as presented later in Eq. (8.22).

<sup>1</sup>  $f^\alpha_a$  are inverted vielbeins to  $e^a_\alpha$  with the properties  $e^a_\alpha f^\alpha_b = \delta^a_b$ ,  $e^a_\alpha f^\beta_a = \delta^\beta_\alpha$ ,  $E = \det(e^a_\alpha)$ . Latin indices  $a, b, \dots, m, n, \dots, s, t, \dots$  denote a tangent space (a flat index), while Greek indices  $\alpha, \beta, \dots, \mu, \nu, \dots, \sigma, \tau, \dots$  denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ( $a, b, c, \dots$  and  $\alpha, \beta, \gamma, \dots$ ), from the middle of both the alphabets the observed dimensions 0, 1, 2, 3 ( $m, n, \dots$  and  $\mu, \nu, \dots$ ), indexes from the bottom of the alphabets indicate the compactified dimensions ( $s, t, \dots$  and  $\sigma, \tau, \dots$ ). We assume the signature  $\eta^{ab} = \text{diag}\{1, -1, -1, \dots, -1\}$ .

We discuss in what follows the first and the second quantization of the fields which depend on the Grassmann coordinates  $\theta^a$ , as well as of the fields which depend on the Clifford coordinates  $\gamma^a$  (or  $\tilde{\gamma}^a$ ) in order to try to understand why "nature has made a choice" of fermions of spins and charges (describable in the *spin-charge-family* theory by subgroups of the Lorentz group expressible with the generators  $S^{ab}$ ) in the fundamental representations of the groups, which interact in the *spin-charge-family* theory through the boson gauge fields (the vielbeins and the spin connections of two kinds). We choose correspondingly either  $\theta^a$ 's or  $\gamma^a$ 's (or  $\tilde{\gamma}^a$ 's, either  $\gamma^a$ 's or  $\tilde{\gamma}^a$ 's [6,7,9]) to describe the internal degrees of freedom of fields to clarify the "choice of nature" and correspondingly also the meaning of *fermionization* of bosons (or *bosonization* of fermions) discussed in Refs. [10] and in this proceedings for any dimension  $d$ .

In all these cases we treat free massless boson and fermion fields; masses of the fields which manifest in  $d = (1 + 3)$  are in the *spin-charge-family* theory due to their interactions with the gravitational fields in  $d > 4$ , described by the scalar vielbeins or spin connection fields

## 8.2 Observations which might be of some help when fermionizing boson fields or bosonizing fermion fields

We present in this section properties of fields with the integer spin in  $d$ -dimensional space, expressed in terms of the Grassmann algebra objects, and the fermion fields, expressed in terms of the Clifford algebra objects. Since the Clifford algebra objects are expressible with the Grassmann algebra objects (Eqs. (8.14, 8.15)), the norms of both are determined by the integral in the Grassmann space, Eqs. (8.24, 8.27).

### a. Fields with the integer spin in the Grassmann space

A point in  $d$ -dimensional Grassmann space of real anticommuting coordinates  $\theta^a$ , ( $a = 0, 1, 2, 3, 5, \dots, d$ ), is determined by a vector  $\{\theta^a\} = (\theta^1, \theta^2, \theta^3, \theta^5, \dots, \theta^d)$ . A linear vector space over the coordinate Grassmann space has correspondingly the dimension  $2^d$ , due to the fact that  $(\theta^{a_i})^2 = 0$  for any  $a_i \in (0, 1, 2, 3, 5, \dots, d)$ .

Correspondingly are fields in the Grassmann space expressed in terms of the Grassmann algebra objects

$$\mathbf{B} = \sum_{k=0}^d a_{a_1 a_2 \dots a_k} \theta^{a_1} \theta^{a_2} \dots \theta^{a_k} |\phi_{og} \rangle, \quad a_i \leq a_{i+1}, \quad (8.4)$$

where  $|\phi_{og} \rangle$  is the vacuum state, here assumed to be  $|\phi_{og} \rangle = |1 \rangle$ , so that  $\frac{\partial}{\partial \theta^a} |\phi_{og} \rangle = 0$  for any  $\theta^a$ . The *Kalb-Ramond* boson fields  $a_{a_1 a_2 \dots a_k}$  are antisymmetric with respect to the permutation of indexes, since the Grassmann coordinates anticommute

$$\{\theta^a, \theta^b\}_+ = 0. \quad (8.5)$$

The left derivative  $\frac{\partial}{\partial \theta_a}$  on vectors of the space of monomials  $\mathbf{B}(\theta)$  is defined as follows

$$\begin{aligned} \frac{\partial}{\partial \theta_a} \mathbf{B}(\theta) &= \frac{\partial \mathbf{B}(\theta)}{\partial \theta_a}, \\ \left\{ \frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b} \right\}_+ \mathbf{B} &= 0, \text{ for all } \mathbf{B}. \end{aligned} \quad (8.6)$$

Defining  $p^{\theta^a} = i \frac{\partial}{\partial \theta_a}$  it correspondingly follows

$$\{p^{\theta^a}, p^{\theta^b}\}_+ = 0, \quad \{p^{\theta^a}, \theta^b\}_+ = i \eta^{ab}, \quad (8.7)$$

The metric tensor  $\eta^{ab}$  ( $= \text{diag}(1, -1, -1, \dots, -1)$ ) lowers the indexes of a vector  $\{\theta^a\}$ :  $\theta_a = \eta_{ab} \theta^b$ , the same metric tensor lowers the indexes of the ordinary vector  $x^a$  of commuting coordinates.

Defining <sup>2</sup>

$$(\theta^a)^\dagger = \frac{\partial}{\partial \theta_a} \eta^{aa} = -i p^{\theta^a} \eta^{aa}, \quad (8.8)$$

it follows

$$\left(\frac{\partial}{\partial \theta_a}\right)^\dagger = \eta^{aa} \theta^a, \quad (p^{\theta^a})^\dagger = -i \eta^{aa} \theta^a. \quad (8.9)$$

By introducing [2] the generators of the infinitesimal Lorentz transformations in the Grassmann space as

$$\mathbf{S}^{ab} = \theta^a p^{\theta^b} - \theta^b p^{\theta^a}, \quad (8.10)$$

one finds

$$\begin{aligned} \{\mathbf{S}^{ab}, \mathbf{S}^{cd}\}_- &= i\{\mathbf{S}^{ad} \eta^{bc} + \mathbf{S}^{bc} \eta^{ad} - \mathbf{S}^{ac} \eta^{bd} - \mathbf{S}^{bd} \eta^{ac}\}, \\ \mathbf{S}^{ab\dagger} &= \eta^{aa} \eta^{bb} \mathbf{S}^{ab}. \end{aligned} \quad (8.11)$$

The basic states in Grassmann space can be arrange into representations [2] with respect to the Cartan subalgebra of the Lorentz algebra, as presented in App. 8.4. The state in d-dimensional space with all the eigenvalues of the Cartan subalgebra of the Lorentz group of Eq. (8.67) equal to either  $i$  or  $1$  is  $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-1} + i\theta^d)|\phi_{oc} \rangle$ , with  $|\phi_{oc} \rangle = |1 \rangle$ .

### b. Fermion fields and the Clifford objects

Let us present as well the properties of the fermion fields with the half integer spin, expressed by the Clifford algebra objects

$$\mathbf{F} = \sum_{k=0}^d a_{a_1 a_2 \dots a_k} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_k} |\psi_{oc} \rangle, \quad a_i \leq a_{i+1}, \quad (8.12)$$

<sup>2</sup> In Ref. [2] the definition of  $\theta^{a\dagger}$  was differently chosen. Correspondingly also the scalar product needed different weight function in Eq. (8.24) is different.

where  $|\psi_{oc} \rangle$  is the vacuum state. The *Kalb-Ramond* fields  $a_{a_1 a_2 \dots a_k}$  are again in general boson fields, which are antisymmetric with respect to the permutation of indexes, since the Clifford objects have the anticommutation relations

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}. \quad (8.13)$$

A linear vector space over the Clifford coordinate space has again the dimension  $2^d$ , due to the fact that  $(\gamma^{a_i})^2 = 0$  for any  $a_i \in (0, 1, 2, 3, 5, \dots, d)$ .

One can see that  $\gamma^a$  are expressible in terms of the Grassmann coordinates and their conjugate momenta as

$$\gamma^a = (\theta^a - i p^{\theta a}). \quad (8.14)$$

We also find  $\tilde{\gamma}^a$

$$\tilde{\gamma}^a = i(\theta^a + i p^{\theta a}), \quad (8.15)$$

with the anticommutation relation of Eq. (8.13) and

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}, \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0. \quad (8.16)$$

Taking into account Eqs. (8.8, 8.14, 8.15) one finds

$$\begin{aligned} (\gamma^a)^\dagger &= \gamma^a \eta^{aa}, \quad (\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a \eta^{aa}, \\ \gamma^a \gamma^a &= \eta^{aa}, \quad \gamma^a (\gamma^a)^\dagger = 1, \quad \tilde{\gamma}^a \tilde{\gamma}^a = \eta^{aa}, \quad \tilde{\gamma}^a (\tilde{\gamma}^a)^\dagger = 1. \end{aligned} \quad (8.17)$$

All three choices for the linear vector space – spanned over either the coordinate Grassmann space, over the vector space of  $\gamma^a$ , as well as over the vector space of  $\tilde{\gamma}^a$  – have the dimension  $2^d$ .

We can express Grassmann coordinates  $\theta^a$  and momenta  $p^{\theta a}$  in terms of  $\gamma^a$  and  $\tilde{\gamma}^a$  as well

$$\begin{aligned} \theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a), \\ \frac{\partial}{\partial \theta^a} &= \frac{1}{2}(\gamma^a + i\tilde{\gamma}^a). \end{aligned} \quad (8.18)$$

It then follows as it should  $\frac{\partial}{\partial \theta^b} \theta^a = \frac{1}{2} \eta_{bc} (\gamma^c + i\tilde{\gamma}^c) \frac{1}{2} (\gamma^c - i\tilde{\gamma}^c) = \delta_b^a$ .

Correspondingly we can use either  $\gamma^a$  as well as  $\tilde{\gamma}^a$  instead of  $\theta^a$  to span the vector space. In this case we change the vacuum from the one with the property  $\frac{\partial}{\partial \theta^a} |\phi_{og} \rangle = 0$  to  $|\psi_{oc} \rangle$  with the property [2,7,9]

$$\begin{aligned} \langle \psi_{oc} | \gamma^a | \psi_{oc} \rangle &= 0, \quad \tilde{\gamma}^a | \psi_{oc} \rangle = i\gamma^a | \psi_{oc} \rangle, \quad \tilde{\gamma}^a \gamma^b | \psi_{oc} \rangle = -i\gamma^b \gamma^a | \psi_{oc} \rangle, \\ \tilde{\gamma}^a \tilde{\gamma}^b | \psi_{oc} \rangle &|_{a \neq b} = -\gamma^a \gamma^b | \psi_{oc} \rangle, \quad \tilde{\gamma}^a \tilde{\gamma}^b | \psi_{oc} \rangle |_{a=b} = \eta^{ab} | \psi_{oc} \rangle. \end{aligned} \quad (8.19)$$

This is in agreement with the requirement

$$\begin{aligned} \gamma^a \mathbf{B}(\gamma) | \psi_{oc} \rangle &:= (a_0 \gamma^a + a_{a_1} \gamma^a \gamma^{a_1} + a_{a_1 a_2} \gamma^a \gamma^{a_1} \gamma^{a_2} + \dots + \\ &\quad a_{a_1 \dots a_d} \gamma^a \gamma^{a_1} \dots \gamma^{a_d}) | \psi_{oc} \rangle, \\ \tilde{\gamma}^a \mathbf{B}(\gamma) | \psi_{oc} \rangle &:= (i a_0 \gamma^a - i a_{a_1} \gamma^{a_1} \gamma^a + i a_{a_1 a_2} \gamma^{a_1} \gamma^{a_2} \gamma^a + \dots + \\ &\quad i(-1)^d a_{a_1 \dots a_d} \gamma^{a_1} \dots \gamma^{a_d} \gamma^a) | \psi_{oc} \rangle. \end{aligned} \quad (8.20)$$

We find the infinitesimal generators of the Lorentz transformations in the Clifford algebra space

$$\begin{aligned} S^{ab} &= \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a), \quad S^{ab\dagger} = \eta^{aa} \eta^{bb} S^{ab}, \\ \tilde{S}^{ab} &= \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a), \quad \tilde{S}^{ab\dagger} = \eta^{aa} \eta^{bb} \tilde{S}^{ab}, \end{aligned} \quad (8.21)$$

with the commutation relations for either  $S^{ab}$  or  $\tilde{S}^{ab}$  of Eq. (8.11), if  $S^{ab}$  is replaced by either  $S^{ab}$  or  $\tilde{S}^{ab}$ , respectively, while

$$\begin{aligned} \mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}, \\ \{S^{ab}, \tilde{S}^{cd}\}_- &= 0. \end{aligned} \quad (8.22)$$

The basic states in the Clifford space can be arranged in representations, in which any state is the eigenstate of the Cartan subalgebra operators of Eq. (8.67). The state in  $d$ -dimensional space with the eigenvalues of either  $S^{03}, S^{12}, S^{56}, \dots, S^{d-1 d}$  or  $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1 d}$  equal to  $\frac{1}{2}(i, 1, 1, \dots, 1)$  is  $(\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6) \dots (\gamma^{d-1} + i\gamma^d)$ , where the states are expressed in terms of  $\gamma^a$ . The states of one representation follow from the starting state obtained by  $S^{ab}$ , which do not belong to the Cartan subalgebra operators, while  $\tilde{S}^{ab}$ , which define family, jumps from the starting family to the new one.

### 8.2.1 Norms of vectors in Grassmann and Clifford space

Let us look for the norm of vectors in Grassmann space

$$\mathbf{B} = \sum_k^d a_{a_1 a_2 \dots a_k} \theta^{a_1} \theta^{a_2} \dots \theta^{a_k} |\phi_{og} >$$

and in Clifford space

$$\mathbf{F} = \sum_k^d a_{a_1 a_2 \dots a_k} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_k} |\psi_{oc} >,$$

where  $|\phi_{og} >$  and  $|\psi_{oc} >$  are the vacuum states in the Grassmann and Clifford case, respectively. In what follows we refer to the Ref. [2].

#### a. Norms of the Grassmann vectors

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates  $\langle \mathbf{B} | \mathbf{C} \rangle$ ,  $\langle \mathbf{B} | \theta \rangle = \langle \theta | \mathbf{B} \rangle^\dagger$ , by requiring

$$\begin{aligned} \{d\theta^a, \theta^b\}_+ &= 0, \quad \int d\theta^a = 0, \quad \int d\theta^a \theta^a = 1, \\ \int d^d \theta \theta^0 \theta^1 \dots \theta^d &= 1, \\ d^d \theta &= d\theta^d \dots d\theta^0, \quad \omega = \prod_{k=0}^d \left( \frac{\partial}{\partial \theta_k} + \theta^k \right), \end{aligned} \quad (8.23)$$



with  $\frac{\partial}{\partial \theta_a} \theta^c = \eta^{ac}$ . The scalar product is defined by the weight function  $\omega = \prod_{k=0}^d (-\frac{\partial}{\partial \theta_k} + \theta^k)$ . It then follows for a scalar product  $\langle \mathbf{B} | \mathbf{C} \rangle$

$$\langle \mathbf{B} | \mathbf{C} \rangle = \int d^d x d^d \theta^a \omega \langle \mathbf{B} | \theta \rangle \langle \theta | \mathbf{C} \rangle = \sum_{k=0}^d \int d^d x b_{b_1 \dots b_k}^* c_{b_1 \dots b_k}, \quad (8.24)$$

where according to Eq. (8.8) follows:

$$\langle \mathbf{B} | \theta \rangle = \langle \phi_{og} | \sum_{p=0}^d (-i)^p a_{a_1 \dots a_p}^* p^{\theta a_p} \eta^{a_p a_p} \dots p^{\theta a_1} \eta^{a_1 a_1}.$$

The vacuum state is chosen to be  $|\phi_{og}\rangle = |1\rangle$ , Eq. (8.4).

The norm  $\langle \mathbf{B} | \mathbf{B} \rangle$  is correspondingly always nonnegative.

### **b. Norms of the Clifford vectors**

Let us look for the norm of vectors, expressed with the Clifford objects  $\mathbf{F} = \sum_k^d a_{a_1 a_2 \dots a_k} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_k} |\psi_{oc}\rangle$ , where  $|\phi_{og}\rangle$  and  $|\psi_{oc}\rangle$  are the two vacuum states when the Grassmann and the Clifford objects are concerned, respectively. By taking into account Eq. (8.17) it follows that

$$(\gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_k})^\dagger = \gamma^{a_k} \eta^{a_k a_k} \dots \gamma^{a_2} \eta^{a_2 a_2} \gamma^{a_1} \eta^{a_1 a_1}, \quad (8.25)$$

while  $\gamma^a \gamma^a = \eta^{aa}$ .

We can use Eqs. (8.23, 8.24) to evaluate the scalar product of two Clifford algebra objects  $\langle \gamma^a | \mathbf{F} \rangle = \langle (\theta^a - i p^{\theta a}) | \mathbf{F} \rangle$  and  $\langle (\theta^b - i p^{\theta b}) | \mathbf{G} \rangle$ . These expressions follow from Eqs. (8.14, 8.15, 8.17)). We must then choose for the vacuum state the one from the Grassmann case –  $|\psi_{oc}\rangle = |\phi_{og}\rangle = |1\rangle$ . We obtain

$$\langle \mathbf{F} | \mathbf{G} \rangle = \int d^d x d^d \theta^a \omega \langle \mathbf{F} | \gamma \rangle \langle \gamma | \mathbf{G} \rangle = \sum_{k=0}^d \int d^d x a_{a_1 \dots a_k}^* b_{b_1 \dots b_k}. \quad (8.26)$$

{Similarly we obtain, if we express  $\tilde{\mathbf{F}} = \sum_{k=0}^d a_{a_1 a_2 \dots a_k} \tilde{\gamma}^{a_1} \tilde{\gamma}^{a_2} \dots \tilde{\gamma}^{a_k} |\phi_{oc}\rangle$  and  $\tilde{\mathbf{G}} = \sum_{k=0}^d b_{b_1 b_2 \dots b_k} \tilde{\gamma}^{b_1} \tilde{\gamma}^{b_2} \dots \tilde{\gamma}^{b_k} |\phi_{oc}\rangle$  and take  $|\psi_{oc}\rangle = |\phi_{og}\rangle = |1\rangle$ , the scalar product

$$\langle \tilde{\mathbf{F}} | \tilde{\mathbf{G}} \rangle = \int d^d x d^d \theta^a \omega \langle \tilde{\mathbf{F}} | \tilde{\gamma} \rangle \langle \tilde{\gamma} | \tilde{\mathbf{G}} \rangle = \sum_{k=0}^d \int d^d x a_{a_1 \dots a_k}^* a_{b_1 \dots b_k} \cdot \} \quad (8.27)$$

Correspondingly we can write

$$(a_{a_1 a_2 \dots a_k} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_k})^\dagger (a_{a_1 a_2 \dots a_k} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_k}) = a_{a_1 a_2 \dots a_k}^* a_{a_1 a_2 \dots a_k}. \quad (8.28)$$

The norm of each scalar term in the sum of  $\mathbf{F}$  is nonnegative.

c. We have learned that in both spaces – Grassmann and Clifford – the norms of basic states can be defined so that the states, which are eigenvectors of the Cartan subalgebra, are orthogonal and normalized using the same integral. Studying the second quantization procedure in Subsect. 8.2.3 we learn that not all  $2^d$  states can be generated by the creation and annihilation operators fulfilling the requirements for the second quantized operators, either for states with integer spins or for states with half integer spin. We also learn that the vacuum state must in the Clifford algebra case be different the one assumed in the first quantization case.

### 8.2.2 Actions in Grassmann and Clifford space

Let us construct actions for states in the Grassmann space, as well as in the Clifford space. While the action in the Clifford space is well known since long [17], the action in the Grassmann space must be found. In both cases we look for actions for free massless states only.

States in Grassmann space as well as states in Clifford space are organized to be – within each of the two spaces – orthogonal and normalized with respect to Eq. (8.23). We choose the states in each of two spaces to be the eigenstates of the Cartan subalgebra – with respect to  $S^{ab}$  in Grassmann space and with respect to  $S^{ab}$  and  $\tilde{S}^{ab}$  in Clifford space, Eq. (8.67).

In both spaces the requirement that states are obtained by the application of creation operators on vacuum states –  $\hat{b}_i^0$  obeying the commutation relations of Eq. (8.40) on the vacuum state  $|\phi_{og} \rangle$  for Grassmann space, and  $\hat{b}_i^\alpha$  obeying the commutation relation of Eq. (8.52) on the vacuum states  $|\psi_{oc} \rangle$ , Eq. (8.59), for Clifford space – reduces the number of states, in the Clifford space more than in the Grassmann space. But while in the Clifford space all physically applicable states are reachable by either  $S^{ab}$  or by  $\tilde{S}^{ab}$ , the states in the Grassmann space, belonging to different representations with respect to the Lorentz generators, seem not to be connected.

#### a. Action in Clifford space

In Clifford space we expect that the action for a free massless object

$$\mathcal{A} = \int d^d x \frac{1}{2} (\psi^\dagger \gamma^0 \gamma^a p_a \psi) + \text{h.c.}, \quad (8.29)$$

is Lorentz invariant, and that it leads to the equations of motion

$$\gamma^a p_a |\psi_i^\alpha \rangle = 0, \quad (8.30)$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi_i^\alpha \rangle = p^a p_a |\psi_i^\alpha \rangle = 0. \quad (8.31)$$

Correspondingly  $\gamma^0$  appears in the action since we pay attention that

$$\begin{aligned} S^{ab\dagger} \gamma^0 &= \gamma^0 S^{ab}, \\ S^\dagger \gamma^0 &= \gamma^0 S^{-1}, \\ S &= e^{-\frac{i}{2} \omega_{ab} (S^{ab} + L^{ab})}. \end{aligned} \quad (8.32)$$

We choose the basic states to be the eigenstates of all the members of the Cartan subalgebra, Eq. (8.67). Correspondingly all the states, belonging to different values of the Cartan subalgebra – at least they differ in one value of either the set of  $S^{ab}$  or the set of  $\tilde{S}^{ab}$ , Eq. (8.67) – are orthogonal with respect to the scalar product for a chosen vacuum state, defined as the integral over the Grassmann coordinates, Eq. (8.23). Correspondingly the states generated by the creation operators, Eq. (8.57), on the vacuum state, Eq. (8.59), are orthogonal as well (both last equations will appear later).

### b. Action in Grassmann space

In Grassmann space we require – similarly as in the Clifford case – that the action for a free massless object

$$\mathcal{A} = \frac{1}{2} \left\{ \int d^d x \, d^d \theta \, \omega \left( \phi^\dagger \left( 1 - 2\theta^0 \frac{\partial}{\partial \theta^0} \right) \theta^a p_a \phi \right) \right\} + \text{h.c.}, \quad (8.33)$$

is Lorentz invariant.  $p_a = i \frac{\partial}{\partial x^a}$ . We use the integral also over  $\theta^a$  coordinates, with the weight function  $\omega$  from Eq. (8.23). Requiring the Lorentz invariance we add after  $\phi^\dagger$  the operator  $(1 - 2\theta^0 \frac{\partial}{\partial \theta^0})$ , which takes care of the Lorentz invariance. Namely

$$\begin{aligned} S^{ab\dagger} (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) S^{ab}, \\ S^\dagger (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) S^{-1}, \\ S &= e^{-\frac{i}{2} \omega_{ab} (L^{ab} + S^{ab})}. \end{aligned} \quad (8.34)$$

We also require that the action leads to the equations of motion

$$\begin{aligned} \theta^a p_a |\phi_i^\theta\rangle &= 0, \\ \frac{\partial}{\partial \theta^a} p_a |\phi_i^\theta\rangle &= 0, \end{aligned} \quad (8.35)$$

both equations leading to the same solution, and also to the Klein-Gordon equation

$$\{\theta^a p_a, \frac{\partial}{\partial \theta^b} p_b\}_+ |\phi_i^\theta\rangle = p^a p_a |\phi_i^\theta\rangle = 0. \quad (8.36)$$

### c. We learned:

In both spaces – in the Clifford and in the Grassmann space – there exists the action, which leads to the equations of motion and to the corresponding Klein-Gordon equation.

We shall see that creation and annihilation operators in both spaces fulfill the anticommutation relations, required for fermions. But while the Clifford algebra

defines spinors with the half integer eigenvalues of the Cartan subalgebra operators of the Lorentz algebra, the Grassmann algebra defines states with the integer eigenvalues of the Cartan subalgebra.

### 8.2.3 Second quantization of Grassmann vectors and Clifford vectors

States in the Grassmann space as well as states in the Clifford space are organized to be – within each of the two spaces – orthogonal and normalized with respect to Eq. (8.23). All the states in each of spaces are chosen to be eigenstates of the Cartan subalgebra – with respect to  $S^{ab}$  in the Grassmann space, and with respect to  $S^{ab}$  and  $\tilde{S}^{ab}$  in the Clifford space, Eq. (8.67).

In both spaces the requirement that states are obtained by the application of creation operators on vacuum states –  $\hat{b}_i^0$  obeying the commutation relations of Eq. (8.40) on the vacuum state  $|\phi_{og} \rangle = |1 \rangle$  for the Grassmann space, and  $\hat{b}_i^\alpha$  obeying the commutation relation of Eq. (8.52) on the vacuum states  $|\psi_{oc} \rangle$ , Eq. (8.59), for the Clifford space – reduces the number of states, in the Clifford space more than in the Grassmann space. But while in the Clifford space all physically applicable states are reachable either by  $S^{ab}$  or by  $\tilde{S}^{ab}$ , the states, belonging to different groups with respect to the Lorentz generators, seems not to be connected by the Lorentz operators in the Grassmann space.

Let us construct the creation and annihilation operators for the cases that we use **a.** the Grassmann vector space, or **b.** the Clifford vector space. We shall see that from  $2^d$  states in either the Grassmann or the Clifford space (all are orthogonal among themselves with respect to the integral, Eq. (8.23)) – separately in each of the two spaces – there are reduced number of sates generated by the corresponding creation and annihilation operators, when products of Grassmann coordinates  $\theta^{a'}$ s and momenta  $\frac{\partial}{\partial \theta^a}$  are required to represent creation and annihilation operators, and only  $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ , Eq.(8.60), when products of nilpotents and projectors, Eq. (8.46), are chosen to generate creation and annihilation operators.

#### a. Quantization in Grassmann space

There are  $2^d$  states in Grassmann space, orthogonal to each other with respect to Eq. (8.23). To any coordinate there exists the conjugate momentum. We pay attention in this paper to  $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1)$  states, Eq. (8.43), when products of the superposition of the Grassmann coordinates, which are eigenstates of the Cartan subalgebra operators, are used to represent creation and their Hermitian conjugatde objects the annihilation operators. Let us see how it goes.

If  $\hat{b}_i^{\theta^\dagger}$  is a creation operator, which creates a state in the Grassmann space, when operating on a vacuum state  $|\psi_{og} \rangle$  and  $\hat{b}_i^0 = (\hat{b}_i^{\theta^\dagger})^\dagger$  is the corresponding annihilation operator, then for a set of creation operators  $\hat{b}_i^{\theta^\dagger}$  and the corresponding annihilation operators  $\hat{b}_i^0$  it must be

$$\begin{aligned}\hat{b}_i^0 |\phi_{og} \rangle &= 0, \\ \hat{b}_i^{\theta^\dagger} |\phi_{og} \rangle &\neq 0.\end{aligned}\tag{8.37}$$

We first pay attention on only the internal degrees of freedom - the spin.

Choosing  $\hat{b}_a^\theta = \frac{\partial}{\partial \theta^a}$  it follows

$$\begin{aligned}\hat{b}_a^{\theta^\dagger} &= \theta^a, \\ \hat{b}_a^\theta &= \frac{\partial}{\partial \theta^a}, \\ \{\hat{b}_a^\theta, \hat{b}_b^{\theta^\dagger}\}_+ &= \delta_b^a, \\ \{\hat{b}_a^\theta, \hat{b}_b^\theta\}_+ &= 0, \\ \{\hat{b}_a^{\theta^\dagger}, \hat{b}_b^{\theta^\dagger}\}_+ &= 0, \\ \hat{b}_a^{\theta^\dagger} |\phi_{og}\rangle &= \theta^a |\phi_{og}\rangle, \\ \hat{b}_a^\theta |\phi_{og}\rangle &= 0.\end{aligned}\tag{8.38}$$

The vacuum state  $|\phi_{og}\rangle$  is in this case  $|1\rangle$ .

The identity  $I$  can not be taken as an creation operator, since its annihilation partner does not fulfill Eq. (8.37).

We can use the products of superposition of  $\theta^a$ 's as creation and products of superposition of  $\frac{\partial}{\partial \theta^a}$ 's as annihilation operators provided that they fulfill the requirements for the creation and annihilation operators, Eq. (8.40), with the vacuum state  $|\phi_{og}\rangle = |1\rangle$ .

It is convenient to take products of superposition of vectors  $\theta^a$  and  $\theta^b$  to construct creation operators so that each factor is the eigenstate of one of the Cartan subalgebra member of the Lorentz algebra (8.67). We can start with the creation operators as products of  $\frac{d}{2}$  states  $\hat{b}_{a_i b_i}^{\theta^\dagger} = \frac{1}{\sqrt{2}}(\theta^{a_i} \pm \epsilon \theta^{b_i})$ . Then the corresponding annihilation operators are  $\frac{d}{2}$  factors of  $\hat{b}_{a_i b_i}^\theta = \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial \theta^{a_i}} \pm \epsilon^* \frac{\partial}{\partial \theta^{b_i}})$ ,  $\epsilon = i$ , if  $\eta^{a_i a_i} = \eta^{b_i b_i}$  and  $\epsilon = -1$ , if  $\eta^{a_i a_i} \neq \eta^{b_i b_i}$ . Starting with the state  $\hat{b}_i^{\theta^\dagger} = (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-1} + i\theta^d)$  the rest of states belonging to the same Lorentz representation follows from the starting state by the application of the operators  $S^{cf}$ , which do not belong to the Cartan subalgebra operators. It follows

$$\begin{aligned}\hat{b}_i^{\theta^\dagger} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-1} + i\theta^d), \\ \hat{b}_i^\theta &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (-\frac{\partial}{\partial \theta^{d-1}} + i\frac{\partial}{\partial \theta^d}) \dots (\frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3}), \\ \hat{b}_j^{\theta^\dagger} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\theta^0 \theta^3 + i\theta^1 \theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-1} + i\theta^d), \\ \hat{b}_j^\theta &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (-\frac{\partial}{\partial \theta^{d-1}} + i\frac{\partial}{\partial \theta^d}) \dots (-\frac{\partial}{\partial \theta^3} \frac{\partial}{\partial \theta^0} - i(\frac{\partial}{\partial \theta^2} \frac{\partial}{\partial \theta^1})). \\ &\dots\end{aligned}\tag{8.39}$$

It is taking into account that  $S^{01}$  transforms  $(\frac{1}{\sqrt{2}})^2(\theta^0 - \theta^3)(\theta^1 + i\theta^2)$  into  $\frac{1}{\sqrt{2}}(\theta^0 \theta^3 + i\theta^1 \theta^2)$  or any  $S^{ac}$ , which does not belong to Cartan subalgebra, Eq.(8.65), transforms  $(\frac{1}{\sqrt{2}})^2(\theta^a + i\theta^b)(\theta^c + i\theta^d)$  into  $i\frac{1}{\sqrt{2}}(\theta^a \theta^b + \theta^c \theta^d)$ .

One finds that  $S^{ab}(\theta^a \pm \epsilon \theta^b) = \mp \frac{\eta^{aa}}{\epsilon}(\theta^a + \epsilon \theta^b)$ , while  $S^{ab}$  applied on  $(\theta^a \theta^b \pm \epsilon \theta^c \theta^d)$  gives zero.

Although all the states, generated by creation operators, which include one  $(I \pm \epsilon \theta^a \theta^b)$  or several  $(I \pm \epsilon \theta^{a_1} \theta^{b_1}) \dots (I \pm \epsilon \theta^{a_k} \theta^{b_k})$ , are orthogonal with respect to the scalar product, Eq.(8.24), such creation operators do not have appropriate annihilation operators since  $(I \pm \epsilon \theta^a \theta^b)$  and  $(I \pm \epsilon^* \frac{\partial}{\partial \theta^b} \frac{\partial}{\partial \theta^a})$  (or several  $(I \pm \epsilon \theta^{a_1} \theta^{b_1}) \dots (I \pm \epsilon \theta^{a_k} \theta^{b_k})$  and  $(I \pm \epsilon^* \frac{\partial}{\partial \theta^{b_k}} \frac{\partial}{\partial \theta^{a_k}}) \dots (I \pm \epsilon^* \frac{\partial}{\partial \theta^{b_1}} \frac{\partial}{\partial \theta^{a_1}})$ ) do not fulfill Eqs. (8.37, 8.38), since  $I$  has no annihilation partner. However, creation operators which are products of one or several, let say  $n$ , of the kind  $\theta^{a_i} \theta^{b_i}$  (at most  $\frac{d}{2}$ , each factor of them is the "eigenstate" of one of the Cartan subalgebra operators –  $S^{ab} \theta^a \theta^b |1\rangle = 0$ ), while the rest,  $\frac{d}{2} - n$ , have the "eigenvalues" either  $(+1$  or  $-1)$  or  $(+i$  or  $-i)$ , fulfill relations

$$\begin{aligned} \{\hat{b}_i^\theta, \hat{b}_j^{\theta^\dagger}\}_+ |\phi_{og}\rangle &= \delta_j^i |\phi_{og}\rangle, \\ \{\hat{b}_i^\theta, \hat{b}_j^\theta\}_+ |\phi_{og}\rangle &= 0 |\phi_{og}\rangle, \\ \{\hat{b}_i^{\theta^\dagger}, \hat{b}_j^{\theta^\dagger}\}_+ |\phi_{og}\rangle &= 0 |\phi_{og}\rangle, \\ \hat{b}_j^{\theta^\dagger} |\phi_{og}\rangle &= |\phi_j\rangle \\ \hat{b}_j^\theta |\phi_{og}\rangle &= 0 |\phi_{og}\rangle. \end{aligned} \quad (8.40)$$

There are in  $(d = 2)$  two creation  $((\theta^0 \mp \theta^1, \text{ for } \eta^{ab} = \text{diag}(1, -1))$  and correspondingly two annihilation operators  $(\frac{\partial}{\partial \theta^0} \mp \frac{\partial}{\partial \theta^1})$ , and one creation operator  $\theta^0 \theta^1$  and the corresponding annihilation operator  $\frac{\partial}{\partial \theta^1} \frac{\partial}{\partial \theta^0}$ , each belonging to its own group with respect to the Lorentz transformation operators, which fulfill Eq. (8.40), in  $(d = 4)$  there are two triplets of the kind presented in Eq. (8.39) of creation and correspondingly two triplets of annihilation operators, and four creation operators with one product of  $\theta^{a_i} \theta^{b_i}$  multiplied by  $(\theta^{c_i} \pm \theta^{d_i})$  and four corresponding annihilation operators as well as the creation operator  $\theta^0 \theta^3 \theta^1 \theta^2$  with the corresponding annihilation operator, they all fulfill Eq. (8.40).

Let us count the number of creation operators, when one starts with the creator, which is the product of  $\frac{d}{2}$  factors, each with the "eigenvalue" of the Cartan subalgebra operators, Eq. (8.67), equal to either  $+i$  or  $+1$ , Eq. (8.39):

$$\hat{b}_0^{\theta^\dagger} = (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d). \quad (8.41)$$

There are  $2^{\frac{d}{2}-1}$  creation operators of this type  $\{(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d), (\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d), (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 - i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d), \dots, (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 - i\theta^6) \dots (\theta^{d-1} - i\theta^d)\}$  with the eigenvalues of the Cartan subalgebra equal to  $\{(+i, +1, +1, \dots, +1, +1), (-i, -1, +1, \dots, +1, +1), (-i, +1, -1, \dots, +1, +1), \dots, (+i, +1, +1, \dots, -1, -1)\}$ , each of the operators distinguishing from the others in one pair of factors with the opposite eigenvalues of the Cartan subalgebra operators.

There are in addition  $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} - 1)/2$  Grassmann odd operators obtained when  $S^{ef}$  apply on  $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)$ ,  $(\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)$  and on the rest of  $2^{\frac{d}{2}-1} - 1$  operators.  $S^{01}$  applied on  $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)$ ,  $(\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)$  gives  $\propto (\theta^0 \theta^3 + i\theta^1 \theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)$ ,  $(\theta^0 + \theta^3)(\theta^1 -$

$i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^5)$ ). Each of these operators have two "eigenvalues" of the Cartan subalgebra equal to zero and all the rest equal to either  $\pm i$  (if one of the two summands has  $\eta^{aa} = 1$ ) or  $\pm 1$  (otherwise). All these creation operators are connected by  $\mathbf{S}^{eg}$ .

There are correspondingly all together  $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1)/2$  creation operators and the same number of annihilation operators (they follow from the creation operators by Hermitian conjugation, Eq. (8.8)), belonging to one group, so that all the operators follow from the starting one by the application of  $\mathbf{S}^{af}$ .

There is additional group of creation and annihilation operators, which follow from the starting one

$$\hat{b}_0^{\theta\dagger} = (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \dots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d). \quad (8.42)$$

(one can chose in the starting creation operator with changed sign in any of factors in the product, in each case the same group will follow). All the rest  $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1)/2$  creation operators can be obtained from the starting one as in the case of the first group.

There is therefore

$$2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1) \quad (8.43)$$

creation and the same number of annihilation operators, which are built on two starting states, presented in Eqs. (8.41, 8.42), divided in two groups, each generating or annihilating states belonging to the same representation of the Lorentz algebra.

The rest of creators (and the corresponding annihilators) have opposite Grassmann character than the ones studied so far – like  $\theta^0\theta^1(\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0})$  in  $d = (1+1)$  and in  $d = (1+3)$   $\theta^0\theta^3(\theta^1 \pm i\theta^2)(\frac{\partial}{\partial\theta^1} \mp i\frac{\partial}{\partial\theta^2})\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}$ ,  $\theta^1\theta^2(\theta^0 \mp i\theta^3)(\frac{\partial}{\partial\theta^0} \pm i\frac{\partial}{\partial\theta^3})\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}$  and  $\theta^0\theta^3\theta^1\theta^2(\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0})$ , which also fulfill the relations of Eq. (8.40).

All the states  $|\phi_i^\theta\rangle$ , generated by the creation operators (presented in Eq. (8.40)) on the vacuum state  $|\phi_{og}\rangle$  are the eigenstates of the Cartan subalgebra operators and are orthogonal and normalized with respect to the norm of Eq. (8.23)

$$\langle \phi_i^\theta | \phi_j^\theta \rangle = \delta^i_j. \quad (8.44)$$

If we now extend the creation and annihilation operators to the ordinary coordinate space, the relation among creation and annihilation operators at one time read

$$\begin{aligned} \{\hat{b}_i^\theta(\vec{x}), \hat{b}_j^{\theta\dagger}(\vec{x}')\}_+ |\phi_{og}\rangle &= \delta_j^i \delta(\vec{x} - \vec{x}') |\phi_{og}\rangle, \\ \{\hat{b}_i^\theta(\vec{x}), \hat{b}_j^\theta(\vec{x}')\}_+ |\phi_{og}\rangle &= 0 |\phi_{og}\rangle, \\ \{\hat{b}_i^{\theta\dagger}(\vec{x}), \hat{b}_j^{\theta\dagger}(\vec{x}')\}_+ |\phi_{og}\rangle &= 0 |\phi_{og}\rangle, \\ \hat{b}_j^{\theta\dagger}(\vec{x}) |\phi_{og}\rangle &= 0 |\phi_{og}\rangle \\ |\phi_{og}\rangle &= |1\rangle. \end{aligned} \quad (8.45)$$

## b. Quantization in Clifford space

In Grassmann space the requirement that products of eigenstates of the Cartan subalgebra operators represent the creation and annihilation operators, obeying the relation Eq. (8.40), reduces the number of states. Let us study what happens, when, let say,  $\gamma^a$ 's are used to create the basis and correspondingly also to create the creation and annihilation operators.

Let us point out that  $\gamma^a$  is expressible with  $\theta^a$  and its derivative ( $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a})$ ), Eq. (8.14), and that we again require that creation (annihilation) operators create (annihilate) states, which are eigenstates of the Cartan subalgebra, Eq. (8.67). We could as well make a choice of  $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a})$ <sup>3</sup>. We shall follow here to some extend Ref. [15].

Making a choice of the Cartan subalgebra eigenstates of  $S^{ab}$ , Eq. (8.67),

$$\begin{aligned} {}^{ab}(\mathbf{k}) &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \\ {}^{ab}[\mathbf{k}] &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \end{aligned} \quad (8.46)$$

where  $k^2 = \eta^{aa}\eta^{bb}$ , recognizing that the Hermitian conjugate values of  ${}^{ab}(\mathbf{k})$  and  ${}^{ab}[\mathbf{k}]$  are

$${}^{ab}(\mathbf{k})^\dagger = \eta^{aa} {}^{ab}(-\mathbf{k}), \quad {}^{ab}[\mathbf{k}]^\dagger = {}^{ab}[\mathbf{k}], \quad (8.47)$$

while the corresponding eigenvalues of  $S^{ab}$ , Eq. (8.48), and  $\tilde{S}^{ab}$ , Eq. (8.85), are

$$\begin{aligned} S^{ab} {}^{ab}(\mathbf{k}) &= \frac{1}{2}k {}^{ab}(\mathbf{k}), \quad S^{ab} {}^{ab}[\mathbf{k}] = \frac{1}{2}k {}^{ab}[\mathbf{k}] \\ \tilde{S}^{ab} {}^{ab}(\mathbf{k}) &= \frac{k}{2} {}^{ab}(\mathbf{k}), \quad \tilde{S}^{ab} {}^{ab}[\mathbf{k}] = -\frac{k}{2} {}^{ab}[\mathbf{k}]. \end{aligned} \quad (8.48)$$

We find in  $d = 2(2n + 1)$  that from the starting state with products of odd number of only nilpotents

$$|\psi_1^1\rangle > |_{2(2n+1)} = (+i)(+)(+)\cdots \begin{matrix} 03 & 12 & 35 & & d-3 & d-2d-1 & d \\ (+) & (+) & (+) & & (+) & (+) & (+) \end{matrix} |\psi_{oc}\rangle, \quad (8.49)$$

having correspondingly an odd Clifford character<sup>4</sup>, all the other states of the same Lorentz representation, there are  $2^{\frac{d}{2}-1}$  members, follow by the application of  $S^{cd}$ <sup>5</sup>, which do not belong to the Cartan subalgebra, Eq. (8.67):  $S^{cd} |\psi_1^1\rangle > |_{2(2n+1)} = |\psi_i^1\rangle > |_{2(2n+1)}$ . The operators  $\tilde{S}^{cd}$ , which do not belong to the Cartan subalgebra of

<sup>3</sup> We choose  $\gamma^a$ 's, Eq.(8.14) to create the basic states. We could instead make a choice of  $\tilde{\gamma}^a$ 's, Eq.(8.15) to create the basic states. In the case of this latter choice the role of  $\tilde{\gamma}^a$  and  $\gamma^a$  should be correspondingly exchanged in Eq. (8.74).

<sup>4</sup> We call the starting state in  $d = 2(2n + 1)$   $|\psi_1^1\rangle > |_{2(2n+1)}$ , and the starting state in  $d = 4n$   $|\psi_1^1\rangle > |_{4n}$ .

<sup>5</sup> The smallest number of all the generators  $S^{ac}$ , which do not belong to the Cartan subalgebra, needed to create from the starting state all the other members is  $2^{\frac{d}{2}-1} - 1$ . This is true for both even dimensional spaces  $-2(2n + 1)$  and  $4n$ .



$\tilde{S}^{ab}$ , Eq. (8.67), generate states with different eigenstates of the Cartan subalgebra ( $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1 d}$ ), we call the eigenvalues of their eigenstates the "family" quantum numbers. There are  $2^{\frac{d}{2}-1}$  families. From the starting new member with a different "family" quantum number the whole Lorentz representation with this "family" quantum number follows by the application of  $S^{ef}$ :  $S^{ef} \tilde{S}^{cd} |\psi_1^1\rangle > |_{2(2n+1)} = |\psi_i^1\rangle > |_{2(2n+1)}$ . All the states of one Lorentz representation of any particular "family" quantum number have an odd Clifford character, since neither  $S^{cd}$  nor  $\tilde{S}^{cd}$ , both with an even Clifford character, can change this character. We shall comment our limitation of states to only those with an odd Clifford character after defining the creation and annihilation operators.

For  $d = 4n$  the starting state must be the product of one projector and  $4n - 1$  nilpotents, since we again limit states to those with an odd Clifford character. Let us start with the state

$$|\psi_1^1\rangle > |_{4n} = \overset{03}{(+i)} \overset{12}{(+)} \overset{35}{(+)} \cdots \overset{d-3}{(+)} \overset{d-2d-1}{(+)} \overset{d}{(+)} |\psi_{oc}\rangle, \quad (8.50)$$

All the other states belonging to the same Lorentz representation follow again by the application of  $S^{cd}$  on this state  $|\psi_1^1\rangle > |_{4n}$ , while a new family starts by the application of  $\tilde{S}^{cd} |\psi_1^1\rangle > |_{4n}$  and from this state all the other members with the same "family" quantum number can be generated by  $S^{ef} \tilde{S}^{cd}$  on  $|\psi_1^1\rangle > |_{4n}$ :  $S^{ef} \tilde{S}^{cd} |\psi_1^1\rangle > |_{4n} = |\psi_i^1\rangle > |_{4n}$ .

All these states in either  $d = 2(2n + 1)$  space or  $d = 4n$  space are orthogonal with respect to Eq. (8.23).

However, let us point out that  $(\gamma^a)^\dagger = \gamma^a \eta^{aa}$ . Correspondingly it follows, Eq. (8.47), that  $\overset{ab}{(k)} = \eta^{aa} \overset{ab}{(-k)}$ , and  $\overset{ab}{[k]} = \overset{ab}{[k]}$ .

Since any projector is Hermitian conjugate to itself, while to any nilpotent  $\overset{ab}{(k)}$  the Hermitian conjugated one has an opposite  $k$ , it is obvious that Hermitian conjugated product to a product of nilpotents and projectors can not be accepted as a new state<sup>6</sup>.

The vacuum state  $|\psi_{oc}\rangle$  ought to be chosen so that  $\langle \psi_{oc} | \psi_{oc} \rangle = 1$ , while all the states belonging to the physically acceptable states, like  $\overset{03}{(+i)} \overset{12}{(+)} \overset{56}{(+)} \overset{78}{(-)} \cdots \overset{d-3}{(+)} \overset{d-2d-1}{(+)} |\psi_{oc}\rangle$ , must not give zero for either  $d = 2(2n + 1)$  or for  $d = 4n$ . We also want that the states, obtained by the application of either  $S^{cd}$  or  $\tilde{S}^{cd}$  or both, are orthogonal. To make a choice of the vacuum it is needed to know the

<sup>6</sup> We could as well start with the state  $|\psi_1^1\rangle > |_{2(2n+1)} = \overset{03}{(+i)} \overset{12}{(+)} \overset{35}{(+)} \cdots \overset{d-3}{(+)} \overset{d-2d-1}{(+)} \overset{d}{(+)} |\psi_{oc}\rangle$  for  $d = 2(2n + 1)$  and with  $|\psi_1^1\rangle > |_{4n} = \overset{03}{(+i)} \overset{12}{(+)} \overset{35}{(+)} \cdots \overset{d-3}{(+)} \overset{d-2d-1}{(+)} |\psi_{oc}\rangle$  in the case of  $d = 4n$ . Then creation and annihilation operators will exchange their roles.

relations of Eq. (8.71). It must be

$$\begin{aligned}
 < \psi_{oc} | \cdots (k)^{ab\dagger} \cdots | \cdots (k')^{ab} \cdots | \psi_{oc} > = \delta_{kk'}, \\
 < \psi_{oc} | \cdots [k]^{ab\dagger} \cdots | \cdots [k']^{ab} \cdots | \psi_{oc} > = \delta_{kk'}, \\
 < \psi_{oc} | \cdots [k]^{ab\dagger} \cdots | \cdots (k')^{ab} \cdots | \psi_{oc} > = 0.
 \end{aligned} \tag{8.51}$$

Our experiences in the case, when states with the integer values of the Cartan subalgebra operators were expressed by Grassmann coordinates, teach us that the requirements, which creation and annihilation operators must fulfill, influence the choice of the number of states, as well as of the vacuum state.

Let us first repeat therefore the requirements which the creation and annihilation operators must fulfill

$$\begin{aligned}
 \{\hat{b}_i^{\alpha\gamma}, \hat{b}_k^{\beta\gamma\dagger}\}_+ | \psi_{oc} > &= \delta_\beta^\alpha \delta_k^i | \psi_{oc} >, \\
 \{\hat{b}_i^{\alpha\gamma}, \hat{b}_k^{\beta\gamma}\}_+ | \psi_{oc} > &= 0 | \psi_{oc} >, \\
 \{\hat{b}_i^{\alpha\gamma\dagger}, \hat{b}_k^{\beta\gamma\dagger}\}_+ | \psi_{oc} > &= 0 | \psi_{oc} >, \\
 \hat{b}_i^{\alpha\gamma\dagger} | \psi_{oc} > &= 0 | \psi_{oc} >,
 \end{aligned} \tag{8.52}$$

paying attention at this stage only at the internal degrees of freedom of the states, that is on their spins. Here  $(\alpha, \beta, \dots)$  represent the family quantum number determined by  $\tilde{S}^{ac}$  and  $(i, j, \dots)$  the quantum number of one representation, determined by  $S^{ac}$ . From Eqs. (8.49, 8.50) is not difficult to extract the creation operators which, when applied on the two vacuum states, generate the starting states.

#### i. One Weyl representation

We define the creation  $\hat{b}_1^{1\dagger}$  – and the corresponding annihilation operator  $\hat{b}_1^1$ ,  $(\hat{b}_1^{1\dagger})^\dagger = \hat{b}_1^1$  – which when applied on the vacuum state  $| \psi_{oc} >$  create a vector of one of the two equations (8.49, 8.50), as follows

$$\begin{aligned}
 \hat{b}_1^{1\dagger} &:= \begin{matrix} 03 & 12 & 56 & \cdots & d-1 & d \\ (+i) & (+) & (+) & \cdots & (+) & \end{matrix}, \\
 \hat{b}_1^1 &:= \begin{matrix} d-1 & d & \cdots & 56 & 12 & 03 \\ (-) & & & (-) & (-) & (-i) \end{matrix}, \\
 &\text{for } d = 2(2n + 1), \\
 \hat{b}_1^{1\dagger} &:= \begin{matrix} 03 & 12 & 56 & \cdots & d-3 & d-2 & d-1 & d \\ (+i) & (+) & (+) & \cdots & (+) & & & (+) \end{matrix}, \\
 \hat{b}_1^1 &:= \begin{matrix} d-1, d & d-2 & d-3 & \cdots & 56 & 12 & 03 \\ (+) & & (-) & & (-) & (-) & (-i) \end{matrix}, \\
 &\text{for } d = 4n.
 \end{aligned} \tag{8.53}$$

We shall call this vector the starting vector of the starting "family".

Now we can make a choice of the vacuum state for this particular "family" taking into account Eq. (8.71)

$$\begin{aligned}
 |\psi_{oc} > &= [-i]^{03} [-]^{12} [-]^{56} \cdots [-]^{d-1} [-]^{d} |0 >, \\
 &\text{for } d = 2(2n + 1), \\
 |\psi_{oc} > &= [-i]^{03} [-]^{12} [-]^{56} \cdots [-]^{d-3} [-]^{d-2} [-]^{d-1} [+]^{d} |0 >, \\
 &\text{for } d = 4n,
 \end{aligned} \tag{8.54}$$

$n$  is a positive integer, so that the requirements of Eq. (8.52) are fulfilled. We see: The creation and annihilation operators of Eq. (8.53) (both are nilpotents,  $(\hat{b}_1^{1\dagger})^2 = 0$  and  $(\hat{b}_1^1)^2 = 0$ ),  $\hat{b}_1^{1\dagger}$  (generating the vector  $|\psi_1^1 >$  when operating on the vacuum state) gives  $\hat{b}_1^{1\dagger}|\psi_{oc} > \neq 0$ , while the annihilation operator annihilates the vacuum state  $\hat{b}_1^1|\psi_0 > = 0$ , giving  $\{\hat{b}_1^1, \hat{b}_1^{1\dagger}\}_+|\psi_{oc} > = |\psi_{oc} >$ , since we choose the appropriate normalization, Eq. (8.46).

All the other creation and annihilation operators, belonging to the same Lorentz representation with the same family quantum number, follow from the starting ones by the application of particular  $S^{ac}$ , which do not belong to the Cartan subalgebra (8.65).

We call  $\hat{b}_2^{1\dagger}$  the one obtained from  $\hat{b}_1^{1\dagger}$  by the application of one of the four generators ( $S^{01}, S^{02}, S^{31}, S^{32}$ ). This creation operator is for  $d = 2(2n + 1)$  equal to  $\hat{b}_2^{1\dagger} = [-i]^{03} [-]^{12} [+]^{35} \cdots [+]^{d-1} [+]^{d}$ , while it is for  $d = 4n$  equal to  $\hat{b}_2^{1\dagger} = [-i]^{03} [-]^{12} [+]^{56} \cdots [+]^{d-1} [+]^{d}$ . All the other family members follow from the starting one by the application of different  $S^{ef}$ , or by the product of several  $S^{gh}$ .

We accordingly have

$$\begin{aligned}
 \hat{b}_i^{1\dagger} &\propto S^{ab} \cdots S^{ef} \hat{b}_1^{1\dagger}, \\
 \hat{b}_i^1 &\propto \hat{b}_1^1 S^{ef} \cdots S^{ab},
 \end{aligned} \tag{8.55}$$

with  $S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}$ . We shall make a choice of the proportionality factors so that the corresponding states  $|\psi_1^1 > = \hat{b}_1^{1\dagger}|\psi_{oc} >$  will be normalized.

We recognize that [15]:

**i.a.**  $(\hat{b}_i^{1\dagger})^2 = 0$  and  $(\hat{b}_i^1)^2 = 0$ , for all  $i$ .

To see this one must recognize that  $S^{ac}$  (or  $S^{bc}, S^{ad}, S^{bd}$ ) transforms  $(+)(+)$  to  $(-)(-)$ , that is an even number of nilpotents  $(+)$  in the starting state is transformed into projectors  $(-)$  in the case of  $d = 2(2n + 1)$ . For  $d = 4n$ ,  $S^{ac}$  (or  $S^{bc}, S^{ad}, S^{bd}$ ) transforms  $(+)(+)$  into  $(-)(-)$ . Therefore for either  $d = 2(2n + 1)$  or  $d = 4n$  at least one of factors, defining a particular creation operator, will be a nilpotent. For  $d = 2(2n + 1)$  there is an odd number of nilpotents, at least one, leading from the starting factor  $((+))$  in the creator. For  $d = 4n$  a nilpotent factor can also be  $(-)$  (since  $(+)$  can be transformed by  $S^{e d-1}$ , for example into  $(-)$ ). A square of at least one nilpotent factor (we started with an odd number of nilpotents, and oddness can not be changed by  $S^{ab}$ ), is enough to guarantee that the square of

the corresponding  $(\hat{b}_i^{1\dagger})^2$  is zero. Since  $\hat{b}_i^1 = (\hat{b}_i^{1\dagger})^\dagger$ , the proof is valid also for annihilation operators.

**i.b.**  $\hat{b}_i^{1\dagger}|\psi_{oc} \rangle \neq 0$  and  $\hat{b}_i^1|\psi_{oc} \rangle = 0$ , for all  $i$ .

To see this in the case  $d = 2(2n+1)$  one must recognize that  $\hat{b}_i^{1\dagger}$  distinguishes from  $\hat{b}_i^{1\dagger}$  in (an even number of) those nilpotents  $(+)$ , which have been transformed into  $[-]$ . When  $[-]$  from  $\hat{b}_i^{1\dagger}$  meets  $[-]$  from  $|\psi_{oc} \rangle$ , the product gives  $[-]$  back, and correspondingly a nonzero contribution. For  $d = 4n$  also the factor  $[-]$  can be transformed. It is transformed into  $[-]$  which, when applied to a vacuum state, gives again a nonzero contribution ( $[-]^{d-1} [d] = [-]$ , Eq.(8.71)).

In the case of  $\hat{b}_i^1$  we recognize that in  $\hat{b}_i^{1\dagger}$  at least one factor is nilpotent; that of the same type as in the starting  $\hat{b}_i^{1\dagger} - (+)$  – or in the case of  $d = 4n$  it can be also  $[-]$ . Performing the Hermitian conjugation  $(\hat{b}_i^{1\dagger})^\dagger$ ,  $(+)$  transforms into  $(-)$ , while  $[-]$  transforms into  $(+)$  in  $\hat{b}_i^1$ . Since  $(-)[-]$  gives zero and  $(+)[+]$  also gives zero,  $\hat{b}_i^1|\psi_{oc} \rangle = 0$ .

**i.c.**  $\{\hat{b}_i^{1\dagger}, \hat{b}_j^{1\dagger}\}_+ = 0$ , for each pair  $(i, j)$ .

There are several possibilities, which we have to discuss. A trivial one is, if both  $\hat{b}_i^{1\dagger}$  and  $\hat{b}_j^{1\dagger}$  have a nilpotent factor (or more than one) for the same pair of indexes, say  $(kl)$ . Then the product of such two  $(+)$  gives zero. It also happens, that  $\hat{b}_i^{1\dagger}$  has a nilpotent at the place  $(kl)$   $([-] \dots (+) \dots [-] \dots)$  while  $\hat{b}_j^{1\dagger}$  has a nilpotent at the place  $(mn)$   $([-] \dots [-] \dots (+) \dots)$ . Then in the term  $\hat{b}_i^{1\dagger} \hat{b}_j^{1\dagger}$  the product  $[-](+)$  makes the term equal to zero, while in the term  $\hat{b}_j^{1\dagger} \hat{b}_i^{1\dagger}$  the product  $[-](+)$  makes the term equal to zero. There is no other possibility in  $d = 2(2n+1)$ . In the case that  $d = 4n$ , it might appear also that  $\hat{b}_i^{1\dagger} = [-] \dots (+) \dots [-]$  and  $\hat{b}_j^{1\dagger} = [-] \dots [-] \dots (-)$ . Then in the term  $\hat{b}_i^{1\dagger} \hat{b}_j^{1\dagger}$  the factor  $[-](-)$  makes it zero, while in  $\hat{b}_j^{1\dagger} \hat{b}_i^{1\dagger}$  the factor  $[-](+)$  makes it zero. Since there are no further possibilities, the proof is complete.

**i.d.**  $\{\hat{b}_i^1, \hat{b}_j^1\}_+ = 0$ , for each pair  $(i, j)$ .

The proof goes similarly as in the case with creation operators. Again we treat several possibilities.  $\hat{b}_i^1$  and  $\hat{b}_j^1$  have a nilpotent factor (or more than one) with the same indexes, say  $(kl)$ . Then the product of such two  $(-)(-)$  gives zero. It also happens, that  $\hat{b}_i^1$  has a nilpotent at the place  $(kl)$   $(\dots [-] \dots (-) \dots [-])$  while  $\hat{b}_j^1$  has a nilpotent at the place  $(mn)$   $(\dots (-) \dots [-] \dots [-])$ . Then in the term  $\hat{b}_i^1 \hat{b}_j^1$  the product  $(-)(-)$  makes the term equal to zero, while in the term  $\hat{b}_j^1 \hat{b}_i^1$  the product  $(-)(-)$  makes the term equal to zero. In the case that  $d = 4n$ , it appears also that

$\hat{b}_i^1 = \overset{d-1}{[+]} \overset{d}{\dots} \overset{ij}{(-)} \dots \overset{03}{[-]}$  and  $\hat{b}_j^1 = \overset{d-1}{(+)} \overset{d}{\dots} \overset{ij}{[-]} \dots \overset{03}{[-]}$ . Then in the term  $\hat{b}_i^1 \hat{b}_j^1$  the factor  $(-)[-]$  makes it zero, while in  $\hat{b}_j^1 \hat{b}_i^1$  the factor  $(+)[+]$  makes it zero.

i.e.  $\{\hat{b}_i^1, \hat{b}_j^{1\dagger}\}_{+|\psi_{oc}} = \delta_{ij}|\psi_{oc} > .$

To prove this we must recognize that  $\hat{b}_i^1 = \hat{b}_1 S^{ef} .. S^{ab}$  and  $\hat{b}_i^{1\dagger} = S^{ab} .. S^{ef} \hat{b}_1$ . Since any  $\hat{b}_i^1 |\psi_{oc} > = 0$ , we only have to treat the term  $\hat{b}_i^1 \hat{b}_j^{1\dagger}$ . We find  $\hat{b}_i^1 \hat{b}_j^{1\dagger} \propto \dots \overset{lm}{(-)} \dots \overset{03}{(-)} S^{ef} \dots S^{ab} S^{lm} \dots S^{pr} \overset{03}{(+)} \dots \overset{lm}{(+)} \dots$ . If we treat the term  $\hat{b}_i^1 \hat{b}_i^{1\dagger}$ , generators  $S^{ef} \dots S^{ab} S^{lm} \dots S^{pr}$  are proportional to a number and we normalize  $< \psi_0 | \hat{b}_i^1 \hat{b}_i^{1\dagger} | \psi_{oc} >$  to one. When  $S^{ef} \dots S^{ab} S^{lm} \dots S^{pr}$  are proportional to several products of  $S^{cd}$ , these generators change  $\hat{b}_i^{1\dagger}$  into  $\overset{03}{(+)} \dots \overset{kl}{[-]} \dots \overset{np}{[-]} \dots$ , making the product  $\hat{b}_i^1 \hat{b}_j^{1\dagger}$  equal to zero, due to factors of the type  $(-)[-]$ . In the case of  $d = 4n$  also a factor  $\overset{d-1}{[+]} \overset{dd-1}{(-)}$  might occur, which also gives zero.

We saw and proved that for the definition of the creation and annihilation operators in Eqs.(8.49,8.50) all the requirements of Eq. (8.52) are fulfilled, provided that creation and correspondingly also the annihilation operators have an odd Clifford character, that is that the number of nilpotents in the product is odd.

For an even number of factors of the nilpotent type in the starting state and accordingly in the starting  $\hat{b}_i^{1\dagger}$ , an annihilation operator  $\hat{b}_i^1$  would appear with all factors of the type  $[-]$ , which on the vacuum state (Eq.(8.54)) would not give zero.

## ii. Families of Weyl representations

Let  $\hat{b}_i^{\alpha\dagger}$  be a creation operator, fulfilling Eq. (8.52), which creates one of the  $(2^{d/2-1})$  Weyl basic states of an  $\alpha$ -th "family", when operating on a vacuum state  $|\psi_{oc} >$  and let  $\hat{b}_i^\alpha = (\hat{b}_i^{\alpha\dagger})^\dagger$  be the corresponding annihilation operator. We shall now proceed to define  $\hat{b}_i^{\alpha\dagger}$  and  $\hat{b}_i^\alpha$  from a chosen starting state (8.49, 8.50), which  $\hat{b}_1^{1\dagger}$  creates on the vacuum state  $|\psi_{oc} >$ .

When treating more than one Weyl representation, that is, more than one "family", we must take into account that: **i.** The vacuum state chosen to fulfill requirements for second quantization of the starting family might not and it will not be the correct one when all the families are taken into account. **ii.** The products of  $\tilde{S}^{ab}$ , which do not belong to the Cartan subalgebra set of the generators  $\tilde{S}^{ab}$  ( $2^{d/2-1} - 1$  of them), when being applied on the starting family  $\psi_1^1$ , generate the starting members  $\psi_1^\alpha$  of all the rest of the families. There are correspondingly the same number of "families" as there is the number of vectors of one Weyl representation, namely  $2^{d/2-1}$ . Then the whole Weyl representations of a particular family  $\psi_1^\alpha$  follows again with the application of  $S^{ef}$ , which do not belong to the Cartan subalgebra of  $S^{ab}$  on this starting family.

Any vector  $|\psi_i^\alpha >$  follows from the starting vector (Eqs.8.49, 8.50) by the application of either  $\tilde{S}^{ef}$ , which change the family quantum number, or  $S^{gh}$ , which change the member of a particular family (as it can be seen from Eqs. (8.73, 8.86)) or with the corresponding product of  $S^{ef}$  and  $\tilde{S}^{ef}$

$$|\psi_i^\alpha > \propto \tilde{S}^{ab} \dots \tilde{S}^{ef} |\psi_1^1 > \propto \tilde{S}^{ab} \dots \tilde{S}^{ef} S^{mn} \dots S^{pr} |\psi_1^1 > . \quad (8.56)$$

Correspondingly we define  $\hat{b}_i^{\alpha\dagger}$  (up to a constant) to be

$$\begin{aligned}\hat{b}_i^{\alpha\dagger} &\propto \tilde{S}^{ab} \dots \tilde{S}^{ef} S^{mn} \dots S^{pr} \hat{b}_1^{1\dagger} \\ &\propto S^{mn} \dots S^{pr} \hat{b}_1^{1\dagger} S^{ab} \dots S^{ef}.\end{aligned}\quad (8.57)$$

This last expression follows due to the property of the Clifford object  $\tilde{\gamma}^a$  and correspondingly of  $\tilde{S}^{ab}$ , presented in Eqs. (8.74, 8.75).

For  $\hat{b}_i^\alpha = (\hat{b}_i^{\alpha\dagger})^\dagger$  we accordingly have

$$\hat{b}_i^\alpha = (\hat{b}_i^{\alpha\dagger})^\dagger \propto S^{ef} \dots S^{ab} \hat{b}_1^1 S^{pr} \dots S^{mn}.\quad (8.58)$$

The proportionality factor will be chosen so that the corresponding states  $|\psi_i^\alpha\rangle = \hat{b}_i^{\alpha\dagger} |\psi_{oc}\rangle$  will be normalized.

We ought to generalize the vacuum state from Eq. (8.54) so that  $\hat{b}_i^{\alpha\dagger} |\psi_{oc}\rangle \neq 0$  and  $\hat{b}_i^\alpha |\psi_{oc}\rangle = 0$  for all the members  $i$  of any family  $\alpha$ . Since any  $\tilde{S}^{eg}$  changes  $(+)^{ef}$  into  $(+)^{gh}$  and  $(+)^{ef}$  into  $(+)^{gh}$  and  $(+)^{ab} \rightarrow (-)^{ab}$ , while  $(+)^{ab} \rightarrow (-)^{ab}$ , the vacuum state  $|\psi_{oc}\rangle$  from Eq. (8.54) must be replaced by

$$\begin{aligned}|\psi_{oc}\rangle &= \\ &[-i]^{03} [-]^{12} [-]^{56} \dots [-]^{d-1} + [+i]^{03} [+ ]^{12} [-]^{56} \dots [-]^{d-1} + [+i]^{03} [-]^{12} [+ ]^{56} \dots [-]^{d-1} + \dots |0\rangle, \\ &\text{for } d = 2(2n+1), \\ |\psi_{oc}\rangle &= \\ &[-i]^{03} [-]^{12} [-]^{35} \dots [-]^{d-3} [-]^{d-2d-1} + [+i]^{03} [+ ]^{12} [-]^{56} \dots [-]^{d-3} [-]^{d-2} [+ ]^{d-1} + \dots |0\rangle, \\ &\text{for } d = 4n,\end{aligned}\quad (8.59)$$

$n$  is a positive integer. There are  $2^{\frac{d}{2}-1}$  summands. since we step by step replace all possible pairs of  $(-)^{ab} \dots (-)^{ef}$  in the starting part  $[-i]^{03} [-]^{12} [-]^{35} \dots [-]^{d-1}$  (or  $[-i]^{03} [-]^{12} [-]^{35} \dots [-]^{d-3} [-]^{d-2d-1}$ ) into  $(+)^{ab} \dots (+)^{ef}$  and include new terms into the vacuum state so that the last  $2n+1$  summands have for  $d = 2(2n+1)$  case,  $n$  is a positive integer, only one factor  $(-)$  and all the rest  $(+)$ , each  $(-)$  at different position. For  $d = 4n$  also the factor  $(+)^{d-1}$  in the starting term  $[-i]^{03} [-]^{12} [-]^{35} \dots [-]^{d-3} [+ ]^{d-2d-1}$  changes to  $(-)^{d-1}$ . The vacuum state has then the normalization factor  $1/\sqrt{2^{d/2-1}}$ .

There is therefore

$$2^{\frac{d}{2}-1} 2^{\frac{d}{2}-1} \quad (8.60)$$

number of creation operators, defining the orthonormalized states when applying on the vacuum state of Eqs. (8.59) and the same number of annihilation operators, which are defined by the creation operators on the vacuum state of Eqs. (8.59).  $\tilde{S}^{ab}$  connect members of different families,  $S^{ab}$  generates all the members of one family.

We recognize that:

**ii.a.** The above creation and annihilation operators are nilpotent –  $(\hat{b}_i^{\alpha\dagger})^2 = 0 =$

$(\hat{b}_i^a)^2$  – since the “starting” creation operator  $\hat{b}_1^{1\dagger}$  and annihilation operator  $\hat{b}_i^a$  are both made of the product of an odd number of nilpotents, while products of either  $S^{ab}$  or  $\tilde{S}^{ab}$  can change an even number of nilpotents into projectors. Any  $\hat{b}_i^{a\dagger}$  is correspondingly a factor of an odd number of nilpotents (at least one) (and an even number of projectors) and its square is zero. The same is true for  $\hat{b}_i^a$ .

**ii.b.** All the creation operators operating on the vacuum state of Eq.(8.59) give a non zero vector –  $\hat{b}_i^{a\dagger}|\psi_{oc} \rangle \neq 0$  – while all the annihilation operators annihilate this vacuum state –  $\hat{b}_i^a|\psi_0 \rangle$  for any  $\alpha$  and any  $i$ .

It is not difficult to see that  $\hat{b}_i^a|\psi_{oc} \rangle = 0$ , for any  $\alpha$  and any  $i$ . First we recognize that whatever the set of factors  $S^{mn} \dots S^{pr}$  appear on the right hand side of the annihilation operator  $\hat{b}_1^1$  in Eq.(8.58), it lives at least one factor  $[-]$  unchanged.

Since  $\hat{b}_1^1$  is the product of only nilpotents  $(-)$  and since  $(-)^{ab}[-] = 0$ , this part of the proof is complete.

Let us prove now that  $\hat{b}_i^{a\dagger}|\psi_{oc} \rangle \neq 0$  for each  $\alpha, i$ . According to Eq.(8.57) the operation  $S^{mn}$  on the left hand side of  $\hat{b}_1^{1\dagger}$ , with  $m, n$ , which does not belong to

the Cartan subalgebra set of indices, transforms the term  $\overset{03}{[-i]} \overset{12}{[-]} \dots \overset{lm}{[-]} \dots \overset{nk}{[-]}$   $\dots \overset{d-1}{[-]} \overset{d}{[-]}$  (or the term  $\overset{03}{[-i]} \overset{12}{[-]} \dots \overset{lm}{[-]} \dots \overset{nk}{[-]} \dots \dots \overset{d-1}{[+]} \overset{d}{[+]}$ ) into the term  $\overset{03}{[-i]} \overset{12}{[-]} \dots \overset{lm}{[+]} \dots \overset{nk}{[+]} \dots \dots \overset{d-1}{[+]} \overset{d}{[+]}$   $\dots \overset{lm}{(+)} \dots \overset{nk}{(+)} \dots \overset{d-1}{[-]} \dots \overset{d}{[+]}$  (or the term  $\overset{03}{[-i]} \overset{12}{[-]} \dots \overset{lm}{(+)} \dots \overset{nk}{(+)} \dots \dots \overset{d-1}{[+]} \overset{d}{[+]}$ ) and  $\hat{b}_1^{1\dagger}$

on such a term gives zero, since  $(+)(+)=0$  and  $(+)(+)=0$ . Let us first assume that  $S^{mn}$  is the only term on the right hand side of  $\hat{b}_1^{1\dagger}$  and that none of the operators from the left hand side of  $\hat{b}_1^{1\dagger}$  in Eq.(8.57) has the indices  $m, n$ . It is only one term among all the summands in the vacuum state (Eq.8.59), which gives non zero contribution in this particular case, namely the term  $\overset{03}{[-i]} \overset{12}{[-]} \dots \overset{lm}{[+]} \dots \overset{nk}{[+]} \dots \overset{d-1}{[+]} \overset{d}{[+]}$  (or the term  $\overset{03}{[-i]} \overset{12}{[-]} \dots \overset{lm}{[+]} \dots \overset{nk}{[+]} \dots \dots \overset{d-1}{[+]} \overset{d}{[+]}$ ).  $S^{mn}$  transforms the part  $\dots \overset{lm}{[+]} \dots \overset{nk}{[+]} \dots$  into  $\dots (-) \dots (-) \dots$  and since  $(+)(-)$  gives  $\eta^{11} [ + ]$ , while for the rest of factors it was already proven that such a factor on  $\hat{b}_1^{1\dagger}$  forms a  $b_i^{1\dagger}$  giving non zero contribution on the vacuum (8.54).

We also proved that what ever other  $S^{ab}$  but  $S^{mn}$  operate on the left hand side of  $\hat{b}_1^{1\dagger}$  the contribution of this particular part of the vacuum state is nonzero. If the operators on the left hand side have the indexes  $m$  or  $n$  or both, the contribution on this term of the vacuum will still be nonzero, since then such a  $S^{mp}$  will transform the factor  $\overset{lm}{(+)}$  in  $\hat{b}_1^{1\dagger}$  into  $\overset{lm}{[-]}$  and  $\overset{lm}{[-]}(-)$  is nonzero, Eq. (8.71).

The vacuum state has a term which guarantees a non zero contribution for any possible set of  $S^{mn} \dots S^{pr}$  operating from the right hand side of  $\hat{b}_1^{1\dagger}$  (that is for each family) (which we achieved just by the transformation of all possible pairs of  $\overset{cd}{[-]}, \overset{gh}{[-]}$  into  $\overset{cd}{[+]}, \overset{gh}{[+]}$ ), the proof that  $\hat{b}_i^{a\dagger}$  operating on the vacuum  $|\psi_{oc} \rangle$  of

Eq. (8.59) gives nonzero contribution. Among  $[-]$  also  $[-i]$  is understood.

It is not difficult to see that for each “family” of  $2^{\frac{d}{2}-1}$  families it is only one term among all the summands in the vacuum state  $|\psi_{oc} \rangle$  of Eq. (8.59), which give a nonzero contribution, since when ever  $[+]$  appears on a wrong position, that

is on the position, so that the product of  $\overset{ab}{+}$  from  $\hat{b}_1^\dagger$  and  $\overset{ab}{+}$  from the vacuum summand appears, the contribution is zero.

**ii.b.** Any two creation operators anti commute —  $\{\hat{b}_i^{\alpha\dagger}, \hat{b}_j^{\beta\dagger}\}_+ = 0$ .

According to Eq.8.57 we can rewrite  $\{\hat{b}_i^{\alpha\dagger}, \hat{b}_j^{\beta\dagger}\}_{++}$  up to a factor, as

$$\{S^{mn} \dots S^{pr} \hat{b}_1^\dagger S^{ab} \dots S^{ef}, S^{m'n'} \dots S^{p'r'} \hat{b}_1^\dagger S^{a'b'} \dots S^{e'f'}\}_+.$$

Whatever the product  $S^{ab} \dots S^{ef} S^{m'n'} \dots S^{p'r'}$  (or  $S^{a'b'} \dots S^{e'f'} S^{mn} \dots S^{pr}$ ) is, it always transforms an even number of  $\overset{ab}{+}$  in  $\hat{b}_1^\dagger$  into  $[-]$ . Since an odd number of nilpotents  $\overset{ab}{+}$  (at least one) stays unchanged in this right  $\hat{b}_1^\dagger$ , after the application of all the  $S^{ab}$  in the product in front of it or  $\overset{d-1}{[+]} \overset{d}{+}$  transforms into  $\overset{d-1}{(-)} \overset{d}{-}$ , and since the left  $\hat{b}_1^\dagger$  is a product of only nilpotents  $\overset{ab}{+}$  or an odd number of nilpotents and  $[-]$  for  $d = 2(2n+1)$  and  $d = 4n$ ,  $n$  is an integer, respectively, while  $\overset{d-1}{[+]} \overset{d}{-} = 0$ , the anticommutator for any two creation operators is zero.

**ii.c.** Any two annihilation operators anticommute —  $\{\hat{b}_i^\alpha, \hat{b}_j^\beta\}_+ = 0$ .

According to Eq.8.58 we can rewrite  $\{\hat{b}_i^\alpha, \hat{b}_j^\beta\}_{++}$  up to a factor, as

$$\{S^{ab} \dots S^{ef} \hat{b}_1 S^{mn} \dots S^{pr}, S^{a'b'} \dots S^{e'f'} \hat{b}_1 S^{m'n'} \dots S^{p'r'}\}_+.$$

Whatever the product  $S^{mn} \dots S^{pr} S^{a'b'} \dots S^{e'f'}$  (or  $S^{m'n'} \dots S^{p'r'} S^{ab} \dots S^{ef}$ ) is, it always transforms an even number of  $\overset{ab}{-}$  in  $\hat{b}_1$  into  $[+]$ . Since an odd number of nilpotents  $\overset{ab}{-}$  (at least one) stays unchanged in this  $\hat{b}_1$ , after the application of all the  $S^{ab}$  in the product in front of it or  $\overset{d-1}{[+]} \overset{d}{+}$  transforms into  $\overset{d-1}{(-)} \overset{d}{-}$ , and since  $\hat{b}_1$  in the left hand side is a product of only nilpotents  $\overset{ab}{-}$  or an odd number of nilpotents and  $[+]$  for  $d = 2(2n+1)$  and  $d = 4n$ ,  $n$  is an integer, respectively, while  $\overset{ab}{(-)} \overset{ab}{(-)} = 0$  and  $\overset{ab}{[+]} \overset{ab}{[-]} = 0$ , the anticommutator of any two annihilation operators is zero.

**ii.d.** For any creation and any annihilation operators it follows:  $\{\hat{b}_i^\alpha, \hat{b}_j^{\beta\dagger}\}_+ |\psi_- \rangle = \delta^{ab} \delta_{ij} |\psi_0 \rangle$ .

Let us prove this. According to Eqs. (8.57,8.58) we may rewrite  $\{\hat{b}_i^\alpha, \hat{b}_j^{\beta\dagger}\}_+$  up to a factor as  $\{S^{ab} \dots S^{ef} \hat{b}_1 S^{mn} \dots S^{pr}, S^{m'n'} \dots S^{p'r'} \hat{b}_1^\dagger S^{a'b'} \dots S^{e'f'}\}_+$ . We distinguish between two cases. It can be that both  $S^{mn} \dots S^{pr} S^{m'n'} \dots S^{p'r'}$  and  $S^{a'b'} \dots S^{e'f'} S^{ab} \dots S^{ef}$  are numbers. This happens when  $\alpha = \beta$  and  $i = j$ . Then we follow **i.b.** We normalize the states so that  $\langle \psi_i^\alpha | \psi_i^\alpha \rangle = 1$ .

The second case is that at least one of

$$S^{mn} \dots S^{pr} S^{m'n'} \dots S^{p'r'} \text{ and } S^{a'b'} \dots S^{e'f'} S^{ab} \dots S^{ef}$$

is not a number. Then the factors like  $\overset{ab}{(-)} \overset{ab}{[-]}$  or  $\overset{ab}{[+]} \overset{ab}{(-)}$  or  $\overset{ab}{(+)} \overset{ab}{[+]}$  make the anticommutator equal zero. And the proof is completed.

**iii. We learned:**

**iii.a.** From  $2^d$  internal states expressed with Grassmann coordinates, which are all orthogonal with respect to the scalar product, Eq.(8.24), not all of  $2^d$  fulfill requirements that the states should be written as product of Grassmann coordinates



on the vacuum state. We payed particular attention on  $2^{\frac{d}{2}-1} (2^{\frac{d}{2}-1} + 1)$ , states, Eqs. (8.41, 8.42). To these creation operators the same number,  $(2^{\frac{d}{2}-1} (2^{\frac{d}{2}-1} + 1))$ , of the corresponding annihilation operators belong, fulfilling the relation for the creation and annihilation operators (8.40), for which we expect that the creation and annihilation operators have to. These states form two (separate) groups of the Lorentz representation: The members of each group are reachable by  $S^{ab}$  (which do not belong to the Cartan subalgebra (8.65)) from one of the state of each group, each with  $(2^{\frac{d}{2}-1} (2^{\frac{d}{2}-1} + 1))/2$  members. The second quantized states have in  $d = 4n$  an even Grassmann caharacter, while in  $d = 2(2n + 1)$  they have an odd Grassmann character. There are in addition creation operators of opposite Grassmann character then these  $2^{\frac{d}{2}-1} (2^{\frac{d}{2}-1} + 1)$  states either in  $d = 4n$  or in  $d = 2(2n + 1)$ . They are products of two, four or at most product of  $d$   $\theta^a$ .

**iii.b.** From  $2^d$  internal states expressed with Clifford coordinates, which again are orthogonal with respect to the scalar product, Eq.(8.24), only  $2^{\frac{d}{2}-1} (2^{\frac{d}{2}-1})$  fulfill requirements that the second quantized states are expressed by products of nilpotents and projectors, which apply on the vacuum state. The products of nillpotents and projectors have to have an odd Clifford character in either  $d = 4n$  or  $d = 2(2n + 1)$ . They form creation operators and annihilation operators, full-filling Eq.(8.52), for which we expect that the creation and annihilation operators have to.

The corresponding states form families of states. Each family members are reachable from any one by  $S^{ab}$ , while any family can be reached by  $\tilde{S}^{ab}$ .

**iii.c.** We pay attention on even-dimensional spaces only.

### 8.3 Conclusions

We have started the present study to understand, why "nature made a choice" of the Clifford algebra, rather than the Grassmann algebra, to describe the internal degrees of freedom of fermion fields, although both spaces enable the second quantization of the internal degrees of freedom of the fermion type. We study as well how to fermionize boson fields (or bosonize fermion fields) in any  $d$  (the reader can find the corresponding contribution in this proceedings) to better understand why and how "nature made choices of the theories and models" in the expansion of the universe.

The creation and annihilation operators fulfill anticommutation relations, desired for fermions either in Grassmann space or in Clifford space, although states in Grassmann space carry integer spins, what leads in the *spinn-charge-family* theory (since spins in  $d \geq 5$  manifest as charges in  $d = (1 + 3)$ ) to the charges in the adjoint representations of the charge groups (the subgroups of the Lorentz group  $SO(1, 13)$ ) while states in the Clifford space carry half integer spin and correspondingly are all the charges in the fundamental representations of the groups.

We want to understand as well how does this choice of whether taking Grassmann or Clifford space, manifest in the breaking of the starting symmetry in  $d$ -dimension down to  $d = (1 + 3)$ . The *spin-charge-family* theory namely starts at  $d = (1 + 13)$  with the simple action in which massless fermions carry only two

kinds of spin described by two kinds of the Clifford algebra objects –  $\gamma^a$  and  $\tilde{\gamma}^a$  – and interact with the gravity only – through vielbeins, the gauge fields of the Poincaré algebra and the two kinds of the spin connection fields, the gauge fields of these two kinds of the Clifford algebra objects. The theory offers the explanation for all the assumptions of the *standard model* of elementary fields, fermions and bosons, with the appearance of families including, explaining also the phenomena like the existence of the dark matter, of the matter-antimatter asymmetry, offering correspondingly the next step beyond both standard models – cosmological one and the one of the elementary fields.

To come to the low energy regime the symmetry must break, first from  $SO(13, 1)$  to  $SO(7, 1) \times SU(3) \times U(1)$  and then further to  $SO(3, 1) \times SU(3) \times U(1)$ . Further study is needed to understand whether the “nature could start” at all with Grassmann space while “recognizing”, when breaking symmetry in steps, the “advantage” of the Clifford degrees of freedom with respect to the Grassmann ones: The covariant momentum of the starting action of the *spin-charge-family* theory, Eq. (8.1), would in the case that the Grassmann coordinates describe the internal degrees of freedom of massless objects with the anticommutation relation of the creation and annihilation operators (Eq.(8.40)) read:  $p_{0\alpha} = p_\alpha - \frac{1}{2} \mathbf{S}^{ab} \Omega_{ab\alpha}$ , where  $\Omega_{ab\alpha}$  are the spin connection gauge fields of  $\mathbf{S}^{ab}$  (of the generators of the Lorentz transformations in the Grassmann space) and  $f^\alpha_a p_{0\alpha}$  would replace the ordinary momentum, when massless objects start to interact with the gravitational field, through the vielbeins and the spin connections in Eq. (8.33).

This contribution is a step towards understanding better the open problems of the elementary particle physics and cosmology.

Although we have not yet learned enough to be able to answer the four questions – **a.** Why is the simple starting action of the *spin-charge-family* theory doing so well in manifesting the observed properties of the fermion and boson fields? **b.** Under which condition can more general action lead to the starting action of Eq. (8.1)? **c.** What would more general action, if leading to the same low energy physics, mean for the history of our Universe? **d.** Could the fermionization procedure of boson fields or the bosonization procedure of fermion fields, discussed in this Proceedings for any dimension  $d$  (by the authors of this contribution, while one of them, H.B.F.N. [11], has succeeded with another author to do the fermionization for  $d = (1 + 1)$ ), tell more about the “decisions” of the universe in the history.

## 8.4 APPENDIX: Lorentz algebra and representations in Grassmann and Clifford space

A Lorentz transformation on vector components  $\theta^a$ ,  $\gamma^a$ , or  $\tilde{\gamma}^a$ , which are used to describe internal degrees of freedom of fields with the fermion nature, and on vector components  $x^a$ , which are real (ordinary) commuting coordinates:

$$\theta'^a = \Lambda^a_b \theta^b, \gamma'^a = \Lambda^a_b \gamma^b, \tilde{\gamma}'^a = \Lambda^a_b \tilde{\gamma}^b \quad \text{and} \quad x^a = \Lambda^a_b x^b,$$

leaves forms

$$a_{a_1 a_2 \dots a_i} \theta^{a_1} \theta^{a_2} \dots \theta^{a_i}, a_{a_1 a_2 \dots a_i} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_i}, a_{a_1 a_2 \dots a_i} \tilde{\gamma}^{a_1} \tilde{\gamma}^{a_2} \dots \tilde{\gamma}^{a_i}$$

and

$$b_{a_1 a_2 \dots a_i} x^{a_1} x^{a_2} \dots x^{a_i}, i = (1, \dots, d)$$

invariant.

While  $b_{a_1 a_2 \dots a_i}$  ( $= \eta_{a_1 b_1} \eta_{a_2 b_2} \dots \eta_{a_i b_i} b^{b_1 b_2 \dots b_i}$ ) is a symmetric tensor field,  $a_{a_1 a_2 \dots a_i}$  ( $= \eta_{a_1 b_1} \eta_{a_2 b_2} \dots \eta_{a_i b_i} a^{b_1 b_2 \dots b_i}$ ) are antisymmetric tensor *Kalb-Ramond* fields. The requirements that  $x'^a x'^b \eta_{ab} = x^c x^d \eta_{cd}$ ,  $\theta'^a \theta'^b \varepsilon_{ab} = \theta^c \theta^d \varepsilon_{cd}$ ,  $\gamma'^a \gamma'^b \varepsilon_{ab} = \gamma^c \gamma^d \varepsilon_{cd}$  and  $\tilde{\gamma}'^a \tilde{\gamma}'^b \varepsilon_{ab} = \tilde{\gamma}^c \tilde{\gamma}^d \varepsilon_{cd}$ , where the metric tensor  $\eta^{ab}$  (in our case  $\eta^{ab} = \text{diag}(1, -1, -1, \dots, -1)$ ) lowers the indices of vectors  $\{x^a\}$  ( $= \eta^{ab} x_b$ ),  $\{\theta^a\}$ : ( $\theta^a = \eta^{ab} \theta_b$ ),  $\{\gamma^a\}$ : ( $\gamma^a = \eta^{ab} \gamma_b$ ) and  $\{\tilde{\gamma}^a\}$ : ( $\tilde{\gamma}^a = \eta^{ab} \tilde{\gamma}_b$ ),  $\varepsilon_{ab}$  is the antisymmetric tensor, lead to  $\Lambda^a_b \Lambda^c_d \eta_{ac} = \eta_{bd}$ . An infinitesimal Lorentz transformation for the case with  $\det \Lambda = 1$ ,  $\Lambda^0_0 \geq 0$  can be written as  $\Lambda^a_b = \delta^a_b + \omega^a_b$ , where  $\omega^a_b + \omega_b^a = 0$ .

According to Eqs. (8.14, 8.15, 8.21) one finds

$$\begin{aligned} \{\gamma^a, \tilde{S}^{cd}\}_- &= 0 = \{\tilde{\gamma}^a, S^{cd}\}_-, \\ \{\gamma^a, S^{cd}\}_- &= \{\gamma^a, S^{cd}\}_- = \frac{i}{2} (\eta^{ac} \gamma^d - \eta^{ad} \gamma^c), \\ \{\tilde{\gamma}^a, S^{cd}\}_- &= \{\tilde{\gamma}^a, \tilde{S}^{cd}\}_- = \frac{i}{2} (\eta^{ac} \tilde{\gamma}^d - \eta^{ad} \tilde{\gamma}^c). \end{aligned} \quad (8.61)$$

*Comments:* In the cases with either the basis  $\theta^a$  or with the basis of  $\gamma^a$  or  $\tilde{\gamma}^a$  the scalar products — the norms —  $\langle \mathbf{B} | \mathbf{B} \rangle < \mathbf{F} | \mathbf{F} \rangle$  are non negative and equal to  $\sum_{k=0}^d \int d^d x b_{b_1 \dots b_k}^* b_{b_1 \dots b_k}$ .

To have the norm which would have fields with the positive and the negative norm one could define the norm as  $\langle \phi_0 | b_{b_1 \dots b_k} \gamma^{b_k} \dots \gamma^{b_1} c_{c_1 \dots c_k} \gamma^{c_1} \dots \gamma^{c_k} | \phi_0 \rangle$ , as it is used in Ref. [21] to obtain the generalized Stueckelberg equation.

#### 8.4.1 Lorentz properties of basic vectors

What follows is taken from Ref. [2] and Ref. [9], Appendix B.

Let us first repeat some properties of the anticommuting Grassmann coordinates.

An infinitesimal Lorentz transformation of the proper orthochronous Lorentz group is then

$$\begin{aligned} \delta \theta^c &= -\frac{i}{2} \omega_{ab} S^{ab} \theta^c = \omega^c_a \theta^a, \\ \delta \gamma^c &= -\frac{i}{2} \omega_{ab} S^{ab} \gamma^c = \omega^c_a \gamma^a, \\ \delta \tilde{\gamma}^c &= -\frac{i}{2} \omega_{ab} \tilde{S}^{ab} \tilde{\gamma}^c = \omega^c_a \tilde{\gamma}^a, \\ \delta x^c &= -\frac{i}{2} \omega_{ab} L^{ab} x^c = \omega^c_a x^a, \end{aligned} \quad (8.62)$$

where  $\omega_{ab}$  are parameters of a transformation and  $\gamma^a$  and  $\tilde{\gamma}^a$  are expressed by  $\theta^a$  and  $\frac{\partial}{\partial \theta^a}$  in Eqs. (8.14, 8.15).

Let us write the operator of finite Lorentz transformations as follows

$$\mathcal{U} = e^{\frac{i}{2} \omega_{ab} (S^{ab} + L^{ab})}. \quad (8.63)$$

We see that the Grassmann  $\theta^a$  and the ordinary  $x^a$  coordinates and the Clifford objects  $\gamma^a$  and  $\tilde{\gamma}^a$  transform as vectors Eq.(8.63)

$$\begin{aligned}\theta'^c &= e^{-\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})}\theta^c e^{\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})} \\ &= \theta^c - \frac{i}{2}\omega_{ab}\{S^{ab}, \theta^c\}_- + \dots = \theta^c + \omega^c_a \theta^a + \dots = \Lambda^c_a \theta^a, \\ x'^c &= \Lambda^c_a x^a, \quad \gamma'^c = \Lambda^c_a \gamma^a, \quad \tilde{\gamma}'^c = \Lambda^c_a \tilde{\gamma}^a.\end{aligned}\quad (8.64)$$

Correspondingly one finds that compositions like  $\gamma^a p_a$  and  $\tilde{\gamma}^a p_a$ , here  $p_a$  are  $p_a^x (= i\frac{\partial}{\partial x^a})$ , transform as scalars (remaining invariants), while  $S^{ab}\omega_{abc}$  and  $\tilde{S}^{ab}\tilde{\omega}_{abc}$  transform as vectors:  $\mathcal{U}^{-1}S^{ab}\omega_{abc}\mathcal{U} = \Lambda_c^d S^{ab}\omega_{abd}$ ,  $\mathcal{U}^{-1}\tilde{S}^{ab}\tilde{\omega}_{abc}\mathcal{U} = \Lambda_c^d \tilde{S}^{ab}\tilde{\omega}_{abd}$ .

Also objects like

$$R = \frac{1}{2}f^{\alpha[a}f^{\beta b]}(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\omega^c_{b\beta})$$

and

$$\tilde{R} = \frac{1}{2}f^{\alpha[a}f^{\beta b]}(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\tilde{\omega}^c_{b\beta})$$

from Eq. (8.1) transform with respect to the Lorentz transformations as scalars.

Making a choice of the Cartan subalgebra set of the algebra  $S^{ab}$ ,  $S^{ab}$  and  $\tilde{S}^{ab}$ , Eqs. (8.10, 8.14, 8.15),

$$\begin{aligned}&S^{03}, S^{12}, S^{56}, \dots, S^{d-1\ d}, \\ &S^{03}, S^{12}, S^{56}, \dots, S^{d-1\ d}, \\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1\ d},\end{aligned}\quad (8.65)$$

one can arrange the basic vectors so that they are eigenstates of the Cartan subalgebra, belonging to representations of  $S^{ab}$ , or of  $S^{ab}$  and  $\tilde{S}^{ab}$ .

## 8.5 APPENDIX: Technique to generate spinor representations in terms of Clifford algebra objects

We shall briefly repeat the main points of the technique for generating spinor representations from Clifford algebra objects, following the reference[12]. We ask the reader to look for details and proofs in this reference.

We assume the objects  $\gamma^a$ , Eq. (8.14), which fulfill the Clifford algebra, Eq (8.13).

$$\{\gamma^a, \gamma^b\}_+ = I\ 2\eta^{ab}, \quad \text{for } a, b \in \{0, 1, 2, 3, 5, \dots, d\}, \quad (8.66)$$

for any  $d$ , even or odd.  $I$  is the unit element in the Clifford algebra, while  $\{\gamma^a, \gamma^b\}_\pm = \gamma^a\gamma^b \pm \gamma^b\gamma^a$ .

We accept the ‘‘Hermiticity’’ property for  $\gamma^a$ ’s, Eq. (8.17),  $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ . leading to  $\gamma^{a\dagger}\gamma^a = I$ .

The Clifford algebra objects  $S^{ab}$  close the Lie algebra of the Lorentz group of Eq. (8.21)  $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$ . One finds from Eq.(8.17) that  $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$  and that  $\{S^{ab}, S^{ac}\}_+ = \frac{1}{2}\eta^{aa}\eta^{bc}$ .

Recognizing that two Clifford algebra objects  $S^{ab}, S^{cd}$  with all indexes different commute, we select (out of infinitely many possibilities) the Cartan sub algebra set of the algebra of the Lorentz group as follows

$$\begin{aligned} S^{0d}, S^{12}, S^{35}, \dots, S^{d-2 \ d-1}, & \quad \text{if } d = 2n, \\ S^{12}, S^{35}, \dots, S^{d-1 \ d}, & \quad \text{if } d = 2n + 1. \end{aligned} \quad (8.67)$$

To make the technique simple, we introduce the graphic representation[12] as follows

$$\begin{aligned} {}^{ab}(\mathbf{k}) &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \\ {}^{ab}[\mathbf{k}] &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \end{aligned} \quad (8.68)$$

where  $k^2 = \eta^{aa}\eta^{bb}$ . One can easily check by taking into account the Clifford algebra relation (Eq.8.66) and the definition of  $S^{ab}$  that if one multiplies from the left hand side by  $S^{ab}$  the Clifford algebra objects  ${}^{ab}(\mathbf{k})$  and  ${}^{ab}[\mathbf{k}]$ , it follows that

$$\begin{aligned} S^{ab} {}^{ab}(\mathbf{k}) &= \frac{1}{2}k {}^{ab}(\mathbf{k}), \\ S^{ab} {}^{ab}[\mathbf{k}] &= \frac{1}{2}k {}^{ab}[\mathbf{k}]. \end{aligned} \quad (8.69)$$

This means that  ${}^{ab}(\mathbf{k})$  and  ${}^{ab}[\mathbf{k}]$  acting from the left hand side on anything (on a vacuum state  $|\psi_0\rangle$ , for example) are eigenvectors of  $S^{ab}$ .

We further find

$$\begin{aligned} \gamma^a {}^{ab}(\mathbf{k}) &= \eta^{aa} {}^{ab}[-\mathbf{k}], \\ \gamma^b {}^{ab}(\mathbf{k}) &= -ik {}^{ab}[-\mathbf{k}], \\ \gamma^a {}^{ab}[\mathbf{k}] &= (-\mathbf{k}), \\ \gamma^b {}^{ab}[\mathbf{k}] &= -ik\eta^{aa} {}^{ab}(-\mathbf{k}) \end{aligned} \quad (8.70)$$

It follows that  $S^{ac} {}^{ab \ cd}(\mathbf{k})(\mathbf{k}) = -\frac{i}{2}\eta^{aa}\eta^{cc} {}^{ab \ cd}[-\mathbf{k}][-\mathbf{k}]$ ,  $S^{ac} {}^{ab \ cd}[\mathbf{k}][\mathbf{k}] = \frac{i}{2} {}^{ab \ cd}(-\mathbf{k})(-\mathbf{k})$ ,  $S^{ac} {}^{ab \ cd}(\mathbf{k})[\mathbf{k}] = -\frac{i}{2}\eta^{aa} {}^{ab \ cd}[-\mathbf{k}](-\mathbf{k})$ ,  $S^{ac} {}^{ab \ cd}[\mathbf{k}](\mathbf{k}) = \frac{i}{2}\eta^{cc} {}^{ab \ cd}(-\mathbf{k})[-\mathbf{k}]$ . It is useful to deduce the following relations

$$\begin{aligned} {}^{ab}(\mathbf{k})({\mathbf{k}}) &= 0, & {}^{ab}(\mathbf{k})(-\mathbf{k}) &= \eta^{aa} {}^{ab}[\mathbf{k}], & {}^{ab}(-\mathbf{k})(\mathbf{k}) &= \eta^{aa} {}^{ab}[-\mathbf{k}], & {}^{ab}(-\mathbf{k})(-\mathbf{k}) &= 0 \\ {}^{ab}[\mathbf{k}][\mathbf{k}] &= {}^{ab}[\mathbf{k}], & {}^{ab}[\mathbf{k}][-\mathbf{k}] &= 0, & {}^{ab}[-\mathbf{k}][\mathbf{k}] &= 0, & {}^{ab}[-\mathbf{k}][-\mathbf{k}] &= {}^{ab}[-\mathbf{k}] \\ {}^{ab}(\mathbf{k})[\mathbf{k}] &= 0, & {}^{ab}[\mathbf{k}](\mathbf{k}) &= {}^{ab}(\mathbf{k}), & {}^{ab}(-\mathbf{k})[\mathbf{k}] &= {}^{ab}(-\mathbf{k}), & {}^{ab}(-\mathbf{k})[-\mathbf{k}] &= 0 \\ {}^{ab}[\mathbf{k}][-\mathbf{k}] &= {}^{ab}(\mathbf{k}), & {}^{ab}[\mathbf{k}](-\mathbf{k}) &= 0, & {}^{ab}[-\mathbf{k}](\mathbf{k}) &= 0, & {}^{ab}[-\mathbf{k}](-\mathbf{k}) &= {}^{ab}(-\mathbf{k}). \end{aligned} \quad (8.71)$$

We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects  $\overset{ab}{(k)}$  and  $\overset{ab}{[k]}$ , respectively.

*Whenever the Clifford algebra objects apply from the left hand side, they always transform  $\overset{ab}{(k)}$  to  $\overset{ab}{[-k]}$ , never to  $\overset{ab}{[k]}$ , and similarly  $\overset{ab}{[k]}$  to  $\overset{ab}{(-k)}$ , never to  $\overset{ab}{(k)}$ .*

We define in Eq. (8.59) a vacuum state  $|\psi_0\rangle$  so that one finds

$$\begin{aligned} \langle \overset{ab}{(k)} | \overset{ab}{(k)} \rangle &= 1, \\ \langle \overset{ab}{[k]} | \overset{ab}{[k]} \rangle &= 1. \end{aligned} \quad (8.72)$$

Taking the above equations into account it is easy to find a Weyl spinor irreducible representation for d-dimensional space, with d even or odd. (We advise the reader to see the reference[12].)

For d even, we simply set the starting state as a product of d/2, let us say, only nilpotents  $\overset{ab}{(k)}$ , one for each  $S^{ab}$  of the Cartan sub algebra elements (Eq.(8.67)), applying it on an (unimportant) vacuum state[12]. Then the generators  $S^{ab}$ , which do not belong to the Cartan sub algebra, applied to the starting state from the left hand side, generate all the members of one Weyl spinor.

$$\begin{aligned} &\overset{0d}{(k_{0d})} \overset{12}{(k_{12})} \overset{35}{(k_{35})} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\overset{0d}{[-k_{0d}]} \overset{12}{(k_{12})} \overset{35}{(k_{35})} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\overset{0d}{[-k_{0d}]} \overset{12}{(k_{12})} \overset{35}{[-k_{35}]} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\vdots \\ &\overset{0d}{[-k_{0d}]} \overset{12}{(k_{12})} \overset{35}{(k_{35})} \cdots \overset{d-1}{[-k_{d-1}]} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\overset{od}{(k_{0d})} \overset{12}{[-k_{12}]} \overset{35}{[-k_{35}]} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\vdots \end{aligned} \quad (8.73)$$

### 8.5.1 Technique to generate "families" of spinor representations in terms of Clifford algebra objects

When all  $2^d$  states are considered as a Hilbert space, we recognize that for d even there are  $2^{d/2}$  "families" and for d odd  $2^{(d+1)/2}$  "families" of spinors [12,13,9]. We shall pay attention of only even d.

One Weyl representation form a left ideal with respect to the multiplication with the Clifford algebra objects. We proved in Ref.[9], and the references therein that there is the application of the Clifford algebra object from the right hand side, which generates "families" of spinors.

Right multiplication with the Clifford algebra objects namely transforms the state of one "family" into the same state with respect to the generators  $S^{ab}$  (when the multiplication from the left hand side is performed) of another "family".

We defined in refs.[13] the Clifford algebra objects  $\tilde{\gamma}^a$ 's as operations which operate formally from the left hand side (as  $\gamma^a$ 's do) on any Clifford algebra object  $A$  as follows

$$\tilde{\gamma}^a A = i(-)^{(A)} A \gamma^a, \quad (8.74)$$

with  $(-)^{(A)} = -1$ , if  $A$  is an odd Clifford algebra object and  $(-)^{(A)} = 1$ , if  $A$  is an even Clifford algebra object.

Then it follows that  $\tilde{\gamma}^a$  obey the same Clifford algebra relation as  $\gamma^a$ .

$$(\tilde{\gamma}^a \tilde{\gamma}^b + \tilde{\gamma}^b \tilde{\gamma}^a) A = -ii((-)^{(A)})^2 A (\gamma^a \gamma^b + \gamma^b \gamma^a) = 2\eta^{ab} A \quad (8.75)$$

and that  $\tilde{\gamma}^a$  and  $\gamma^a$  anticommute

$$(\tilde{\gamma}^a \gamma^b + \gamma^b \tilde{\gamma}^a) A = i(-)^{(A)} (-\gamma^b A \gamma^a + \gamma^b A \gamma^a) = 0. \quad (8.76)$$

We may write

$$\{\tilde{\gamma}^a, \gamma^b\}_+ = 0, \quad \text{while} \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}. \quad (8.77)$$

One accordingly finds

$$\tilde{\gamma}^a \stackrel{ab}{(k)} := -i \stackrel{ab}{(k)} \gamma^a = -i\eta^{aa} \stackrel{ab}{[k]}, \quad (8.78)$$

$$\tilde{\gamma}^b \stackrel{ab}{(k)} := -i \stackrel{ab}{(k)} \gamma^b = -k \stackrel{ab}{[k]},$$

$$\tilde{\gamma}^a \stackrel{ab}{[k]} := i \stackrel{ab}{[k]} \gamma^a = i \stackrel{ab}{(k)}, \quad (8.79)$$

$$\tilde{\gamma}^b \stackrel{ab}{[k]} := i \stackrel{ab}{[k]} \gamma^b = -k\eta^{aa} \stackrel{ab}{(k)}. \quad (8.80)$$

If we define

$$\tilde{S}^{ab} = \frac{i}{4} [\tilde{\gamma}^a, \tilde{\gamma}^b] = \frac{1}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a), \quad (8.81)$$

it follows

$$\tilde{S}^{ab} A = A \frac{1}{4} (\gamma^b \gamma^a - \gamma^a \gamma^b), \quad (8.82)$$

manifesting accordingly that  $\tilde{S}^{ab}$  fulfil the Lorentz algebra relation as  $S^{ab}$  do. Taking into account Eq.(8.74), we further find

$$\{\tilde{S}^{ab}, S^{ab}\}_- = 0, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0, \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0. \quad (8.83)$$

One also finds

$$\begin{aligned} \{\tilde{S}^{ab}, \Gamma\}_- &= 0, \quad \{\tilde{\gamma}^a, \Gamma\}_- = 0, \quad \text{for } d \text{ even,} \\ \{\tilde{S}^{ab}, \Gamma\}_- &= 0, \quad \{\tilde{\gamma}^a, \Gamma\}_+ = 0, \quad \text{for } d \text{ odd,} \end{aligned} \quad (8.84)$$

which means that in  $d$  even transforming one "family" into another with either  $\tilde{S}^{ab}$  or  $\tilde{\gamma}^a$  leaves handedness  $\Gamma$  unchanged. (The transformation to another "family"

in  $d$  odd with  $\tilde{\gamma}^a$  changes the handedness of states, namely the factor  $\frac{1}{2}(1 \pm \Gamma)$  changes to  $\frac{1}{2}(1 \mp \Gamma)$  in accordance with what we know from before: In spaces with odd  $d$  changing the handedness means changing the "family".)

We advise the reader also to read [2] where the two kinds of Clifford algebra objects follow as two different superpositions of a Grassmann coordinate and its conjugate momentum.

We present for  $\tilde{S}^{ab}$  some useful relations

$$\begin{aligned}
 \tilde{S}^{ab} (k) &= \frac{k}{2} \overset{ab}{(k)}, \\
 \tilde{S}^{ab} [k] &= -\frac{k}{2} \overset{ab}{[k]}, \\
 \tilde{S}^{ac} (k)(k) &= \frac{i}{2} \eta^{aa} \eta^{cc} \overset{ab}{[k]} \overset{cd}{[k]}, \\
 \tilde{S}^{ac} \overset{ab}{[k]} \overset{cd}{[k]} &= -\frac{i}{2} (k)(k) \overset{ab}{[k]} \overset{cd}{[k]}, \\
 \tilde{S}^{ac} (k)[k] &= -\frac{i}{2} \eta^{aa} \overset{ab}{[k]} \overset{cd}{(k)}, \\
 \tilde{S}^{ac} \overset{ab}{[k]} \overset{cd}{(k)} &= \frac{i}{2} \eta^{cc} \overset{ab}{(k)} \overset{cd}{[k]}.
 \end{aligned} \tag{8.85}$$

We transform the state of one "family" to the state of another "family" by the application of  $\tilde{\gamma}^a$  or  $\tilde{S}^{ac}$  (formally from the left hand side) on a state of the first "family" for a chosen  $a$  or  $a, c$ . To transform all the states of one "family" into states of another "family", we apply  $\tilde{\gamma}^a$  or  $\tilde{S}^{ac}$  to each state of the starting "family". It is, of course, sufficient to apply  $\tilde{\gamma}^a$  or  $\tilde{S}^{ac}$  to only one state of a "family" and then use generators of the Lorentz group ( $S^{ab}$ ), and for  $d$  even also  $\gamma^{a'}$ s, to generate all the states of one Dirac spinor.

One must notice that nilpotents  $\overset{ab}{(k)}$  and projectors  $\overset{ab}{[k]}$  are eigenvectors not only of the Cartan subalgebra  $S^{ab}$  but also of  $\tilde{S}^{ab}$ . Accordingly only  $\tilde{S}^{ac}$ , which do not carry the Cartan subalgebra indices, cause the transition from one "family" to another "family".

The starting state of Eq.(8.73) can change, for example, to

$$\overset{0d}{[k_{0d}]} \overset{12}{[k_{12}]} \overset{35}{(k_{35})} \cdots (k_{d-1} \overset{d-2}{d-2}), \tag{8.86}$$

if  $\tilde{S}^{01}$  was chosen to transform the Weyl spinor of Eq.(8.73) to the Weyl spinor of another "family".

## References

1. N. Mankoč Borštnik, "Spin connection as a superpartner of a vielbein", Phys. Lett. B 292 (1992) 25-29.
2. N. Mankoč Borštnik, J. of Math. Phys. **34** (1993), 3731-3745.
3. N.S. Mankoč Borštnik, "Spin-charge-family theory is offering next step in understanding elementary particles and fields and correspondingly universe", Proceedings to



- the Conference on Cosmology, Gravitational Waves and Particles, IARD conferences, Ljubljana, 6-9 June 2016, The 10<sup>th</sup> Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields, J. Phys.: Conf. Ser. 845 (2017) 012017 [arXiv:1607.01618v2]
4. N.S. Mankoč Borštnik, Phys. Rev. **D 91** 065004 (2015) [arxiv:1409.7791].
  5. N.S. Mankoč Borštnik, D. Lukman, Eur. Phys. J. C **77** (2017) 231.
  6. N.S. Mankoč Borštnik, "Do we have the explanation for the Higgs and Yukawa couplings of the *standard model*" [arXiv:1212.3184v2, arXiv:1207.6233], in *Proceedings to the 15<sup>th</sup> Workshop "What comes beyond the standard models"*, Bled, 9-19 of July, 2012, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2012, p.56-71 [arXiv:1302.4305].
  7. N.S. Mankoč Borštnik N S, J. of Modern Phys. **4** (2013) 823 [arxiv:1312.1542].
  8. N.S. Mankoč Borštnik, H.B.F. Nielsen, Fortschritte der Physik, Progress of Physics (2017) 1700046.
  9. N.S. Mankoč Borštnik, J. of Mod. Physics **6** (2015) 2244-2274, <http://dx.org./10.4236/jmp.2015.615230> [arXiv:1409.4981].
  10. N.S. Mankoč Borštnik, H.B.F. Nielsen, "Fermionization in an Arbitrary Number of Dimensions", Proceedings to the 18<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 11-19 of July, 2015, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2015, p. 111-128 [<http://arxiv.org/abs/1602.03175>]
  11. H. Aratyn, H.B. Nielsen, "Constraints On Bosonization In Higher Dimensions", NBI-HE-83-36, Conference: C83-10-10.2 (Ahrenshoop Sympos.1983:0260), p.0260 Proceed-ings.
  12. N.S. Mankoč Borštnik, H.B.F. Nielsen, J. of Math. Phys. **43**, 5782 (2002) [hep-th/0111257].
  13. N.S. Mankoč Borštnik, H.B.F. Nielsen, J. of Math. Phys. **44** 4817 (2003) [hep-th/0303224].
  14. N.S. Mankoč Borštnik and H.B. Nielsen, Phys. Rev. **D 62**, 044010 (2000) [hep-th/9911032].
  15. N.S. Mankoč Borštnik and H.B. Nielsen, Proceedings to the 8<sup>th</sup> Workshop "What Comes Beyond the Standard Models", Bled, July 19 - 29, 2005, Ed. by Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman, DMFA Založništvo, Ljubljana December 2005, p.63-71, hep-ph/0512061.
  16. N.S. Mankoč Borštnik and H.B.F. Nielsen, JHEP 04:165, 2014 [arXiv:1212.2362].
  17. P.A/M. Dirac, Proc. Roy. Soc. (London), **A 117** (1928) 610.
  18. D. Lukman, N.S. Mankoč Borštnik and H.B. Nielsen, New J. Phys. **13**:103027, 2011.
  19. D. Lukman and N.S. Mankoč Borštnik, J. Phys. A: Math. Theor. **45**:465401, 2012 [arxiv:1205.1714, arxiv:1312.541, hep-ph/0412208 p.64-84].
  20. D. Lukman, N.S. Mankoč Borštnik and H.B. Nielsen, "Families of spinors in  $d = (1 + 5)$  with a zweibein and two kinds of spin connection fields on an almost  $S^2$ ", Proceedings to the 15<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 9-19 of July, 2012, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2012, 157-166, arxiv.1302.4305.
  21. M. Pavšič, "Quantized fields á la Clifford and unification" [arXiv:1707.05695].



## 9 Reality from Maximizing Overlap in the Future-included theories

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**Abstract.** In the future-included complex and real action theories whose paths run over not only the past but also the future, we briefly review the theorem on the normalized matrix element of an operator  $\hat{O}$ , which is defined in terms of the future and past states with a proper inner product  $I_Q$  that makes a given Hamiltonian normal. The theorem states that, provided that the operator  $\hat{O}$  is Q-Hermitian, i.e. Hermitian with regard to the proper inner product  $I_Q$ , the normalized matrix element becomes real and time-develops under a Q-Hermitian Hamiltonian for the past and future states selected such that the absolute value of the transition amplitude from the past state to the future state is maximized. Discussing what the theorem implicates, we speculate that the future-included complex action theory would be the most elegant quantum theory.

**Povzetek.** Avtorja obravnavata teorijo z realno in kompleksno akcijo, ki poleg preteklosti vključi tudi prihodnost. Na kratko predstavita izrek o normaliziranih matričnih elementih operatorja  $\hat{O}$ , ki operira na stanja preteklosti in prihodnosti tako, da je v primerno izbranem skalarnem produktu  $I_Q$  hamiltonka normalna. Če je operator  $\hat{O}$  hermitski glede na ta skalarni produkt, so normalizirani matrični elementi realni, njihov časovni razvoj pa poteka po tistih preteklih in prihodnjih stanjih, za katere je absolutna vrednost amplitude prehoda iz pretekega v prihodnje stanje maksimizirana. Obravnavata posledice izreka in domevata, da je najbolj elegantna kvantna teorija prav teorija kompleksne akcije, ki vključuje prihodnost.

**Keywords:** Complex action theories, Future-included action theories, Influence from the future

### 9.1 Introduction

Quantum theory is formulated via the Feynman path integral (FPI). Usually an action in the FPI is taken to be real. However, there is a possibility that the action

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is complex at the fundamental level but looks real effectively. If we pursue a fundamental theory, it is better to require less conditions imposed on it at first. In this sense such a complex action theory (CAT) is preferable to the usual real action theory (RAT), because the former has less conditions at least by one: there is no reality condition on the action. Based on this speculation the CAT has been investigated with the expectation that the imaginary part of the action would give some falsifiable predictions [1–4], and various interesting suggestions have been made for Higgs mass [5], quantum mechanical philosophy [6–8], some fine-tuning problems [9,10], black holes [11], de Broglie-Bohm particles and a cut-off in loop diagrams [12]. Also, in Ref. [13], introducing what we call the proper inner product  $I_Q$  so that a given non-normal Hamiltonian becomes normal with respect to it, we proposed a mechanism to effectively obtain a Hamiltonian which is  $Q$ -Hermitian, i.e., Hermitian with respect to the proper inner product, after a long time development. Furthermore, using the complex coordinate formalism [14], we explicitly derived the momentum relation  $p = m\dot{q}$ , where  $m$  is a complex mass, via the FPI [15]. In general, the CAT<sup>1</sup> could be classified into two types: one is the future-not-included theory [21]<sup>2</sup>, i.e., the theory including only a past time as an integration interval of time, and the other one is the future-included theory [1], in which not only the past state  $|A(T_A)\rangle$  at the initial time  $T_A$  but also the future state  $|B(T_B)\rangle$  at the final time  $T_B$  is given at first, and the time integration is performed over the whole period from the past to the future.

In the future-included theory, the normalized matrix element [1]  $\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{O}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$ , where  $t$  is an arbitrary time ( $T_A \leq t \leq T_B$ ), seems to have a role of an expectation value of the operator  $\hat{O}$ . Indeed, in Refs. [23,24] we argued in the case of the action being complex that, if we regard  $\langle \hat{O} \rangle^{BA}$  as an expectation value in the future-included theory, we obtain the Heisenberg equation, Ehrenfest's theorem, and a conserved probability current density. So  $\langle \hat{O} \rangle^{BA}$  is a strong candidate for the expectation value in the future-included theory. The normalized matrix element  $\langle \hat{O} \rangle^{BA}$  is called the weak value [25] in the context of the future-included RAT, and it has been intensively studied. The details are found in Ref. [26] and references therein.

In Ref. [27], we considered a slightly modified normalized matrix element  $\langle \hat{O} \rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q \hat{O} |A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}$ , where  $\langle B(t)|_Q \equiv \langle B(t)|Q$ , and  $Q$  is a Hermitian operator that is appropriately chosen to define the proper inner product  $I_Q$ . This matrix element is obtained just by changing the notation of  $\langle B(t)|$  as  $\langle B(t)| \rightarrow \langle B(t)|_Q$  in  $\langle \hat{O} \rangle^{BA}$ . We proposed a theorem in the future-included CAT, which states that, provided that an operator  $\hat{O}$  is  $Q$ -Hermitian,  $\langle \hat{O} \rangle_Q^{BA}$  becomes real and time-develops under a  $Q$ -Hermitian Hamiltonian for the future and past states selected such that the absolute value of the transition amplitude defined with  $I_Q$  from the past state

<sup>1</sup> The corresponding Hamiltonian  $\hat{H}$  is generically non-normal. So the set of the Hamiltonians we consider is much larger than that of the PT-symmetric non-Hermitian Hamiltonians, which has been intensively studied [16–20].

<sup>2</sup> In our recent study [22], we have pointed out that, if a theory is described with a complex action, then such a theory is suggested to be the future-included theory rather than the future-not-included theory.

to the future state is maximized. We call this way of thinking the maximization principle. This theorem applies to not only the CAT but also the RAT. In Ref. [27], we proved this theorem only in the CAT, i.e., in the case of non-Hermitian Hamiltonians, by finding that essentially only terms associated with the largest imaginary parts of the eigenvalues of the Hamiltonian  $\hat{H}^3$  contribute significantly to the absolute value of the transition amplitude defined with  $I_Q$ , and that  $\langle \hat{O} \rangle_Q^{BA}$  for such maximizing states becomes an expression similar to an expectation value defined with  $I_Q$  in the future-not-included theory. This proof is based on the existence of imaginary parts of the eigenvalues of  $\hat{H}$ , so it cannot be applied to the RAT. In Ref. [28], we presented another theorem particular to the case of Hermitian Hamiltonians, i.e., the RAT case for simplicity, and proved it. In this paper, we review the maximization principle and clarify what the theorems implicate based on Refs. [27–29].

This paper is organized as follows. In section 2 we briefly review the proper inner product and the future-included theory. In section 3 we present the theorems, and prove them in section 4. Section 5 is devoted to discussion.

## 9.2 Proper inner product and future-included complex action theory

We suppose that our system that could be the whole world is described by a non-normal diagonalizable Hamiltonian  $\hat{H}$  such that  $[\hat{H}, \hat{H}^\dagger] \neq 0$ . Based on Refs.[13,14,29], we first review the proper inner product for  $\hat{H}$  which makes  $\hat{H}$  normal with respect to it. We define the eigenstates  $|\lambda_i\rangle$  ( $i = 1, 2, \dots$ ) of  $\hat{H}$  such that

$$\hat{H}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle, \quad (9.1)$$

where  $\lambda_i$  ( $i = 1, 2, \dots$ ) are the eigenvalues of  $\hat{H}$ , and introduce the diagonalizing operator  $P = (|\lambda_1\rangle, |\lambda_2\rangle, \dots)$ , so that  $\hat{H}$  is diagonalized as  $\hat{H} = PDP^{-1}$ , where  $D$  is given by  $\text{diag}(\lambda_1, \lambda_2, \dots)$ . Let us consider a transition from an eigenstate  $|\lambda_i\rangle$  to another  $|\lambda_j\rangle$  ( $i \neq j$ ) fast in time  $\Delta t$ . Since  $|\lambda_i\rangle$  are not orthogonal to each other in the usual inner product  $I$ ,  $I(|\lambda_i\rangle, |\lambda_j\rangle) \equiv \langle \lambda_i | \lambda_j \rangle \neq \delta_{ij}$ , the transition can be measured, i.e.,  $|I(|\lambda_j\rangle, \exp(-\frac{i}{\hbar}\hat{H}\Delta t)|\lambda_i\rangle)|^2 \neq 0$ , though  $\hat{H}$  cannot bring the system from  $|\lambda_i\rangle$  to  $|\lambda_j\rangle$  ( $i \neq j$ ). In any reasonable theories, such an unphysical transition from an eigenstate to another one with a different eigenvalue should be prohibited. In order to have reasonable probabilistic results, we introduce a proper inner product [13,14]<sup>4</sup> for arbitrary kets  $|u\rangle$  and  $|v\rangle$  as

$$I_Q(|u\rangle, |v\rangle) \equiv \langle u|_Q|v\rangle \equiv \langle u|Q|v\rangle, \quad (9.2)$$

where  $Q$  is a Hermitian operator chosen as  $Q = (P^\dagger)^{-1}P^{-1}$ , so that  $|\lambda_i\rangle$  get orthogonal to each other with regard to  $I_Q$ ,

$$\langle \lambda_i |_Q | \lambda_j \rangle = \delta_{ij}. \quad (9.3)$$

<sup>3</sup> In the CAT the imaginary parts of the eigenvalues of  $\hat{H}$  are supposed to be bounded from above to avoid the Feynman path integral  $\int e^{\frac{i}{\hbar}S} \mathcal{D}\text{path}$  being divergently meaningless.

<sup>4</sup> Similar inner products are studied also in refs.[30,19,20].

This implies the orthogonality relation  $\sum_i |\lambda_i\rangle\langle\lambda_i|_Q = 1$ . In the special case of  $\hat{H}$  being hermitian,  $Q$  is the unit operator. We introduce the “ $Q$ -Hermitian” conjugate  $\dagger^Q$  of an operator  $A$  by  $\langle u|_Q A|v\rangle^* \equiv \langle v|_Q A^{\dagger^Q}|u\rangle$ , so

$$A^{\dagger^Q} \equiv Q^{-1} A^\dagger Q. \quad (9.4)$$

If  $A$  obeys  $A^{\dagger^Q} = A$ ,  $A$  is  $Q$ -Hermitian. We also define  $\dagger^Q$  for kets and bras as

$$|u\rangle^{\dagger^Q} \equiv \langle u|_Q \text{ and } (\langle u|_Q)^{\dagger^Q} \equiv |u\rangle. \text{ In addition, } P^{-1} = \begin{pmatrix} \langle\lambda_1|_Q \\ \langle\lambda_2|_Q \\ \vdots \end{pmatrix} \text{ satisfies } P^{-1} \hat{H} P =$$

$D$  and  $P^{-1} \hat{H}^{\dagger^Q} P = D^\dagger$ , so  $\hat{H}$  is “ $Q$ -normal”,  $[\hat{H}, \hat{H}^{\dagger^Q}] = P[D, D^\dagger]P^{-1} = 0$ . Thus the inner product  $I_Q$  makes  $\hat{H}$   $Q$ -normal. We note that  $\hat{H}$  can be decomposed as  $\hat{H} = \hat{H}_{Qh} + \hat{H}_{Qa}$ , where  $\hat{H}_{Qh} = \frac{\hat{H} + \hat{H}^{\dagger^Q}}{2}$  and  $\hat{H}_{Qa} = \frac{\hat{H} - \hat{H}^{\dagger^Q}}{2}$  are  $Q$ -Hermitian and anti- $Q$ -Hermitian parts of  $\hat{H}$  respectively.

In Refs.[1,23,24], the future-included theory is described by using the future state  $|B(T_B)\rangle$  at the final time  $T_B$  and the past state  $|A(T_A)\rangle$  at the initial time  $T_A$ , where  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  time-develop as follows,

$$i\hbar \frac{d}{dt} |A(t)\rangle = \hat{H} |A(t)\rangle, \quad (9.5)$$

$$-i\hbar \frac{d}{dt} \langle B(t)| = \langle B(t)| \hat{H}, \quad (9.6)$$

and the normalized matrix element  $\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t)| \hat{O} |A(t)\rangle}{\langle B(t)| A(t)\rangle}$  is studied. The quantity  $\langle \hat{O} \rangle^{BA}$  is called the weak value[25,26] in the RAT. In refs.[23,24], we investigated  $\langle \hat{O} \rangle^{BA}$  and found that, if we regard  $\langle \hat{O} \rangle^{BA}$  as an expectation value in the future-included theory, then we obtain the Heisenberg equation, Ehrenfest’s theorem, and a conserved probability current density. Therefore,  $\langle \hat{O} \rangle^{BA}$  seems to have a role of an expectation value in the future-included theory.

In the following, we adopt the proper inner product  $I_Q$  for all quantities. Hence we change the notation of the final state  $\langle B(T_B)|$  as  $\langle B(T_B)| \rightarrow \langle B(T_B)|_Q$  so that the Hermitian operator  $Q$  pops out and the usual inner product  $I$  is replaced with  $I_Q$ . Then  $\langle B(T_B)|$  time-develops according not to eq.(9.6) but to

$$-i\hbar \frac{d}{dt} \langle B(t)|_Q = \langle B(t)|_Q \hat{H} \Leftrightarrow i\hbar \frac{d}{dt} |B(t)\rangle = \hat{H}^{\dagger^Q} |B(t)\rangle, \quad (9.7)$$

and the normalized matrix element is expressed as

$$\langle \hat{O} \rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q \hat{O} |A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}. \quad (9.8)$$

In addition, we suppose that  $|A(T_A)\rangle$  and  $\langle B(T_B)|$  are  $Q$ -normalized by  $\langle A(T_A)|_Q A(T_A)\rangle = 1$  and  $\langle B(T_B)|_Q B(T_B)\rangle = 1$ . In the RAT, since  $Q = 1$ ,  $\langle \hat{O} \rangle_Q^{BA}$  corresponds to  $\langle \hat{O} \rangle^{BA}$ .

### 9.3 Theorems of the maximization principle

In Ref. [27] we proposed the following theorem :

**Theorem 1. Maximization principle in the future-included CAT**

*As a prerequisite, assume that a given Hamiltonian  $\hat{H}$  is non-normal but diagonalizable and that the imaginary parts of the eigenvalues of  $\hat{H}$  are bounded from above, and define a modified inner product  $I_Q$  by means of a Hermitian operator  $Q$  arranged so that  $\hat{H}$  becomes normal with respect to  $I_Q$ . Let the two states  $|A(t)\rangle$  and  $|B(t)\rangle$  time-develop according to the Schrödinger equations with  $\hat{H}$  and  $\hat{H}^{\dagger Q}$  respectively:  $|A(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-T_A)}|A(T_A)\rangle$ ,  $|B(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}^{\dagger Q}(t-T_B)}|B(T_B)\rangle$ , and be normalized with  $I_Q$  at the initial time  $T_A$  and the final time  $T_B$  respectively:  $\langle A(T_A)|_Q A(T_A)\rangle = 1$ ,  $\langle B(T_B)|_Q B(T_B)\rangle = 1$ . Next determine  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  so as to maximize the absolute value of the transition amplitude  $|\langle B(t)|_Q A(t)\rangle| = |\langle B(T_B)|_Q \exp(-i\hat{H}(T_B - T_A))|A(T_A)\rangle|$ . Then, provided that an operator  $\hat{O}$  is  $Q$ -Hermitian, i.e., Hermitian with respect to the inner product  $I_Q$ ,  $\hat{O}^{\dagger Q} = \hat{O}$ , the normalized matrix element of the operator  $\hat{O}$  defined by  $\langle \hat{O} \rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q \hat{O} |A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}$  becomes real and time-develops under a  $Q$ -Hermitian Hamiltonian.*

We call this way of thinking the maximization principle. This theorem means that the normalized matrix element  $\langle \hat{O} \rangle_Q^{BA}$ , which is taken as an average for an operator  $\hat{O}$  obeying  $\hat{O}^{\dagger Q} = \hat{O}$ , turns out to be real almost unavoidably. Also, in the case of non-normal Hamiltonians, it is nontrivial to obtain the emerging  $Q$ -hermiticity for the Hamiltonian by the maximization principle. The theorem is given for systems defined with such general Hamiltonians that they do not even have to be normal, so it can also be used for normal Hamiltonians in addition to non-normal Hamiltonians. For a normal Hamiltonian  $\hat{H}$ ,  $Q$  is the unit operator. In such a case the above theorem becomes simpler with  $Q = 1$ . There are two possibilities for such a case: one is that  $\hat{H}$  is non-Hermitian but normal, and the other is that  $\hat{H}$  is Hermitian. In both cases  $Q = 1$ , but there is a significant difference between them. In the former case, there are imaginary parts of the eigenvalues of  $\hat{H}$ ,  $\text{Im}\lambda_i$ , and the eigenstates having the largest  $\text{Im}\lambda_i$  blow up and contribute most to the absolute value of the transition amplitude  $|\langle B(t)|_Q A(t)\rangle|$ . In the latter case, there are no  $\text{Im}\lambda_i$ , and the full set of the eigenstates of  $\hat{H}$  can contribute to  $|\langle B(t)|_Q A(t)\rangle|$ . So we need to investigate them separately.

In the special case where the Hamiltonian is Hermitian, i.e., in the future-included RAT, we can consider three possibilities: One is that  $|A(T_A)\rangle$  is given at first, and  $|B(T_B)\rangle$  is chosen by the maximization principle. Another is the reverse. The other is that both  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  are partly given and chosen. Since we know empirically the second law of thermodynamics, we choose the first option in the future-included RAT. We suppose that  $|A(t)\rangle$  is a given fixed state, and only  $|B(t)\rangle$  is a random state, which should be chosen appropriately by the maximization principle, though in the future-included CAT both  $|A(t)\rangle$  and  $|B(t)\rangle$  are supposed to be random states at first. In addition, in the future-included RAT the hermiticity of the Hamiltonian is given at first, so we write the theorem particular to the case of Hermitian Hamiltonians as follows:

**Theorem 2. Maximization principle in the future-included RAT**

*As a prerequisite, assume that a given Hamiltonian  $\hat{H}$  is diagonalizable and Hermitian. Let the two states  $|A(t)\rangle$  and  $|B(t)\rangle$  time-develop according to the Schrödinger equation*

with  $\hat{H}$ :  $|A(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-T_A)}|A(T_A)\rangle$ ,  $|B(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-T_B)}|B(T_B)\rangle$ , and be normalized at the initial time  $T_A$  and the final time  $T_B$  respectively:  $\langle A(T_A)|A(T_A)\rangle = 1$ ,  $\langle B(T_B)|B(T_B)\rangle = 1$ . Next determine  $|B(T_B)\rangle$  for the given  $|A(T_A)\rangle$  so as to maximize the absolute value of the transition amplitude  $|\langle B(t)|A(t)\rangle| = |\langle B(T_B)|\exp(-\frac{i}{\hbar}\hat{H}(T_B - T_A))|A(T_A)\rangle|$ . Then, provided that an operator  $\hat{O}$  is Hermitian,  $\hat{O}^\dagger = \hat{O}$ , the normalized matrix element of the operator  $\hat{O}$  defined by  $\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{O}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$  becomes real and time-develops under the given Hermitian Hamiltonian.

We investigate the above theorems separately.

## 9.4 Proof of the theorems

To prove the theorems we expand  $|A(t)\rangle$  and  $|B(t)\rangle$  in terms of the eigenstates  $|\lambda_i\rangle$  as follows:

$$|A(t)\rangle = \sum_i a_i(t)|\lambda_i\rangle, \quad (9.9)$$

$$|B(t)\rangle = \sum_i b_i(t)|\lambda_i\rangle, \quad (9.10)$$

where

$$a_i(t) = a_i(T_A)e^{-\frac{i}{\hbar}\lambda_i(t-T_A)}, \quad (9.11)$$

$$b_i(t) = b_i(T_B)e^{-\frac{i}{\hbar}\lambda_i^*(t-T_B)}. \quad (9.12)$$

We express  $a_i(T_A)$  and  $b_i(T_B)$  as

$$a_i(T_A) = |a_i(T_A)|e^{i\theta_{a_i}}, \quad (9.13)$$

$$b_i(T_B) = |b_i(T_B)|e^{i\theta_{b_i}}, \quad (9.14)$$

and introduce

$$T \equiv T_B - T_A, \quad (9.15)$$

$$\Theta_i \equiv \theta_{a_i} - \theta_{b_i} - \frac{1}{\hbar}T\text{Re}\lambda_i, \quad (9.16)$$

$$R_i \equiv |a_i(T_A)||b_i(T_B)|e^{\frac{1}{\hbar}T\text{Im}\lambda_i}. \quad (9.17)$$

Then, since  $\langle B(t)|_Q A(t)\rangle$  is expressed as

$$\langle B(t)|_Q A(t)\rangle = \sum_i R_i e^{i\Theta_i}, \quad (9.18)$$

$|\langle B(t)|_Q A(t)\rangle|^2$  is calculated as

$$|\langle B(t)|_Q A(t)\rangle|^2 = \sum_i R_i^2 + 2 \sum_{i < j} R_i R_j \cos(\Theta_i - \Theta_j). \quad (9.19)$$

The normalization conditions for  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  are expressed as

$$\sum_i |a_i(T_A)|^2 = \sum_i |b_i(T_B)|^2 = 1. \quad (9.20)$$

We proceed with this study separately according to whether the given Hamiltonian  $\hat{H}$  is non-Hermitian or Hermitian.

### 9.4.1 Non-Hermitian Hamiltonians case

In the case of non-Hermitian Hamiltonians, there exist imaginary parts of the eigenvalues of the Hamiltonian,  $\text{Im}\lambda_i$ , which are supposed to be bounded from above to avoid the Feynman path integral  $\int e^{\frac{i}{\hbar}S} \mathcal{D}\text{path}$  being divergently meaningless. We can imagine that some of  $\text{Im}\lambda_i$  take the maximal value  $B$ , and denote the corresponding subset of  $\{i\}$  as  $A$ . Then, since  $R_i \geq 0$ ,  $|\langle B(t)|_Q A(t)\rangle|$  can take a maximal value only under the following conditions:

$$|a_i(T_A)| = |b_i(T_B)| = 0 \quad \text{for } \forall i \notin A, \quad (9.21)$$

$$\Theta_i \equiv \Theta_c \quad \text{for } \forall i \in A, \quad (9.22)$$

$$\sum_{i \in A} |a_i(T_A)|^2 = \sum_{i \in A} |b_i(T_B)|^2 = 1, \quad (9.23)$$

and  $|\langle B(t)|_Q A(t)\rangle|^2$  is estimated as

$$\begin{aligned} |\langle B(t)|_Q A(t)\rangle|^2 &= \left( \sum_{i \in A} R_i \right)^2 \\ &= e^{\frac{2B\Gamma}{\hbar}} \left( \sum_{i \in A} |a_i(T_A)| |b_i(T_B)| \right)^2 \\ &\leq e^{\frac{2B\Gamma}{\hbar}} \left\{ \sum_{i \in A} \left( \frac{|a_i(T_A)| + |b_i(T_B)|}{2} \right)^2 \right\}^2 \\ &= e^{\frac{2}{\hbar} B\Gamma}, \end{aligned} \quad (9.24)$$

where the third equality is realized for

$$|a_i(T_A)| = |b_i(T_B)| \quad \text{for } \forall i \in A. \quad (9.25)$$

In the last equality we have used this relation and Eq.(9.23). The maximization condition of  $|\langle B(t)|_Q A(t)\rangle|$  is represented by Eqs.(9.21)-(9.23) and (9.25). That is to say, the states to maximize  $|\langle B(t)|_Q A(t)\rangle|$ ,  $|A(t)\rangle_{\max}$  and  $|B(t)\rangle_{\max}$ , are expressed as

$$|A(t)\rangle_{\max} = \sum_{i \in A} a_i(t) |\lambda_i\rangle, \quad (9.26)$$

$$|B(t)\rangle_{\max} = \sum_{i \in A} b_i(t) |\lambda_i\rangle, \quad (9.27)$$

where  $a_i(t)$  and  $b_i(t)$  obey Eqs.(9.22), (9.23), and (9.25).

To evaluate  $\langle \hat{\mathcal{O}} \rangle_Q^{BA}$  for  $|A(t)\rangle_{\max}$  and  $|B(t)\rangle_{\max}$ , utilizing the Q-Hermitian part of  $\hat{H}$ ,  $\hat{H}_{Qh} \equiv \frac{\hat{H} + \hat{H}^\dagger Q}{2}$ , we define the following state:

$$|\tilde{A}(t)\rangle \equiv e^{-\frac{i}{\hbar}(t-T_A)\hat{H}_{Qh}} |A(T_A)\rangle_{\max}, \quad (9.28)$$

which is normalized as  $\langle \tilde{A}(t)|_Q \tilde{A}(t)\rangle = 1$  and obeys the Schrödinger equation

$$i\hbar \frac{d}{dt} |\tilde{A}(t)\rangle = \hat{H}_{Qh} |\tilde{A}(t)\rangle. \quad (9.29)$$



Using Eqs.(9.21)-(9.23) and (9.25), we obtain

$$\max \langle B(t) |_Q A(t) \rangle_{\max} = e^{i\Theta_c} \sum_{i \in A} R_i = e^{i\Theta_c} e^{\frac{B \cdot T}{\hbar}}, \quad (9.30)$$

and

$$\begin{aligned} & \max \langle B(t) |_Q \hat{O} | A(t) \rangle_{\max} \\ &= e^{i\Theta_c} e^{\frac{B \cdot T}{\hbar}} \sum_{i,j \in A} a_j(T_A)^* a_i(T_A) e^{\frac{i}{\hbar}(t-T_A)(\text{Re}\lambda_j - \text{Re}\lambda_i)} \langle \lambda_j |_Q \hat{O} | \lambda_i \rangle \\ &= e^{i\Theta_c} e^{\frac{B \cdot T}{\hbar}} \langle \tilde{A}(t) |_Q \hat{O} | \tilde{A}(t) \rangle. \end{aligned} \quad (9.31)$$

Thus  $\langle \hat{O} \rangle_Q^{BA}$  for  $|A(t)\rangle_{\max}$  and  $|B(t)\rangle_{\max}$  is evaluated as

$$\langle \hat{O} \rangle_Q^{B_{\max} A_{\max}} = \langle \tilde{A}(t) |_Q \hat{O} | \tilde{A}(t) \rangle \equiv \langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}}. \quad (9.32)$$

Since  $\left\{ \langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}} \right\}^* = \langle \hat{O}^{\dagger Q} \rangle_Q^{\tilde{A}\tilde{A}}$ ,  $\langle \hat{O} \rangle_Q^{BA}$  for  $|A(t)\rangle_{\max}$  and  $|B(t)\rangle_{\max}$  has been shown to be real for Q-Hermitian  $\hat{O}$ .

Next we study the time development of  $\langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}}$ . We express  $\langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}}$  as  $\langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}} = \langle \tilde{A}(T_A) |_Q \hat{O}_H(t, T_A) | \tilde{A}(T_A) \rangle$ , where we have introduced the Heisenberg operator  $\hat{O}_H(t, T_A) \equiv e^{\frac{i}{\hbar} \hat{H}_{QH}(t-T_A)} \hat{O} e^{-\frac{i}{\hbar} \hat{H}_{QH}(t-T_A)}$ . This operator  $\hat{O}_H(t, T_A)$  obeys the Heisenberg equation  $i\hbar \frac{d}{dt} \hat{O}_H(t, T_A) = [\hat{O}_H(t, T_A), \hat{H}_{QH}]$ , so we find that  $\langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}}$  time-develops under the Q-Hermitian Hamiltonian  $\hat{H}_{QH}$  as

$$\frac{d}{dt} \langle \hat{O} \rangle_Q^{\tilde{A}\tilde{A}} = \frac{i}{\hbar} \langle [\hat{H}_{QH}, \hat{O}] \rangle_Q^{\tilde{A}\tilde{A}}. \quad (9.33)$$

Thus Theorem 1 has been proven, and the maximization principle provides both the reality of  $\langle \hat{O} \rangle_Q^{BA}$  for Q-Hermitian  $\hat{O}$  and the Q-Hermitian Hamiltonian.

#### 9.4.2 Hermitian Hamiltonians case

Theorem 2 can be proven more simply than Theorem 1. Since the norms of  $|A(t)\rangle$  and  $|B(t)\rangle$  are constant in time in the case of Hermitian Hamiltonians,

$$\langle A(t) | A(t) \rangle = \langle A(T_A) | A(T_A) \rangle = 1, \quad (9.34)$$

$$\langle B(t) | B(t) \rangle = \langle B(T_B) | B(T_B) \rangle = 1, \quad (9.35)$$

we can directly use an elementary property of linear space, and find that the final state to maximize  $|\langle B(t) | A(t) \rangle|$ ,  $|B(T_B)\rangle_{\max}$ , is the same as  $|A(t)\rangle$  up to a constant phase factor:

$$|B(t)\rangle_{\max} = e^{-i\Theta_c} |A(t)\rangle. \quad (9.36)$$

This phase factor presents the ambiguity of the maximizing state  $|B(t)\rangle_{\max}$ , and shows that  $|B(t)\rangle_{\max}$  is not determined uniquely. We note that this is quite in contrast to the case of non-Hermitian Hamiltonians, where only a unique class of

$|A(t)\rangle$  and  $|B(t)\rangle$  is chosen by the maximization principle. The normalized matrix element  $\langle\hat{O}\rangle^{BA}$  for the given  $|A(t)\rangle$  and  $|B(t)\rangle_{\max}$  becomes

$$\begin{aligned}\langle\hat{O}\rangle^{B_{\max}A} &= \frac{\max\langle B(t)|\hat{O}|A(t)\rangle}{\max\langle B(t)|A(t)\rangle} \\ &= \langle A(t)|\hat{O}|A(t)\rangle \\ &\equiv \langle\hat{O}\rangle^{AA},\end{aligned}\tag{9.37}$$

where in the second equality we have used Eqs.(9.36) and (9.34). Thus  $\langle\hat{O}\rangle^{BA}$  for the given  $|A(t)\rangle$  and  $|B(t)\rangle_{\max}$  has become the form of a usual average  $\langle\hat{O}\rangle^{AA}$ , and so it becomes real for Hermitian  $\hat{O}$ . In addition,  $\langle\hat{O}\rangle^{AA}$  time-develops under the Hermitian Hamiltonian  $\hat{H}$  as

$$\frac{d}{dt}\langle\hat{O}\rangle^{AA} = \frac{i}{\hbar}\langle[\hat{H}, \hat{O}]\rangle^{AA}.\tag{9.38}$$

We emphasize that the maximization principle provides the reality of  $\langle\hat{O}\rangle^{BA}$  for Hermitian  $\hat{O}$ , though  $\langle\hat{O}\rangle^{BA}$  is generically complex by definition.

To see the differences from the case of non-Hermitian Hamiltonians more explicitly, we investigate Theorem 2 by expanding  $|A(t)\rangle$  and  $|B(t)\rangle$  in the same way as Eqs.(9.9)-(9.12). Then we can make use of Eqs.(9.13)-(9.20) just by noting that Eqs.(9.17)-(9.19) are expressed as

$$R_i \equiv |\alpha_i(T_A)||b_i(T_B)|,\tag{9.39}$$

$$\langle B(t)|A(t)\rangle = \sum_i R_i e^{i\Theta_i},\tag{9.40}$$

$$|\langle B(t)|A(t)\rangle|^2 = \sum_i R_i^2 + 2 \sum_{i<j} R_i R_j \cos(\Theta_i - \Theta_j),\tag{9.41}$$

since  $\text{Im}\lambda_i = 0$  and  $Q = 1$ . Then, since  $R_i \geq 0$ ,  $|\langle B(t)|A(t)\rangle|$  can take a maximal value only under the condition:

$$\Theta_i = \Theta_c \quad \text{for } \forall i,\tag{9.42}$$

and  $|\langle B(t)|A(t)\rangle|^2$  is estimated as

$$\begin{aligned}|\langle B(t)|A(t)\rangle|^2 &= \left(\sum_i R_i\right)^2 \\ &= \left(\sum_i |\alpha_i(T_A)||b_i(T_B)|\right)^2 \\ &\leq \left\{\sum_i \left(\frac{|\alpha_i(T_A)| + |b_i(T_B)|}{2}\right)^2\right\}^2 \\ &= 1,\end{aligned}\tag{9.43}$$

where the third equality is realized for

$$|\alpha_i(T_A)| = |b_i(T_B)| \quad \text{for } \forall i.\tag{9.44}$$

In the last equality we have used this relation and Eq.(9.20). The condition for maximizing  $|\langle B(t)|A(t)\rangle|$  is represented by Eqs.(9.42) and (9.44). In the case of non-Hermitian Hamiltonians, the condition for maximizing  $|\langle B(t)|_Q A(t)\rangle|$  is represented by Eqs.(9.21)-(9.23) and (9.25), and essentially only the subset having the largest imaginary parts of the eigenvalues of  $\hat{H}$  contributes most to the absolute value of the transition amplitude  $|\langle B(t)|_Q A(t)\rangle|$ , as we saw in Subsection 9.4.1. This is quite in contrast to the present study in the case of Hermitian Hamiltonians, where the full set of the eigenstates of  $\hat{H}$  can contribute to  $|\langle B(t)|A(t)\rangle|$ . Thus the final state to maximize  $|\langle B(t)|A(t)\rangle|$ ,  $|B(T_B)\rangle_{\max}$ , is expressed as

$$|B(T_B)\rangle_{\max} = \sum_i b_i^{\max}(T_B)|\lambda_i\rangle, \quad (9.45)$$

where

$$b_i^{\max}(T_B) \equiv |a_i(T_A)|e^{i(\theta_{a_i} - \frac{1}{\hbar}T\lambda_i - \Theta_c)} \quad (9.46)$$

obeys

$$\sum_i |b_i^{\max}(T_B)|^2 = 1. \quad (9.47)$$

Hence  $|B(t)\rangle_{\max}$  is expressed as

$$|B(t)\rangle_{\max} = e^{-\frac{i}{\hbar}\hat{H}(t-T_B)}|B(T_B)\rangle_{\max} = \sum_i b_i^{\max}(t)|\lambda_i\rangle, \quad (9.48)$$

where  $b_i^{\max}(t)$  is given by

$$b_i^{\max}(t) = b_i^{\max}(T_B)e^{-\frac{i}{\hbar}\lambda_i(t-T_B)} = a_i(t)e^{-i\Theta_c}. \quad (9.49)$$

In the second equality we have used Eq.(9.46). Consequently,  $|B(t)\rangle_{\max}$  is found to be the same as  $|A(t)\rangle$  up to the constant phase factor, as we saw in Eq.(9.36).

## 9.5 Discussion

In this paper, after briefly explaining the proper inner product  $I_Q$ , which makes a given non-normal Hamiltonian normal, and also the future-included CAT, we have reviewed the theorem on the normalized matrix element of  $\hat{\mathcal{O}}$ ,  $\langle\hat{\mathcal{O}}\rangle_Q^{BA}$ , which seems to have a role of an expectation value in the future-included CAT and RAT. Assuming that a given Hamiltonian  $\hat{H}$  is non-normal but diagonalizable, and that the imaginary parts of the eigenvalues of  $\hat{H}$  are bounded from above, we presented a theorem that states that, provided that  $\hat{\mathcal{O}}$  is Q-Hermitian, i.e.,  $\hat{\mathcal{O}}^{\dagger Q} = \hat{\mathcal{O}}$ , and that  $|A(t)\rangle$  and  $|B(t)\rangle$  time-develop according to the Schrödinger equations with  $\hat{H}$  and  $\hat{H}^{\dagger Q}$  and are Q-normalized at the initial time  $T_A$  and at the final time  $T_B$ , respectively,  $\langle\hat{\mathcal{O}}\rangle_Q^{BA}$  becomes real and time-develops under a Q-Hermitian Hamiltonian for  $|A(t)\rangle$  and  $|B(t)\rangle$  such that the absolute value of the transition amplitude  $|\langle B(t)|_Q A(t)\rangle|$  is maximized. First we proved the theorem in the case of non-Hermitian Hamiltonians based on Refs. [27,29]. Next we provided

another theorem particular to the case of Hermitian Hamiltonians, and proved it, based on Refs. [28,29]. It is noteworthy that, both in the future-included CAT and RAT, we have obtained a real average for  $\hat{O}$  at any time  $t$  by means of the simple expression  $\langle \hat{O} \rangle_Q^{BA}$ , though it is generically complex by definition. In addition, we emphasize that, in the case of non-Hermitian Hamiltonians, we have obtained a Q-Hermitian Hamiltonian.

In the usual theory, i.e., the future-not-included RAT, the expectation value of  $\hat{O}$ ,  $\langle \hat{O} \rangle^{AA}$ , is constructed to be real for a Hermitian operator  $\hat{O}$  by definition. Similarly, even in the future-not-included CAT,  $\langle \hat{O} \rangle_Q^{AA}$  is real for a Q-Hermitian operator  $\hat{O}$ . On the other hand, in the future-included CAT and RAT,  $\langle \hat{O} \rangle_Q^{BA}$  is not adjusted so, but it becomes real by our natural way of thinking, the maximization principle. In addition,  $\langle \hat{O} \rangle_Q^{BA}$  is expressed more elegantly than  $\langle \hat{O} \rangle_Q^{AA}$  in the functional integral form:

$$\langle \hat{O} \rangle_Q^{BA} = \frac{\int \mathcal{D}\text{path } \psi_B^* \psi_A Q \mathcal{O} e^{\frac{i}{\hbar} S[\text{path}]}}{\int \mathcal{D}\text{path } \psi_B^* \psi_A Q e^{\frac{i}{\hbar} S[\text{path}]}}. \quad (9.50)$$

In the future-not-included theories  $\langle \hat{O} \rangle_Q^{AA}$  does not have such a full functional integral expression for all time. Therefore,  $\langle \hat{O} \rangle_Q^{BA}$  seems to be more natural than  $\langle \hat{O} \rangle_Q^{AA}$ , and we can speculate that the fundamental physics is given by  $\langle \hat{O} \rangle_Q^{BA}$  in the future-included theories rather than by  $\langle \hat{O} \rangle_Q^{AA}$  in the future-not-included theories. This interpretation provides a more direct connection of functional integrals to measurable physics.

In such future-included theories we are naturally motivated to consider the maximization principle. If we do not use it,  $\langle \hat{O} \rangle_Q^{BA}$ , which is expected to have a role of an expectation value in the future-included theories, is generically complex by definition not only in the CAT but also in the RAT. This situation is analogous to the usual classical physics, where classical solutions are generically complex, unless we impose an initial condition giving the reality. Therefore, the maximization principle could be regarded as a special type of initial (or final) condition. Indeed, in the case of the future-included CAT, it specifies a unique class of combinations of  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$ . On the other hand, in the case of the future-included RAT, the maximization principle does not specify such a unique class, but only gives the proportionality relation: Eq.(9.36), and thus leaves the initial condition to be chosen arbitrarily. This is in contrast to the case of the future-included CAT. Thus the specification of the future and past states by the maximization principle is more ambiguous in the RAT than in the CAT. In this sense, the future-included CAT seems to be nicer than the future-included RAT, though it still requires a bit of phenomenological adjustment of the imaginary part of the action to get a cosmologically or experimentally good initial condition, and also suggests a periodic universe.

Therefore, we speculate that the functional integral formalism of quantum theory would be most elegant in the future-included CAT. Though the future-included CAT looks very exotic, it cannot be excluded from a phenomenological point of view[23,24]. Only the maximization principle would be needed in addition to the imaginary part of the action. The future-included CAT supplemented with

the maximization principle could provide a unification of an initial condition prediction and an equation of motion.

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## References

1. H. B. Nielsen and M. Ninomiya, Proc. Bled 2006: What Comes Beyond the Standard Models, pp.87-124 (2006) [arXiv:hep-ph/0612250].
2. H. B. Nielsen and M. Ninomiya, Int. J. Mod. Phys. A **23**, 919 (2008).
3. H. B. Nielsen and M. Ninomiya, Int. J. Mod. Phys. A **24**, 3945 (2009).
4. H. B. Nielsen and M. Ninomiya, Prog. Theor. Phys. **116**, 851 (2007).
5. H. B. Nielsen and M. Ninomiya, Proc. Bled 2007: What Comes Beyond the Standard Models, pp.144-185 (2007) [arXiv:0711.3080 [hep-ph]].
6. H. B. Nielsen and M. Ninomiya, arXiv:0910.0359 [hep-ph].
7. H. B. Nielsen, Found. Phys. **41**, 608 (2011) [arXiv:0911.4005[quant-ph]].
8. H. B. Nielsen and M. Ninomiya, Proc. Bled 2010: What Comes Beyond the Standard Models, pp.138-157 (2010) [arXiv:1008.0464 [physics.gen-ph]].
9. H. B. Nielsen, arXiv:1006.2455 [physics.gen-ph].
10. H. B. Nielsen and M. Ninomiya, arXiv:hep-th/0701018.
11. H. B. Nielsen, arXiv:0911.3859 [gr-qc].
12. H. B. Nielsen, M. S. Mankoc Borstnik, K. Nagao, and G. Moulataka, Proc. Bled 2010: What Comes Beyond the Standard Models, pp.211-216 (2010) [arXiv:1012.0224 [hep-ph]].
13. K. Nagao and H. B. Nielsen, Prog. Theor. Phys. **125**, 633 (2011).
14. K. Nagao and H. B. Nielsen, Prog. Theor. Phys. **126**, 1021 (2011); **127**, 1131 (2012) [erratum].
15. K. Nagao and H. B. Nielsen, Int. J. Mod. Phys. A **27**, 1250076 (2012).
16. C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
17. C. M. Bender, S. Boettcher, and P. Meisinger, J. Math. Phys. **40**, 2201 (1999).
18. C. M. Bender and P. D. Mannheim, Phys. Rev. D **84**, 105038 (2011).
19. A. Mostafazadeh, J. Math. Phys. **43**, 3944 (2002).
20. A. Mostafazadeh, J. Math. Phys. **44**, 974 (2003).
21. K. Nagao and H. B. Nielsen, Prog. Theor. Exp. Phys. **2013**, 073A03 (2013).
22. K. Nagao and H. B. Nielsen, arXiv:1709.10179 [quant-ph].
23. K. Nagao and H. B. Nielsen, Prog. Theor. Exp. Phys. **2013**, 023B04 (2013).
24. K. Nagao and H. B. Nielsen, Proc. Bled 2012: What Comes Beyond the Standard Models, pp.86-93 (2012) [arXiv:1211.7269 [quant-ph]].
25. Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
26. Y. Aharonov, S. Popescu, and J. Tollaksen, Phys. Today **63**, 27 (2010).
27. K. Nagao and H. B. Nielsen, Prog. Theor. Exp. Phys. **2015**, 051B01 (2015).
28. K. Nagao and H. B. Nielsen, Prog. Theor. Exp. Phys. **2017**, 081B01 (2017).
29. K. Nagao and H. B. Nielsen, "Fundamentals of Quantum Complex Action Theory", to be published by Lambert Academic Publishing.
30. F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).



## 10 Bosons Being Their Own Antiparticles in Dirac Formulation <sup>★</sup>

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**Abstract.** Using our earlier formalism of extending the idea of the Dirac sea (negative energy states) to also for Bosons, we construct a formalism for Bosons which are their own antiparticles. Since antiparticles in formalisms with a Dirac sea are at first formulated as holes, they are a priori formally a bit different from the particles themselves. To set up a formalism/a theory for Majorana fermions and for Bosons in which particles are their own antiparticles is thus at first non-trivial. We here develop this not totally trivial formalism for what one could call extending the name for the fermions, “Majorana-bosons”. Because in our earlier work had what we called “different sectors” we got there some formal extensions of the theory which did not even have positive definite metric. Although such unphysical sectors may a priori be of no physical interest one could hope that they could be helpful for some pedagogical deeper understanding so that also the formalism for particles in such unphysical sectors of their own antiparticles would be of some “academic” interest.

**Povzetek.** Avtorja razširita pojm Diracovega morja (stanj z negativno energijo) na bozone, ki so sami sebi antidelci. V kontekstu Diracovega morja so antidelci vrzeli v morju, tedaj formalno različni od delcev. Konstrukcija formalizma za Majoranine fermione in bozone, v kateri so delci enaki antidelcem, je zato vsaj na prvi pogled netrivialna. V tem prispevku avtorja napravita ta netrivialni korak in gledata fermione kot “Majoranine bozone”. V predhodnem delu avtorjev definirata “sektorje”, v katerih metrika ni pozitivno definitna. Taki sektorji sicer morda nimajo fizikalnega pomena, lahko pa pomagajo pedagoško globlje razumeti obravnavani formalizem.

Keywords: Dirac sea, Majorana fermion, Dirac sea for Bosons, Quantum field theory, String field theory

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## 10.1 Introduction

Majorana [1] put forward the idea of fermions having the property of being their own antiparticles and such fermions are now usually called Majorana fermions or Majorana particles.

The Majorana field  $\psi_M$  is defined in general as real or hermitean  $\psi_M^\dagger = \psi_M$ .

Among the known bosons we have more commonly bosons, which are their own antiparticles, and which we could be tempted to call analogously “Majorana bosons” (a more usual name is “real neutral particles” [2]), such as the photon,  $Z^0$ ,  $\pi^0$ , ... particles. Now the present authors extended [3,4] the Dirac sea idea [5] of having negative energy electron single particle states in the second quantized theory being already filled in vacuum, also to Bosons. This extension of the Dirac sea idea to Bosons has a couple of new features:

- 1) We had to introduce the concept of having a negative number of bosons in a single particle state. We described that by considering the analogy of a single particle state in which a variable number of bosons can be present to a harmonic oscillator, and then extend their wave functions from normalizable to only be analytical. The harmonic oscillator with wave functions allowed to be non-normalizable and only required to be analytical has indeed a spectrum of energies  $E_n = (n + \frac{1}{2})\omega$  where now  $n$  can be all integers  $n = \dots, -3, -2, -1, 0, 1, 2, \dots$ . So it corresponds to that there can be a negative number of bosons in a single particle state.
- 2) It turns out though that these states - of say the “analytical wave function harmonic oscillator” corresponding to negative numbers of bosons have alternating norm square: For  $n \geq 1$  we have as usual  $\langle n|n \rangle = 1$  for  $n \geq 0$  (by normalization) but for  $n \leq -1$  we have instead  $\langle n|n \rangle = c \cdot (-1)^n$  for  $n$  negative. ( $c$  is just a constant we put say  $c = +1$ .) This variation of norm square is needed to uphold the usual rules for the creation  $a^+$  and annihilation  $a$  operators

$$\begin{aligned} a^+|n\rangle &= \sqrt{1+n}|n+1\rangle \\ a|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned} \tag{10.1}$$

to be valid also for negative  $n$ .

- 3) With the relations (10.1) it is easily seen that there is a “barrier” between  $n = -1$  and  $n = 0$  in the sense that the creation and annihilation operators  $a^+$ , and  $a$  cannot bring you across from the space spanned by the  $n = 0, 1, 2, \dots$  states to the one spanned by the  $n = -1, -2, \dots$  one or opposite. It is indeed best to consider the usual space spanned by the  $|n\rangle$ 's with  $n = 0, 1, 2, \dots$  as one separate “sector” the “positive sector” and the one spanned by the  $|n\rangle$  states with  $n = -1, -2, \dots$  as another “sector” called the “negative sector”. Since in the harmonic oscillator with the wave functions only required to be analytical but not normalizable the states in the “positive sector” are not truly orthogonal to those in the “negative sector” but rather have divergent or ill-defined inner products with each other, it is best not even to (allow) consider inner products like say

$$\begin{aligned}\langle 0 | -1 \rangle &= \text{ill defined} \\ \langle n | p \rangle &= \text{ill defined}\end{aligned}\tag{10.2}$$

when  $n \leq -1$  and  $p \geq 0$  or opposite.

Basically we shall consider only one sector at a time.

- 4) The use of our formalism with negative number of particles to connect to the usual and physically correct description of bosons with some charge or at least an (at first) conserved particle number comes by constructing a “Dirac sea for bosons”. That is to say one first notes that e.g. the free Klein-Gordon equation

$$\square \phi = 0$$

has both positive and negative energy solutions, and that the inner product

$$\langle \varphi_1 | \varphi_2 \rangle = \int \varphi_1^* \overleftrightarrow{\partial}_0 \varphi_2 d^3 \vec{x}\tag{10.3}$$

gives negative norm square for negative energy eigenstates and positive norm square for positive energy eigenstates.

Then the physical or true world is achieved by using for the negative energy single particle states the “negative sector” (see point 3) above) while one for the positive energy single particle states use the “positive sector”. That is to say that in the physical world there is (already) a negative number of bosons in the negative energy single particle states. In the vacuum, for example, there is just  $-1$  boson in each negative energy single particle state.

This is analogous to that for fermions there is in the Dirac sea just  $+1$  fermion in each negative energy single particle state. For bosons – where we have  $-1$  instead  $+1$  particle– we just rather emptied Dirac sea by one boson in each single particle negative energy state being removed from a thought upon situation with with 0 particles everywhere. (Really it is not so nice to think on this removal because the “removal” cross the barrier from the “positive sector” to the “negative sector” and strictly speaking we should only look at one sector at a time (as mentioned in 3).)

- 5) It is rather remarkable that the case with the “emptied out Dirac sea” described in 4) – when we keep to positive sector for positive energy and negative sector for negative energy– we obtain a positive definite Fock space. This Fock space also has only positive energy of its excitations as possibilities. Indeed we hereby obtained exactly a Fock space for a theory with bosons, that are different from their antiparticles.

In the present paper we like to study how to present a theory for bosons which are their own antiparticles, Majorana bosons so to speak, in this formalism with the “emptied” Dirac sea.

Since in the Dirac sea formalisms – both for fermions and for bosons – an antiparticle is the removal of a particle from the Dirac sea, an antiparticle a priori



is something quite different from a particle with say positive energy. Therefore to make a theory / a formalism for a theory with particle being identified with its antiparticles – as for Majorana fermions or for the photon,  $Z^0, \pi^0$  – is in our or Dirac's Dirac sea formalisms a priori not trivial. Therefore this article. Of course it is at the end pretty trivial, but we think it has value for our understanding to develop the formalism of going from the Dirac sea type picture to the theories with particles being their own antiparticles ("Majorana theories").

One point that makes such a study more interesting is that we do not have to only consider the physical model in the boson case with using positive sector for positive single particle energy states and negative sector for negative energy single particle states. Rather we could – as a play – consider the sectors being chosen in a non-physical way. For example we could avoid "emptying" the Dirac sea in the boson-case and use the positive sector for both negative and positive energy single boson eigenstates. In this case the Fock space would not have positive norm square. Rather the states with an odd number of negative energy bosons would have negative norm square, and of course allowing a positive number of negative energy bosons leads to their being no bottom in the Hamiltonian for such a Fock space.

The main point of the present article is to set up a formalism for particles that are their own antiparticles (call them "Majorana") on the basis of a formalism for somehow charged particles further formulated with the Dirac sea. That is to say we consider as our main subject how to restrict the theory with the Dirac sea – and at first essentially charged particle – to a theory in which the particles and antiparticles move in the same way and are identified with each other.

For example to describe a one-particle state of a "Majorana" particle one would naturally think that one should use a state related to either the particle or the antiparticle for instance being a superposition of a particle and antiparticle state.

So the states of the Fock space  $H_{\text{Maj}}$  for describing the particles which are their own antiparticles shall be below identified with some corresponding states in the theory with Dirac sea. However, there are more degrees of freedom in a theory with charged particles (as the Dirac sea one) than in a corresponding theory for particles which are their own antiparticle. Thus the states in the with Dirac sea Fock space cannot all be transferred to the Fock space from "Majorana" particles. So only a certain subspace of the Fockspace for the with Dirac sea theory can be identified with states of some number of Majorana particles.

To develop our formalism for this transition from the Dirac sea theory to the one for Majorana particles, we therefore need a specification of which subspace is the one to be used to describe the "Majorana particles". Below we shall argue for that this subspace  $H_{\text{Maj}}$  becomes

$$H_{\text{Maj}} = \{ | \rangle | (a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)) | \rangle = 0, \text{ for all } \vec{p} \} \quad (10.4)$$

where we used the notation of  $a(\vec{p}, E < 0)$  for the annihilation operator for a particle with momentum  $\vec{p}$  and energy  $E$  corresponding to that being positive i.e.  $E > 0 \Rightarrow E = \sqrt{m^2 + \vec{p}^2}$ . Correspondingly the annihilation operator  $a(\vec{p}, E < 0)$

annihilates a particle with energy  $E = -\sqrt{m^2 + \vec{p}^2}$ . The corresponding creation operators just have the dagger  $\dagger$  attached to the annihilation operator, as usual. We define

$$r(\vec{p}) = \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)) \quad (10.5)$$

This then shall mean, that we should identify a basis, the basis elements of which have a certain number of the “Majorana bosons”, say, with some momenta – in a physical world we only expect conventional particles with positive energy – for the subspace  $H_{\text{Maj}}$  contained in the full space with Dirac sea.

We thus have to construct below creation  $b^\dagger(\vec{p})$  and annihilation  $b(\vec{p})$  operators for the particles which are their own antiparticles (“Majoranas”). These  $b^\dagger(\vec{p})$  and  $b(\vec{p})$  should now in our work be presented by formulas giving them in terms of the creation and annihilation operators for the theory with Dirac sea (and thus acting on the Fock space  $H$  of this “full” theory). In fact we shall argue for (below)

$$b_n^\dagger = \begin{cases} \frac{(a^\dagger(a^\dagger \vec{p}, E > 0) - a(-\vec{p}, E < 0))}{\sqrt{2}} & \text{(on pos. sec for pos. E, neg sec for neg E)} \\ \frac{(a^\dagger(a^\dagger \vec{p}, E > 0) + a(-\vec{p}, E < 0))}{\sqrt{2}} & \text{for both sectors} \\ \dots & \end{cases} \quad (10.6)$$

and then of course it has to be so that these  $b^\dagger(\vec{p})$  and  $b(\vec{p})$  do not bring a Hilbert vector out of the subspace  $H_{\text{Maj}}$  but let it stay there once it is there. It would be the easiest to realize such a keeping inside  $H_{\text{Maj}}$  by action with  $b^\dagger(\vec{p})$  – and we shall have it that way – if we arrange the commutation rules

$$\begin{aligned} [r(\vec{p}), b^\dagger(\vec{p}')] &= 0 \\ [r(\vec{p}), b(\vec{p}')] &= 0 \end{aligned} \quad (10.7)$$

(Here the commutation for  $\vec{p} \neq \vec{p}'$  is trivial because it then concerns different d.o.f. but the  $[r(\vec{p}), b^\dagger(\vec{p}')] = 0$  and  $[r(\vec{p}), b(\vec{p}')] = 0$  are the nontrivial relations to be arranged (below))

Indeed we shall find below

$$\begin{aligned} b^\dagger(\vec{p}) &= \frac{1}{\sqrt{2}} (a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \quad (\text{defined on both pos.}) \\ b(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)) \end{aligned} \quad (10.8)$$

It is then that we shall arrange that if we extrapolate to define also the  $b^{(\dagger)}(\vec{p}, E < 0)$  and not only for positive energy  $b^{(\dagger)}(\vec{p}, E > 0) = b^{(\dagger)}(\vec{p})$  we should obtain the formula usual in conventional description of Majorana particle theories

$$\begin{aligned} b^\dagger(\vec{p}) &= b^\dagger(\vec{p}, E > 0) = b(-\vec{p}, E < 0) \\ b(\vec{p}) &= b(\vec{p}, E > 0) = b^\dagger(-\vec{p}, E < 0) \end{aligned} \quad (10.9)$$

For fermions we simply do construct these  $r(\vec{p})$  and  $b(\vec{p})$  rather trivially and it must be known in some notation to everybody. For bosons, however almost nobody but ourselves work with Dirac sea at all, and therefore it must be a bit more new to get particles which are their own antiparticles into such a scheme. For bosons also we have already alluded to the phenomenon of different “sectors” (see 3) above) being called for due to our need for negative numbers of particles. We therefore in the present article as something also new have to see what becomes of the theory with bosons being their own antiparticles when we go to the unphysical sector-combinations. (The physical combination of sectors means as described in point 4) above, but if we e.g. have the positive sector both for negative and positive energy single particle states, this is a unphysical sector-combination.) This is a priori only a discussion though of academic interest, since the truly physical world corresponds to the physical combination described in point 4) with the Dirac sea “emptied out”.

However, in our attempts to describe string field theory in a novel way we raised to a problem that seemed formally to have solution using such on unphysical sector-combination.

In the following section 10.2 we just, as a little warm up, discuss the introduction in the Majorana fermion theory on a subspace of the Fock space of a fermion theory in Dirac sea formulation.

In section 10.3 we then review with more formalism our “Dirac sea for bosons” theory.

Then in section 10.4 we introduce the formalism  $r(\vec{p})$ ,  $b^\dagger(\vec{p})$  and  $b(\vec{p})$  relevant for the Majorana rather theory or for particles which are their own antiparticles. The operators  $r(\vec{p})$  defined in (10.5) are the operators defined to be used for singling out the Majorana subspace, and  $b^\dagger(\vec{p})$  and  $b(\vec{p})$  are the creation and annihilation operators for “Majorana-bosons”.

In section 10.5 we go to the unphysical sector combinations to study the presumably only of academic interest problems there.

In section 10.6 we bring conclusion and outlook.

## 10.2 Warming up by Fermion

### 10.2.1 Fermion Warm Up Introduction

As the warming up consider that we have a fermion theory at first described by making naively (as if nonrelativistically, but we consider relativity) creation  $a^\dagger(n, \vec{p}, E > 0)$  and  $a^\dagger(n, \vec{p}, E < 0)$  for respectively positive and negative energy  $E$  of the single particle state. Also we consider the corresponding annihilation operators  $a(\sigma, \vec{p}, E > 0)$  and  $a(\sigma, \vec{p}, E < 0)$

The physically relevant second quantized system takes its outset in the physical vacuum in which all the negative energy  $E < 0$  single particle states are filled while the positive energy ones are empty.

$$| \text{vac phys} \rangle = \prod_{\sigma, \vec{p}} a^\dagger(\sigma, \vec{p}, E < 0) | 0 \text{ totally empty} \rangle \quad (10.10)$$

Of course in modern practice you may ignore the Dirac sea and just start from the physical vacuum  $|\text{vac phys}\rangle$  and operate on that with creation and annihilation operators. If you want to say create on antiparticle with momentum  $\vec{p}$  (and of course physically wanted positive energy) you operate on  $|\text{vac phys}\rangle$  with

$$a_{\text{anti}}^\dagger(\sigma, \vec{p}, E > 0) = a(\sigma^1, -\vec{p}, E < 0) \quad (10.11)$$

i.e. the antiparticle creation operator  $a_{\text{anti}}^\dagger(\sigma, \vec{p}, E > 0)$  is equal to the annihilation operator  $a(\sigma^1, -\vec{p}, E < 0)$  with the “opposite” quantum numbers.

### 10.2.2 Constructing Majorana

Now the main interest of the present article is how to construct a theory of particles being their own antiparticle (“Majorana”) from the theory with essentially charged particles – carrying at least a particle-number “charge” – by appropriate projection out of a sub-Fock space and by constructing creation and annihilation operators for the Majorana –in this section– fermions.

Let us remark that this problem is so simple, that we can do it for momentum value, and if we like to simplify this way we could decide to consider only one single value of the momentum  $\vec{p}$  and spin. Then there would be only two creation and two annihilation operators to think about

$$\begin{aligned} a^\dagger(E > 0) &= a^\dagger(\sigma, \vec{p}, E > 0) \\ a^\dagger(E < 0) &= a^\dagger(\sigma, \vec{p}, E < 0) \end{aligned} \quad (10.12)$$

and thus the whole Fock space, we should play with would only have  $2 \cdot 2 = 4$  states, defined by having filled or empty the two single particle states being the only ones considered in this simplifying description just denoted by “ $E > 0$ ” and “ $E < 0$ ”.

In fact the construction of a full Majorana-formalism will namely be obtained by making the construction of the Majorana Fock (or Hilbert) space for each momentum  $\vec{p}$  and spin and then take the Cartesian product of all the obtained Majorana-Fock spaces, a couple for each spin and momentum combination.

The four basis states in the Fock space after throwing away all but one momentum and one spin-state are:

$$\begin{aligned} |1 \text{ antiparticle}\rangle &= |\text{vac totally empty}\rangle \\ |\text{vac phys}\rangle &= a^\dagger(E < 0) |\text{vac totally empty}\rangle \\ |1 \text{ fermion in phys}\rangle &= a^\dagger(E > 0) a^\dagger(E < 0) |\text{vac totally empty}\rangle \\ |\text{both particle and antip.}\rangle &= a^\dagger(E > 0) |\text{vac totally empty}\rangle \end{aligned} \quad (10.13)$$

Considering the situation from the point of view of the physical vacuum

$$|\text{vac phys}\rangle = a^\dagger(E < 0) |\text{vac totally empty}\rangle \quad (10.14)$$

creating a Majorana particle should at least either a particle or an antiparticle or some superposition of the two (but not both).

So the one Majorana particle state should be a superposition of

$$|1 \text{ fermion in phys}\rangle = a^\dagger(E > 0)a^\dagger(E < 0) | \text{vac totally empty}\rangle \quad (10.15)$$

and

$$|1 \text{ antiferm in phys}\rangle = | \text{vac totally empty}\rangle \quad (10.16)$$

The most symmetric state would naturally be to take with coefficients  $\frac{1}{\sqrt{2}}$  these two states with equal amplitude:

$$|1 \text{ Majorana}\rangle = \frac{1}{\sqrt{2}} (a^\dagger(E > 0)a^\dagger(E < 0) + 1) | \text{vac totally empty}\rangle \quad (10.17)$$

We should then construct a creation operators  $b^\dagger(\sigma, \vec{p})$  or just  $b^\dagger$  so that

$$b^\dagger | \text{vac phys}\rangle = |1 \text{ Majorana}\rangle \quad (10.18)$$

Indeed we see that

$$b^\dagger = \frac{1}{\sqrt{2}} (a^\dagger(E > 0) + a(E < 0)) \quad (10.19)$$

will do the job.

If we use  $b^\dagger$  and

$$b = \frac{1}{\sqrt{2}} (a(E > 0) + a^\dagger(E < 0)) \quad (10.20)$$

it turns out that states needed are the – superpositions of –

$$| \text{vac phys}\rangle = a^\dagger(E < 0) | \text{vac totally empty}\rangle \quad (10.21)$$

and

$$b^\dagger | \text{vac phys}\rangle = |1 \text{ Majorana}\rangle \quad (10.22)$$

This subspace which in our simplification of ignoring all but one momentum and spin state actually represents the whole space  $H_{\text{Maj}}$  used to describe the Majorana theory has only 2 dimensions contrary to the full Hilbert space  $H$  which in our only one momentum and spin consideration has 4 dimensions.

So it is a genuine subspace and we shall look for an operator  $r = r(\hbar, \vec{p})$  which gives zero when acting on  $H_{\text{Maj}}$  but not when it acts on the rest of  $H$ .

It is easily seen that

$$r = \frac{1}{\sqrt{2}} (a(E > 0) - a^\dagger(E < 0)) \quad (10.23)$$

will do the job. Thus we can claim that

$$H_{\text{Maj}} = \{ | \rangle \mid r | \rangle = 0 \} \quad (10.24)$$

Written for the full theory with all the momenta and spins we rather have

$$H_{\text{Maj}} = \{ | \rangle \in H \mid \forall_h \vec{p} [r(\vec{p}, h) | \rangle = 0] \} \quad (10.25)$$

where

$$r(\vec{p}, h) = \frac{1}{\sqrt{2}} (a(\vec{p}, h, E > 0) - a^\dagger(-\vec{p}, h, E < 0)) \quad (10.26)$$

and  $a(\vec{p}, h, E > 0)$  is the annihilation operator for a fermion with momentum  $\vec{p}$  and eigenstate  $h$  of the normalized helicity

$$h \sim \vec{\Sigma} \cdot \vec{p} / |\vec{p}| \quad (10.27)$$

where  $\vec{\Sigma}$  is the spin angular momentum, and the energy  $E = +\sqrt{\vec{p}^2 + m^2}$ .

The fully described creation operator for a Majorana particle fermion with momentum  $\vec{p}$  and helicity  $h$ ,

$$b^\dagger(\vec{p}, h) = \frac{1}{\sqrt{2}} (a^\dagger(\vec{p}, h, E > 0) + a(-\vec{p}, h, E < 0)) \quad (10.28)$$

and the corresponding annihilation operator

$$b(\vec{p}, h) = \frac{1}{\sqrt{2}} (a(\vec{p}, h, E > 0) + a^\dagger(-\vec{p}, h, E < 0)) \quad (10.29)$$

One easily checks that the operation with these operators  $b(\vec{p}, h)$  and  $b^\dagger(\vec{p}, h)$  map  $H_{\text{Maj}}$  on  $H_{\text{Maj}}$  because

$$\begin{aligned} \left\{ r(\vec{p}', h'), b(\vec{p}, h) \right\}_+ &= 0 \\ \left\{ r(\vec{p}', h'), b^\dagger(\vec{p}, h) \right\}_+ &= 0 \end{aligned} \quad (10.30)$$

### 10.3 Review of Dirac Sea for Bosons

Considering any relativistically invariant dispersion relation for a single particle it is, by analyticity or better by having a finite order differential equation, impossible to avoid that there will be both negative and positive energy (eigen) solutions. This is true no matter whether you think of integer or half integer spin or on bosons or fermions (the latter of course cannot matter at all for a single particle theory). In fact this unavoidability of also negative energy single particle states is what is behind the unavoidable CPT-theorem.

There is for each type of equation a corresponding inner product for single particle states, so that for instance the Klein-Gordon equation and the Dirac equation have respectively

$$\langle \varphi_1 | \varphi_2 \rangle = \int \varphi_1^* \frac{\overleftrightarrow{\partial}}{\partial t} \varphi_2 d^3\vec{X} \text{ (Klein Gordon)} \quad (10.31)$$

and

$$\langle \psi_1 | \psi_2 \rangle = \int \psi_1^\dagger \psi_2 d^3\vec{X} = \int \vec{\psi}_1 \gamma^0 \psi_2 d^3\vec{X} \text{ (for Dirac equation)} \quad (10.32)$$

( see e.g. [7])

At least in these examples –but it works more generally– the inner product of a single particle state with itself, the norm square, gets negative for integer spin and remains positive for the half integer spin particles, when going to the negative energy states.

For integer spin particles (according to spin statistics theorem taken to be bosons) as for example a scalar we thus have negative norm square for the negative energy single particle states. This means that for all the states for which we want to make an analogy to the filling of the Dirac sea, we have to have in mind, that we have this negative norm square.

That is to say, that thinking of second quantizing the norm square of a multiple particle state in the Fock space would a priori alternate depending on whether the number of particles (bosons) with negative energy is even or odd.

Physically we do not want such a Fock space, which has non-positive-definite norm –since for the purpose of getting positive probabilities we need a positive definite inner product –.

The resolution to this norm square problem in our “Dirac sea for bosons” – model is to compensate the negative norm square by another negative norm square which appears, when one puts into a single particle state a negative number of bosons.

This is then the major idea of our ‘Dirac sea for bosons’-work, that we formally –really of course our whole model in this work is a formal game – assume that it is possible to have a negative number of particles (bosons) in a single particle state. That is to say we extend the usual idea of the Fock space so as to not as usual have its basic vectors described by putting various non-negative numbers of bosons into each single particle state, but allow also to have a negative number of bosons.

Rather we allow also as Fock-space basis vector states corresponding to that there could be negative integer numbers of bosons. So altogether we can have any integer number of bosons in each of the single particle states (whether it has positive or negative energy at first does not matter, you can put any integer number of bosons in it anyway).

In our “Dirac-sea for Bosons” –paper [3] we present the development to include negative numbers of particles via the analogy with an harmonic oscillator. It is well-known that a single particle state with a non-negative number of bosons in it is in perfect correspondance with a usual harmonic oscillator[6] in which

the number of excitations can be any positive number or zero. If one extend the harmonic oscillator to have in the full complex plan extending the position variable  $q$  (say) and the wave function  $\psi(q)$  to be formally analytical wave function only, but give up requiring normalizability, it turns out that the number of excitations  $n$  extends to  $n \in \mathbb{Z}$ , i.e. to  $n$  being any integer. This analogy to extend harmonic oscillator can be used to suggest how to build up a formalism with the creation  $a^\dagger$  and annihilation operators  $a$  and an inner product for a single particle states in which one can have any integer number of bosons.

It is not necessary to use extended harmonic oscillator. In fact one could instead just write down the usual relations for creation and annihilation operators first for a single particle state say

$$a^\dagger(\vec{p}, E > 0) |k(\vec{p}, E > 0)\rangle = \sqrt{k(\vec{p}, E > 0) + 1} |k(\vec{p}, E > 0) + 1\rangle \quad (10.33)$$

and

$$a(\vec{p}, E > 0) |k(\vec{p}, E > 0)\rangle = \sqrt{k(\vec{p}, E > 0)} |k(\vec{p}, E > 0) - 1\rangle \quad (10.34)$$

or the analogous ones for a negative energy single particle state

$$a^\dagger(\vec{p}, E < 0) |k(\vec{p}, E < 0)\rangle = \sqrt{k(\vec{p}, E < 0) + 1} |k(\vec{p}, E < 0) + 1\rangle \quad (10.35)$$

and

$$a(\vec{p}, E < 0) |k(\vec{p}, E < 0)\rangle = \sqrt{k(\vec{p}, E < 0)} |k(\vec{p}, E < 0) - 1\rangle \quad (10.36)$$

and then extend them - formally by allowing the number  $k(\vec{p}, E > 0)$  of bosons in say a positive energy single particle state with momentum  $\vec{p}$  and (positive energy) to be also allowed to be negative. You shall also allow the numbers  $k(\vec{p}, E < 0)$  in a negative energy single particle state with momentum  $\vec{p}$  to be both positive or zero and negative.

Then there are a couple of very important consequences:

- A)** You see from these stepping formulas that there is a "barriere" between the number of bosons  $k$  being  $k = -1$  and  $k = 0$ . Operating with the annihilation operator  $a$  on a state with  $k = 0$  particles give zero

$$a |k = 0\rangle = 0 \quad (10.37)$$

and thus does not give the  $|k = -1\rangle$  as expected from simple stepping. Similar one cannot with the creation operator  $a^\dagger$  cross the barriere in the opposite direction, since

$$a^\dagger |k = -1\rangle = \sqrt{-1 + 1} |k = 0\rangle = 0 \quad (10.38)$$



Thus we have that the states describing the number of bosons  $k$  in a given single particle state are not connected by a finite number of operations – of creation and annihilation operators.

Really this means that we make best by considering the positive sector of the space of positive or zero number of bosons and another sector formed from the  $|k\rangle$  states with  $k = -1, -2, -3, \dots$  being a negative number of bosons. By ordinary creation and annihilation operators, as they would occur in some interaction Hamiltonian, one cannot cross the barrier. This means that if to begin with one has say a negative number of boson in a given single particle state, then an ordinary interaction cannot change that fact.

Thus we take it that one can choose once forever to put some single particles states in their positive sector and others in their negative sector, and they then will stay even under operation of an interaction Hamiltonian. If one for example make the ansatz that all the negative energy single particle states have a negative number of bosons while the positive energy states have zero or a positive number of bosons in them, then this ansatz can be kept forever. This special choice we call the “physical choice” and we saw already[3] – and shall see very soon here – that this choice gives us a positive definite Fock space.

- B)** The norm square of the states  $|k\rangle$  (with  $k = -1, -2, \dots$ ) i.e. with negative numbers  $k$  of bosons have to vary alternatingly with  $k$  even,  $k$  odd. Using the writing of a negative  $k$

$$|k\rangle \sim a^{|k|-1} |k = -1\rangle \quad (10.39)$$

We may evaluate  $\langle k | k \rangle \sim \langle -1 | (a^\dagger)^{|k|-1} a^{|k|-1} | -1 \rangle$  for  $k \leq -1$ .

Now using still the usual commutation rule

$$[a^\dagger, a] = -1 \quad (10.40)$$

you easily see that we normalize by putting

$$|k\rangle = \frac{1}{\sqrt{(|k|-1)!}} a^{|k|-1} | -1 \rangle \quad (10.41)$$

and

$$\langle k | k \rangle = (-1)^{|k|} \quad (10.42)$$

say for  $k \leq -1$  (having taken  $\langle -1 | -1 \rangle = -1$ .) while of course for  $k = 0, 1, 2, \dots$  you have  $\langle k | k \rangle = 1$ .

The major success of our “Dirac sea for bosons” is that one can arrange the sign alternation with (10.42) with the total number of negative energy bosons to cancel against the sign from in (10.31) so as to achieve, if we choose the “physical sector”, to get in total the Fock space, which has positive norm square. This “physical sector” corresponds to that negative energy single particle states are in the negative sectors, while the positive energy single particle states are in the positive sector.

The basis vectors of the full Fock space for the physical sector are thus of the form

$$|\dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}, E < 0), \dots\rangle \quad (10.43)$$

where the dots ... denotes that we have one integer number for every momentum vector -value ( $\vec{p}$  or  $\vec{p}'$ ), but now the numbers  $k(\vec{p}, E > 0)$  of particles in a positive energy are- in the physical sector-combination- restricted to be non-negative while the numbers of bosons in the negative energy single particle states are restricted to be negative

$$\begin{aligned} k(\vec{p}, E > 0) &= 0, 1, 2, \dots \\ k(\vec{p}, E < 0) &= -1, -2, -3, \dots \end{aligned} \quad (10.44)$$

These basis vectors (10.43) are all orthogonal to each other, and so the inner product is alone given by their norm squares

$$\begin{aligned} &\langle \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots | \\ &| \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\ &= (-1)^{\#(\text{neg energy b})} \prod_{\vec{p}'} (-1)^{|k(\vec{p}', E < 0)|} = 1 \end{aligned} \quad (10.45)$$

Here  $\#(\text{neg energy b})$  means the total number of negative energy bosons i.e.

$$\#(\text{neg energy b}) = \sum_{\vec{p}'} k(\vec{p}', E < 0) \quad (10.46)$$

(a negative number in our physical sector-combination). The factor

$$(-1)^{\#(\text{neg energy b.})}$$

comes from (10.31) which gives negative norm square for single particle states with negative energy, because  $\frac{\partial}{\partial t}$  is essentially the energy. The other factor  $\prod_{\vec{p}'} (-1)^{|k(\vec{p}', E < 0)|}$  comes from (10.42) one factor for each negative single particle energy state, i.e. each  $\vec{p}'$ . Had we here chosen another sector-combination, e.g. to take  $k(\vec{p}, E < 0)$  non-negative as well as  $k(\vec{p}, E > 0)$ , then we would have instead

$$\left. \begin{aligned} k(\vec{p}, E > 0) &= 0, 1, 2, \dots \\ k(\vec{p}', E < 0) &= 0, 1, 2, \dots \end{aligned} \right\} (\text{both pos sectors.}) \quad (10.47)$$

and the inner with themselves, norm squares product for the still mutually orthogonal basis vectors would be

$$\begin{aligned} &\langle \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots | \\ &| \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\ &= (-1)^{\#(\text{neg energy b})} \\ &(\text{for both positive sectors}) \end{aligned} \quad (10.48)$$

and that for this case ("sector combination"), the inner product is not positive definite.

Such strange sector combination is of course mainly of academical interest. But for instance this last mentioned "both positive sector" sector-combination, can have easily position eigenstate particles in the Fock space description. Normally positon is not possible to be well defined in relativistic theories.

As already mentioned above, we have a slightly complicated inner product in as far as we have sign-factors in the inner product coming from two different sides:

- 1) The inner product sign-factor from the single particle wave function coming from (10.31) gives a minus for negative energy particles, ending up being  $(-1)^{\#(\text{neg. energy b})}$  in (10.45).

- 2) The other inner product sign factor comes from (10.42).

In the above, we have used the dagger symbol " $\dagger$ " on  $a^\dagger$  to denote the Hermitian conjugate w.r.t. only the inner product coming from (10.42), but have not included the factor from 1) meaning from (10.31). Thus we strictly speaking must consider also a full dagger (full  $\dagger_f$ ) meaning hermitian conjugation corresponding the full inner product also including 1), i.e. the (10.31) extra minus for the negative energy states. So although we have not changed  $a(\vec{p}, E > 0)$  nor  $a(\vec{p}, E < 0)$  we have to distinguish two different  $a^\dagger$ 's namely  $a^\dagger$  and  $a^{\dagger_f}$ . In fact we obtain with this notation of two different  $\dagger$ (')s.

$$a^{\dagger_f}(\vec{p}, E > 0) = a^\dagger(\vec{p}, E > 0) \quad (10.49)$$

but

$$a^{\dagger_f}(\vec{p}, E < 0) = -a^\dagger(\vec{p}, E < 0) \quad (10.50)$$

Since at the end, the physical/usual second quantized Boson-theory has as its inner product the full inner product one should, in the physical use, use the Hermitian conjugation  $\dagger_f$ . So the creation operators to be identified with creation operators are respectively:

For a particle;

$$a_{\text{usual}}^\dagger(\vec{p}) = a^{\dagger_f}(\vec{p}, E > 0) \quad (10.51)$$

while for an antiparticle of momentum  $\vec{p}$  it is;

$$a_{\text{usual anti}}^\dagger(\vec{p}) = a^\dagger(-\vec{p}, E < 0) \quad (10.52)$$

Similarly:

$$\begin{aligned} a_{\text{usual}}(\vec{p}) &= a(\vec{p}, E > 0) \\ a_{\text{usual anti}}(\vec{p}) &= a^{\dagger_f}(-\vec{p}, E < 0) = -a^\dagger(-\vec{p}, E < 0) \end{aligned} \quad (10.53)$$

Using the extended commutation rules

$$[a(\vec{p}, \geq E), a^\dagger(\vec{p}', \geq E)] = \delta_{\vec{p}'\vec{p}} \cdot \begin{cases} 1 & \text{for same } < \text{ or } > \\ 0 & \text{for different } < \text{ or } > \end{cases} \quad (10.54)$$

so that for instance

$$[a(\vec{p}, < E), a^\dagger(\vec{p}', < E)] = \delta_{\vec{p}\vec{p}'} \quad (10.55)$$

We quickly derive the corresponding commutation rules using the “full dagger”

$$[a(\vec{p}, E > 0), a^\dagger(\vec{p}', E > 0)] = \delta_{\vec{p}\vec{p}'} \quad (10.56)$$

and

$$[a(\vec{p}, E < 0), a^\dagger(\vec{p}', E < 0)] = \delta_{\vec{p}\vec{p}'} \quad (10.57)$$

## 10.4 “Majorana-bosons”

We shall now in this section analogously to what we did in sections 10.2 for Fermions as a warm up exercise from our Fock space defined in section 10.3 for e.g. the physical sector- combination extract a subspace  $H_{\text{Maj}}$  and on that find a description of now bosons which are their own antiparticles. There would be some meaning in analogy to the Fermion case to call such bosons which are their own antiparticles by the nickname “Majorana-bosons”.

As for the fermions we shall expect a state with say  $k_{\text{Maj}}(\vec{p})$  “Majorana-bosons” with momentum equal to  $\vec{p}$  to be presented as a superposition of a number of the “essentially charged” bosons or antibosons of the type discussed in foregoing section. Here an antiparticle of course means that one has made the number of bosons in a negative energy single particle state one unit more negative. Typically since the physical vacuum has  $k(\vec{p}, E < 0) = -1$  for all momenta and an antiparticle of momentum  $\vec{p}$  would mean that  $k(-\vec{p}, E < 0)$  gets decreased from  $-1$  to  $-2$ . If you have several antiparticles  $l$  say in the same state with momentum  $\vec{p}$  of course you decrease  $k(-\vec{p}, E < 0)$  to  $-1 - l$ ,  $k(-\vec{p}, E < 0) = -1 - l$  (for  $l$  antiparticles).

In other words we expect a state with say  $l_{\text{Maj}}$  “Majorana-bosons” with momentum  $\vec{p}$  to be a superposition of states in the Fock space with the number of antiparticles running from  $l = 0$  to  $l = l_{\text{Maj}}$  while correspondingly the number with momentum  $\vec{p}$  is made to  $l_{\text{Maj}} - l$  so that there are together in the representing state just equally many particles or antiparticles as the number of “Majorana-bosons”  $l_{\text{Maj}}$  wanted.

We actually hope –and we shall see we shall succeed– that we can construct a “Majorana-boson” creation operator for say a “Majorana-boson” with momentum  $\vec{p}$ ,  $b^\dagger(\vec{p})$  analogously to the expressions (10.19) and (10.20)  
 $b^\dagger(\vec{p}) = \frac{1}{\sqrt{2}} (a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0))$  and  $b(\vec{p}) = \frac{1}{\sqrt{2}} (a(E > 0) + a^\dagger(E < 0)).$

Since an extra phase on the basis states does not matter so much we could also choose for the boson the “Majorana boson” creation and annihilation operators to

be

$$\begin{aligned}
 b^{\dagger f}(\vec{p}) &= \frac{1}{\sqrt{2}} (a^{\dagger}(E > 0) + a(-\vec{p}, E < 0)) \\
 &= \frac{1}{\sqrt{2}} (a^{\dagger f}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \\
 b(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^{\dagger f}(-\vec{p}, E < 0)) \\
 &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0)). \tag{10.58}
 \end{aligned}$$

One must of course then check –first on the physical sector-combination but later on others– that  $b^{\dagger}(\vec{p})$  and  $b(\vec{p})$  obey the usual commutation rules

$$\begin{aligned}
 [b(\vec{p}), b(\vec{p}')] &= 0 \\
 [b^{\dagger f}(\vec{p}), b^{\dagger f}(\vec{p}')] &= 0 \\
 [b(\vec{p}), b^{\dagger f}(\vec{p}')] &= \delta_{\vec{p}\vec{p}'}. \tag{10.59}
 \end{aligned}$$

We also have to have a vacuum for the “Majorana-boson” theory, but for that we use in the physical sector-combination theory the same state in the Fock space as the one for the “essentially charged” boson system. This common physical vacuum state (in the Fock space) is characterized as the basis vector

$$| \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}, E < 0), \dots \rangle \tag{10.60}$$

with

$$k(\vec{p}, E > 0) = 0 \text{ for all } \vec{p} \tag{10.61}$$

and

$$k(\vec{p}, E < 0) = -1 \text{ for all } \vec{p} \tag{10.62}$$

Indeed we also can check then of course that defining

$$| \text{vac phys} \rangle = | \dots, k(\vec{p}, E > 0) = 0, \dots; \dots, k(\vec{p}', E < 0) = -1, \dots \rangle \tag{10.63}$$

we have

$$b(\vec{p}) | \text{vac phys} \rangle = 0 \tag{10.64}$$

On the other hand, we can also see that e.g.

$$\begin{aligned}
 &\frac{1}{\sqrt{l_{\text{Maj}}!}} (b^{\dagger f}(\vec{p}))^{l_{\text{Maj}}} | \text{vac phys} \rangle \\
 &= \frac{1}{2^{l_{\text{Maj}}/2}} \cdot \Sigma_l \binom{l_{\text{Maj}}}{l} (a^{\dagger}(\vec{p}, E > 0))^l a(-\vec{p}, E < 0)^{l_{\text{Maj}}-l} | \text{vac phys} \rangle \\
 &= \Sigma_l \binom{l_{\text{Maj}}}{l} | \dots, k(\vec{p}, E > 0) = l, \dots; \dots, k(-\vec{p}, E < 0) = l_{\text{Maj}} - l, \dots \rangle \\
 &\cdot \sqrt{l!} \sqrt{(l_{\text{Maj}} - l)!} \cdot \frac{1}{\sqrt{l_{\text{Maj}}!}} \\
 &= \frac{1}{2^{l_{\text{Maj}}/2}} \cdot \Sigma_l \sqrt{\binom{l_{\text{Maj}}}{l}} | \dots, k(\vec{p}, E > 0), \dots; \dots, k(-\vec{p}, E < 0), \dots \rangle \tag{10.65}
 \end{aligned}$$

If we only put the Majorana-boson particles into the momentum  $\vec{p}$  state of course only  $k(\vec{p}, E > 0)$  and  $k(-\vec{p}, E < 0)$  will be different from their  $|\text{phys vac}\rangle$  values 0 and  $-1$  respectively for  $E > 0$  and  $E < 0$ . But really the extension to put “Majorana-bosons” in any number of momentum states is trivial.

We now have also to construct the analogous operator to the  $r(\vec{p})$  for the fermions so that we can characterize the subspace  $H_{Maj}$  to be for the boson case

$$H_{Maj} = \{|\rangle \mid r(\vec{p}) |\rangle = 0\}. \quad (10.66)$$

We in fact shall see that the proposal

$$\begin{aligned} r(\vec{p}) &= \frac{1}{\sqrt{2}} (a(+\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)) \\ &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0)) \end{aligned} \quad (10.67)$$

does the job.

Now we check (using (10.58))

$$\begin{aligned} [r(\vec{p}), b^{\dagger_f}(\vec{p})] &= \\ &= \left[ \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)), \frac{1}{\sqrt{2}} (a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \right] \\ \text{or} \\ &= \frac{1}{2} [(a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0)), (a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0))] \\ &= \frac{1}{2} (1 - 1) = 0 \end{aligned} \quad (10.68)$$

and also

$$\begin{aligned} [r(\vec{p}), b(\vec{p})] &= \\ &= \left[ \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0)), \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)) \right] \\ &= \left[ \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)), \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^\dagger(-\vec{p}, E < 0)) \right] \\ &= 0 \end{aligned} \quad (10.69)$$

We should also check that the physical vacuum

$$|\text{phys vac}\rangle = |\dots, k(\vec{p}, E > 0) = 0, \dots; \dots, k(\vec{p}', E < 0) = -1, \dots\rangle \quad (10.70)$$

in which there is in all negative energy (with momentum  $\vec{p}'$  say) single particle states  $k(\vec{p}', E < 0) = -1$  bosons, and in all positive energy single particle states  $k(\vec{p}, E > 0) = 0$  bosons is annihilated by the  $r(\vec{p})$  operators.

Now indeed

$$\begin{aligned}
 & r(\vec{p}) | \text{phys vac} \rangle \\
 &= \frac{1}{2} (a(\vec{p}, E > 0) - a^{\dagger r}(-\vec{p}, E, 0)) | \text{phys vac} \rangle \\
 &= \frac{1}{2} (a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E, 0)) \cdot | \dots, k(\vec{p}, E > 0) = 0, \dots; \dots, k(\vec{p}, E < 0) \\
 &= -1, \dots \rangle \\
 &= 0
 \end{aligned} \tag{10.71}$$

basically because of the barriere, meaning the square roots in the formulas (10.34,10.35) became zero.

The result of this physical section for the most attractive formalism with  $b(\vec{p})$  and  $b^{\dagger r}(\vec{p})$  annihilating and creating operators for the Boson-type particle being its own antiparticle (=Majorana Boson) and to them corresponding useful state condition operator  $r_f(\vec{p})$  is summarized as:

$$\begin{aligned}
 b(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger r}(-\vec{p}, E < 0)) \\
 b^{\dagger r}(\vec{p}) &= \frac{1}{\sqrt{2}} (a^{\dagger r}(\vec{p}, E > 0) - a(-\vec{p}, E < 0)) \\
 | \text{phys vac} \rangle &= | \dots, k(\text{all } \vec{p}, E > 0) = 0, \dots; \dots, k(\text{all } \vec{p}, E < 0) = -1, \dots \rangle \\
 r_f(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^{\dagger r}(-\vec{p}, E < 0)) \tag{10.72}
 \end{aligned}$$

the useful subspace for bosons being their own antiparticles being

$$H_{\text{Majorana}} = \{ | \rangle \mid \forall_{\vec{p}} r_f(\vec{p}) | \rangle = 0 \} \tag{10.73}$$

(One should note that whether one chooses our  $r(\vec{p})$ 's or the  $r_f(\vec{p})$ 's to define makes no difference for the space  $H_{\text{Majorana}}$  rather than  $H_f$ , since we actually have  $r(\vec{p}) = r_f(\vec{p})$  the two expressions being just expressed in terms of different  $a^{\dagger}(\vec{p}, E < 0)$  and  $a^{\dagger r}(\vec{p}, E < 0)$  say)

We can easily check that our explicit state expressions (10.65) indeed are annihilated by  $r(\vec{p})$  It were formally left out the  $E > 0$  or  $E < 0$  for the  $b(\vec{p})$  and  $b^{\dagger r}(\vec{p})$  it being understood that  $E > 0$ , but formally we can extrapolate also to  $E < 0$  and it turns of  $b(\vec{p}, E > 0) = b^{\dagger r}(-\vec{p}, E < 0)$ ?

#### 10.4.1 Charge Conjugation Operation

Since we discuss so much bosons being their own antiparticles coming out of a formalism in which the bosons –at first– have antiparticles different from themselves, we should here define a charge conjugation operator  $C$  that transform a boson into its antiparticle:

That is to say we want this operator acting on the Fock space to have the commutation properties with our creation and annihilation operators

$$\mathbf{C}^{-1} \mathbf{a}(\vec{p}, E > 0) \mathbf{C} = \mathbf{a}^{\dagger_f}(-\vec{p}, E < 0) \quad (10.74)$$

and

$$\mathbf{C}^{-1} \mathbf{a}^{\dagger_f}(\vec{p}, E > 0) \mathbf{C} = \mathbf{a}(-\vec{p}, E < 0). \quad (10.75)$$

We also have

$$\mathbf{C}^{-1} \mathbf{a}(\vec{p}, E < 0) \mathbf{C} = \mathbf{a}^{\dagger_f}(-\vec{p}, E > 0) \quad (10.76)$$

and

$$\mathbf{C}^{-1} \mathbf{a}^{\dagger_f}(\vec{p}, E < 0) \mathbf{C} = \mathbf{a}(-\vec{p}, E > 0) \quad (10.77)$$

These requirements suggest that we on the basis of (10.43) for the Fock space have the operation

$$\begin{aligned} \mathbf{C} | \dots \tilde{k}(\vec{p}, E > 0), \dots; \dots, \tilde{k}(\vec{p}', E < 0), \dots \rangle \\ = | \dots, k(\vec{p}, E > 0) = -\tilde{k}(-\vec{p}', E < 0) + 1, \dots; \\ \dots, k(\vec{p}', E < 0) = -1 - \tilde{k}(-\vec{p}', E > 0), \dots \rangle. \end{aligned} \quad (10.78)$$

Using the “full inner product” this  $\mathbf{C}$  operation conserves the norm, and in fact it is unitary under the full inner product corresponding hermitean conjugation  $\dagger_f$  i.e.

$$\mathbf{C}^{\dagger_f} \mathbf{C} = \mathbf{1} = \mathbf{C} \mathbf{C}^{\dagger_f} \quad (10.79)$$

But if we used the not full inner product, so that the norm squares for basis vector would be given by (10.81) and therefore corresponding hermitean conjugation  $\dagger$ , then if  $\mathbf{C}$  acts on a state in which the difference of the number of positive and negative energy bosons is odd, the norm square would change sign under the operation with  $\mathbf{C}$ .

So under  $\dagger$  the charge conjugation operator could not possibly be unitary:

$$\mathbf{C}^{\dagger} \mathbf{C} \neq \mathbf{1} \neq \mathbf{C} \mathbf{C}^{\dagger} \quad (10.80)$$

## 10.5 “Majorana boson” in unphysical sector-combination

As an example of one of the unphysical sector-combination we could take what in our earlier work “Dirac sea for Bosons” were said to be based on the naive vacuum. This naive vacuum theory means a theory in which we do not make any emptying vacuum but rather let there be in both positive and negative single particle energy states a positive or zero number of particles. So in this naive vacuum attached sector combination we can completely ignore the extrapolated negative number of boson possibilities; we so to speak could use the analogue of the harmonic oscillator with normalized states only.



This means that the inner product excluding the negative single particle state normalization using (10.31) will be for this naive vacuum sector combination completely positive definite.

However, including the negative norm factor for the negative energy states from (10.31) so as to get the full inner product we do no longer have the positive definite Hilbert inner product on the Fock space. Now rather we have for basis vectors (10.43) instead of (10.45) that the norm squares

$$\begin{aligned} \langle \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\ = (-1)^{\#(\text{neg. energy b.})} \end{aligned} \quad (10.81)$$

This means that the norm square of a basis vector is positive when the number of negative energy bosons is even, but negative when the number of negative energy bosons is odd.

In the naive vacuum sector combination the vacuum analogue Fock space state is the “naive vacuum”,

$$| \text{naive vac.} \rangle = | \dots, k(\vec{p}, E > 0) = 0, \dots; \dots, k(\vec{p}, E < 0) = 0, \dots \rangle. \quad (10.82)$$

In analogy to what we did in the foregoing section 10.4, we should then construct the states with various numbers of bosons of the Majorana type being their own antiparticles by means of some creation and annihilation operators  $b^{\dagger}(\vec{p})$  and  $b(\vec{p})$ , but first one needs a vacuum that is its own “anti state” so to speak, meaning that the charge conjugation operator  $\mathbf{C}$  acting on it gives it back. i.e. one need a vacuum  $| \text{vac?} \rangle$  so that

$$\mathbf{C} | \text{vac?} \rangle = | \text{vac?} \rangle \quad (10.83)$$

But this is a trouble! The “naive vacuum”  $| \text{naive vac.} \rangle$  is not left invariant under the charge conjugation operator  $\mathbf{C}$  defined in the last subsection of Section 10.4 by (10.78).

Rather this naive vacuum is by  $\mathbf{C}$  transformed into a quite different sector combination, namely in that sector combination, in which there is a negative number of bosons in both positive and negative energy single particle eigenstates. i.e. the charge conjugation operates between one sector combination and another one! But this then means, that we cannot make a representation of a theory with (only) bosons being their own antiparticles unless we use more than just the naive vacuum sector combination. i.e. we must include also the both number of particles being negative sector combination.

In spite of this need for having the two sector combinations –both the naive all positive particle number and the opposite all negative numbers of particles– in order that the charge conjugation operator should stay inside the system –Fock space, we should still have in mind that the creation and annihilation operators cannot pass the barriers and thus can not go from sectors, also the inner product between different sector combinations are divergent and ill defined (and we should either avoid such inner products or define them arbitrarily).

So if we construct “Majorana boson” creation and annihilation operators analogously to the  $b(\vec{p})$  and  $b^{\dagger}(\vec{p})$  in foregoing section as a linear combination of

$a^{(\dagger)}(\vec{p}, \geq E)$  operators operating with such  $b(\vec{p})$  and  $b^{\dagger}(\vec{p}')$ s will stay inside one sector combination. For instance such  $b(\vec{p})$  and  $b^{\dagger}(\vec{p})$  constructed analogously to the physical sector ones formally would operate around staying inside the naive vacuum sector combination if one starts there, e.g. on the naive vacuum  $| \text{naive vac.} \rangle$ . In this –slightly cheating way– we could then effectively build up a formalism for bosons which are their own antiparticles inside just one sector combination. When we say that it is “slightly cheating” to make this construction on only one sector combination it is because we cannot have the true antiparticles if we keep to a sector combination only, which is not mapped into itself by the charge conjugation operator  $C$ . It namely then would mean that the true antiparticle cannot be in the same sector combination.

Nevertheless let us in this section 5 study precisely this “slightly cheating” formalism of keeping to the naive vacuum sector combination with positive numbers of particles only.

We then after all simply use the naive vacuum  $| \text{naive vac.} \rangle$  defined by (10.82) as the “Majorana boson”-vacuum although it is not invariant under  $C$ , which we must ignore or redefine, if this shall be o.k.

We may e.g. build up a formalism for the slightly cheating Majorana bosons by starting from the  $| \text{naive vac.} \rangle$  (10.82) and build up with  $b^{\dagger}(\vec{p})$  taken to be the same as

$$b^{\dagger}(\vec{p}) = \frac{1}{\sqrt{2}} (a^{\dagger}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \quad (10.84)$$

and

$$\begin{aligned} b(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0)) \\ &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0)) \end{aligned} \quad (10.85)$$

We have already checked that for all sector combinations we have

$$[b(\vec{p}), b^{\dagger}(\vec{p}')] = \delta_{\vec{p}\vec{p}'} \quad (10.86)$$

and of course

$$\begin{aligned} [b(\vec{p}), b(\vec{p}')] &= 0 \\ &= [b^{\dagger}(\vec{p}), b^{\dagger}(\vec{p}')] \end{aligned} \quad (10.87)$$

So we see that we can build up using  $b(\vec{p})$  and  $b^{\dagger}(\vec{p})$  a tower of states with any nonnegative number of what we can call the Majorana bosons for any momentum  $\vec{p}$ .

We can also in all the sector combinations use the already constructed

$$\begin{aligned} r(\vec{p}) &= \frac{1}{2} (a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0)) \\ &= \frac{1}{2} (a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0)) \end{aligned} \quad (10.88)$$

to fulfill the commutation conditions

$$\begin{aligned} [r(\vec{p}), b^{\dagger}(\vec{p})] &= 0 \\ [r(\vec{p}), b(\vec{p}')] &= 0 \end{aligned} \quad (10.89)$$

and we even have

$$r(\vec{p}) | \text{naive vac.} \rangle = 0 \quad (10.90)$$

So indeed we have gotten a seemingly full theory of “Majorana Bosons” inside the naive vacuum sector combination subspace

$$H_{\text{Maj}} = \{ | \rangle \mid \forall_{\vec{p}} (r(\vec{p}) | \rangle = 0) \} \quad (10.91)$$

but it is not kept under the  $\mathbf{C}$  as expected.

But really what we ended up constructing were only a system of positive energy particle states since the creation with  $b^{\dagger}(\vec{p}) = b^{\dagger}(\vec{p})$  starting from the naive vacuum only produces positive energy particles in as far as the  $a(-\vec{p}, E < 0)$  contained in  $b^{\dagger}(\vec{p})$  just gives zero on the naive vacuum.

So this a “bit cheating” formalism really just presented for us the “essentially charged” positive energy particles as “the Majorana-bosons”.

That is to say this a bit cheating formalism suggests us to use in the naive vacuum sector combination the “essentially charged particles” as were they their own antiparticles.

If we similarly built a Majorana boson Fock space system of the

$$\begin{aligned} \mathbf{C} | \text{naive vac.} \rangle &= | \text{vac. with both } E > 0 \text{ and } E < 0 \text{ emptied out} \rangle \\ &= | \dots, k(\vec{p}, E > 0) = -1, \dots; \dots, k(\vec{p}, E < 0) = -1, \dots \rangle, \end{aligned} \quad (10.92)$$

we would obtain a series of essentially antiparticles (with positive energies) constructed in the “both numbers of bosons negative” sector combination.

What we truly should have done were to start from the superposition

$$\begin{aligned} | \text{self copy vac.} \rangle &\triangleq \frac{1}{\sqrt{2}} (| \text{naive vac.} \rangle + \mathbf{C} | \text{naive vac.} \rangle) \\ &= \frac{1}{\sqrt{2}} (| \dots, k(\vec{p}, E > 0) = 0, \dots; \dots, k(\vec{p}, E < 0) = 0, \dots \rangle \\ &\quad + | \dots, k(\vec{p}, E > 0) = -1, \dots; \dots, k(\vec{p}, E < 0) = -1, \dots \rangle) \end{aligned} \quad (10.93)$$

and then as we would successively go up the latter with  $b^{\dagger}(\vec{p})$  operators we would successively fill equally many positive energy particles into the  $| \text{naive vac.} \rangle$  and positive energy antiparticles in  $\mathbf{C} | \text{naive vac.} \rangle$ . Note that analogously to the above called “a bit cheating” Majorana-boson construction using *only the positive energy single particle states* we obtain here only use of the positive energy states for the

naive vacuum sector combination and only the negative energy single particle states for the Charge conjugation to the naive vacuum sector combination. Also it should not be misunderstood: The filling in is *not* running parallel in the sense that the sectors truly follow each other. Rather one has to look for if there is Majorana boson by looking into *both* sector-combination- projections.

So we see that what is the true Majorana boson theory built on the two unphysical sector combinations having respectively nonzero numbers of particles (the naive vacuum construction) and negative particles number in both positive and negative energies is the following:

A basis state with  $n(\vec{p})$  Majorana bosons with momentum  $\vec{p}$ , -and as we always have for Majorana's positive energy- gets described as a superposition of two states –one from each of the two sector combinations– with just  $n(\vec{p})$  ordinary (positive energy) essentially charged bosons (of the original types of our construction created by  $a^\dagger$ ..) and a corresponding Fock space state from the other sector, now with  $n(\vec{p})$  antiparticles in the other sector combination (the one built from  $C | \text{naive vac.} \rangle$ ).

Both of these separate sector combinations have for the used states a positive definite Hilbert space.

As already stated the overlap between different sector combinations vectors are divergent and illdefined.

We can check this rather simple way of getting the Majorana bosons described in our on the state  $\frac{1}{2} (| \text{naive vac.} \rangle + C | \text{naive vac.} \rangle)$  built system of states by noting what the condition  $r(\vec{p}) | \rangle = 0$  tells us the two sector combinations:

On a linear combination of basis vectors of the naive vacuum construction type

$$| \rangle = \sum | k(\vec{p}, E > 0) \geq 0, \dots; \dots, k(\vec{p}, E < 0) \geq 0, \dots \rangle \\ C \dots k(\vec{p}, E > 0) \dots \dots \dots \bar{k}(\vec{p}, E < 0) \dots \quad (10.94)$$

the requirement

$$r(\vec{p}) | \rangle = 0 \quad (10.95)$$

relates coefficients which correspond to basis states being connected by  $k(\vec{p}, E > 0)$  going one up while  $k(-\vec{p}, E < 0)$  going one unit down or opposite. As we get the relation

$$\sqrt{1 + k(\vec{p}, E > 0)} C \dots k(\vec{p}, E > 0) + 1, \dots, \dots, k(-\vec{p}, E < 0), \dots \\ + C \dots, k(\vec{p}', E > 0), \dots, \dots, k(-\vec{p}, E < 0) - 1, \dots \cdot \sqrt{k(-\vec{p}, E < 0)} = 0 \quad (10.96)$$

we can easily see that the states being annihilated are of the form

$$\sum (-1)^{k(\vec{p}, E > 0)} \frac{\sqrt{k(-\vec{p}, E < 0)!}}{\sqrt{k(\vec{p}, E > 0)!}} \cdot | \dots, k(\vec{p}, E > 0), \dots; \dots, k(-\vec{p}, E < 0), \dots \rangle, \quad (10.97)$$

where we sum over  $k(\vec{p}, E > 0)$  and the difference  $d = k(\vec{p}, E > 0) - k(-\vec{p}, E < 0)$  is FIXED.

As a special case we might look at possibility that the difference

$$d = k(\vec{p}, E > 0) - k(-\vec{p}, E < 0) \quad (10.98)$$

were 0. In this case the  $|\text{naive vac.}\rangle$  itself would be in the series. In this case the solution (10.97) reduces to

$$\sum_{k=0}^{\infty} (-1)^k |\dots, k(\vec{p}, E > 0) = k, \dots; \dots, k(-\vec{p}, E < 0) = k, \dots\rangle \quad (10.99)$$

But it is now the problem that this series does not converge. But for appropriate values of the difference  $d$ ,

$$d \geq 2, \quad (10.100)$$

the series (10.97) converge.

For the convergent cases we can estimate the norm square of a state (10.97) to go proportional to

$$\| |\rangle \|^2 \propto \sum_{k=0}^{\infty} \frac{(k-d)!}{k!} (-1)^{k-d} \quad (10.101)$$

where the  $(-1)^{k-d}$  now comes from the alternating “full” norm square due to the factor  $(-1)^{\#(\text{neg. energy b.})}$ . This expression in turn is proportional to

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{k-d}{-d} (-1)^{k-d} &= \sum_{n=-d}^{\infty} \binom{n}{-d} (-1)^n \quad (n = k-d) \\ &= \frac{(-1)^{-d}}{(1 - (-1))^{-d+1}} \end{aligned} \quad (10.102)$$

which is zero for  $d - 1 \geq 1$ .

So indeed it is seen that the basis states in  $H_{\text{Maj}}$  part inside the naive vacuum sector combination has zero norm. Since the states with different numbers of Majorana-bosons are represented by mutually orthogonal it means that the whole part of the naive vacuum sector combination used to represent the Majorana-bosons has totally zero inner product. Basically that means that the inner product transferred from the original theory with its “essentially charged bosons” to the for Majorana bosons in subspace  $H_{\text{Maj}}$  turns out to be zero.

This result means –extrapolating to suppose zero norm also in the divergent cases– that in the unphysical sector combination we get no non-trivial inner product for the Majorana-bosons.

If ones use the true Majorana boson description by as necessary combining two sector combinations, one could use the ambiguity (and divergence) of the inner product of states from different sectors to make up instead a non trivial inner product.

### 10.5.1 Overview of All four Sector Combinations

Strictly speaking we could make an infinite number of sector combinations, because we for every single particle state – meaning for every combination of a spin state and a momentum say  $\vec{p}$  – could choose for just that single particle state to postulate the second quantized system considered to be started at such a side of the “barriers” that just this special single particle state had always a negative number of bosons in it. For another one we could instead choose to have only a non-negative number of bosons. Using all the choice possibilities of this type would lead us so to speak to the infinite number of sector combinations  $2^{\#}$  “single particle states”, where  $\#$  “single particle states” means the number of single particle states. But most of these enormously many sector combinations would not be Lorentz invariant nor rotational invariant. Really, since the sector combination should presumably rather be considered a part of the initial state condition than of the laws of Nature, it might be o.k. that it be not Lorentz nor rotational invariant. Nevertheless we strongly suspect that it is the most important to consider the Lorentz and rotational invariant sector-combination-choices. Restricting to the latter we can only choose a separate sector for the positive energy states and for the negative energy sector, and then there would be only  $2^2 = 4$  sector combinations.

Quite generally we have the usual rules for creation and annihilation operators, but you have to have in mind that we have two different hermitean conjugations denoted respectively by  $\dagger$  and by  $\dagger_f$ , and that the creation operators constructed from the same annihilation operators are related

$$\begin{aligned} a^{\dagger_f}(\vec{p}, E > 0) &= a^{\dagger}(\vec{p}, E > 0) \\ a^{\dagger_f}(\vec{p}, E < 0) &= -a^{\dagger}(\vec{p}, E < 0) \end{aligned} \quad (10.103)$$

These “usual” relations are

$$\begin{aligned} [a(\vec{p}, E > 0), a^{\dagger}(\vec{p}', E > 0)] &= \delta_{\vec{p}\vec{p}'} \\ [a(\vec{p}, E < 0), a^{\dagger}(\vec{p}', E < 0)] &= \delta_{\vec{p}\vec{p}'} \\ [a(\vec{p}, E > 0), a^{\dagger_f}(\vec{p}', E > 0)] &= \delta_{\vec{p}\vec{p}'} \\ [a(\vec{p}, E < 0), a^{\dagger_f}(\vec{p}', E < 0)] &= -\delta_{\vec{p}\vec{p}'} \end{aligned} \quad (10.104)$$

while we have exact commutation for  $a$  with  $a$  or for  $a^{\dagger}$  or  $a^{\dagger_f}$  with  $a^{\dagger}$  or  $a^{\dagger_f}$ . Each  $a(\vec{p}, E \geq 0)$  or  $a^{\dagger_f}$  or  $a^{\dagger}$  act changing only the number of particle in just the single relevant single particle state, meaning it changes only  $k(\vec{p}, E \geq 0)$ ; the rules are as

seen analytical continuations generally

$$\begin{aligned}
& \alpha^\dagger(\vec{p}, E > 0) | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
&= \sqrt{k(\vec{p}, E > 0) + 1} | \dots, k(\vec{p}, E > 0) + 1, \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
& \alpha^\dagger(\vec{p}', E < 0) | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
&= \sqrt{k(\vec{p}', E < 0) + 1} | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0) + 1, \dots \rangle \\
& \alpha^{\dagger r}(\vec{p}, E > 0) | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
&= \sqrt{k(\vec{p}, E > 0) + 1} | \dots, k(\vec{p}, E > 0) + 1, \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
& \alpha^{\dagger r}(\vec{p}', E < 0) | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
&= -\sqrt{k(\vec{p}', E < 0) + 1} | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0) + 1, \dots \rangle \\
& \alpha(\vec{p}, E > 0) | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
&= \sqrt{k(\vec{p}, E > 0)} | \dots, k(\vec{p}, E > 0) - 1, \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
& \alpha(\vec{p}', E < 0) | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\
&= \sqrt{k(\vec{p}', E < 0)} | \dots, k(\vec{p}, E > 0) + 1, \dots; \dots, k(\vec{p}', E < 0) - 1, \dots \rangle \quad (10.105)
\end{aligned}$$

The four sector combination with the same sector for the same sign of the energy  $E$  of the single particle states were called:

1)The “physical sector” has

$$\begin{aligned}
k(\vec{p}, E > 0) &= 0, 1, 2, \dots \\
k(\vec{p}, E < 0) &= -1, -2, -3 \dots
\end{aligned} \quad (10.106)$$

2)The “sector-combination constructed from the naive vacuum” has

$$\begin{aligned}
k(\vec{p}, E > 0) &= 0, 1, 2, \dots \\
k(\vec{p}, E < 0) &= 0, 1, 2, \dots
\end{aligned} \quad (10.107)$$

3)The “both sectors with negative numbers” sector-combination has

$$\begin{aligned}
k(\vec{p}, E > 0) &= -1, -2, -3 \dots \\
k(\vec{p}, E < 0) &= -1, -2, -3 \dots
\end{aligned} \quad (10.108)$$

4)The “a positive number with negative energy and vice versa” has

$$\begin{aligned}
k(\vec{p}, E > 0) &= -1, -2, -3 \dots \\
k(\vec{p}, E < 0) &= 0, 1, 2, \dots
\end{aligned} \quad (10.109)$$

In the physical sector combination the Fock space ends up having positive definite norm square and so this sector-combination is the one usual taken for being the in nature realized one.

### 10.5.2 Formulas for “Majorana particles”

The theory of Majorana fermions may be so well known that we had nothing to say, but it were written about it in section 2.

For the boson case we introduced for each (vectorial) value of the momentum an operator acting on the Fock space called  $r(\vec{p})$  defined by (10.88)

$$\begin{aligned} r(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0)) \\ &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0)) \end{aligned} \quad (10.110)$$

with the properties

$$\begin{aligned} [r(\vec{p}), b(\vec{p}')] &= 0 \\ [r(\vec{p}), b^{\dagger}(\vec{p}')] &= 0 \\ [r(\vec{p}), b^{\dagger}(\vec{p}')] &= 0 \end{aligned} \quad (10.111)$$

where the creation  $b^{\dagger}(\vec{p})$  ( $= b^{\dagger}(\vec{p})$ ) and annihilation  $b(\vec{p})$  operators for the “Majorana bosons” (i.e. boson being its own antiparticle) were defined in terms of the  $a$ ’s as

$$b^{\dagger}(\vec{p}) = \frac{1}{\sqrt{2}} (a^{\dagger}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \quad (10.112)$$

and

$$b(\vec{p}) = \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0)) \quad (10.113)$$

$$= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0)) \quad (10.114)$$

These operators obey (see(10.86) and (10.87))

$$\begin{aligned} [b(\vec{p}), b^{\dagger}(\vec{p}')] &= \delta_{\vec{p}\vec{p}'} \\ [b(\vec{p}), b(\vec{p}')] &= 0 \\ [b^{\dagger}(\vec{p}), b^{\dagger}(\vec{p}')] &= 0 \end{aligned} \quad (10.115)$$

and so these operators are suitable for creating and annihilation of particles, and indeed these particles are the “Majorana bosons”. As a replacement for the in usual formalism for “Majorana bosons” say

$$b(\vec{p}, E) = b^{\dagger}(-\vec{p}, -E) \quad (10.116)$$



we have in our notation

$$b(-\vec{p}) \big|_{\substack{\text{with} \\ >\leftrightarrow<}} = b^{\dagger}(\vec{p}) \quad (10.117)$$

as is easily seen from (10.113) and (10.112) just above.

But now we need also a vacuum from which to start the creation of the “Majorana bosons” with  $b^{\dagger}(\vec{p})$ . In the two sector-combinations 1) the physical one and 4) “a positive number with negative energy and vice versa” there are the suitable vacua:

In 1)

$$|\text{physical vac}\rangle = |\dots, k(\vec{p}, E > 0) = 0, \dots; \dots, k(\vec{p}, E < 0) = -1, \dots\rangle \quad (10.118)$$

and in 4)

$$\begin{array}{l} \text{pos in } E < 0 \\ \text{neg in } E > 0 \end{array} \rangle = 1 \dots, k(\vec{p}, E > 0) = -1, \dots; \dots, k(\vec{p}, E < 0) = 0, \dots \rangle \quad (10.119)$$

In the sector-combinations 2) and 3), however, there are no charge conjugation symmetric states to use as the vacuum state for a “Majorana-boson” formalism. In this case the vacuum of 2) goes under charge conjugation  $C$  into that of 3).

## 10.6 Outlook on String Field Theory Motivation

One of our own motivations for developping the sort of boson Dirac sea theory for bosons being their own antiparticles, i.e. a theory with Dirac sea, were to use it in our own so called “novel string field theory”[8–11].

In this “novel string field theory” we sought to rewrite the whole of string theory[12–15,28,29] (see also modified cubic theory [24])- although we did not yet come to superstrings[25,17,26] although that should be relatively easy - into a formalism in which there seems a priori to be no strings. The strings only come out of our novel string field theory [18–23] by a rather complicated special way of looking at it. In fact our basic model in this novel string field theory is rather like a system of /a Fock space for massless scalar particles, which we call “objects” in our formulation, but they have much although not all properties similar to scalar massless particles. These particles/objects we think must be in an abstract way what we here called Majorana bosons. This means they should be their own antiparticles to the extend that they have antiparticles.

But their being put into cyclically ordered orientable chains may put a need for a deeper understanding of the Majoraneness for these “objects”.

The reason for the objects, that in our novel string field theory are a kind of constituents, for the strings being supposed to a nature reminiscent of the Majorana particles or being their own antiparticles, is that they carry in themselves no particle number or charge, except that they can have (26)-momentum. (For complete consistency of the bosonic string theory it is wellknown that 26 space time dimensions are required.) The bulk of the string (in string theory) can namely

be shrunk or expanded ad libitum, and it is therefore not in itself charged, although it can carry some conserved quantum numbers such as the momentum densities.

We take this to mean that the string as just bulk string should be considered to be equal to its own antimaterial. If we think of splitting up the string into small pieces like Thorn[27], or we split the right and left mover parts separately like we did ourselves, one would in both cases say that the pieces of Thorn's or the objects of ours should be their own antiparticles. With our a bit joking notation: they should be Majorana. Thus we a priori could speculate that, if for some reason we should also like to think our objects as particles, then from the analytical properties of the single particle in relativistic theories must have both positive and negative energy states. Then a treatment of particles being their own antiparticles in the Dirac sea formulation could - at least superficially - look to be relevant.

One could then ask, what we learned above, that could be of any help suggesting, how to treat long series of "objects", if these objects are to be considered bosons that are their own antiparticles:

- 1. In the novel string field theory of ours it is important for the association to the strings, that one considers ring shaped chains of objects. We called such ring shaped chains of objects for "cyclically ordered chains". Now such ordering of our "objects" (as we call them), or of any type of particles, into chains in which each particle (or "object") can be assigned a number (although in our special model only a number modulo some large number  $N$ ) is o.k. for particles with an *individuality*. However, if we have particles (or "objects") that are say *bosons*, then all particles are identical - or one could say any allowed state is a superposition of states in which all possible permutations on the particles have been performed and a superposition of the results of all these permutations with same amplitude only is presented as the final state -. But this then means that *one cannot order them*, because you cannot say, which is before which in the ordering, because you cannot *name* the single particle. You could only say, that some particle A is, say, just before some particle B in a (cyclic) ordering, if you characterize A as being the particle with a certain combination of coordinates (or other properties) and B as being the one with a certain other combination of coordinates (and other properties). Unless you somehow specify by e.g. some approximate coordinates (or other characteristic) which particle you think about, it has no meaning to express some relation involving the relative ordering, say, of two bosons.
- 2. The problem just mentioned in assigning order to bosons means, that the concept of "cyclically ordered chains" of objects - or for that matter building up any string from particle pieces like Thorn say - cannot be done once the particles or objects are bosons, but rather should be preferably formulated *before one symmetrize the wave function* under the particle permutation so as to implement that they are bosons. One shall so to speak go back in the "pedagogical" development of boson-theory and think in the way before the symmetry principle under permutations making the particles bosons were imposed. In this earlier stage of the description the cyclically ordered chains, or any type of ordering, which one might wish, makes sense. So here it looks

that going back and postponing the boson constraint is needed for ordering chains.

- 3. But seeking to go back prior to boson or fermion formulation makes a problem for the Dirac sea - in both boson and fermion cases -: If we want to consider the case of individual particles or objects fully, we have to imagine that we have given *names (or numbers) to all the particles in the Dirac sea!*

For this problem we may think of a couple of solutions:

- a. We could imagine an interaction that would organize the particles in the ground state (to be considered a replacement for the physical vacuum) or that some especially important state for the Fock space obtained by imposing some other principle is postulated to make up a kind of vacuum state. Then one could hope or arrange for the interaction or state-selecting principle chosen, that the vacuum state becomes such, that the objects (or particles) in the Dirac sea goes into such a state, that these objects have such positions or momenta, that it due to this state becomes possible to recognize such structure that their ordering in the wanted chain becomes obvious. If so, then the (cyclic) ordering can come to make sense.

This solution to the problem may be attractive a priori, because we then in principle using the now somewhat complicated state of the vacuum can assign orderings to the whole Dirac sea, and thus in principle give an individuality even to the Dirac sea particles and missing particles / the holes can make sense, too. They so to speak can inherit their individuality from the particles missing, which before being removed were sitting in the chains of the vacuum. We have thus at least got allowance to talk about a chain ordering for pieces of chains for the holes. There is so to speak an ordering of the holes given by the ordering of the particles removed from the Dirac sea originating from the chain postulated to have appeared from the interaction or from some special selection principle for the vacuum state.

A little technical worry about the “gauge choice” in our novel string field theory: In our novel string field theory we had made a gauge choice for the parametrization of the strings, that led to the objects having a special component of their momenta  $p^+$ , or in the language of our papers on this string field theory  $J^+(I)$  for the  $I$ th object in the chain *fixed* to a chosen small value  $\alpha\alpha'/2$ . Since the argument for there having to be negative energy solutions (to say the Dirac equation) and thus a need for a Dirac sea at all is actually analyticity of the equation of motion, we would suppose that also for our objects one should keep “analyticity” in developing ones picture of the “negative energy states” and thereby of the Dirac sea. But then the  $p^+$  or  $J^+$ , which is fixed to constant could hardly get continued to anything else than the same constant? This sounds a bit unpleasant, if we imagine the  $p^+$  be lightlike or timelike, because then we cannot find the negative energy state with the chosen gauge condition, and the whole reason for the Dirac sea seems to have disappeared. And thus the discussion of Majorana may also have lost its ground. But if we imagine the gauge choice fixed component to be spacelike, then we obtain, that

the gauge condition surface intersects the light cone in two disconnected pieces that are actually having respectively positive and negative energy. So assuming the gauge choice done with a space-like component we have indeed the possibility of the Majorana discussion! And also in this case of a spacelike  $p$  or  $J$  component being fixed (by gauge choice) our construction of the Majorana bosons makes perfect sense.

Now it gets again severely complicated by the chains postulated in the vacuum. In the space-like gauge fixing case it also becomes of course complicated, but the complication is due to the complicated state rather than to the gauge fixing alone.

Let us, however, stress again: To make a ordering of the objects in the Dirac sea into say cyclically ordered chains a much more complicated state in the Fock space is needed than the simple say physical vacuum.

To figure out how to think about such a situation with a "complicated" vacuum state replacing the, say, "physical vacuum" as discussed above, we might think about the analogous situation with the fermions. When one has a quantum field theory with fermions having interactions, it means that the interaction part of the Hamiltonian has caused that the ground state for the full Hamiltonian is no longer the state with just the Dirac sea filled and the positive energy single particle states empty. Rather it is a "complicated" superposition of states in the Fock space, most of which would in the free theory have positive energy. These are states which can be described as states with some - infinite - number of positive energy fermions and some anti-fermions present (in addition to the vacuum with just the Dirac sea filled). The presence of anti-fermions (holes) means, that if one acts with a creation operators  $b^\dagger(\vec{p}, E < 0)$  for inserting a fermion with a negative energy ( $E < 0$ ), then one shall not necessarily get 0 as in the free theory vacuum, because one has the possibility(chance) of hitting a single particle state in which there is a *hole*. The Fock-space state created by such an action will have higher full Hamiltonian energy than the "interaction vacuum", because the latter is by definition the lowest energy state, but one has anyway succeeded in inserting a fermion in a state which from the free theory counted has a negative energy. It should be absolutely possible that such an inserted in the just mentioned sense negative energy particle could be part of the construction of say a bound state or some composite object resonance or so. Similarly it could on top of a "complicated vacuum" (meaning a ground state e.g. for the full Hamiltonian but not for the free one) be possible to remove with an annihilation operator  $a(\vec{p}, E > 0)$  a particle from a single particle state (having with the free Hamiltonian) positive energy ( $E > 0$ ). One could namely have the chance of hitting a positive energy single particle state, in which there already is a particle in the "complicated vacuum". Such a removal or hole in a positive single particle state is what we ought to call a "negative energy anti-particle".

We here sought to argue, that if one for some reason or another (because of interaction and taking the ground state, or because one has postulated some "complicated vacuum" just to make ordering make sense) use a

“complicated vacuum”, then it becomes possible formally to add particles or anti-particles with *negative energy*.

Especially we want to stress the possibility that, if one wants to describe properly a resonance or a bound state composed of several particles (e.g. fermions) then one might need to assign some of the constituents negative energy in the sense just alluded to here.

Strictly speaking it comes to look in the “complicated vacuum” as if one has got *doubled the number of species of effective particle*, because one now by acting with e.g.  $\alpha^\dagger(\vec{p}, E > 0)$  both can risk to produce a positive energy particle, and can risk to fill in a hole in positive energy single particle state and thereby creating a negative energy anti-particle. So operating with the same operator we risk *two different results*, which may be interpreted as if one had effectively had two different types of operators and thereby doubly as many types of particles as we started with. We have so to speak - in the case of non-Majorana particles - gotten both positive and negative energy particles and also both positive and negative anti-particles effective on the “complicated vacuum”.

If we go to make our particles Majorana, we reduce the number of species by a factor two (as expected in as far as Majorana means that particle and anti-particle gets identified.)

In the case of the “complicated vacuum” the transition to Majorana also reduce the number of species by a factor 2 and thus compensates for the effect of the “complicated vacuum”. With Majorana the particles and anti-particles are no longer distinguished, but with the “complicated vacuum” we obtain both positive and negative energy (Majorana)particles. It essentially functions as if the particle were no more Majorana. The “complicated vacuum”, so to speak, removed the Majorananness.

We hope in later publication to be able to check that the just delivered story of the interaction vacuum increasing the number of species effectively by the factor two, is found when using the Bethe-Salpeter equation to describe bound states. Then there ought according to the just said to be effectively both negative and positive states relevant for the “constituent” particles in the Bethe-Salpeter equation.

Applying the just put forward point of view on the objects in our novel string field theory we should imagine that in this formulation with the “complicated vacuum” being one with chains in it it is possible for some objects to have their energy negative. Nevertheless a whole chain formed from them might end up with positive energy by necessity.

Such a possibility of negative energy for single objects that can nevertheless be put onto the vacuum might be very important for complete annihilation of pieces of one chain put onto the vacuum with part on an other one also put onto that vacuum. If we did not have such possibility for both signs along the chains, then we could not arrange that two incomming cyclically ordered chains could partly annihilate, because energy conservation locally along the chains would prevent that.

At least in principle it must though be admitted, that such a picture based on an interaction or by some restriction of the state of the whole world makes a complicated vacuum is a bit complicated technically.

But physically it is wellknown, that the vacuum in quantum field theories is a very complicated state, and so we might also expect that in string theory a similarly complicated vacuum would be needed. And that should even be the case in our novel string field theory in spite of the statement, often stated about this theory, that it has no interaction properly; all the seeming interactions being fake. But we could circumvent the need for an interaction to produce the complicated vacuum, we seemingly need by claiming that we instead have a restriction on the *Fock space states* of the system of objects, that is allowed. Such a constraint could force the vacuum to be more complicated, and thus in succession lead to that it becomes allowed in the more complicated vacuum to have some of the objects having even negative energy, which in turn could allow a *complete annihilation* of objects from one cyclically ordered chain and another set up in the same state (built on the complicated vacuum)

- b. We give up seeing any chain structure in the vacuum as a whole, but rather attempt to be satisfied with ordering the missing particles, (or may be the antiparticles?).

Naturally we would start imagining that we can have a Majorana boson, if we wish, represented by -1 negative energy boson, because the Majorana boson is a superposition of a particle and an antiparticle, and the latter really can be considered -1 particle of negative energy.

At first one might think that having two bound states or two strings, which would like to partially annihilate -as it seems that we need in our derivation of Veneziano amplitude in our novel string field theory - could be indeed achieved by having part of one of these composed structures treated or thought upon as consisting of antiparticles, since one would say that particle and anti-particle can annihilate. However, when anti-particle and particle both with positive energy annihilate, then at least some energy is in excess and they therefore cannot annihilate completely into nothing. Rather there would have to some emitted material left over to take away the energy. If we therefore as it seems that we would to get the terms missing in our novel string field theory to get the correct three term Veneziano amplitude should have a total annihilation without left over such positive energy particles and antiparticles are not sufficient. Therefore this b. alternative seems not to truly help us with the problem of our novel string field theory to reproduce the Veneziano model fully.

- 4. In our formalism above - taken in the physical vacuum - the "Majorana-boson" became a superposition of being a hole and a genuine positive energy particle. The hole meant it were in part of the superposition - i.e. with some probability 50 % - -1 particle with negative energy. So one would with significant probability be able to consider that the "Majorana-boson" were indeed a lack of a negative energy original particle. For calculating amplitudes of some

sort one would then imagine that we might even have to add up contributions from the holes and contributions from the positive energy particles.

For each object, say, we should think we should have both a contribution in which it is considered a particle (with positive energy) and one in which it is a hole.

- 5. From the construction of the creation and annihilation operators for the “Bosons being their own antiparticles” - the  $b$ ’s - being constructed as containing the quite analogous contributions from a hole part and a particle part, it looks that in building up states with many Majorana- bosons one gets an analogous built up for both the holes and the particles and with say the analogous momenta.

Here analogous means that the holes are holes for states with opposite momentum, but since it is holes it becomes the same net momentum for the hole as from the particle analogous to it.

- 6. With any sort of even formal interaction one would think that a hole and a particle can annihilate as stuff annihilate anti-matter. But if you have a pair of positive energy particles or anti-particles, they can only annihilate into some other particles of some sort. They cannot just disappear together. That is however, possible, if you have a negative energy particle and a positive energy one of just opposite four(or 26) momenta.
- 7. If one would say choose a gauge so that the particles get as in our gauge choice in our papers on the novel string theory that a certain momentum component,  $p^+$  say, is specified to be a fixed value  $\alpha\alpha'/2$  as we choose, then one would have to let the particle, the state of which is made the hole have its  $p^+ = -\alpha\alpha'/2$ , i.e. the opposite value. (Then if one has negative numbers of such particles, of course they contribute a positive  $p^+$  again.) If one has indeed completely opposite four momenta - including energy - then an annihilation without left over is possible, otherwise not. It is therefore it is so crucial with negative energy constituents, if any such total disappearance of a pair is needed/wanted.

But if we have physically only the free simple vacuum in which one has just for bosons emptied the negative energy states and for fermions just filled the negative energy states and no more, then all modifications will even particle for particle have positive energy. It will either be a removal of a negative energy particle meaning an antiparticle created or an insertion of a positive energy particle. Both these modifications would mean insertion of positive energy and they could not annihilate with each other without leaving decay material. So to have a piece of a cyclically ordered chain annihilate *without decay material* with another piece, it is needed that we *do not just have the free theory vacuum*. We need instead something like a “complicated vacuum” such as can be gotten by the effect of either interactions, or from some more complicated postulate as to what the vacuum state should be.

In our novel string field theory, in which it is claimed that there are no interactions in the object formulation, we cannot refer to interactions. Rather we must refer to making a postulate about what the “complicated vacuum state” should be. As already mentioned above we need in order that ordering

into the cyclically ordered chains can make sense to have as the (vacuum) state a state in which the various objects can have so different single particle states that we can use their single particle state characteristic to mark them so as to give them sufficient individuality. Really we should postulate such a “complicated vacuum state” that there would for each object be an effectively unique successor lying as neighbor for the first one. But such restrictions to somewhat welldefined positions relative to neighbors in a chain must mean that it cannot at all be so, that there are just, 1 for fermions, -1 for bosons, particles in the negative energy states and zero in the positive ones. Rather it means, that considering such a free vacuum as starting point the state with the chains organized into the “complicated vacuum” is strongly excited. So there are many both particles and anti particles present in this “complicated vacuum” needed to have chains inside the vacuum.

But as already said such “complicated vacuum” can give the possibility of having effectively negative energy constituents. Since our objects are essentially constituents, this also means that our objects in a complicated vacuum can get allowed to be of negative energy. We must arrange that by allowing them in our gauge choice to get the  $J^+$  have both signs. If so we may enjoy the full annihilation without left over material.

- 8. To construct an operator creating a chain (or series) of Majorana particles - in our novel SFT we mean the objects - we strictly speaking should use a specific linear combination of the hole and the positive energy particle (or object) for every Majorana particle created along the chain, but if we project out at the end the constructed Fock space state into the subspace used for the Majorana boson description, it is not so important to use precisely the correct linear combination. We shall namely obtain the right linear combination, since in that case it comes out of such a projection automatically.

But trusting that projecting into the Majorana-describing sub-space will do the job, we can just choose at will whether we use a series of positive energy particle (or object) creation operator or instead the corresponding hole creating (destruction of negative energy) operator.

Since our objects are a priori Majorana ones, it may at the end due to the doubling of state-types mentioned get them rather described effective as non-Majorana, in the way that they can be in both positive and negative energy single particle states. This actually reminds us more about the “naive vacuum” sector combination. But now it is the result of the “complicated vacuum” and of the thereby associated “doubling of the number of species effectively”.

### 10.6.1 The “Rough Dirac Sea” in General

Let us extract and stress the idea, which we suppose will be very important for our formulation of the scattering amplitude for strings in our novel string field theory, but which could also be imagined to deliver an approximation that could be useful especially for bound states with many constituents, “the (very) rough Dirac sea”. This rough Dirac sea is really the same as what we called above the “complicated vacuum”.



The picture of true rough sea (a rough sea is the opposite of a calm sea, and it means that there lots of high waves may actually) be a very good one to pedagogically promote the idea of the effects of the “complicated vacuum” or the “rough Dirac sea” leading to that we effectively get negative energy particles and antiparticles.

In this picture the “calm Dirac sea” means the free approximation vacuum, in which - in the physical choice of sector combination, which is what one normally will have in mind - the negative energy states are filled for the fermion case, while “emptied out” in the boson case. In any case this calm Dirac sea is the picture for the theory vacuum in the unperturbed approximation (the free vacuum). But in interacting quantum field theories the vacuum gets perturbed by the interaction and becomes a more complicated state “the complicated vacuum”, and it is for this “complicated vacuum” that the analogy with the rough sea is very good. There should have been near the surface -at the average surface height - a region in heights, in which you find with some probability water and with some probability air. Just at the should-have-been surface (= average surface) one expects that the probability for finding water is 50 % and for finding air in a given point is 50 %.

Now imagine: we come with an extra water molecule (or may be just a tiny bit of water) and want to insert it into the sea or the air not too far from the “should-have-been surface”. Now if there happen to be a wave of water present, where you want to or attempt to insert such an extra tiny bit of water, you will not succeed, and that is analogous to getting zero, when you want to create a particle with a creation operator into a state that is already filled (say, we think for simplicity on the fermion case). If, however, there happen to be a valley in the waves, you will succeed in inserting a tiny bit of water *even if it is under the average water height!* This corresponds to inserting a negative energy particle into the “rough Dirac sea” or the “complicated vacuum”. You may also think about removing a droplet of water. That will of course only succeed, if there *is* some water in the point in space, wherein you want to do it. Again it is not guaranteed that you can remove a bit of water in the rough sea, even if you attempt to remove it deeper than the average water height, because there might be a valley among the waves. Also if you hit a wave you might be able to remove a bit of water from a height above the average height.

In this way we see that you can produce sometimes a hole in the water both with positive and negative height (analogous to the both positive and negative (single particle) energy). Similarly you may produce both above and below extra bubbles of water.

This means that we have got a kind of doubling: While in the calm Dirac sea you can only make droplets ( particles) above the average surface and only holes ( antiparticles) below, we now in *the rough sea can do all four combinations*.

### 10.6.2 Infinite Momentum Frame Wrong, in Rough Dirac Sea?

With “rough Dirac sea”-thinking we arrived at the idea, that one might describe for instance a bound state or resonance as composed of constituent particles not all having positive energy; but some of the constituents could have *negative* energy.

It must be legal to choose to describe a bound state or resonance state by a linear combination - weighted with what is essentially a wave function for the constituents in the bound state or resonance - of creation operators and annihilation operators (for describing the contained anti-particles among the constituent particles) and let it act on the vacuum. We might, say, think of an operator of the form

$$A^\dagger(\text{bound state}) = \quad (10.120)$$

$$= \int \Psi(\vec{p}_1, h_1, s_1; \dots; \vec{p}_N, h_N, s_N) \\ * \prod_{h_1, s_1} (a^\dagger(\vec{p}_1, h_1, s_1) d^3 \vec{p}_{1, h_1, s_1}) \cdots \prod_{h_N, s_N} (a^\dagger(\vec{p}_N, h_N, s_N) d^3 \vec{p}_N); \quad (10.121)$$

$$|\text{bound state (Fock)state}\rangle = \quad (10.122)$$

$$= A^\dagger(\text{bound state})| \text{"complicated vacuum"} \rangle \quad (10.123)$$

$$= \int \Psi(\vec{p}_1, h_1, s_1; \dots; \vec{p}_N, h_N, s_N) \quad (10.124)$$

$$\prod_{h_1, s_1} (a^\dagger(\vec{p}_1, h_1, s_1) d^3 \vec{p}_{1, h_1, s_1}) \cdots \prod_{h_N, s_N} (a^\dagger(\vec{p}_N, h_N, s_N) d^3 \vec{p}_N) \quad (10.125)$$

$$| \text{"complicated vacuum"} \rangle \quad (10.126)$$

where  $\Psi(\vec{p}_1, h_1, s_1; \dots; \vec{p}_N, h_N, s_N)$  is (essentially) the wave function for a bounds state of  $N$  constituents numbered from 1 to  $N$ . The momenta of the constituents are denoted by  $\vec{p}_i$  with  $i = 1, 2, \dots, N$ , while the internal quantum numbers are denoted  $h_i$ , and then *there is the symbol  $s_i$  that can be  $s_i = \text{"positive"} = (E > 0)$  or  $s_i = \text{"negative"} = (E < 0)$* , meaning that the single particle energy of the constituent here is allowed to be both positive and negative, it being denoted by  $s_i$ , which of these two possibilities is realized for constituent number  $i$ . In this expression (10.126) we took just  $N$  constituents, but it is trivial to write formally also the possiblility of the bound state being in a state, that is a superposition of states with different values of the number  $N$  of constituents:

$$A^\dagger(\text{bound state}) = \quad (10.127)$$

$$= \sum_{N=1,2,\dots} \int \Psi_N(\vec{p}_1, h_1, s_1; \dots; \vec{p}_N, h_N, s_N) \\ * \prod_{h_1, s_1} (a^\dagger(\vec{p}_1, h_1, s_1) d^3 \vec{p}_{1, h_1, s_1}) \cdots \prod_{h_N, s_N} (a^\dagger(\vec{p}_N, h_N, s_N) d^3 \vec{p}_N). \quad (10.128)$$

In this way we could describe a (bound) state inserted on the background of the true ("complicated") vacuum with a superposition of different numbers of constituents. In principle we could find a wave function set,  $\Psi_N(\vec{p}_1, h_1, s_1; \dots; \vec{p}_N, h_N, s_N)$  for  $N = 1, 2, \dots$ , that could precisely produce the (bound) state or resonance in question. It might because of the allowance of both negative and positive energy constituents be possible to construct in this way a given state in more than one way. But one could well imagine, that if we would like to have the wave function reasonably smooth, then it would be hard to quite avoid the

negative energy constituents contributions - they are of course only relevant by giving nonzero contributions to the state created provided the Dirac sea is rough - and thus it looks like being essentially needed to use wave functions with also negative energy constituents, unless one is willing to give up the accuracy in which the influence from the interaction on the vacuum must be included.

But if we thus accept a description with negative constituent energy, the usual thinking on the “infinite momentum frame”[31] seems wrong:

If we in fact have constituents with single particle state negative energy, then boosting such a state ever so much in the longitudinal momentum direction cannot bring these negative energy constituents to get positive longitudinal and thereby positive Bjorken  $x$ . So the usual story that provided we boost enough all constituents obtain positive  $x$  cannot be kept in our rough Dirac sea scenario with its negative energy constituents!

This may be the reason for the trouble in our novel string field theory which triggered us into the present work. In this novel string field theory formulation we namely used infinite momentum frame and actually took it, that all the there called objects - which are essentially constituents - had their  $J^+ = \alpha\alpha'/2$ . But now the 26-momentum, which is proportional to the  $J^\mu$ , should then for all the objects have the  $+$  component positive. But now the notation is so, that this  $+$  component means the longitudinal momentum in the infinite momentum frame. So we assumed a gauge choice in our formulation of this novel string field theory which is inconsistent with the negative energy constituent story arising from rough Dirac sea.

This “mistake” is very likely to be the explanation for the strange fact, that we in deriving the Veneziano model from our novel string field theory formalism only got *one out of the three terms we would have expected*.

The suggested solution to our trouble would then be to allow also for constituents with the  $J^+$  being negative. That would mean we could not keep to the simple gauge choice enforcing a positive value to  $J^+$  but would have to allow also negative values for this  $J^+$ .

That in turn might then allow constituent pairs from say different bound states - or different strings as it would be in our formalism - to totally annihilate meaning without leaving any material after them, because no excess energy would have to be there after the annihilation. Negative energy and positive energy together have the chance of such total annihilation.

## 10.7 Conclusion and Outlook

The in many ways intuitively nice and appealing language of the Dirac sea, which we have in an earlier work extended also to be applicable for bosons, is at first not so well suited for particles – “Majorana particles” – which are identical to their own antiparticles. In the present article we have nevertheless developed precisely this question of how to describe particles – bosons or fermions – which are, as we call it, “Majorana”. We use also this terminology “Majorana” even for bosons to mean that a particle is its own antiparticle. The fermion case is rather well known. So our main story was first to review, how it were at all possible to make (a free) theory

for bosons based on a Dirac sea, and secondly the new features of this Dirac sea for boson theory as follows:

- a) negative norm squares
- b) negative number of particles.

The main point then became how to get what we call a Majorana-boson theory through these new features. This comes about by constructing in terms of the creation and annihilation operators  $a^{\dagger_f}(\vec{p}, E > 0)$  and  $a(\vec{p}, E < 0)$  for a type of boson that might have a charge, some creation and annihilation operators  $b(\vec{p})$  and  $b^{\dagger_f}(\vec{p})$  for the Majorana boson, which is really a superposition of a boson and an anti-boson of the type described by  $a$  and  $a^{\dagger}$ .

### 10.7.1 The old Dirac sea for bosons

The Dirac sea for boson theory is based on having a Fock space, for which a basis consists of states with a number of bosons  $k(\vec{p}, E \geq 0)$ , which can be both positive, zero and negative integer, in both positive and negative energy  $E$  single particle states for each 3-momentum  $\vec{p}$ ,

$$|\dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots\rangle \in \text{Fock space.} \quad (10.129)$$

Because of the complication that the inner product

$$\langle \varphi_1 |_{(f)} \varphi_2 \rangle = \int \varphi_1^* \overleftrightarrow{\frac{\partial}{\partial t}} \varphi_2 d^3\vec{X} \quad (10.130)$$

for a (single particle) boson is not positive definite we have to distinguish two different inner products  $|$  and  $|_f$  say and thus also the two thereto responding hermitean conjugations  $\dagger$  and  $\dagger_f$ , meaning respectively without and with the  $\int \varphi_1^* \overleftrightarrow{\frac{\partial}{\partial t}} \varphi_2 d^3\vec{X}$  included. In fact we have for the norm square for these two inner products

$$\begin{aligned} & \langle \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots |_f \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\ &= (-1)^{\sharp(\text{neg. energy b.})} \langle \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots | \dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}', E < 0), \dots \rangle \\ &= (-1)^{\sharp(\text{neg. energy b.})} \prod_{(\vec{p}, E \geq 0) \text{ for which } k \leq -1} (-1)^{|k|} \end{aligned} \quad (10.131)$$

### 10.7.2 Main Success of Our Previous Dirac Sea (also) for Bosons:

The remarkable feature of the sector with the emptied out Dirac sea for bosons - what we called the physical sector - is that one has arranged the sign alternation (10.42) with the total number of negative energy bosons to cancel the sign from (10.31) so as to achieve that the total Fock space has positive norm square. This

“physical sector” corresponds to that negative energy single particle states are in the negative sectors, while the positive energy single particle states are in the positive sector.

Thus the basis vectors of the full Fock space for the physical sector are of the form

$$|\dots, k(\vec{p}, E > 0), \dots; \dots, k(\vec{p}, E < 0), \dots\rangle \quad (10.132)$$

where the dots ... denotes that we have one integer number for every momentum vector –value ( $\vec{p}$  or  $\vec{p}'$ ), but now the numbers  $k(\vec{p}, E > 0)$  of particles in a positive energy are– in the physical sector–combination– restricted to be non-negative while the numbers of bosons in the negative energy single particle states are restricted to be negative

$$\begin{aligned} k(\vec{p}, E > 0) &= 0, 1, 2, \dots \\ k(\vec{p}, E < 0) &= -1, -2, -3, \dots \end{aligned} \quad (10.133)$$

**In this physical sector our Dirac Sea formalism is completely equivalent to the conventional formalism for quantizing Bosons with “charge” (i.e. Bosons that are not their own antiparticles), say e.g.  $\pi^+$  and  $\pi^-$ .**

But let us remind ourselves that this idea of using Dirac sea **allows one to not fill the Dirac sea**, if one should wish to think of such world. With our extension of the idea of the Dirac sea to also include Bosons one also gets allowed to not empty out to have  $-1$  boson in each negative energy single particle state. But for bosons you have the further strange feature of the phantasy world with the Dirac sea not treated as it should be to get physical, that one even gets negative norm square states, in addition to like in the fermion case having lost the bottom in the energy.

### 10.7.3 Present Article Main Point were to Allow for Bosons being their own Antiparticles also in Dirac sea Formalism

We could construct a “Majorana-boson” creation operator for say a “Majorana-boson” with momentum  $\vec{p}$ ,  $b^\dagger(\vec{p})$  analogously to the expressions (10.19) and (10.20).

$$b^\dagger(\vec{p}) = \frac{1}{\sqrt{2}} (a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \text{ and } b(\vec{p}) = \frac{1}{\sqrt{2}} (a(E > 0) + a^\dagger(E < 0))$$

Since an extra phase on the basis states does not matter so much we could also choose for the bosons the “Majorana boson” creation and annihilation operators to be

$$\begin{aligned} b^{\dagger_f}(\vec{p}) &= \frac{1}{\sqrt{2}} (a^\dagger(E > 0) + a(-\vec{p}, E < 0)) \\ &= \frac{1}{\sqrt{2}} (a^{\dagger_f}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)) \\ b(\vec{p}) &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) + a^{\dagger_f}(-\vec{p}, E < 0)) \\ &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^\dagger(-\vec{p}, E < 0)) \end{aligned} \quad (10.134)$$

Such creation operators  $b^{\dagger}(\vec{p})$  and their corresponding annihilation operators  $b(\vec{p})$  make up the completely usual creation and annihilation operator algebra for Bosons that are their own antiparticles in the case of the “physical sector combination”. This “physical sector combination” means that we emptied out the Dirac sea in the sense that in the “vacuum” put just  $-1$  boson in each negative energy single particle state. This correspondence means that our formalism is for this “physical sector combination” completely equivalent to how one usually describes Bosons - naturally without charge - which are their own antiparticles. But our formalism is to put into the framework of starting with a priori “charged” Bosons which then quite analogously to fermions have the possibility of having negative energy (as single particles). We then treat the analogous problem(s) to the Dirac sea for Fermions, by “putting *minus* one boson in each of the negative energy single particle states. That a bit miraculously solves both the problem of negative norm squares and negative second quantized energy, and even we can on top of that *restrict the theory*, if we so wish, *to enforce the bosons to be identified with their own antiparticles*.

We saw above that

- 1. We obtain the Fock-space (Hilbert-space) for the Bosons being their own antiparticles by *restriction* to a subspace

$$H_{\text{Maj}} = \{|\rangle \mid r(\vec{p}) |\rangle = 0\}, \quad (10.135)$$

where we have defined

$$\begin{aligned} r(\vec{p}) &= \frac{1}{\sqrt{2}} (a(+\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0)) \\ &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0)). \end{aligned} \quad (10.136)$$

Of course when one forces in the original Dirac sea formalism the antiparticles differnt from the particles to behave the same way in detail it means a drastic reduction of the degrees of freedom for the second quantized system - the Fock space-, and thus it is of course quite natural that we only use the subspace  $H_{\text{Maj}}$  being of much less (but still infinite) dimension than the original one.

- 2. In our formalism - since we use to write the creation operator for the boson being its own antiparticle as a sum of creation of a particle and of a hole (10.134) - a “Majorana-boson” is physically described as statistically or in superposition being with some chance a particle and with some chance a hole. Really it is obvious, that it is 50 % chance for each. So the physical picture is that the “Majorana-boson” is a superposition of a hole and an original positive energy particle in the “physical sector combination”.
- 3. We could construct a charge conjugation operation  $C$  which on our Fock space with both negative and positive energy states present as possibilities obtained the definition:

$$\begin{aligned} C |\dots \tilde{k}(\vec{p}, E > 0), \dots; \dots, \tilde{k}(\vec{p}', E < 0), \dots\rangle \\ = |\dots, k(\vec{p}, E > 0) = -\tilde{k}(-\vec{p}', E < 0) + 1, \dots; \\ \dots, k(\vec{p}', E < 0) = -1 - \tilde{k}(-\vec{p}, E > 0), \dots\rangle. \end{aligned} \quad (10.137)$$

Of course the state of the system of negative single particle comes to depend on that of the positive energy system after the charge conjugation and oppositely. With (10.58) or (10.134) one sees that on the whole system or Fock space of Bosons being their own antiparticles is left invariant under the charge conjugation operator  $C$ . This is as expected since these “Majorana Bosons” should be invariant under  $C$ .

#### 10.7.4 The Unphysical Sector Combinations and Boson-theories therein with Bosons being their own antiparticles

As a curiosity - but perhaps the most new in the present article - we have not only the physical sector combination, which so successfully just gives the usual formalism for both “charged” bosons and for what we called Majorana-bosons (the ones of their own antiparticles) but three more “sector-combinations” meaning combinations of whether one allows only negative numbers of bosons, or only non-negative numbers for the positive and the negative single particle states. The reader should have in mind that there is what we called the barrier, meaning that the creation and annihilation operators cannot cross from a negative number of particles in a single particle state to a positive one or opposite, and thus we can consider the theories in which a given single particle state has a positive or zero number of particles in it as a completely different theory from one in which one has a negative number of bosons in that single particle state. For simplicity we had chosen to only impose that we only considered that all single particle states with one sign of the single particle energy would have their number of particles being on the same side of the barrier. But even with this simplifying choice there remained  $2^2 = 4$  different sector-combinations. One of these sector-combinations - and of course the most important one because it matches the usual and physical formalism - were the “physical sector combination” characterized by their being a non-negative number  $k$  of bosons in all the positive energy single particle states (i.e. for  $E > 0$ ), while the number  $k$  of bosons in the negative energy single particle states (i.e. for  $E < 0$ ) is restricted to be genuinely negative  $-1, -2, -3, \dots$

The sector combination possibility 4) in our enumeration above the Fock space gets negative definite instead of as the one for the physical sector combination which gets positive definite. But these sector combinations are analogous or isomorphic with the appropriate sign changes allowed. Also our charge conjugation operator  $C$  operates inside both the “physical sector combination” and inside the sector combination number 4), which is characterized as having just the opposite to those of the physical sector, meaning that in sector combination 4) one has a negative number of particles in each positive energy (single particle)state, while there is a positive or zero number in the negative energy states. Thus the construction of particles being their own antiparticles would be rather analogous to that in the physical sector combination.

Less trivial is it to think about the two sector combinations 2) and 3) because now the charge conjugation operator  $C$  goes *between them*: Acting with the charge conjugation operator  $C$  on a state in the section combination 2) which we called the “naive vacuum sector combination” one gets a result of the operation in the *different*

*sector combination* namely 3). You can say that the charge conjugation operator does not respect the barrier, it is only the creation and annihilation operators which respect this barrier. A priori one would therefore now expect that one should construct the formalism for the boson being its own antiparticle for these sector combinations 2) and 3) based on a Fock space covering both parts of the sector combination 2) and part of 3). To realize that one gets eigenstates of the charge conjugation operator such a combination of the the two sector combinations is of course also needed. However, if one just wanted to realize an algebra of the creation and annihilation operators that could be interpreted as a formalism for the boson type being its own antiparticle, one might throw away one of the two sector combinations, say combination 3), and keep only the “naive vacuum sector combination” 2). Since the creation and annihilation operators *cannot* cross the barrier from one sector combination into the other one, such a keeping to only one of the two sectors between the charge conjugation operator goes back and forth would not make much difference for the creation and annihilation operators. We did in fact develop such a formalism for bosons being their own antiparticles in this way in alone “the naive vacuum sector combination”. Interestingly it now turned out that keeping to only one sector combination the whole Fock space constructed for the boson being its own antiparticle became of zero norm square. Really we should say Hilbert inner product became completely zero for the subsector of the Fock space - of this unphysical “naive vacuum sector combination” -. This is of course at least possible since the sector combinations 2)(=the naive vacuum one) and 3) have both positive and negative norm square states- so that no-zero Hilbert vectors can be formed as linear combinations of positive and negative normsquare Hilbert-vectors. (In the physical sector combination nor the sector combination 4) zero norm states cannot be found because the Hilbert innerproduct is respectively positively and negatively definite.).

### 10.7.5 Speculations Bound States, Rough Dirac Sea etc.

Then in the last section above we have some to the rest more weakly connected speculations meant to be of help for the original problem bringing us to the considerations in this article, namely our “novel string field theory”. A major suggestion, that came out of these considerations were to have in mind that, when you have an interacting quantum field theory, the vacuum gets into a rather complicated superposition of Fock space states, that makes descriptions as the “rough Dirac sea” or “the complicated vacuum” appropriate. While in say the “physical vacuum” - descussed in the article - you can only remove particles from negative energy states and only add particles to the positive single particle states, one does *not* have this restriction in the interacting vacuum, or say the “rough Dirac sea” vacuum. This point of view suggests that to make a proper description of a bound state or a resonance by means of a wave function in a relativistic quantum field theory, describing how to add or remove constituents from the “rough Dirac sea”-vacuum one should include *also negative energy possibilities* for the particles or antiparticle constituents.



These considerations are also hoped to be helpful for the problems we have for the moment with obtaining the full Veneziano model amplitude from our novel string field theory.

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## References

1. Majorana, Ettore; Maiani, Luciano (2006). "A symmetric theory of electrons and positrons". In Bassani, Giuseppe Franco. Ettore Majorana Scientific Papers. pp. 201E3. doi:10.1007/978-3-540-48095-2\_10. ISBN 978-3-540-48091-4. Translated from: Majorana, Ettore (1937). "Teoria simmetrica dell'elettrone e del positrone". Il Nuovo Cimento (in Italian) 14 (4): 171E4. doi:10.1007/bf02961314.
2. Griffiths, David J. (2008). Elementary Particles (Second, Revised ed.). WILEY-VCH. p. 149. ISBN 978-3-527-40601-2)
3. Nielsen, H. B.; Ninomiya, M., Dirac Sea for Bosons. I—Formulation of Negative Energy Sea for Bosons— Progress of Theoretical Physics, Vol. 113, No. 3, pp. 603-624 DOI: 10.1143/PTP.113.603 Bibliographic Code: 2005PThPh.113..603N  
Holger B. Nielsen and Masao Ninomiya, Progress of Theoretical Physics Volume 113, Issue 3 pp. 625-643.  
Dirac Sea for Bosons. II E Study of the Naive Vacuum Theory for the Toy Model World Prior to Filling the Negative Energy Sea
4. Yoshinobu Habara, Yukinori Nagatani, Holger B. Nielsen and Masao Ninomiya  
Dirac Sea and Hole Theory for Bosons II: Renormalization Approach  
Int. J. Mod. Phys. A 23, 2771 (2008). DOI: 10.1142/S0217751X08040354
5. Dirac, P A M - 1931 - Quantized Singularities In The Electromagnetic Fields (Published by api-3773171) P.A.M. Dirac, Proc. Roy. Soc.A 133, 60 1931.  
See also e.g. Alvarez-Gaume, Luis; Vazquez-Mozo, Miguel A. (2005). "Introductory Lectures on Quantum Field Theory". CERN Yellow Report CERN-, pp. - 1 (96): 2010001. arXiv:hep-th/0510040. Bibcode:2005hep.th...10040A.
6. Feynman, Richard P. (1998) [1972]. Statistical Mechanics: A Set of Lectures (2nd ed.). Reading, Massachusetts: Addison-Wesley. ISBN 978-0-201-36076-9.
7. arXiv:quant-ph/0602151v2 19 Feb 2006 Quantum Mechanics of Klein-Gordon Fields I: Hilbert Space, Localized States, and Chiral Symmetry A. Mostafazadeh and F. Zamani
8. H. B. Nielsen and M. Ninomiya, *An Idea of New String Field Theory – Liberating Right and Left movers –*, in Proceedings of the 14th Workshop, "What Comes Beyond the Standard Models" Bled, July 11-21, 2011, eds. N. M. Borstnik, H. B. Nielsen and D. Luckman; arXiv: 1112. 542[hep-th].

9. H. B. Nielsen and M. Ninomiya, *A New Type of String Field Theory*, in Proceedings of the 10th Tohwa International Symposium in String Theory", July 3-7, 2001, Fukuoka Japan, AIP conference Proc vol.607, p.185-201; arXiv: hep-th/0111240.v1, Nov.2001
10. H.B. Nielsen and M. Ninomiya, "A Novel String Field Theory Solving String Theory by Liberating Left and Right Movers" JHEP\_202P\_05131 v2 (2013)
11. H. B. Nielsen and M. Ninomiya, "Deriving Veneziano Model in a Novel String Field Theory Solving String Theory by Liberating Right and Left Movers," arXiv:1410.1048 [hep-th].
12. String references are cited in References [12]-[15], [17]  
H. B. Nielsen, *A Physical interpretation of the integrand of the n-point Veneziano model*, (1969)(Nordita preprint...); a later version "An almost Physical interpretation of the integrand of the n-point Veneziano model" preprint at Niels Bohr Institute; a paper presented at the 15th International Conference on High Energy Physics, Kiev 1970 (see p.445 in Veneziano's talk.)
13. Y. Nambu, *Quark Model and the factorization of the Veneziano amplitude*, in Proceedings of the International Conference on Symmetries and Quark Models, June 18-20, 1969 ed. by Chand, R. (Gordon and Breach), New York, 269-277; reprinted in Broken Symmetry Selected Papers of Y. Nambu eds. T. Eguchi and K. Nishijima, (World Scientific, Singapore, 1995)258-277.
14. Y. Nambu, *Duality and hadrondynamics*, Lecture notes prepared for Copenhagen summer school, 1970; It is reproduced in "Broken Symmetry", Selected Papers of Y. Nambu eds. T. Eguchi, and K. Nishijima, (World Scientific, Singapore, 1995)280.
15. L. Susskind, *Structure of hadrons implied by duality*, Phys. Rev. D1(1970)1182-1186(1970); L. Susskind, *Dual symmetric theory of hadrons*, 1. Nuovo Cimento A69,457-496(1970).
16. For many original papers of early string theory found in *The Birth of String Theory" eds. by Andrea Cappelletti, Elena Castellani, Flippo Colomo and Paolo Di Vecchia*, Cambridge University Press 2012. Cambridge, New York, Melbourne.
17. M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, 1-2, Cambridge University Press 1987.
18. As for bosonic string field theory in the light-cone gauge:  
M. Kaku and K. Kikkawa, Phys. Rev. D10(1974)1110; M. Kaku and K. Kikkawa, Phys. Rev. D10(1974)1823; S. Mandelstam, Nucl. Phys. B64(1973)205; E. Cremmer and J.-L. Gervais, Nucl. Phys. B90(1975)410.
19. Witten type mid-point interaction of covariant bosonic string field theory: for open string  
E. Witten, Nucl. Phys. B268(1986)253.
20. Closed string version of ref.[19] of the Witten theory:  
M. Saadi and B. Zwiebach, Ann. Phys. 192(1989)213; T. Kugo, H. Kunitomo and K. Suehiro, Phys. Lett. B226(1989)48; T. Kugo and K. Suehiro, Nucl. Phys. B337(1990)434; B. Zwiebach, Nucl. Phys. B390(1993)33.
21. Joining-Splitting light-cone type: for open string  
H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Rev. D34(1986)2360.
22. Closed string of Joining-Splitting light-cone type:  
H. Hata, K. Itoh, H. Kunitomo and K. Ogawa, Phys. Rev. D35(1987)1318;  
H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Rev. D35(1987)1356.
23. Supersymmetrized covariant open Witten theory  
E. Witten, Nucl. Phys. B276(1986)291; I. Arefeva and P. Medvedev, Phys. Lett. B212(1988)299; C. Wendt, Nucl. Phys. B314(1989)209; N. Berkovits, Nucl. Phys. B450(1995)90 [Errata, B459(1996)439] hep-th/9503099.

24. Modified cubic theory:  
I. Arefeva, P. Medvedev and A. Zubarev, Nucl. Phys. B341(1990)464; C. R. Preitschopf, C. B. Thorn and S. A. Yost, Nucl. Phys. B337(1990)363.
25. Supersymmetrized Green-Schwarz theory in the light-cone gauge:  
M. B. Green and J. H. Schwarz, Nucl. Phys. B218(1983)43; M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B219(1983)437; M. B. Green, and J. H. Schwarz, Nucl. Phys. B243(1984)437; J. Greensite and F. R. Klinkhamer, Nucl. Phys. B281(1987)269; Nucl. Phys. B291(1987)557; Nucl. Phys. B304(1988)108; M. B. Green and N. Seiberg, Nucl. Phys. B299(1988)559; S.-J. Sin, Nucl. Phys. B313(1989)165.
26. For review articles of string field theory:  
W. Taylor and B. Zwiebach, *D-branes, Tachyons, and String Field Theory*, hep-th/0311017 <http://jp.arxiv.org/abs/hep-th/0311017>; K. Ohmori, *A Review on Tachyon Condensation in Open String Theories*, hep-th/0102085 <http://jp.arxiv.org/abs/hep-th/0102085>.
27. C. B. Thorn, *Derivational model from field theory II*, Phys. Rev. D17(1978)1037, and references therein.
28. G. Veneziano, *Construction of a crossing-symmetric, Regge behaved amplitude for linearly rising trajectories*, Nuovo Cimento A57 190(1968).
29. For comprehensive reviews of string theory see  
e.g. J. Polchinski, *String Theory, vol.I-II*, Cambridge University Press 1998.
30. H. Bethe, E. Salpeter (1951). "A Relativistic Equation for Bound-State Problems". Physical Review 84 (6): 1232. Bibcode:1951PhRv...84.1232S. doi:10.1103/PhysRev.84.1232.
31. S. Fubini and G. Furlan, Physics, 229.  
S. Weinberg, Phys. Rev. 150, 1313.  
L. Susskind, Phys. Rev. 165, 1535.  
L. Susskind, Infinite Momentum Frames and Particle Dynamics, in Lectures in Theoretical Physics, Vol. XI-D: Mathematical Methods in Theoretical Physics, K. Mahanthappa and W. Brittin, ed.; Gordon and Breach, New York.



## 11 UV complete Model With a Composite Higgs Sector for Baryogenesis, DM, and Neutrino masses

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**Abstract.** We propose a UV complete model based on SUSY  $SU(2)_H$  gauge theory with confinement. New  $Z_2$  discrete symmetry and  $Z_2$ -odd right-handed neutrino superfields are also introduced to the model. Its low-energy effective theory can provide solutions for Baryogenesis, DM candidate, and origin of neutrino masses. Below a confinement scale, the Higgs sector is described in terms of mesonic superfields of fundamental  $SU(2)_H$  doublets. We also discuss how to test the scenario by the future collider experiments in a benchmark scenario.

**Povzetek.** Avtor predlaga model za konfinirane kvarke, ki temelji na supersimetrični umeritveni teoriji  $SU(2)_H$ , dopolnjeni z diskretno simetrijo  $Z_2$ . Tudi za nevtrinska superpolja uporabi  $Z_2$  diskretno simetrijo. V limiti nizkih energij lahko model ponudi odgovore za nastanek barionov, kandidate za temno snov in pojasni izvor nevtrinskih mas. Na energijski skali pod kromodinamskim faznim prehodom opiše Higgsove skalarje z mezonskimi superpolji osnovnega dubleta  $SU(2)_H$ . Obravnava tudi možnosti preverbe modela na bodočih poskusih na pospeševalnikih.

Keywords: New Physics, Composite Higgs sector, SUSY

### 11.1 Introduction

A Higgs boson was discovered in 2012 at LHC experiments, and it has been confirmed that its properties are consistent with the Higgs boson in the Standard Model (SM). However, it is not the end of the story. The SM has still serious problems. For example, there is no successful mechanism of Baryogenesis, there is no candidate of the Dark Matter (DM), there is no natural explanation of tiny neutrino masses, and so on. On the other hand, we have not fully understood the Higgs sector yet. There are still several fundamental questions. For example, how many Higgs bosons are there?, Whether is the Higgs boson a elementary scalar or a composite state? What is the origin of the negative mass squared of the Higgs boson? and so on. In many models, extension of the SM for explaining unsolved problems, such as Baryogenesis, DM, neutrino masses, etc lead to an

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extended Higgs sector. Thus, we can say that the Higgs sector will be a probe of new physics.

In this talk, we consider a SUSY model[1,2] with additional  $SU(2)_H$  gauge symmetry to the SM gauge group and three matter fields (and three anti-matter fields) which are fundamental representations under the  $SU(2)_H$ . In the low energy effective theory of this model, the Higgs sector is described by mesonic fields of those six fields. We then show that this effective theory can provide enough enhancement of the first order electroweak phase transition (1stOPT) which is required by successful electroweak baryogenesis scenario[3], DM candidates, and mechanism to generate tiny neutrino masses through radiative corrections.

## 11.2 Model

In SUSY  $SU(N_c)$  gauge theory with  $N_c + 1$  flavour fields, confinement occurs at some scale[6]. The simplest example is  $N_c = 2$  case. Utilising this setup, we propose a model with  $SU(2)_H$  symmetry with three flavour fields which are fundamental representations of  $SU(2)_H$ . There should also be three anti-matter fields for each fundamental representation matter fields. We described these six fields as  $T_i (i = 1, \dots, 6)$ . This setup is almost same as one in the minimal SUSY fat Higgs model[7]. In the minimal SUSY fat Higgs model, two doublets and one singlet mesonic fields are light in the low energy effective theory by introducing additional fields. In our model, in contrast, all the mesonic fields appears in the low energy effective theory.

We here introduce a right-handed neutrino (RHN) which is singlet under  $SU(2)_H$  as well as the SM gauge symmetry. The model also has an unbroken discrete symmetry  $Z_2$  in order to forbid tree level contributions to neutrino masses. The RHN has an odd charge under the  $Z_2$  parity. We show the charge assignment of  $T_i$  and the RHN  $N_R^c$  under the SM gauge symmetry,  $SU(2)_H$ , and the  $Z_2$  parity in Table 11.1-(I). The fifteen mesonic fields below a certain scale  $\Lambda_H$  which are canonically normalized as  $H_{ij} \simeq \frac{1}{4\pi\Lambda_H} T_i T_j (i \neq j)$  are listed in the Table 11.1-(II).

The superpotential of the Higgs sector below  $\Lambda_H$  is given by

$$\begin{aligned}
 W_{\text{eff}} = & \lambda N (H_u H_d + v_0^2) \\
 & + \lambda N_\Phi (\Phi_u \Phi_d + v_\Phi^2) \\
 & + \lambda N_\Omega (\Omega_+ \Omega_- - \zeta \eta + v_\Omega^2) \\
 & + \lambda \{ \zeta H_d \Phi_u + \eta H_u \Phi_d - \Omega_+ H_d \Phi_d - \Omega_- H_u \Phi_u - N N_\Phi N_\Omega \} . \quad (11.1)
 \end{aligned}$$

(I)

Superfield	SU(2) <sub>H</sub>	SU(3) <sub>C</sub>	SU(2) <sub>L</sub>	U(1) <sub>Y</sub>	Z <sub>2</sub>
$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$	2	1	2	0	+1
T <sub>3</sub>	2	1	1	+1/2	+1
T <sub>4</sub>	2	1	1	-1/2	+1
T <sub>5</sub>	2	1	1	+1/2	-1
T <sub>6</sub>	2	1	1	-1/2	-1
N <sub>R</sub> <sup>c</sup>	1	1	1	0	-1

(II)

Superfield	SU(3) <sub>C</sub>	SU(2) <sub>L</sub>	U(1) <sub>Y</sub>	Z <sub>2</sub>
H <sub>d</sub> $\equiv$ $\begin{pmatrix} H_{14} \\ H_{24} \end{pmatrix}$	1	2	-1/2	+1
H <sub>u</sub> $\equiv$ $\begin{pmatrix} H_{13} \\ H_{23} \end{pmatrix}$	1	2	+1/2	+1
Φ <sub>d</sub> $\equiv$ $\begin{pmatrix} H_{15} \\ H_{25} \end{pmatrix}$	1	2	-1/2	-1
Φ <sub>u</sub> $\equiv$ $\begin{pmatrix} H_{16} \\ H_{26} \end{pmatrix}$	1	2	+1/2	-1
Ω <sub>-</sub> $\equiv$ H <sub>46</sub>	1	1	-1	-1
Ω <sub>+</sub> $\equiv$ H <sub>35</sub>	1	1	+1	-1
N $\equiv$ H <sub>56</sub> , N <sub>Φ</sub> $\equiv$ H <sub>34</sub> , N <sub>Ω</sub> $\equiv$ H <sub>12</sub>	1	1	0	+1
ζ $\equiv$ H <sub>36</sub> , η $\equiv$ H <sub>45</sub>	1	1	0	-1

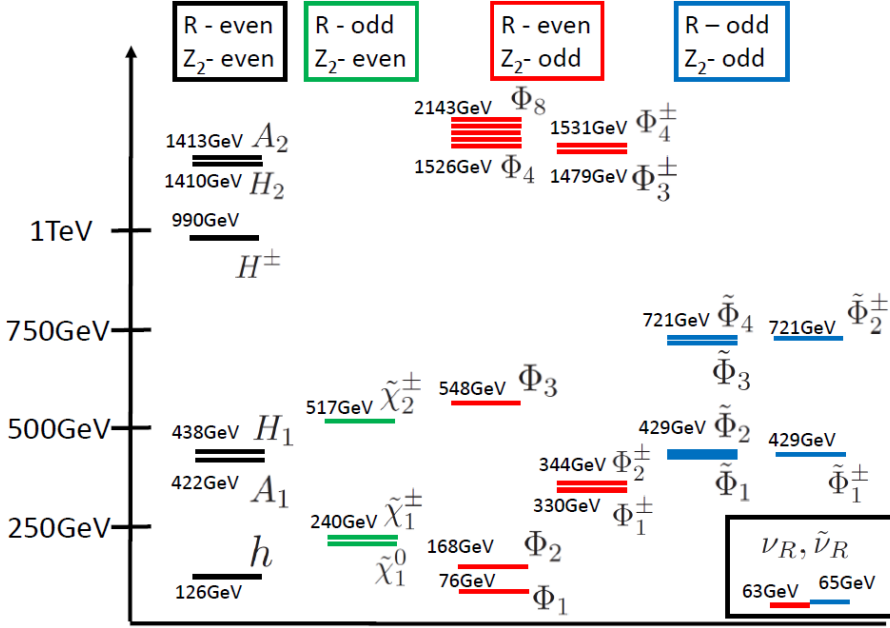
**Table 11.1.** (I) The charge assignment of the SU(2)<sub>H</sub> doublets T<sub>i</sub> and the RHN N<sub>R</sub><sup>c</sup> under the SM gauge group SU(3)<sub>c</sub> × SU(2)<sub>L</sub> × U(1)<sub>Y</sub> and the Z<sub>2</sub> parity. (II) The field content of the extended Higgs sector in the low energy effective theory below the scale Λ<sub>H</sub>.

By the Naive Dimensional Analysis,  $\lambda \simeq 4\pi$  is naively expected at the confinement scale Λ<sub>H</sub>. The relevant soft SUSY breaking Lagrangian terms are given by

$$\begin{aligned}
\mathcal{L}_H = & -m_{H_u}^2 H_u^\dagger H_u - m_{H_d}^2 H_d^\dagger H_d - m_{\Phi_u}^2 \Phi_u^\dagger \Phi_u - m_{\Phi_d}^2 \Phi_d^\dagger \Phi_d \\
& - m_N^2 N^* N - m_{N_\Phi}^2 N_\Phi^* N_\Phi - m_{N_\Omega}^2 N_\Omega^* N_\Omega - m_{\Omega_+}^2 \Omega_+^* \Omega_+ - m_{\Omega_-}^2 \Omega_-^* \Omega_- \\
& - m_\zeta^2 \zeta^* \zeta - m_\eta^2 \eta^* \eta - \left\{ m_{\zeta\eta}^2 \eta^* \zeta + \frac{B_\zeta^2}{2} \zeta^2 + \frac{B_\eta^2}{2} \eta^2 + \text{h.c.} \right\} \\
& - \{ C \lambda v_0^2 N + C_\Phi \lambda v_\Phi^2 N_\Phi + C_\Omega \lambda v_\Omega^2 N_\Omega + \text{h.c.} \} \\
& - \{ B \mu H_u H_d + B_\Phi \mu_\Phi \Phi_u \Phi_d + B_\Omega \mu_\Omega (\Omega_+ \Omega_- + \zeta \eta) + \text{h.c.} \} \\
& - \lambda \{ A_N H_u H_d N + A_{N_\Phi} \Phi_u \Phi_d N_\Phi + A_{N_\Omega} (\Omega_+ \Omega_- - \eta \zeta) N_\Omega + A_\zeta H_d \Phi_u \zeta \\
& \quad + A_\eta H_u \Phi_d \eta + A_{\Omega_-} H_u \Phi_u \Omega_- + A_{\Omega_+} H_d \Phi_d \Omega_+ + \text{h.c.} \} . \tag{11.2}
\end{aligned}$$

By the vacuum expectation values (vev's) of Z<sub>2</sub>-even singlet fields N, N<sub>Φ</sub> and N<sub>Ω</sub>, the mass parameters  $\mu = \lambda \langle N \rangle$ ,  $\mu_\Phi = \lambda \langle N_\Phi \rangle$  and  $\mu_\Omega = \lambda \langle N_\Omega \rangle$  are induced. The





**Fig. 11.2.** The mass spectrum of the relevant particles in the bench mark scenario. The figure is taken from Ref.[1].

Though this point is already excluded by the direct detection experiment of the DM[9], we discuss phenomenological consequences of this benchmark scenario, because we can see some general features of our model in the scenario. In Fig. 11.2, the mass spectrum of the relevant particles in this benchmark scenario is shown. The  $Z_2$ -even part of the spectrum is similar to one in nMSSM. A significant size of mass splitting between the charged Higgs boson and the heavy Higgs bosons is required for obtaining the large mixing between doublet fields and a singlet field, which is necessary to reproduce the relic abundance of the DM. By looking at such a large splitting in the spectrum of extra Higgs bosons, the  $Z_2$ -even part of our scenario can be distinguished from the MSSM. In this benchmark scenario,  $\varphi_c/T_c$  is enhanced by the loop effect of  $\Phi_u$  and  $\Omega_-$ . The loop effect can also significantly affect the  $h\text{-}\gamma\text{-}\gamma$  coupling and the triple Higgs boson coupling as shown in Table 11.2. By using precise measurement of the SM-like Higgs boson couplings at future collider experiment such as ILC[10], our benchmark scenario can be distinguished from nMSSM too.

Couplings	$hWW$	$hZZ$	$h\bar{u}u$	$h\bar{d}d$	$h\bar{\ell}\ell$	$h\gamma\gamma$	$hhh$
$\kappa_{h\phi\phi} = g_{h\phi\phi}/g_{h\phi\phi}^{\text{SM}}$	0.990	0.990	0.990	0.978	0.978	0.88	1.2

**Table 11.2.** The deviations in the coupling constants from the SM values in the benchmark scenario defined in Ref. [1].



It is also interesting to discuss phenomenology in the  $Z_2$ -odd sector. By the direct search of inert doublet particles[11] and inert charged singlet searches[12] at ILC, it is expected to get a strong hint on the  $Z_2$ -odd sector of the scenario.

## 11.4 Conclusion

We have attempted to construct a simple model to solve the three problems such as baryogenesis, DM, and tiny neutrino mass, which cannot be explained in the SM. We have succeeded to find such a UV model based on SUSY  $SU(2)_H$  gauge theory with confinement. In its low energy effective theory, we have shown that the 1stOPT is enhanced strongly enough for successful electroweak baryogenesis, multi-components DM scenario is realised, and tiny neutrino masses are generated via one-loop and three-loop diagrams. We have also introduced a benchmark scenario and we have discussed how to test it at future collider experiments. In this benchmark scenario, the spin-independent cross section of DM's are above the latest result of the DM direct detection experiments, so that we should look for a new benchmark scenario. In addition, we focus only on the 1stOPT for the baryogenesis. For complete analysis, new CP violation phases should be taken into account.

Recently, effects of CP violation in the singlet-doublet dark matter model is discussed and it is shown that the spin-independent cross section can be suppressed with a certain CP violation in the dark sector[13]. Therefore, it will be important to take CP phases in to account for evading the strong constraint from the direct detection of DMs as well as for complete analysis of the baryogenesis scenario.

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## References

1. S. Kanemura, N. Machida and T. Shindou, Phys. Lett. B **738** (2014) 178
2. S. Kanemura, T. Shindou and T. Yamada, Phys. Rev. D **86** (2012) 055023; S. Kanemura, E. Senaha, T. Shindou and T. Yamada, JHEP **1305** (2013) 066; S. Kanemura, N. Machida, T. Shindou and T. Yamada, Phys. Rev. D **89** (2014) 1, 013005.
3. V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **155** (1985) 36; A. G. Cohen, D. B. Kaplan and A. E. Nelson, Ann. Rev. Nucl. Part. Sci. **43** (1993) 27; M. Quiros, Helv. Phys. Acta **67** (1994) 451; V. A. Rubakov and M. E. Shaposhnikov, Usp. Fiz. Nauk **166** (1996) 493 [Phys. Usp. **39** (1996) 461]; K. Funakubo, Prog. Theor. Phys. **96** (1996) 475; M. Trodden, Rev. Mod. Phys. **71** (1999) 1463; W. Bernreuther, Lect. Notes Phys. **591** (2002) 237; J. M. Cline, hep-ph/0609145; D. E. Morrissey and M. J. Ramsey-Musolf, New J. Phys. **14** (2012) 125003.
4. E. Ma, Phys. Rev. D **73** (2006) 077301.

5. M. Aoki, S. Kanemura and O. Seto, Phys. Rev. Lett. **102** (2009) 051805; M. Aoki, S. Kanemura and O. Seto, Phys. Rev. D **80** (2009) 033007; M. Aoki, S. Kanemura and K. Yagyu, Phys. Rev. D **83** (2011) 075016.
6. K. A. Intriligator and N. Seiberg, Nucl. Phys. Proc. Suppl. **45BC** (1996) 1.
7. R. Harnik, G. D. Kribs, D. T. Larson and H. Murayama, Phys. Rev. D **70** (2004) 015002.
8. M. Dine, P. Huet, R. L. Singleton, Jr and L. Susskind, Phys. Lett. B **257** (1991) 351; A. G. Cohen and A. E. Nelson, Phys. Lett. B **297** (1992) 111.
9. D. S. Akerib *et al.* [LUX Collaboration], Phys. Rev. Lett. **118** (2017) no.2, 021303.
10. H. Baer, T. Barklow, K. Fujii, Y. Gao, A. Hoang, S. Kanemura, J. List and H. E. Logan *et al.*, arXiv:1306.6352 [hep-ph]; D. M. Asner, T. Barklow, C. Calancha, K. Fujii, N. Graf, H. E. Haber, A. Ishikawa and S. Kanemura *et al.*, arXiv:1310.0763 [hep-ph].
11. M. Aoki, S. Kanemura and H. Yokoya, Phys. Lett. B **725** (2013) 302.
12. M. Aoki and S. Kanemura, Phys. Lett. B **689** (2010) 28.
13. T. Abe, Phys. Lett. B **771** (2017) 125.



## 12 Structure of Quantum Corrections in $\mathcal{N} = 1$ Supersymmetric Gauge Theories

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**Abstract.** Some recent research of quantum corrections in  $\mathcal{N} = 1$  supersymmetric theories is briefly reviewed. The most attention is paid to the theories regularized by higher covariant derivatives. In particular, we discuss, how the NSVZ and NSVZ-like relations appear with this regularization and how one can construct the NSVZ scheme in all orders.

**Povzetek.** Avtor na kratko poroča o nedavnih raziskavah kvantnih popravkov v supersimetričnih teorijah tipa  $\mathcal{N} = 1$  s posebnim poudarkom na teorijah regulariziranih z višjimi kovariantnimi odvodi. Predstavi, kaj se zgodi z relacijami tipa NSVZ in tej podobnimi v tej regularizaciji in kako poteka konstrukcija NSVZ sheme v vseh redih.

**Keywords:** Supersymmetric theories, Quantum corrections, Regularization,  $\mathcal{N} = 1$  supersymmetric extensions of the Standard Model

### 12.1 Introduction

$\mathcal{N} = 1$  supersymmetric extensions of the Standard Model (SM) are very interesting candidates for describing physics beyond it [1]. In these theories there are no quadratically divergent quantum corrections to the Higgs mass, the running of coupling constants agrees with the predictions of the Grand Unified Theories, and the proton lifetime (proportional to  $M_X^4$ ) is much larger than in the non-supersymmetric case. This makes them very attractive from the phenomenological point of view. However, the supersymmetric extensions of SM predict a lot of new particles, which are superpartners of quarks, leptons, gauge bosons and Higgs bosons. Supersymmetry also requires two Higgs doublets, which produces  $2 \times 2 \times 2 - 3 = 5$  Higgs bosons. To make masses of superpartners sufficiently large, it is necessary to break supersymmetry. Although it is highly desirable to break supersymmetry spontaneously, the simplest models (like MSSM) include soft terms, which explicitly break supersymmetry, but do not produce quadratic divergences. Investigation of quantum corrections in supersymmetric theories and theories with softly broken supersymmetry and comparing them with experimental data can provide information about physics beyond SM.

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It is convenient to describe  $\mathcal{N} = 1$  supersymmetric theories in  $\mathcal{N} = 1$  superspace, because in this case supersymmetry is a manifest symmetry. In this language, the renormalizable  $\mathcal{N} = 1$  SYM theory (with a simple gauge group  $G$ , for simplicity) is described by the action

$$S = \frac{1}{2e_0^2} \text{Re tr} \int d^4x d^2\theta W^a W_a + \frac{1}{4} \int d^4x d^4\theta \phi^{*i} (e^{2V})_i{}^j \phi_j \\ + \left\{ \int d^4x d^2\theta \left( \frac{1}{4} m_0^{ij} \phi_i \phi_j + \frac{1}{6} \lambda_0^{ijk} \phi_i \phi_j \phi_k \right) + \text{c.c.} \right\},$$

where  $\theta$  denotes auxiliary Grassmannian coordinates. The real superfield  $V(x, \theta, \bar{\theta})$  is the gauge superfield, and the supersymmetric gauge field strength is defined as  $W_a = \bar{D}^2 (e^{-2V} D_a e^{2V}) / 8$ . The matter superfields  $\phi_i$  are chiral,  $\bar{D}_a \phi_i = 0$ , where in our notation  $D_a$  and  $\bar{D}_a$  denote the right and left supersymmetric covariant derivatives, respectively. In terms of superfields the gauge transformations can be written as

$$\phi \rightarrow e^A \phi; \quad e^{2V} \rightarrow e^{-A^+} e^{2V} e^{-A}, \quad (12.1)$$

and are parameterized by a chiral superfield  $A = ie_0 A^B T^B$ .

Quantum behaviour of supersymmetric theories is better than in the non-supersymmetric case. For example, in the most interesting for phenomenology case of  $\mathcal{N} = 1$  supersymmetry, there are no divergent quantum corrections to the superpotential [2]. Consequently, the renormalization of masses and Yukawa couplings in such theories is related to the renormalization of the chiral matter superfields. As a non-renormalization theorem one can also consider a relation between the  $\beta$ -function and the anomalous dimensions of the chiral matter superfields which takes place in  $\mathcal{N} = 1$  supersymmetric theories [3–6],

$$\beta(\alpha, \lambda) = - \frac{\alpha^2 \left( 3C_2 - T(R) + C(R)_i{}^j \gamma_j{}^i(\alpha, \lambda)/r \right)}{2\pi(1 - C_2 \alpha/2\pi)}. \quad (12.2)$$

In our notation  $r = \dim G$ , and  $T^A$  are the generators of the representation  $R$  to which the chiral matter superfields belong, such that  $\text{tr}(T^A T^B) = T(R) \delta^{AB}$  and  $(T^A T^A)_i{}^j \equiv C(R)_i{}^j$ . For the adjoint representation  $T(\text{Adj}) = C_2$ , where  $f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}$ . The relation (12.2) is usually called the exact NSVZ  $\beta$ -function, because for the pure  $\mathcal{N} = 1$  SYM theory it gives the exact expression for the  $\beta$ -function. In this paper (following Ref. [7]) we will also discuss the relation between the NSVZ  $\beta$ -function and the non-renormalization theorem for the triple gauge-ghost vertices. This theorem claims that in  $\mathcal{N} = 1$  SYM theories three-point vertices with two ghost legs and one leg of the quantum gauge superfield are finite.

Although a lot of general arguments can be used for obtaining Eq. (12.2), see, e.g., [8–10], it is not so trivial to establish how the NSVZ relation appears in perturbative calculations. Certainly, for doing such calculations the theory should be properly regularized, and the way of removing divergences should be specified. By other words, it is necessary to fix a subtraction scheme. The calculations done

with the dimensional reduction [11] in the  $\overline{\text{DR}}$ -scheme in the three- and four-loop approximations [12–14,16] demonstrated that Eq. (12.2) does not take place starting from the three-loop approximation. However, one can explain the disagreement by the scheme dependence of the NSVZ relation [17,18]. A possibility of this explanation is non-trivial due to some scheme-independent consequences of the NSVZ relation [18,19]. Thus, with the dimensional reduction the NSVZ equation should be obtained by a special tuning of the subtraction scheme in every order, while the general all-order prescription giving the NSVZ scheme is absent.

Also it should be noted that the dimensional reduction is not mathematically consistent [20], and can break supersymmetry in higher orders [21,22]. That is why the use of other regularizations is also reasonable and interesting. In this paper we will mostly discuss various application of the Slavnov higher covariant derivative regularization [23,24] to calculating quantum corrections in  $\mathcal{N} = 1$  supersymmetric theories. Unlike the dimensional reduction, this regularization is consistent and can be formulated in a manifestly  $\mathcal{N} = 1$  supersymmetric way [25,26]. It is also applicable to theories with  $\mathcal{N} = 2$  supersymmetry [27–29]. The main idea of this regularization is to add a term with higher degrees of covariant derivatives to the action of a theory. Then divergences beyond the one-loop approximation disappear, while the remaining one-loop divergences are regularized by inserting the Pauli–Villars determinants into the generating functional [30]. In this paper we will demonstrate that this regularization allows to reveal some interesting features of quantum corrections in supersymmetric theories which are missed in the case of using the dimensional technique.

## 12.2 NSVZ relation in $\mathcal{N} = 1$ SQED

### 12.2.1 Higher derivative regularization in the Abelian case

We will start with the simplest  $\mathcal{N} = 1$  supersymmetric gauge theory, namely, the  $\mathcal{N} = 1$  supersymmetric electrodynamics (SQED) with  $N_f$  flavors. In the massless case this theory is described by the action

$$S = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a + \sum_{f=1}^{N_f} \frac{1}{4} \int d^4x d^4\theta \left( \phi_f^* e^{2V} \phi_f + \tilde{\phi}_f^* e^{-2V} \tilde{\phi}_f \right), \quad (12.3)$$

which is written in terms of  $\mathcal{N} = 1$  superfields. In this formalism supersymmetry is a manifest symmetry of the theory. The usual gauge field is now a component of the real gauge superfield  $V$ . The terms containing the chiral matter superfields  $\phi_f$  and  $\tilde{\phi}_f$  produce Dirac fermions and the other terms needed for supersymmetry invariance. In the Abelian case the supersymmetric gauge field strength is described by the chiral spinor superfield  $W_a = \bar{D}^2 D_a V/4$ . For the theory (12.3) the NSVZ  $\beta$ -function (12.2) takes the form [31,32]

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi} \left( 1 - \gamma(\alpha) \right). \quad (12.4)$$

To regularize the theory (12.3) by the Slavnov higher derivatives method, we add the term

$$S_\Lambda = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a \left( R(\partial^2/\Lambda^2) - 1 \right) W_a \quad (12.5)$$

to the classical action, where the function  $R(\partial^2/\Lambda^2)$  contains higher degrees of derivatives. Note that for Abelian theories one should use the usual derivatives (instead of the covariant ones). In the simplest case it is possible to choose  $R = 1 + \partial^{2n}/\Lambda^{2n}$ . Due to the presence of the higher derivative term, the propagator of the gauge superfield contains higher degrees of the momentum in the denominator, and all diagrams beyond the one-loop approximation become finite. For removing the remaining one-loop divergences, following Ref. [30], we insert into the generating functional the Pauli–Villars determinants,

$$Z = \int D\mu \prod_I \left( \det PV(V, M_I) \right)^{N_f c_I} \exp \left\{ iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{Sources}} \right\}, \quad (12.6)$$

with the constants  $c_I$  satisfying the conditions  $\sum_I c_I = 1$ ;  $\sum_I c_I M_I^2 = 0$ . Here  $M_I = a_I \Lambda$  (where  $a_I$  are constants independent of  $\alpha_0$ ) are masses of the Pauli–Villars superfields proportional to the parameter  $\Lambda$  which enters the regulator function  $R$ .

Below we will see that the NSVZ equation follows from the underlying relation between the two-point Green functions. In  $\mathcal{N} = 1$  SQED these two-point Green functions are related to the corresponding part of the effective action by the equation

$$\begin{aligned} \Gamma^{(2)} = & \int \frac{d^4p}{(2\pi)^4} d^4\theta \left( -\frac{1}{16\pi} V(-p) \partial^2 \Pi_{1/2} V(p) d^{-1}(\alpha_0, \Lambda/p) \right. \\ & \left. + \frac{1}{4} \sum_{f=1}^{N_f} \left( \phi_f^*(-p, \theta) \phi_f(p, \theta) + \tilde{\phi}_f^*(-p, \theta) \tilde{\phi}_f(p, \theta) \right) G(\alpha_0, \Lambda/p) \right). \end{aligned} \quad (12.7)$$

Here  $\partial^2 \Pi_{1/2} \equiv -D^a \bar{D}^2 D_a / 8$  is a supersymmetric transversal projection operator, and the transversality of the gauge superfield two-point function follows from the Slavnov–Taylor identities.

The function  $d^{-1}$  expressed in terms of the renormalized coupling constant  $\alpha(\alpha_0, \Lambda/\mu)$  should be finite in the limit  $\Lambda \rightarrow \infty$ . The charge renormalization constant  $Z_3$  is then defined as  $Z_3(\alpha, \Lambda/\mu) \equiv \alpha/\alpha_0$ . To construct the renormalization constant  $Z$  for the chiral matter superfields, we require finiteness of the function  $Z(\alpha, \Lambda/\mu) G(\alpha_0, \Lambda/p)$  in the limit  $\Lambda \rightarrow \infty$ .

According to [33], it is important to distinguish the renormalization group functions (RGF) defined in terms of the bare coupling constant and the ones defined in terms of the renormalized coupling constant. In terms of the bare coupling constant RGF are defined by the equations

$$\beta(\alpha_0) \equiv \left. \frac{d\alpha_0}{d \ln \Lambda} \right|_{\alpha=\text{const}}; \quad \gamma(\alpha_0) \equiv - \left. \frac{d \ln Z}{d \ln \Lambda} \right|_{\alpha=\text{const}}. \quad (12.8)$$

They are independent of a renormalization prescription for a fixed regularization, see, e.g., [33], but depend on the regularization. Below we will see that for the theory (12.3) these RGF satisfy the NSVZ relation in all loops in the case of using the above described version of the higher derivative regularization.

### 12.2.2 Charge renormalization in the lowest loops

Explicit calculations in the lowest loops made with the higher covariant derivative regularization demonstrated that loop integrals giving a  $\beta$ -function defined in terms of the bare coupling constant are integrals of total derivatives [34]. They can be also presented as integrals of double total derivatives [35]. The  $\beta$ -function of  $\mathcal{N} = 1$  SQED with  $N_f$  flavours, regularized by higher derivatives, is calculated by the help of the equation

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \Big|_{p=0}. \quad (12.9)$$

By other words, we calculate the two-point Green function of the gauge superfield and differentiate it with respect to  $\ln \Lambda$  in the limit of the vanishing external momentum. For example, the two-loop result for the  $\beta$ -function written as the integral of double total derivatives has the form

$$\begin{aligned} \frac{\beta(\alpha_0)}{\alpha_0^2} &= 2\pi N_f \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \left\{ \sum_I c_I \frac{\ln(q^2 + M_I^2)}{q^2} + \int \frac{d^4 k}{(2\pi)^4} \right. \\ &\times \left. \frac{2e^2}{k^2 R_k} \left( \frac{1}{q^2(k+q)^2} - \sum_I c_I \frac{1}{(q^2 + M_I^2)((k+q)^2 + M_I^2)} \right) \right\} + O(e^4). \end{aligned} \quad (12.10)$$

The (essentially larger) three-loop expression can be found, e.g., in [36]. Note that the  $\beta$ -function does not vanish because of integrand singularities. This can be illustrated by a simple example: consider a nonsingular function  $f(q^2)$  rapidly decreasing at infinity. Then

$$\int \frac{d^4 q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \left( \frac{q^\mu}{q^4} f(q^2) \right) = -\frac{1}{8\pi^2} f(0). \quad (12.11)$$

Doing similar calculations it is possible to decrease the number of integrations in Eq. (12.10) and reduce this expression to the integral giving the one-loop anomalous dimension of the matter superfield (also defined in terms of the bare coupling constant),

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} \left( 1 - \frac{d}{d \ln \Lambda} \ln G(\alpha_0, \Lambda/q) \Big|_{q=0} \right) = \frac{N_f}{\pi} (1 - \gamma(\alpha_0)). \quad (12.12)$$

### 12.2.3 NSVZ relation in all loops

The all-loop derivation of the NSVZ relation for RGF defined in terms of the bare coupling constant by the direct summing of supergraphs for  $\mathcal{N} = 1$  SQED regularized by higher derivatives has been made in [37,38] and verified at the three-loop level in [39]. Here we briefly explain the main ideas of the method of Ref. [37].

First, it is necessary to prove that all loop integrals for the  $\beta$ -function defined in terms of the bare coupling constant are integrals of double total derivatives. For this purpose it is convenient to use the background field method which (in the Abelian case) is introduced by making the replacement  $V \rightarrow V + \mathbf{V}$ , where  $\mathbf{V}$  is the background gauge superfield, in the action. Then we make the formal substitution  $V \rightarrow \theta^4$ , after which

$$\left. \frac{d\Delta\Gamma_{\mathbf{V}}^{(2)}}{d\ln\Lambda} \right|_{V=\theta^4} = \frac{1}{2\pi} \mathcal{V}_4 \cdot \frac{d}{d\ln\Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) = \frac{1}{2\pi} \mathcal{V}_4 \cdot \frac{\beta(\alpha_0)}{\alpha_0^2}, \quad (12.13)$$

where  $\mathcal{V}_4$  is the (properly regularized) volume of the space-time.

For  $\mathcal{N} = 1$  SQED the functional integrals over the matter superfield are Gaussian and can be calculated exactly. This allows operating with some expressions valid in all loops. In particular, it is possible to find the formal expression for the two-point function of the background gauge superfield. Then after the substitution  $V \rightarrow \theta^4$  we try to present the result as an integral of double total derivatives. In the coordinate representation an integral of a total derivative is written as

$$\text{Tr}([x^\mu, \text{Something}]) - \text{Singularities} = -\text{Singularities}. \quad (12.14)$$

After some non-trivial transformations the result for the expression (12.13) can be presented as a trace of double commutator, i.e. as an integral of a double total derivative. The details of this calculation are described in Ref. [37]. The result does not vanish due to singularities of the integrand, which can be summed in all orders. This gives

$$\left. \frac{d\Delta\Gamma^{(2)}}{d\ln\Lambda} \right|_{V=\theta^4} = \frac{N_f}{2\pi^2} \mathcal{V}_4 \left( 1 - \left. \frac{d\ln G}{d\ln\Lambda} \right|_{q=0} \right) = \frac{N_f}{2\pi^2} \mathcal{V}_4 \left( 1 - \gamma(\alpha_0) \right), \quad (12.15)$$

and we obtain the exact all-order result

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} \left( 1 - \gamma(\alpha_0) \right). \quad (12.16)$$

Note that this equation is valid for an arbitrary renormalization prescription in the case of using the higher derivative regularization, because RGF entering it are defined in terms of the bare coupling constant.

In graphical language, this result can be explained as follows [35] (see also [40]): If we have a supergraph without external lines, then a contribution to the



$\beta$ -function can be constructed by attaching two external lines of the background gauge superfield  $V$  to it, while a contribution to the anomalous dimension is obtained by cutting matter lines in the considered supergraph. The equation (12.16) relates both these contributions.

#### 12.2.4 How to construct the NSVZ scheme in $\mathcal{N} = 1$ SQED

Eq. (12.16) is valid for RGF defined in terms of the bare coupling constant. However, RGF are standardly defined by a different way, in terms of the renormalized coupling constant,

$$\tilde{\beta}(\alpha) \equiv \left. \frac{d\alpha}{d \ln \mu} \right|_{\alpha_0 = \text{const}}; \quad \tilde{\gamma}(\alpha) \equiv \left. \frac{d \ln Z}{d \ln \mu} \right|_{\alpha_0 = \text{const}}, \quad (12.17)$$

and are scheme-dependent. However, both definitions of RGF give the same functions, if the conditions

$$Z_3(\alpha, x_0) = 1; \quad Z(\alpha, x_0) = 1 \quad (12.18)$$

are imposed on the renormalization constants, in which  $x_0$  is a fixed value of  $x = \ln \Lambda / \mu$  [18,19,33]:  $\tilde{\beta}(\alpha_0) = \beta(\alpha_0)$ ;  $\tilde{\gamma}(\alpha_0) = \gamma(\alpha_0)$ .

$\tilde{\beta}$  and  $\tilde{\gamma}$  are scheme-dependent and satisfy the NSVZ equation only in a certain (NSVZ) scheme. Now, from Eq. (12.16) and the above arguments it is evident that for the theory regularized by higher derivatives this NSVZ scheme is fixed in all loops by the boundary conditions (12.18).

The general statements discussed above can be verified by explicit calculations in the lowest loops. They are non-trivial starting from the three-loop approximation, because the  $\beta$ -function and the anomalous dimension are scheme-dependent starting from the three- and two-loop order, respectively.

For the higher derivative regulator  $R_k = 1 + k^{2n} / \Lambda^{2n}$

$$\begin{aligned} \frac{1}{\alpha_0} = & \frac{1}{\alpha} - \frac{N_f}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) - \frac{\alpha N_f}{\pi^2} \left( \ln \frac{\Lambda}{\mu} + b_2 \right) - \frac{\alpha^2 N_f}{\pi^3} \left( \frac{N_f}{2} \ln^2 \frac{\Lambda}{\mu} \right. \\ & \left. - \ln \frac{\Lambda}{\mu} \left( N_f \sum_I c_I \ln a_I + N_f + \frac{1}{2} - N_f b_1 \right) + b_3 \right) + O(\alpha^3); \end{aligned} \quad (12.19)$$

$$\begin{aligned} Z = & 1 + \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) + \frac{\alpha^2 (N_f + 1)}{2\pi^2} \ln^2 \frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} \\ & \times \left( N_f \sum_I c_I \ln a_I - N_f b_1 + N_f + \frac{1}{2} - g_1 \right) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3), \end{aligned} \quad (12.20)$$

where  $b_i$  and  $g_i$  are arbitrary finite constants, which fix a subtraction scheme. Differentiating Eqs. (12.19) and (12.20) with respect to  $\ln \Lambda$  we construct RGF defined in terms of the bare coupling constant,

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3} \left( N_f \sum_I c_I \ln a_I + N_f + \frac{1}{2} \right) + O(\alpha_0^3); \quad (12.21)$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2} \left( N_f \sum_I c_I \ln a_I + N_f + \frac{1}{2} \right) + O(\alpha_0^3), \quad (12.22)$$

which appear to be independent of the constants  $b_i$  and  $g_i$  and to satisfy the NSVZ relation. However, RGF defined in terms of the renormalized coupling constant,

$$\begin{aligned} \frac{\tilde{\beta}(\alpha)}{\alpha^2} &= \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3} \left( N_f \sum_I c_I \ln a_I + N_f + \frac{1}{2} + N_f(b_2 - b_1) \right) \\ &+ O(\alpha^3); \end{aligned} \quad (12.23)$$

$$\tilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} \left( N_f + \frac{1}{2} + N_f \sum_I c_I \ln a_I - N_f b_1 + N_f g_1 \right) + O(\alpha^3) \quad (12.24)$$

depend on these constants and, therefore, on a subtraction scheme. This subtraction scheme can be fixing, e.g., by imposing the conditions (12.18). Choosing  $\chi_0 = 0$ , from these equations we obtain  $g_2 = b_1 = b_2 = b_3 = 0$ . Therefore, in this scheme only powers of  $\ln \Lambda/\mu$  are included into the renormalization constants, while all finite constants vanish. Thus, the considered scheme looks very similar to the minimal subtractions. However, now we use the higher derivative regularization, so that it is reasonable to call this scheme HD + MSL, where MSL is the abbreviation for Minimal Subtraction of Logarithms. Substituting the above values of the finite constants into Eqs. (12.23) and (12.24), it is easy to see that in this scheme these RGF satisfy the NSVZ relation.

### 12.2.5 Quantum corrections with the dimensional reduction

It is well known [12–14] that in the  $\overline{\text{DR}}$ -scheme the NSVZ relation is not valid starting from the three-loop approximation. However, to obtain it, one can specially tune a subtraction scheme in each order. It is also possible to try making calculations similarly to the higher derivative case [41,42]. However, the corresponding relation between the functions  $d^{-1}$  and  $G$  (which is at present obtained only in the lowest orders) has a more complicated form, than for the higher derivative case. The boundary conditions analogous to (12.18) can also be written, but the right hand side of one of them is a series in  $\alpha$ . It was demonstrated that such a structure agrees with the results obtained in [13,14].

### 12.2.6 NSVZ-like relation in softly broken $\mathcal{N} = 1$ SQED regularized by higher derivatives

NSVZ-like relations [43–45] also exist in theories with softly broken supersymmetry for renormalization of the gaugino mass. Their origin is the same as in the case of rigid theories. For example, the exact equation describing the renormalization of the photino mass in softly broken  $\mathcal{N} = 1$  SQED, regularized by higher derivatives,

$$\gamma_m(\alpha_0) = \frac{\alpha_0 N_f}{\pi} \left[ 1 - \frac{d}{d\alpha_0} \left( \alpha_0 \gamma(\alpha_0) \right) \right], \quad (12.25)$$

is obtained by exactly the same method as the NSVZ  $\beta$ -function in the case of rigid  $\mathcal{N} = 1$  SQED [46]. For RGF defined in terms of the renormalized coupling constant this relation is also valid in the HD + MSL scheme [47].

### 12.3 Adler D-function in $\mathcal{N} = 1$ SQCD

NSVZ-like expression can be also written for the Adler D-function [48] in (massless)  $\mathcal{N} = 1$  SQCD interacting with the Abelian gauge field [49,50],

$$\begin{aligned} S = & \frac{1}{2g_0^2} \text{tr Re} \int d^4x d^2\theta W^a W_a + \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a \\ & + \sum_{f=1}^{N_f} \frac{1}{4} \int d^4x d^4\theta \left( \phi_f^+ e^{2q_f V + 2V} \phi_f + \tilde{\phi}_f^+ e^{-2q_f V - 2V^\dagger} \tilde{\phi}_f \right). \end{aligned} \quad (12.26)$$

This theory is invariant under the  $SU(N_c) \times U(1)$  gauge transformations. The chiral matter superfields  $\phi_f$  and  $\tilde{\phi}_f$  belong to the fundamental representation of  $SU(N_c)$  and have the charges  $q_f e$  and  $-q_f e$  with respect to the group  $U(1)$ , respectively. In our notation  $V$  is the non-Abelian  $SU(N_c)$  gauge superfield and  $V$  is the Abelian  $U(1)$  gauge superfield. Evidently, the theory contains two coupling constants,  $\alpha_s = g^2/4\pi$  and  $\alpha = e^2/4\pi$ .

The D-function encodes quantum corrections to the electromagnetic coupling constant  $\alpha$  which appear due to the strong interaction. In the supersymmetric case this implies that the electromagnetic gauge superfield  $V$  is treated as an external field. Due to the Ward identity the two-point Green function of this superfield is transversal,

$$\Delta\Gamma^{(2)} = -\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} d^4\theta V \partial^2 \Pi_{1/2} V \left( d^{-1}(\alpha_0, \alpha_{0s}, \Lambda/p) - \alpha_0^{-1} \right). \quad (12.27)$$

The Adler function can be defined in terms of the bare coupling constant by the equation

$$D(\alpha_{0s}) = \frac{3\pi}{2} \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \alpha_{0s}, \Lambda/p) - \alpha_0^{-1} \right) \Big|_{p=0} = \frac{3\pi}{2\alpha_0^2} \frac{d\alpha_0}{d \ln \Lambda}. \quad (12.28)$$

Again, this function depends on regularization, but is independent of a renormalization prescription for a fixed regularization.

According to [49,50], in the case of using the higher covariant derivative regularization<sup>1</sup> the exact expression for the Adler function for the considered theory can be written in the NSVZ-like form

$$D(\alpha_{0s}) = \frac{3}{2} \sum_f q_f^2 \cdot N_c \left(1 - \gamma(\alpha_{0s})\right). \quad (12.29)$$

It looks very similar to the NSVZ  $\beta$ -function in  $\mathcal{N} = 1$  SQED and is derived in all loops by exactly the same method. However, Eq. (12.29) contains the anomalous dimension of the *non-Abelian* theory, and this is a very essential difference from the  $\mathcal{N} = 1$  SQED case. Recently this expression has been confirmed by an explicit three-loop calculation in Ref. [51].

## 12.4 Non-Abelian $\mathcal{N} = 1$ supersymmetric theories

### 12.4.1 Regularization and renormalization

Let us consider the theory described by the action (12.1) in the massless limit. It is convenient to do calculations using the background field method introduced by replacement  $e^{2V} \rightarrow e^{\Omega^+} e^{2V} e^{\Omega}$ . The background gauge superfield  $V$  is then related to  $\Omega$  and  $\Omega^+$  by the equation  $e^{2V} = e^{\Omega^+} e^{\Omega}$ . The higher derivative term in this case can be written in the form

$$S_\Lambda = \frac{1}{2e_0^2} \text{Re tr} \int d^4x d^2\theta e^{\Omega} e^{\Omega} W^a e^{-\Omega} e^{-\Omega} \left[ R\left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2}\right) - 1 \right]_{\text{Adj}} e^{\Omega} e^{\Omega} \\ \times W_a e^{-\Omega} e^{-\Omega} + \frac{1}{4} \int d^4x d^4\theta \phi^+ e^{\Omega^+} e^{\Omega^+} \left[ F\left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2}\right) - 1 \right] e^{\Omega} e^{\Omega} \phi, \quad (12.30)$$

where the functions  $R(x)$  and  $F(x)$  rapidly increase at infinity and satisfy the condition  $R(0) = F(0) = 1$ . It is convenient to fix a gauge without breaking the background gauge invariance. For this purpose it is possible to use the gauge fixing term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2} \text{tr} \int d^4x d^4\theta \nabla^2 V K\left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2}\right)_{\text{Adj}} \bar{\nabla}^2 V, \quad (12.31)$$

where  $K(0) = 1$  and  $K(x)$  also rapidly grows at infinity. The corresponding actions for ghosts and the Pauli–Villars determinants can be found in Ref. [52], where they are discussed in all details. The renormalization constants are introduced by the equations

<sup>1</sup> The higher derivative term for the considered theory should contain covariant derivatives  $\nabla_a = e^{-\Omega^+} D_a e^{\Omega^+}$ ;  $\bar{\nabla}_{\dot{a}} = e^{\Omega} \bar{D}_{\dot{a}} e^{-\Omega}$ , where  $e^{2V} = e^{\Omega^+} e^{\Omega}$ .

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \quad V = Z_V Z_\alpha^{-1/2} V_R; \quad \bar{c}c = Z_c Z_\alpha^{-1} \bar{c}_R c_R; \quad \phi_i = (\sqrt{Z_\phi})_i^j (\phi_R)_j, \quad (12.32)$$

where  $\bar{c}$  and  $c$  are the chiral Faddeev–Popov ghost superfields.

#### 12.4.2 Finiteness of the triple gauge-ghost vertices

In  $\mathcal{N} = 1$  gauge supersymmetric theories the three-point gauge-ghost vertices ( $\bar{c} V c$ ,  $\bar{c}^+ V c$ ,  $\bar{c} V c^+$ , and  $\bar{c}^+ V c^+$ ) with two ghost legs and a single leg of the quantum gauge superfield are finite [7], so that

$$\frac{d}{d \ln \Lambda} (Z_\alpha^{-1/2} Z_c Z_V) = 0. \quad (12.33)$$

(At the one-loop level it was found in [52].) This theorem is derived by the help of the Slavnov–Taylor identities, which can be obtained using the standard methods [53,54]. To write the identity for the considered three-point functions, we introduce the chiral source  $\mathcal{J}$  and the source term

$$-\frac{e_0}{2} \int d^4 x d^2 \theta f^{ABC} \mathcal{J}^A c^B c^C + \text{c.c.} \quad (12.34)$$

Then using the superspace Feynman rules it is possible to prove that the effective vertex

$$\frac{\delta^3 \Gamma}{\delta c_z^C \delta c_w^D \delta \mathcal{J}_y^B} = \frac{e_0}{4} f^{BCD} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} H(p, q) \bar{D}_z^2 \delta_{zy}^8(q + p) \bar{D}_w^2 \delta_{yw}^8(q) \quad (12.35)$$

is finite in all orders. Really, we can present the corresponding superdiagrams as integrals over the total superspace, which include integration over

$$\int d^4 \theta = -\frac{1}{2} \int d^2 \theta \bar{D}^2 + \text{total derivatives in the coordinate space.} \quad (12.36)$$

Consequently, due to chirality of all external legs the non-vanishing result can be obtained only if two right spinor derivatives also act to the external legs. Thus, commuting supersymmetric covariant derivatives, we see that the result should be proportional to, at least, second degree of the external momenta and is finite in the ultraviolet region.

From dimensional and chirality considerations one can write the following expression for the one of triple gauge-ghost Green functions,

$$\begin{aligned} \frac{\delta^3 \Gamma}{\delta \bar{c}_x^{*A} \delta V_y^B \delta c_z^C} &= -\frac{ie_0}{16} f^{ABC} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left( f(p, q) \partial^2 \Pi_{1/2} \right. \\ &\quad \left. - F_\mu(p, q) (\gamma^\mu)_a{}^b \bar{D}^a D_b + F(p, q) \right)_y \left( D_x^2 \delta_{xy}^8(q + p) \bar{D}_z^2 \delta_{yz}^8(q) \right), \end{aligned} \quad (12.37)$$

where  $\delta_{xy}^8(p) \equiv \delta^4(\theta_x - \theta_y) e^{ip_\alpha(x^\alpha - y^\alpha)}$ . Then the Slavnov–Taylor identity can be written in the form

$$G_c(q)F(q, p) + G_c(p)F(p, q) = 2G_c(q + p)H(-q - p, q), \quad (12.38)$$

where  $G_c(q)$  is the two-point Green function for the Faddeev–Popov ghosts. Multiplying this equation to  $Z_c$ , differentiating the result with respect to  $\ln \Lambda$  and setting  $p = -q$  we obtain finiteness of the function  $F(-q, q)$ , which follows from the finiteness of  $(G_c)_R$  and  $H$  in the limit  $\Lambda \rightarrow \infty$ . This means that the corresponding renormalization constant is finite, see Eq. (12.33). Consequently, all three-point ghost-gauge vertices are also finite.

### 12.4.3 $V\bar{c}c$ -vertices in the one-loop approximation

In the one-loop approximation (after the Wick rotation)

$$\begin{aligned} F(p, q) = 1 + \frac{e_0^2 C_2}{4} \int \frac{d^4 k}{(2\pi)^4} \left\{ -\frac{(q + p)^2}{R_k k^2 (k + p)^2 (k - q)^2} - \frac{\xi_0 p^2}{K_k k^2 (k + q)^2} \right. \\ \times \frac{1}{(k + q + p)^2} + \frac{\xi_0 q^2}{K_k k^2 (k + p)^2 (k + q + p)^2} + \left( \frac{\xi_0}{K_k} - \frac{1}{R_k} \right) \left( -\frac{1}{k^2 (k + q)^2} \right. \\ \left. \left. - \frac{1}{k^2 (k + p)^2} + \frac{2}{k^2 (k + q + p)^2} - \frac{2(q + p)^2}{k^4 (k + q + p)^2} \right) \right\} + O(\alpha_0^2, \alpha_0 \lambda_0^2). \quad (12.39) \end{aligned}$$

It is easy to see that this expression is finite in the UV region. The other functions in Eq. (12.37) are also finite, see [7]. The finiteness of the function  $H$ , defined in Eq. (12.35), at the one-loop level has also been demonstrated,

$$\begin{aligned} H(p, q) = 1 - \frac{e_0^2 C_2}{4} \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{p^2}{R_k k^2 (k + q)^2 (k + q + p)^2} + \frac{(q + p)^2}{k^4 (k + q + p)^2} \right. \\ \left. \times \left( \frac{\xi_0}{K_k} - \frac{1}{R_k} \right) + \frac{q^2}{k^4 (k + q)^2} \left( \frac{\xi_0}{K_k} - \frac{1}{R_k} \right) \right\} + O(e_0^4, e_0^2 \lambda_0^2). \quad (12.40) \end{aligned}$$

### 12.4.4 New form of the NSVZ relation

Let write the NSVZ relation (12.2) for RGF defined in terms of the bare couplings (see the definitions in Ref. [7]) in the form

$$\frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R) i^j (\gamma_\phi)_j i^i (\alpha_0, \lambda_0)/r}{2\pi} + \frac{C_2}{2\pi} \cdot \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0} \quad (12.41)$$

and take into account that the  $\beta$ -function can be related to the renormalization constant  $Z_{\alpha}$ ,

$$\beta(\alpha_0, \lambda_0) = \left. \frac{d\alpha_0(\alpha, \lambda, \Lambda/\mu)}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}} = -\alpha_0 \left. \frac{d \ln Z_\alpha}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}}. \quad (12.42)$$

Then the right hand side of Eq. (12.41) can be expressed in terms of  $\gamma_c$  and  $\gamma_V$  by the help of Eq. (12.33),

$$\beta(\alpha_0, \lambda_0) = -2\alpha_0 \left. \frac{d \ln(Z_c Z_V)}{d \ln \Lambda} \right|_{\alpha, \lambda = \text{const}} = 2\alpha_0 \left( \gamma_c(\alpha_0, \lambda_0) + \gamma_V(\alpha_0, \lambda_0) \right). \quad (12.43)$$

Substituting this identity into Eq. (12.41) we rewrite the exact NSVZ  $\beta$ -function in a different form,

$$\begin{aligned} \frac{\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = & -\frac{1}{2\pi} \left( 3C_2 - T(R) - 2C_2\gamma_c(\alpha_0, \lambda_0) - 2C_2\gamma_V(\alpha_0, \lambda_0) \right. \\ & \left. + C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha_0, \lambda_0)/r \right). \end{aligned} \quad (12.44)$$

Eq. (12.44) admits a simple graphical interpretation similar to the Abelian case. Consider a supergraph without external lines. By attaching two external legs of the superfield  $V$  we obtain a set of diagrams contributing to the  $\beta$ -function. From the other side, cutting internal lines gives superdiagrams contributing to the anomalous dimensions of the Faddeev–Popov ghosts, of the quantum gauge superfield, and of the matter superfields. Eq. (12.44) relates these two sets of superdiagrams.

#### 12.4.5 The NSVZ scheme for non-Abelian gauge theories

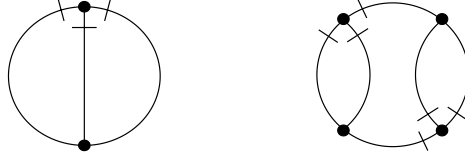
The RGF standardly defined in terms of the renormalized couplings (we again denote them by tildes) are scheme-dependent and satisfy the NSVZ relation only in a certain (NSVZ) subtraction scheme. Let us suggest that, similar to the Abelian case, RGF defined in terms of the bare couplings satisfy the NSVZ relation (12.44) in the case of using the higher covariant derivative regularization. Really, the qualitative way of its derivation looks exactly as in  $\mathcal{N} = 1$  SQED and the factorization into total derivatives [55,56] and double total derivatives [57] also takes place at least in the lowest orders. Then, repeating the argumentation of Ref. [33], one can prove that in the non-Abelian case both definitions of RGF give the same result (for coinciding arguments) if the renormalization constants satisfy the conditions

$$Z_\alpha(\alpha, \lambda, x_0) = 1; \quad (Z_\phi)_i{}^j(\alpha, \lambda, x_0) = \delta_i{}^j; \quad Z_c(\alpha, \lambda, x_0) = 1. \quad (12.45)$$

Thus, under the assumption that the NSVZ relation is valid for RGF defined in terms of the bare couplings with the higher derivative regularization, the NSVZ scheme is given by the boundary conditions (12.45). Again, it is easy to see that for  $x_0 = 0$  in this scheme only powers of  $\ln \Lambda/\mu$  are included into the renormalization constants, so that the NSVZ scheme coincides with HD + MSL. Certainly, it is also assumed that  $Z_V = Z_\alpha^{1/2} Z_c^{-1}$  due to the non-renormalization of the  $V\bar{c}c$ -vertices.

### 12.4.6 Checking the new form of the NSVZ relation by explicit calculations

To check the above results, we consider terms quartic in the Yukawa couplings [58] corresponding to the graphs presented in Fig. 12.1.



**Fig. 12.1.** The terms in the NSVZ relation which are investigated here are obtained from these two graphs.

Attaching two external lines of the background gauge superfield gives a large number of two- and three-loop diagrams contributing to the  $\beta$ -function. The corresponding diagrams for the anomalous dimension are obtained by cutting internal lines in the considered graphs. The result for the considered part of the  $\beta$ -function defined in terms of the bare couplings can be presented as an integral of double total derivatives,

$$\begin{aligned}
 \frac{\Delta\beta(\alpha_0, \lambda_0)}{\alpha_0^2} &= -\frac{2\pi}{r} C(R)_i^j \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \lambda_0^{imn} \lambda_{0jmn}^* \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q^\mu} \\
 &\times \left( \frac{1}{k^2 F_k q^2 F_q (q+k)^2 F_{q+k}} \right) + \frac{4\pi}{r} C(R)_i^j \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \\
 &\times \left( \lambda_0^{iab} \lambda_{0kab}^* \lambda_0^{kcd} \lambda_{0jcd}^* \left( \frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k^\mu} - \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q^\mu} \right) + 2\lambda_0^{iab} \lambda_{0jac}^* \lambda_0^{cde} \lambda_{0bde}^* \right. \\
 &\times \left. \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q^\mu} \right) \frac{1}{k^2 F_k^2 q^2 F_q (q+k)^2 F_{q+k} l^2 F_l (l+k)^2 F_{l+k}}. \quad (12.46)
 \end{aligned}$$

Taking one of loop integrals it is possible to relate this expression to the corresponding contribution to the anomalous dimension of the matter superfield (defined in terms of the bare couplings),

$$\frac{\Delta\beta(\alpha_0, \lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi r} C(R)_i^j \Delta\gamma_\phi(\lambda_0)_j^i. \quad (12.47)$$

This equation completely agrees with Eq. (12.44), so that the NSVZ relation is satisfied for terms of the considered structure.

For  $F(k^2/\Lambda^2) = 1 + k^2/\Lambda^2$  all loop integrals can be calculated,

$$\Delta\gamma_\phi(\alpha_0, \lambda_0)_j^i = \frac{1}{4\pi^2} \lambda_0^{iab} \lambda_{0jab}^* - \frac{1}{16\pi^4} \lambda_0^{iab} \lambda_{0jac}^* \lambda_0^{cde} \lambda_{0bde}^*. \quad (12.48)$$



Scheme-dependent RGF defined in terms of the renormalized couplings have been calculated in Ref. [58]. The contribution to the  $\beta$ -function depends on some finite constants  $g_1$  and  $b_2$ , which appear due to arbitrariness of choosing a subtraction scheme,

$$\begin{aligned} \tilde{\gamma}_\Phi(\alpha, \lambda)_j^i &= \frac{1}{4\pi^2} \lambda^{iab} \lambda_{jab}^* - \frac{1}{16\pi^4} \lambda^{iab} \lambda_{jac}^* \lambda^{cde} \lambda_{bde}^* + O(\alpha) + O(\lambda^6); \quad (12.49) \\ \frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} &= -\frac{1}{2\pi} (3C_2 - T(R)) + \frac{1}{2\pi r} C(R)_i^j \left[ -\frac{1}{4\pi^2} \lambda^{iab} \lambda_{jab}^* + \frac{1}{16\pi^4} \right. \\ &\quad \times \lambda^{iab} \lambda_{kab}^* \lambda^{kcd} \lambda_{jcd}^* (b_2 - g_1) + \frac{1}{16\pi^4} \lambda^{iab} \lambda_{jac}^* \lambda^{cde} \lambda_{bde}^* (1 + 2b_2 - 2g_1) \Big] \\ &\quad + O(\alpha) + O(\lambda^6). \end{aligned} \quad (12.50)$$

We see that for an arbitrary values of  $g_1$  and  $b_2$  the NSVZ relation is not valid. However, the values of  $g_1$  and  $b_2$  can be fixed by imposing the conditions (12.45). In this case  $g_1 = b_2 = -x_0$ , so that  $b_2 - g_1 = 0$ . Therefore, in this scheme

$$\frac{\tilde{\beta}(\alpha, \lambda)}{\alpha^2} = -\frac{1}{2\pi} (3C_2 - T(R)) - \frac{1}{2\pi r} C(R)_i^j \tilde{\gamma}_\Phi(\alpha, \lambda)_i^j + O(\alpha) + O(\lambda^6). \quad (12.51)$$

This confirms the guess that Eq. (12.45) gives the NSVZ scheme in the non-Abelian case.

Note that recently [59] the identity (12.44) has been completely checked in the two-loop approximation in the case of using the non-invariant version of the higher covariant derivative regularization supplemented by a special subtraction procedure which restores the Slavnov–Taylor identities [60].

## 12.5 Conclusion

The  $\beta$ -function defined in terms of the bare coupling constant for  $\mathcal{N} = 1$  supersymmetric gauge theories regularized by higher derivatives is given by integrals of double total derivatives. In some cases it has been proved in all loops, but for general non-Abelian SYM theories at present there are only strong evidences in favour of this. Such a structure of quantum corrections naturally leads to the NSVZ relation for RGF defined in terms of the bare coupling constant, which is obtained after taking the integral of the total derivative and is valid independently of the subtraction scheme. Note that in the non-Abelian case an important ingredient of the derivation is the finiteness of the three-point ghost-gauge vertices, which allows rewriting the NSVZ equation in a different form.

The RGF defined in terms of the renormalized couplings satisfy the NSVZ relation only in a certain (NSVZ) scheme, which is obtained with the higher derivative regularization by minimal subtraction of logarithms. This means that only powers of  $\ln \Lambda/\mu$  are included into various renormalization constants. This

prescription can be also reformulated by imposing simple boundary conditions on the renormalization constants.

All general statements considered here are confirmed by explicit perturbative calculations. Note that some of them are made in the three-loop approximation and are highly non-trivial.

## Acknowledgements

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## References

1. R.N.Mohapatra, *"Unification And Supersymmetry. The Frontiers Of Quark - Lepton Physics: The Frontiers Of Quark-lepton Physics,"* New York, USA: Springer (2003) 421 p.
2. M.T.Grisaru, W.Siegel and M.Rocek, Nucl. Phys. B **159** (1979) 429.
3. V.A.Novikov, M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Nucl. Phys. B **229**, (1983) 381.
4. D.R.T.Jones, Phys. Lett. B **123**, (1983) 45.
5. V.A.Novikov, M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Phys. Lett. B **166**, (1986) 329; Sov. J. Nucl. Phys. **43** (1986) 294.
6. M.A.Shifman and A.I.Vainshtein, Nucl. Phys. B **277**, (1986) 456; Sov. Phys. JETP **64**, (1986) 428.
7. K.V.Stepanyantz, Nucl. Phys. B **909**, (2016) 316.
8. M.A.Shifman and A.I.Vainshtein, *"Instantons versus supersymmetry: Fifteen years later,"* In \*Shifman, M.A.: ITEP lectures on particle physics and field theory, vol. 2\* 485-647.
9. N.Arkani-Hamed and H.Murayama, JHEP **0006**, (2000) 030.
10. E.Kraus, C.Rupp and K.Sibold, Nucl. Phys. B **661**, (2003) 83.
11. W.Siegel, Phys. Lett. B **84**, (1979) 193.
12. L.V.Avdeev and O.V.Tarasov, Phys. Lett. **112B**, (1982) 356.
13. I.Jack, D.R.T.Jones and C.G.North, Phys. Lett. B **386**, (1996) 138.
14. I.Jack, D.R.T.Jones and C.G.North, Nucl. Phys. B **486**, (1997) 479.
15. I.Jack, D.R.T.Jones and A.Pickering, Phys. Lett. B **435**, (1998) 61.
16. R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila and M.Steinhauser, JHEP **0612**, (2006) 024.
17. D.Kutasov and A.Schwimmer, Nucl. Phys. B **702**, (2004) 369.
18. A.L.Kataev and K.V.Stepanyantz, Theor. Math. Phys. **181**, (2014) 1531.
19. A.L.Kataev and K.V.Stepanyantz, Phys. Lett. B **730**, (2014) 184.
20. W.Siegel, Phys. Lett. B **94**, (1980) 37.
21. L.V.Avdeev, Phys. Lett. B **117**, (1982) 317.
22. L.V.Avdeev and A.A.Vladimirov, Nucl. Phys. B **219**, (1983) 262.
23. A.A.Slavnov, Nucl. Phys. B **31**, (1971) 301.
24. A.A.Slavnov, Theor.Math.Phys. **13**, (1972) 1064.
25. V.K.Krivoshchekov, Theor. Math. Phys. **36**, (1978) 745.
26. P.C.West, Nucl. Phys. B **268**, (1986) 113.

27. V.K.Krivoshchekov, Phys. Lett. B **149**, (1984) 128.
28. I.L.Buchbinder and K.V.Stepanyantz, Nucl. Phys. B **883**, (2014) 20.
29. I.L.Buchbinder, N.G.Pletnev and K.V.Stepanyantz, Phys. Lett. B **751**, (2015) 434.
30. A.A.Slavnov, Theor. Math. Phys. **33**, (1977) 977.
31. A.I.Vainshtein, V.I.Zakharov and M.A.Shifman, JETP Lett. **42**, (1985) 224.
32. M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Phys. Lett. B **166** (1986) 334.
33. A.L.Kataev and K.V.Stepanyantz, Nucl. Phys. B **875**, (2013) 459.
34. A.A.Soloshenko and K.V.Stepanyantz, Theor. Math. Phys. **140**, (2004) 1264.
35. A.V.Smilga and A.Vainshtein, Nucl. Phys. B **704**, (2005) 445.
36. K.V.Stepanyantz, J. Phys. Conf. Ser. **343**, (2012) 012115.
37. K.V.Stepanyantz, Nucl. Phys. B **852**, (2011) 71.
38. K.V.Stepanyantz, JHEP **1408**, (2014) 096.
39. A.E.Kazantsev and K.V.Stepanyantz, J. Exp. Theor. Phys. **120**, (2015) 618.
40. A.B.Pimenov and K.V.Stepanyantz, Theor. Math. Phys. **147**, (2006) 687.
41. S.S.Aleshin, A.L.Kataev and K.V.Stepanyantz, JETP Lett. **103**, (2016) no.2, 77.
42. S.S.Aleshin, I.O.Goriachuk, A.L.Kataev and K.V.Stepanyantz, Phys. Lett. B **764**, (2017) 222.
43. J.Hisano and M.A.Shifman, Phys. Rev. D **56**, (1997) 5475.
44. I.Jack and D.R.T.Jones, Phys. Lett. B **415**, (1997) 383.
45. L.V.Avdeev, D.I.Kazakov and I.N.Kondrashuk, Nucl. Phys. B **510**, (1998) 289.
46. I.V.Nartsev and K.V.Stepanyantz, JHEP **1704**, (2017) 047.
47. I.V.Nartsev and K.V.Stepanyantz, JETP Lett. **105**, (2017) no.2, 69.
48. S.L.Adler, Phys. Rev. D **10**, (1974) 3714.
49. M.Shifman and K.Stepanyantz, Phys. Rev. Lett. **114**, (2015) 051601.
50. M.Shifman and K.V.Stepanyantz, Phys. Rev. D **91**, (2015) 105008.
51. A.L.Kataev, A.E.Kazantsev and K.V.Stepanyantz, arXiv:1710.03941 [hep-th].
52. S.S.Aleshin, A.E.Kazantsev, M.B.Skoptsov and K.V.Stepanyantz, JHEP **1605**, (2016) 014.
53. J.C.Taylor, Nucl. Phys. B **33**, (1971) 436.
54. A.A.Slavnov, Theor. Math. Phys. **10**, (1972) 99.
55. A.B.Pimenov, E.S.Shevtsova and K.V.Stepanyantz, Phys. Lett. B **686**, (2010) 293.
56. K.V.Stepanyantz, Proceedings of the Steklov Institute of Mathematics, **272**, (2011) 256.
57. K.V.Stepanyantz, arXiv:1108.1491 [hep-th].
58. V.Y.Shakhmanov and K.V.Stepanyantz, Nucl. Phys. B **920**, (2017) 345.
59. V.Y.Shakhmanov and K.V.Stepanyantz, arXiv:1711.03899 [hep-th].
60. A.A.Slavnov and K.V.Stepanyantz, Theor. Math. Phys. **139**, (2004) 599.



## Discussion Section

The discussion section is reserved for those open problems presented and discussed during the workshop, that they might start new collaboration among participants or at least stimulate participants to start to think about possible solutions of particular open problems in a different way, or to invite new collaborators on the problems. Since the time between the workshop and the deadline for contributions for the proceedings is very short and includes for most of participants also their holidays, it is not so easy to prepare there presentations or besides their presentations at the workshop also the common contributions to the discussion section.

However, the discussions, even if not presented as a contribution to this section, influenced participants' contributions, published in the main section. Contributions in this section might not be written yet in a shape that look like a normal paper, although they even might be very innovative and correspondingly valuable.

As it is happening every year also this year quite a lot of started discussions have not succeeded to appear in this proceedings. Organizers hope that they will be developed enough to appear among the next year talks, or will just stimulate the works of the participants.

There are two contributions in this section this year. One contribution is treating the fermionization of bosons of the Kalb-Ramond type boson fields to better "understand why Nature" has decided to use fermions besides bosons. It is extending the old theorem of Aratyn and Nielsen. Another is proving that the symmetry of the  $4 \times 4$  family matrix, predicted by the spin-charge-family theory of one of the authors on the tree level, is kept in all orders of loop corrections, lowering the number of free parameters of the mass matrices and enabling correspondingly to predict the masses and the mixing matrix elements of the fourth family to the observed three.

All discussion contributions are arranged alphabetically with respect to the authors' names.

Ta razdelek je namenjen odprtim vprašanjem, o katerih smo med delavnico izčrpno razpravljali. Problemi, o katerih smo razpravljali, bodo morda privedli do novih sodelovanj med udeleženci, ali pa so pripravili udeležence, da razmislijo o možnih rešitvah odprtih vprašanj na drugačne načine, ali pa bodo k sodelovanju pritegnili katerega od udeležencev. Ker je čas med delavnico in rokom za oddajo prispevkov zelo kratek, vmes pa so poletne počitnice, je zelo težko pripraviti prispevek in še težje poleg prispevka, v katerem vsak udeleženec predstavi lastno delo, pripraviti še prispevek k temu razdelku.

Tako se velik del diskusij ne bo pojavil v letošnjem zborniku. So pa gotovo vplivale na prispevek marsikaterega udeleženca. Organizatorji upamo, da bodo te diskusije do prihodnje delavnice dozorele do oblike, da jih bo mogoče na njej predstaviti.

Prispevki v tem razdelku niso nujno napisani v običajni obliki članka, kar pa ne pomeni, da niso zelo inovativni in posledično dragoceni.

Letos sta v tem razdelku dva prispevka. Eden išče pot, kako bozone tipa Kalb-Ramond "predstaviti" kot fermione z namenom, da bi avtorja prispevka bolje razumela, zakaj se je "narava odločila" uporabiti poleg bozonov tudi fermione. Prinaša posplošitev teorema Aratyna in Nielsena. Drugi prispevek dokazuje, da se simetrija masne matrike  $4 \times 4$ , ki jo napove teorija spina-nabojev-družin ene od avtorjev prispevka, ohranja v popravkih v vseh redih ter tako omeji število prostih parametrov masnih matrik vsakega člana družine, kar omogoči napovedi mase in matričnih elementov četrte družine.

Prispevki v tej sekciji so, tako kot prispevki v glavnem delu, urejeni po abecednem redu priimkov avtorjev.



## 13 The Symmetry of $4 \times 4$ Mass Matrices Predicted by the *Spin-charge-family* Theory — $SU(2) \times SU(2) \times U(1)$ — Remains in All Loop Corrections

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**Abstract.** The *spin-charge-family* theory [1–11,14–22] predicts the existence of the fourth family to the observed three. The  $4 \times 4$  mass matrices manifest the symmetry  $SU(2) \times SU(2) \times U(1)$ , determined on the tree level by the nonzero vacuum expectation values of several scalar fields – the three singlets with the family members quantum numbers (belonging to  $U(1)$ ) and the two triplets with the family quantum numbers (belonging to  $SU(2) \times SU(2)$ ) with the weak and the hyper charge of the *standard model* higgs field ( $\pm \frac{1}{2}, \mp \frac{1}{2}$ , respectively). It is demonstrated, using the massless spinor basis, on several cases that (why) the symmetry of  $4 \times 4$  mass matrices remains the same in all loop corrections.

**Povzetek.** Teorija *spinov-nabojev-družin* [1–11,14–22] napove obstoj četrte družine k opazženim trem. Masne matrike  $4 \times 4$  kažejo simetrijo  $SU(2) \times SU(2) \times U(1)$ , ki je na drevesnem nivoju določena z neničelnimi vakuumskimi pričakovanimi vrednostmi več skalarnih polj — treh singletov s kvantnimi števili družin (v  $U(1)$ ) in dveh tripletov s kvantnimi števili družin (v  $SU(2) \times SU(2)$ ), ki imajo šibki in hipernaboj higgsovega polja *standardnega modela*, (enak  $\pm \frac{1}{2}$  in  $\mp \frac{1}{2}$ ). Avtorja pokažeta, da (zakaj) se v bazi brezmasnih spinorjev, v več primerih, simetrija masnih matrik  $4 \times 4$  ohranja v vseh redih.

**Keywords:** Unifying theories, Beyond the standard model, Origin of families, Origin of mass matrices of leptons and quarks, Properties of scalar fields, The fourth family, Origin and properties of gauge bosons, Flavour symmetry, Kaluza-Klein-like theories

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### 13.1 Introduction

The *spin-charge-family* theory [1–11,14–22] predicts before the electroweak break four - rather than the observed three - coupled massless families of quarks and leptons.

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The  $4 \times 4$  mass matrices of all the family members demonstrate in this theory the same symmetry [1,5,4,19,20], determined by the scalar fields: the two triplets — the gauge fields of the two family groups  $\widetilde{\text{SU}}(2) \times \widetilde{\text{SU}}(2)$  operating among families — and the three singlets — the gauge fields of the three charges (Q, Q' and Y') distinguishing among family members. All these scalar fields carry the weak and the hyper charge as does the scalar of the *standard model*: ( $\pm \frac{1}{2}$  and  $\mp \frac{1}{2}$ , respectively) [1,4,22].

Although there is no direct observations of the fourth family quarks masses below 1 TeV, while the fourth family quarks with masses above 1 TeV would contribute according to the *standard model* (the *standard model* Yukawa couplings of the quarks with the scalar higgs is proportional to  $\frac{m_4^\alpha}{v}$ , where  $m_4^\alpha$  is the fourth family member ( $\alpha = u, d$ ) mass and  $v$  the vacuum expectation value of the scalar) to either the quark-gluon fusion production of the scalar field (the higgs) or to the scalar field decay too much in comparison with the observations, the high energy physicists do not expect the existence of the fourth family members at all [23,24].

One of the authors (N.S.M.B) discusses in Refs. ([1], Sect. 4.2.) that the *standard model* estimation with one higgs scalar might not be the right way to evaluate whether the fourth family, coupled to the observed three, does exist or not. The  $u_i$ -quarks and  $d_i$ -quarks of an  $i^{\text{th}}$  family, namely, if they couple with the opposite sign (with respect to the " $\pm$ " degree of freedom) to the scalar fields carrying the family ( $\tilde{A}, i$ ) quantum numbers and have the same masses, do not contribute to either the quark-gluon fusion production of the scalar fields with the family quantum numbers or to the decay of these scalars into two photons:

The strong influence of the scalar fields carrying the family members quantum numbers to the masses of the lower (observed) three families manifests in the huge differences in the masses of the family members, let say  $u_i$  and  $d_i$ ,  $i = (1, 2, 3)$ , and families ( $i$ ). For the fourth family quarks, which are more and more decoupled from the observed three families the higher are their masses [20,19], the influence of the scalar fields carrying the family members quantum numbers on their masses is expected to be much weaker. Correspondingly the  $u_4$  and  $d_4$  masses become closer to each other the higher are their masses and the weaker are their couplings (the mixing matrix elements) to the lower three families. For  $u_4$ -quarks and  $d_4$ -quarks with the similar masses the observations might consequently not be in contradiction with the *spin-charge-family* theory prediction that there exists the fourth family coupled to the observed three ([26], which is in preparation).

We demonstrate in the main Sect. 13.2 why the symmetry, which the mass matrices demonstrate on the tree level, keeps the same in all loop corrections.

We present shortly the *spin-charge-family* theory and its achievements so far in Sect. 13.4. All the mathematical support appears in appendices.

Let be here stressed what supports the *spin-charge-family* theory to be the right next step beyond the *standard model*. This theory can not only explain - while starting from the very simple action in  $d \geq (13 + 1)$ , Eqs. (13.20) in App. 13.4, with the massless fermions (with the spin of the two kinds  $\gamma^a$  and  $\tilde{\gamma}^a$ , one kind taking care of the spin and of the charges of the family members (Eq. (13.2)), the second kind taking care of the families (Eqs. (13.19, 13.35))) coupled only to the gravity (through the vielbeins and the two kinds of the corresponding spin connections



fields  $\omega_{ab\alpha} f^\alpha_c$  and  $\tilde{\omega}_{ab\alpha} f^\alpha_c$ , the gauge fields of  $S^{ab}$  and  $\tilde{S}^{ab}$  (Eqs. (13.20)) - all the assumptions of the *standard model*, but also answers several open questions beyond the *standard model*. It offers the explanation for [4–6,1,7–11,14–22]:

- a. the appearance of all the charges of the left and right handed family members and for their families and their properties,
- b. the appearance of all the corresponding vector and scalar gauge fields and their properties (explaining the appearance of higgs and Yukawa couplings),
- c. the appearance and properties of the dark matter,
- d. the appearance of the matter/antimatter asymmetry in the universe.

The theory predicts for the low energy regime:

- i. The existence of the fourth family to the observed three.
- ii. The existence of twice two triplets and three singlets of scalars, all with the properties of the higgs with respect to the weak and hyper charges, what explains the origin of the Yukawa couplings.
- iii. There are several other predictions, not directly connected with the topic of this paper.

The fact that the fourth family quarks have not yet been observed - directly or indirectly - pushes the fourth family quarks masses to values higher than 1 TeV.

Since the experimental accuracy of the  $(3 \times 3)$  submatrix of the  $4 \times 4$  mixing matrices is not yet high enough [30], it is not possible to calculate the mixing matrix elements among the fourth family and the observed three. Correspondingly it is not possible to estimate masses of the fourth family members by fitting the experimental data to the parameters of mass matrices, determined by the symmetry predicted by the *spin-charge-family* [20,19].

But assuming the masses of the fourth family members the matrix elements can be estimated from the existing  $3 \times 3$  submatrix of the  $4 \times 4$  matrix.

The more effort and work is put into the *spin-charge-family* theory, the more explanations of the observed phenomena and the more predictions for the future observations follow out of it. Offering the explanation for so many observed phenomena - keeping in mind that all the explanations for the observed phenomena originate in a simple starting action - qualifies the *spin-charge-family* theory as the candidate for the next step beyond the *standard model*.

The reader is kindly asked to learn more about the *spin-charge-family* theory in Refs. [2–4,1,5,6] and the references there in. We shall point out sections in these references, which might be of particular help, when needed.

## 13.2 The symmetry of family members mass matrices keeps unchanged in all orders of loop corrections

It is demonstrated in this main section that the symmetry  $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$  of the mass term, which manifests in the starting action 13.20 of the *spin-charge-family* theory [4,1,5,6], remains unchanged in all orders of loop corrections. The massless basis will be used for this purpose.

Let us rewrite formally the fermion part of the starting action, Eq. (13.20), in the way that it manifests, Eq. (13.1), the kinetic and the interaction term in

$d = (3+1)$  (the first line,  $m = (0, 1, 2, 3)$ ), the mass term (the second line,  $s = (7, 8)$ ) and the rest (the third line,  $t = (5, 6, 9, 10, \dots, 14)$ ).

$$\begin{aligned} \mathcal{L}_f = & \bar{\Psi} \gamma^m (p_m - \sum_{A,i} g^{Ai} \tau^{Ai} A_m^{Ai}) \Psi + \\ & \{ \sum_{s=7,8} \bar{\Psi} \gamma^s p_{0s} \Psi \} + \\ & \{ \sum_{t=5,6,9,\dots,14} \bar{\Psi} \gamma^t p_{0t} \Psi \}, \end{aligned} \quad (13.1)$$

where  $p_{0s} = p_s - \frac{1}{2} S^{s's''} \omega_{s's''s} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{abs}$ ,  $p_{0t} = p_t - \frac{1}{2} S^{t't''} \omega_{t't''t} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{abt}$ <sup>1</sup>, with  $m \in (0, 1, 2, 3)$ ,  $s \in (7, 8)$ ,  $(s', s'') \in (5, 6, 7, 8)$ ,  $(a, b)$  (appearing in  $S^{ab}$ ) run within either  $(0, 1, 2, 3)$  or  $(5, 6, 7, 8)$ ,  $t$  runs  $\in (5, \dots, 14)$ ,  $(t', t'')$  run either  $\in (5, 6, 7, 8)$  or  $\in (9, 10, \dots, 14)$ . The spinor function  $\psi$  represents all family members, presented on Table 13.3 of all the  $2^{\frac{Z+1}{2}-1} = 8$  families, presented on Table 13.4.

The first line of Eq. (13.1) determines (in  $d = (3+1)$ ) the kinematics and dynamics of spinor (fermion) fields, coupled to the vector gauge fields. The generators  $\tau^{Ai}$  of the charge groups are expressible in terms of  $S^{ab}$  through the complex coefficients  $c^{Ai}_{ab}$  (the coefficients  $c^{Ai}_{ab}$  of  $\tau^{Ai}$  can be found in Eqs. (13.23, 13.24)<sup>2</sup>,

$$\tau^{Ai} = \sum_{a,b} c^{Ai}_{ab} S^{ab}, \quad (13.2)$$

fulfilling the commutation relations

$$\{\tau^{Ai}, \tau^{Bj}\}_{-} = i \delta^{AB} f^{Aijk} \tau^{Ak}. \quad (13.3)$$

They represent the colour ( $\tau^{3i}$ ), the weak ( $\tau^{1i}$ ) and the hyper (Y) charges, as well as the  $SU(2)_{II}$  ( $\tau^{2i}$ ) and  $U(1)_{II}$  ( $\tau^4$ ) charges, the gauge fields of these last two groups gain masses interacting with the condensate, Table 13.5. The condensate leaves massless, besides the colour and gravity gauge fields, the weak and the hyper charge vector gauge fields. The corresponding vector gauge fields  $A_m^{Ai}$  are expressible with the spin connection fields  $\omega_{stm}$  Eq. (13.29)

$$A_m^{Ai} = \sum_{s,t} c^{Ai}_{st} \omega^{st}_m. \quad (13.4)$$

The scalar gauge fields of the charges, Eq. (13.30), are expressible with the spin connections and vielbeins [2].

<sup>1</sup> If there are no fermions present, then either  $\omega_{abc}$  or  $\tilde{\omega}_{abc}$  are expressible by vielbeins  $f^\alpha_a$  [[2,5], and the references therein]. We assume that there are spinor fields which determine spin connection fields –  $\omega_{abc}$  and  $\tilde{\omega}_{abc}$ . In general one would have [6]:  $p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\}_{-}$ ,  $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{s's''} \omega_{s's''\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ . Since the term  $\frac{1}{2E} \{p_\alpha, E f^\alpha_a\}_{-}$  does not influence the symmetry of mass matrices, we do not treat it in this paper.

<sup>2</sup> Before the electroweak break there are the conserved charges  $\vec{\tau}^1, \vec{\tau}^3$  and  $Y := \tau^4 + \tau^{23}$ , and the non conserved charge  $Y' := -\tau^4 \tan^2 \vartheta_2 + \tau^{23}$ , where  $\vartheta_2$  is the angle of the break of  $SU(2)_{II}$  from  $SU(2)_I \times SU(2)_{II} \times U(1)_{II}$  to  $SU(2)_I \times U(1)_I$ . After the electroweak break the conserved charges are  $\vec{\tau}^3$  and  $Q := Y + \tau^{13}$ , the non conserved charge is  $Q' := -Y \tan^2 \vartheta_1 + \tau^{13}$ , where  $\vartheta_1$  is the electroweak angle.

The groups  $SO(3, 1)$ ,  $SU(3)$ ,  $SU(2)_I$ ,  $SU(2)_{II}$  and  $U(1)_{II}$  determine spin and charges of fermions, the groups  $\widetilde{SO}(3, 1)$ ,  $\widetilde{SU}(2)_I$ ,  $\widetilde{SU}(2)_{II}$  and  $\widetilde{U}(1)_{II}$  determine family quantum numbers<sup>3</sup>.

The generators of these groups are expressible by  $\tilde{S}^{\tilde{a}b}$

$$\tilde{\tau}^{Ai} = \sum_{a,b} c^{Ai}_{ab} \tilde{S}^{ab}, \quad (13.5)$$

fulfilling again the commutation relations

$$\{\tilde{\tau}^{Ai}, \tilde{\tau}^{Bj}\}_- = i\delta^{AB} f^{Aijk} \tilde{\tau}^{Ak}, \quad (13.6)$$

while

$$\{\tau^{Ai}, \tilde{\tau}^{Bj}\}_- = 0. \quad (13.7)$$

The scalar gauge fields of the groups  $\widetilde{SU}(2)_I$ ,  $\widetilde{SU}(2)_I$  and  $U(1)$  are presented in Eq. (13.30), the application of the generators of  $\tilde{\tau}^1$ , Eq. (13.26),  $\tilde{N}_L$ , Eq. (13.25), which distinguish among families and are the same for all the family members, are presented in Eq. (13.12). The application of the family members generators  $Q$ ,  $Y$ ,  $\tau^4$  and  $Y'$  on the family members of any family is presented on Table 13.1.

R	$Q_{L,R}$	$Y$	$\tau^4_{L,R}$	$Y'$	$Q'$	L	$Y$	$Y'$	$Q'$
$u_R^i$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{2}(1 - \frac{1}{3}\tan^2\theta_2)$	$-\frac{2}{3}\tan^2\theta_1$	$u_L^i$	$\frac{1}{6}$	$-\frac{1}{6}\tan^2\theta_2$	$\frac{1}{2}(1 - \frac{1}{3}\tan^2\theta_1)$
$d_R^i$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{2}(1 + \frac{1}{3}\tan^2\theta_2)$	$\frac{1}{3}\tan^2\theta_1$	$d_L^i$	$\frac{1}{6}$	$-\frac{1}{6}\tan^2\theta_2$	$-\frac{1}{2}(1 + \tan^2\theta_1)$
$\nu_R^i$	0	0	$-\frac{1}{2}$	$\frac{1}{2}(1 + \tan^2\theta_2)$	0	$\nu_L^i$	$-\frac{1}{2}$	$\frac{1}{2}\tan^2\theta_2$	$\frac{1}{2}(1 + \tan^2\theta_1)$
$e_R$	-1	-1	$-\frac{1}{2}$	$\frac{1}{2}(-1 + \tan^2\theta_2)$	$\tan^2\theta_1$	$e_L$	$-\frac{1}{2}$	$\frac{1}{2}\tan^2\theta_2$	$-\frac{1}{2}(1 - \tan^2\theta_1)$

**Table 13.1.** The quantum numbers  $Q$ ,  $Y$ ,  $\tau^4$ ,  $Y'$ ,  $Q'$ , Eq. (13.28), of the members of one family (anyone) [6]. Left and right handed members of any family have the same  $Q$  and  $\tau^4$ , the right handed members have  $\tau^{13} = 0$  and  $\tau^{23} = \frac{1}{2}$ , while the left handed members have  $\tau^{13} = \frac{1}{2}$  and  $\tau^{23} = 0$ .

There are in the *spin-charge-family* theory  $2^{\frac{(1+7)}{2}-1} = 8$  families, which split in two groups of four families, due to the break of the symmetry from  $\widetilde{SO}(1, 7)$  into  $\widetilde{SO}(1, 3) \times \widetilde{SO}(4)$ . Each of these two groups manifests  $\widetilde{SU}(2)_{\widetilde{SO}(1,3)} \times \widetilde{SU}(2)_{\widetilde{SO}(4)}$  [6]. These decoupled twice four families are presented in Table 13.4

The lowest of the upper four families, forming neutral clusters with respect to the electromagnetic and colour charges, is the candidate to forms the dark matter [18].

We discuss in this paper symmetry properties of the lower four families, presented in Table 13.4 in the first four lines. We repeat in Table 13.2 the representation and the family quantum numbers of the left and right handed members of the lower four families. Since any of the family members ( $u_{L,R}^i$ ,  $d_{L,R}^i$ ,  $\nu_{L,R}^i$ ,  $e_{L,R}^i$ )

<sup>3</sup>  $\widetilde{SU}(3)$  do not contribute to the families at low energies [34].

behave equivalently with respect to all the operators concerning the family groups  $\widetilde{\text{SU}}(2)_{\widetilde{\text{SO}}(1,3)} \times \widetilde{\text{SU}}(2)_{\widetilde{\text{SO}}(4)}$ , we use a common notation  $|\psi^i\rangle$ .

The interaction, which is responsible for the appearance of masses of fermions, is presented in Eq. (13.1) in the second line

$$\begin{aligned}\mathcal{L}_{\text{mass}} &= \frac{1}{2} \sum_{+,-} \{\psi_L^\dagger \gamma^0 \begin{smallmatrix} 78 \\ (\pm) \end{smallmatrix} (- \sum_A \tau^A A_\pm^A - \sum_{\tilde{A}i} \tilde{\tau}^{Ai} A_\pm^{Ai}) \psi_R\} + \text{h.c.}, \\ \tau^A &= (Q, Q', Y'), \quad \tilde{\tau}^{Ai} = (\tilde{N}_L^i, \tilde{\tau}^1, \tilde{\tau}^4), \\ \gamma^0 \begin{smallmatrix} 78 \\ (\pm) \end{smallmatrix} &= \gamma^0 \frac{1}{2} (\gamma^7 \pm i \gamma^8), \\ A_\pm^A &= \sum_{st} c_{st}^A \omega^{st}_\pm, \quad \omega^{st}_\pm = \omega^{st}_7 \mp i \omega^{st}_8, \\ \tilde{A}_\pm^A &= \sum_{ab} c_{ab}^A \tilde{\omega}^{ab}_\pm, \quad \tilde{\omega}^{ab}_\pm = \tilde{\omega}^{ab}_7 \mp i \tilde{\omega}^{ab}_8.\end{aligned}\tag{13.8}$$

In Eq. (13.8) the  $p_s$  is left out since at low energies its contribution is negligible,  $A$  determines operators, which distinguish among family members –  $(Q, Y, \tau^4)$ , the values are presented in Table 13.1 –  $(\tilde{A}, i)$  represent the family operators, determined in Eqs. (13.25, 13.26, 13.27). The detailed explanation can be found in Refs. [4,5,1].

Operators  $\tau^{Ai}$  are Hermitian,  $\gamma^0 \begin{smallmatrix} 78 \\ (\pm) \end{smallmatrix} = \gamma^0 \begin{smallmatrix} 78 \\ (\mp) \end{smallmatrix}$ . In what follows it is assumed that the scalar fields  $A_s^{Ai}$  are Hermitian as well and consequently it follows  $(A_\pm^{Ai})^\dagger = A_\mp^{Ai}$ .

While the family operators  $\tilde{\tau}^{1i}$  and  $\tilde{N}_L^i$  commute with  $\gamma^0 \begin{smallmatrix} 78 \\ (\pm) \end{smallmatrix}$ , the family members operators  $(Y, Y', Q')$  do not, since  $S^{78}$  does not ( $S^{78} \gamma^0 \begin{smallmatrix} 78 \\ (\mp) \end{smallmatrix} = -\gamma^0 \begin{smallmatrix} 78 \\ (\mp) \end{smallmatrix} S^{78}$ ). However

$$\begin{aligned}[\psi_L^{k\dagger} \gamma^0 \begin{smallmatrix} 78 \\ (\mp) \end{smallmatrix} (Q, Q', Y') A_\mp^{(Q, Q', Y')} \psi_R^l]^\dagger &= \\ &= \psi_R^{l\dagger} (Q, Q', Y') A_\pm^{(Q, Q', Y')\dagger} \gamma^0 \begin{smallmatrix} 78 \\ (\pm) \end{smallmatrix} \psi_L^k \delta_{k,l} = \\ &= \psi_R^{l\dagger} (Q_R^k, Q_R'^k, Y_R'^k) A_\pm^{(Q, Q', Y')} \psi_R^k \delta_{k,l},\end{aligned}\tag{13.9}$$

where  $(Q_R^k, Q_R'^k, Y_R'^k)$  denote the eigenvalues of the corresponding operators on the spinor state  $\psi_R^k$ . This means that we evaluate in both cases quantum numbers of the right handed partners.

In Table 13.2 four families of spinors belonging to the group with the nonzero values of  $\tilde{N}_L$  and  $\tilde{\tau}^1$  are presented in the *technique* 13.5. These are the lower four families, presented in Table 13.4. There are indeed the four families of  $\psi_{u_R}^i$  and  $\psi_{u_L}^i$ . All the  $2^{\frac{13+1}{2}-1}$  members of the first family are represented in Table 13.3. The scalar fields  $\gamma^0 \begin{smallmatrix} 78 \\ (\mp) \end{smallmatrix} (Q, Q', Y') A_\mp^{(Q, Q', Y')}$  are "diagonal"; They transform a right handed member of one family into the left handed member of the same family, or they transform a left handed member of one family into the right handed member

of the same family. These terms are different for different family members but the same for all the families of the same family member.

*We shall prove that the symmetry of mass term keep the same in all the orders of loop corrections in the massless basis.*

Since  $Q = (\tau^{13} + \tau^{23} + \tau^4) = (S^{56} + \tau^4)$ ,  $Y' = (-\tau^4 \tan^2 \theta_1 + \tau^{23})$  and  $Q' = (-\tau^4 + \tau^{23}) \tan^2 \theta_1 + \tau^{13}$ , we can use as well the operators  $(\gamma^0 (\pm) \tau^4 A_{\pm}^4, \gamma^0 (\pm) \tau^{23} A_{\pm}^{23}, \gamma^0 (\pm) \tau^{13} A_{\pm}^{13})$ . In either case we denote the contributions of these terms as  $-a_0^\alpha$

$$\begin{aligned}
 -a_0^\alpha &= \\
 &= -\frac{1}{2} \{\psi_L^{\dagger} \sum_{+,-} (\gamma^0 (\pm) \tau^4 A_{\pm}^4 + \gamma^0 (\pm) \tau^{23} A_{\pm}^{23} + \gamma^0 (\pm) \tau^{13} A_{\pm}^{13}) \psi_R^j\} \delta^{ij} + \text{h.c.},
 \end{aligned}
 \tag{13.10}$$

where  $\alpha$  means that a particular family member ( $\alpha = (u, d, v, e)$ ) is studied. We could make different superposition of these terms. Our proof does not depend on this choice, although each family member has a different value for  $a_0^\alpha$ .

*Transitions among families for any family member are caused by  $(\tilde{N}_L^i$  and  $\tilde{\tau}^{1i})$ , which manifest the symmetry  $\widetilde{SU}_{N_L}(2) \times \widetilde{SU}_{\tau^1}(2)$ .*

				$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}_L^3$	$\tilde{N}_R^3$	$\tilde{\tau}^4$
$\psi_{u_R^c}^1$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) &   & (+) & (+) \end{smallmatrix} \parallel \dots$	$\psi_{u_L^c}^1$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ -[-i] & (+) &   & (+) & [-] \end{smallmatrix} \parallel \dots$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\psi_{u_R^c}^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ [+i] & (+) &   & (+) & (+) \end{smallmatrix} \parallel \dots$	$\psi_{u_L^c}^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ -(-i) & (+) &   & (+) & [-] \end{smallmatrix} \parallel \dots$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\psi_{u_R^c}^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) &   & (+) & (+) \end{smallmatrix} \parallel \dots$	$\psi_{u_L^c}^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ -[-i] & (+) &   & (+) & (-) \end{smallmatrix} \parallel \dots$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\psi_{u_R^c}^4$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ [+i] & (+) &   & (+) & (+) \end{smallmatrix} \parallel \dots$	$\psi_{u_L^c}^4$	$\begin{smallmatrix} 03 & 12 & 56 & 78 \\ -(-i) & (+) &   & (+) & (-) \end{smallmatrix} \parallel \dots$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$

**Table 13.2.** Four families of the right handed  $u_R^c$  and of the left handed  $u_L^c$  quarks with spin  $\frac{1}{2}$  and the colour charge ( $\tau^{33} = 1/2, \tau^{38} = 1/(2\sqrt{3})$ ) (the definition of the operators is presented in Eqs. (13.23,13.24) are presented (1<sup>st</sup> and 7<sup>th</sup> line in Table 13.3). A few examples how to calculate the application of the operators on the states written as products of nilpotents and projectors on the vacuum state can be found in Sect. 13.5. The spin and charges, which distinguish among family members, are not shown in this table, since they commute with  $\tilde{N}_L^i, \tilde{\tau}^{1i}$  and  $\tilde{\tau}^4$ , and are correspondingly the same for all the families.

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i	$ \alpha \psi_i\rangle$ (Anti)octet, $\Gamma(7,1) = (-1)1$ , $\Gamma(6) = (1) - 1$ of (anti)quarks and (anti)leptons	$\Gamma(3,1)$	$S^{12}$	$\tau^{13}$	$\tau^{23}$	$\tau^{33}$	$\tau^{38}$	$\tau^4$	Y	Q
40	$\bar{u}_R^1$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ [-i] & (-) & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (+) & (+) & (+) \end{bmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}_L^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ [-i] & (+) & (+) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (+) & (+) & (+) \end{bmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\bar{d}_L^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (-) & (+) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (+) & (+) & (+) \end{bmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}_L^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & [-i] & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\bar{u}_L^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & (+i) & (-) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\bar{d}_R^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\bar{d}_R^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & [-i] & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\bar{u}_R^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\bar{u}_R^2$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (-i) & (-) & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\bar{d}_L^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ [-i] & (+) & (+) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\bar{d}_L^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (-) & (+) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\bar{u}_L^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & [-i] & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\bar{u}_L^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & (+i) & (-) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\bar{d}_R^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\bar{d}_R^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & [-i] & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\bar{u}_R^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\bar{u}_R^3$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (-i) & (-) & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{bmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$e_L$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ [-i] & (+) & (+) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (-) & (-) & (-) \end{bmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$e_L$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (-) & (+) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (-) & (-) & (-) \end{bmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\nu_L$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & [-i] & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\nu_L$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & (+i) & (-) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (+) \end{bmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\nu_R$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) & (-) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (-) & (-) & (-) \end{bmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\nu_R$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ - & [-i] & (-) & (+) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (-) & (-) & (-) \end{bmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$e_R$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (+i) & (+) & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (-) & (-) & (-) \end{bmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$e_R$ $\begin{bmatrix} 03 & 12 & 56 & 78 \\ (-i) & (-) & (+) & (-) \end{bmatrix}    \begin{bmatrix} 9 & 10 & 11 & 12 \\ (-) & (-) & (-) & (-) \end{bmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

**Table 13.3.** The left handed ( $\Gamma(13,1) = -1$ , Eq. (13.38)) multiplet of spinors—the members of the fundamental representation of the  $S O(13,1)$  group, manifesting the subgroup  $S O(7,1)$  of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons—is presented in the massless basis using the technique presented in App. 13.5. It contains the left handed ( $\Gamma(3,1) = -1$ ) weak ( $S U(2)_I$ ) charged ( $\tau^{13} = \pm \frac{1}{2}$ , Eq. (13.23)), and  $S U(2)_{II}$  chargeless ( $\tau^{23} = 0$ , Eq. (13.23)) quarks and leptons and the right handed ( $\Gamma(3,1) = 1$ , Sect. 13.5) weak ( $S U(2)_I$ ) chargeless and  $S U(2)_{II}$  charged ( $\tau^{23} = \pm \frac{1}{2}$ ) quarks and leptons, both with the spin  $S^{12}$  up and down ( $\pm \frac{1}{2}$ , respectively). Quarks distinguish from leptons only in the  $S U(3) \times U(1)$  part: Quarks are triplets of three colours ( $c^i = (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$ , Eq. (13.24)) carrying the “fermion charge” ( $\tau^4 = \frac{1}{6}$ , Eq. (13.24)). The colourless leptons carry the “fermion charge” ( $\tau^4 = -\frac{1}{2}$ ). The same multiplet contains also the left handed weak ( $S U(2)_I$ ) chargeless and  $S U(2)_{II}$  charged anti-quarks and anti-leptons and the right handed weak ( $S U(2)_I$ ) charged and  $S U(2)_{II}$  chargeless anti-quarks and anti-leptons. Anti-quarks distinguish from anti-leptons again only in the  $S U(3) \times U(1)$  part: Anti-quarks are anti-triplets, carrying the “fermion charge” ( $\tau^4 = -\frac{1}{6}$ ). The anti-colourless anti-leptons carry the “fermion charge” ( $\tau^4 = \frac{1}{2}$ ).  $Y = (\tau^{23} + \tau^4)$  is the hyper charge, the electromagnetic charge is  $Q = (\tau^{13} + Y)$ . The states of opposite charges (anti-particle states) are reachable from the particle states (besides by  $S^{ab}$ ) also by the application of the discrete symmetry operator  $C_N P_N$ , presented in Refs. [41,42] and in Sect. 13.5. The vacuum state, on which the nilpotents and projectors operate, is not shown. The reader can find this Weyl representation also in Refs. [5,14,15,4] and in the references therein.

Taking into account Table 13.3 and Eqs. (13.34, 13.43) one easily finds what do operators  $\gamma^0$  ( $\pm$ ) do on the left handed and the right handed members of any

family  $i = (1, 2, 3, 4)$ .

$$\begin{aligned}
 \gamma^0 \begin{pmatrix} 78 \\ - \end{pmatrix} |\psi_{u_R, v_R}^i\rangle &= -|\psi_{u_L, v_L}^i\rangle, \\
 \gamma^0 \begin{pmatrix} 78 \\ + \end{pmatrix} |\psi_{u_L, v_L}^i\rangle &= |\psi_{u_R, v_R}^i\rangle, \\
 \gamma^0 \begin{pmatrix} 78 \\ + \end{pmatrix} |\psi_{d_R, e_R}^i\rangle &= |\psi_{d_L, e_L}^i\rangle, \\
 \gamma^0 \begin{pmatrix} 78 \\ - \end{pmatrix} |\psi_{d_L, e_L}^i\rangle &= |\psi_{d_R, e_R}^i\rangle.
 \end{aligned} \tag{13.11}$$

We need to know also what do operators  $(\tilde{\tau}^{1\pm} = \tilde{\tau}^{11} \pm i\tilde{\tau}^{12}, \tilde{\tau}^{13})$  and  $(\tilde{N}_L^\pm = \tilde{N}_L^1 \pm i\tilde{N}_L^2, \tilde{N}_L^3)$  do when operating on any member  $(u_{L,R}, v_{L,R}, d_{L,R}, e_{L,R})$  of a particular family  $\psi^i, i = (1, 2, 3, 4)$ .

Taking into account, Eqs. (13.32, 13.33, 13.43, 13.45, 13.36, 13.25, 13.26),

$$\begin{aligned}
 \tilde{N}_L^\pm &= -\frac{03}{(\mp i)(\pm)}, \quad \tilde{\tau}^{1\pm} = (\mp) \frac{56}{(\pm)} \frac{78}{(\mp)}, \\
 \tilde{N}_L^3 &= \frac{1}{2}(\tilde{S}^{12} + i\tilde{S}^{03}), \quad \tilde{\tau}^{13} = \frac{1}{2}(\tilde{S}^{56} - \tilde{S}^{78}), \\
 \frac{ab}{(-k)} \frac{ab}{(k)} &= -i\eta^{aa} \frac{ab}{[k]}, \quad \frac{ab}{(k)} \frac{ab}{(k)} = 0, \\
 \frac{ab}{(k)} \frac{ab}{[k]} &= i \frac{ab}{(k)}, \quad \frac{ab}{(k)} \frac{ab}{[-k]} = 0, \\
 \frac{ab}{(k)} &= \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik} \tilde{\gamma}^b), \quad \frac{ab}{[k]} = \frac{1}{2}(1 + \frac{i}{k} \tilde{\gamma}^a \tilde{\gamma}^b),
 \end{aligned} \tag{13.12}$$

one finds

$$\begin{aligned}
 \tilde{N}_L^+ |\psi^1\rangle &= |\psi^2\rangle, \quad \tilde{N}_L^+ |\psi^2\rangle = 0, \\
 \tilde{N}_L^- |\psi^2\rangle &= |\psi^1\rangle, \quad \tilde{N}_L^- |\psi^1\rangle = 0, \\
 \tilde{N}_L^+ |\psi^3\rangle &= |\psi^4\rangle, \quad \tilde{N}_L^+ |\psi^4\rangle = 0, \\
 \tilde{N}_L^- |\psi^4\rangle &= |\psi^3\rangle, \quad \tilde{N}_L^- |\psi^3\rangle = 0, \\
 \tilde{\tau}^{1+} |\psi^1\rangle &= |\psi^3\rangle, \quad \tilde{\tau}^{1+} |\psi^3\rangle = 0, \\
 \tilde{\tau}^{1-} |\psi^3\rangle &= |\psi^1\rangle, \quad \tilde{\tau}^{1-} |\psi^1\rangle = 0, \\
 \tilde{\tau}^{1-} |\psi^4\rangle &= |\psi^2\rangle, \quad \tilde{\tau}^{1-} |\psi^2\rangle = 0, \\
 \tilde{\tau}^{1+} |\psi^2\rangle &= |\psi^4\rangle, \quad \tilde{\tau}^{1+} |\psi^4\rangle = 0, \\
 \tilde{N}_L^3 |\psi^1\rangle &= -\frac{1}{2} |\psi^1\rangle, \quad \tilde{N}_L^3 |\psi^2\rangle = +\frac{1}{2} |\psi^2\rangle, \\
 \tilde{N}_L^3 |\psi^3\rangle &= -\frac{1}{2} |\psi^3\rangle, \quad \tilde{N}_L^3 |\psi^4\rangle = +\frac{1}{2} |\psi^4\rangle, \\
 \tilde{\tau}^{13} |\psi^1\rangle &= -\frac{1}{2} |\psi^1\rangle, \quad \tilde{\tau}^{13} |\psi^2\rangle = -\frac{1}{2} |\psi^2\rangle, \\
 \tilde{\tau}^{13} |\psi^3\rangle &= +\frac{1}{2} |\psi^3\rangle, \quad \tilde{\tau}^{13} |\psi^4\rangle = +\frac{1}{2} |\psi^4\rangle.
 \end{aligned} \tag{13.13}$$

Let the scalars  $(\tilde{A}_{(\pm)}^{N_L \boxplus}, \tilde{A}_{(\pm)}^{N_L^3}, \tilde{A}_{(\pm)}^{1\boxplus}, \tilde{A}_{(\pm)}^{13})$  be the scalar gauge fields of the operators  $(\tilde{N}_L^\pm, \tilde{N}_L^3, \tilde{\tau}^{1\pm}, \tilde{\tau}^{13})$ , respectively. Here  $\tilde{A}_{(\pm)} = \tilde{A}_7 \mp i\tilde{A}_8$  for all the scalar gauge



fields, while  $\tilde{A}_{(\pm)}^{N_L \Box} = \frac{1}{2} (\tilde{A}_{(\pm)}^{N_L 1} \mp i \tilde{A}_{(\pm)}^{N_L 2})$ , respectively, and  $\tilde{A}_{(\pm)}^{1\Box} = \frac{1}{2} (\tilde{A}_{(\pm)}^{11} \mp i \tilde{A}_{(\pm)}^{12})$ , respectively. All these fields can be expressed by  $\tilde{\omega}_{abc}$ , as presented in Eq. (13.30).

We are prepared now to calculate the mass matrix elements for any of the family members. Let us notice that the operators  $\gamma^0 \begin{smallmatrix} 78 \\ \mp \end{smallmatrix}$ , as well as the operators of spin and charges, distinguish between  $|\psi_L^i\rangle$  and  $|\psi_R^i\rangle$ . Correspondingly all the diagrams must have an odd number of contribution.

We use the massless basis  $|\psi_{L,R}^i\rangle$ . We shall simplify the calculation by making a choice of the  $\frac{1}{\sqrt{2}} (|\psi_L^i\rangle + |\psi_R^i\rangle)$ , keeping in mind that we must have an odd number of contributions

We can calculate the mass matrix for any family member using Eqs. (13.13). Below we present the mass matrix on the tree level, where  $(\tilde{a}_1, \tilde{a}_2, a_\alpha)$  represent the vacuum expectation values of  $\frac{1}{2} \frac{1}{\sqrt{2}} (\tilde{A}_{(+)}^{13} + \tilde{A}_{(-)}^{13})$ ,  $\frac{1}{2} \frac{1}{\sqrt{2}} (\tilde{A}_{(+)}^{N_L 3} + \tilde{A}_{(-)}^{N_L 3})$ ,  $\frac{1}{\sqrt{2}} (A_{(+)}^\alpha + A_{(-)}^\alpha)$ , respectively and where to  $A_{(\pm)}^\alpha$  the sum of  $\tau^{4\alpha} A_{(\pm)}^4$ ,  $\tau^{13\alpha} A_{(\pm)}^{13}$  and  $\tau^{23\alpha} A_{(\pm)}^{23}$ , Eq. (13.10), is contributing.

We use the notation  $\langle \tilde{A}^{N_L \Box} \rangle = \frac{1}{\sqrt{2}} (\langle \tilde{A}_{(+)}^{N_L \Box} \rangle + \langle \tilde{A}_{(-)}^{N_L \Box} \rangle)$  and  $\langle \tilde{A}^{1\Box} \rangle = \frac{1}{\sqrt{2}} (\langle \tilde{A}_{(+)}^{1\Box} \rangle + \langle \tilde{A}_{(-)}^{1\Box} \rangle)$ , since we use the basis  $\frac{1}{\sqrt{2}} (|\psi_L^i\rangle + |\psi_R^i\rangle)$ .

On the tree level is the contribution to the matrix elements  $\langle \psi^1 | \dots | \psi^4 \rangle$ ,  $\langle \psi^2 | \dots | \psi^3 \rangle$ ,  $\langle \psi^3 | \dots | \psi^2 \rangle$  and  $\langle \psi^4 | \dots | \psi^1 \rangle$  equal to zero. One can come, however, from  $\langle \psi^1 | \dots | \psi^4 \rangle$  in three steps (not two, due to the left right jumps in each step):  $\langle \psi^4 | \sum_{+,-} \tilde{\tau}^{1\Box} \tilde{A}^{1\Box} \sum_k |\psi^k\rangle \langle \psi^k | \sum_{+,-} \tilde{N}_L^{\Box} \tilde{A}^{N_L \Box} | \psi^4 \rangle \langle \psi^4 | (\tilde{a}_1 + \tilde{a}_2 + a^\alpha) | \psi^4 \rangle$ , there are all together six such terms, since the diagonal term appears also at the beginning as  $(-\tilde{a}_1 - \tilde{a}_2 + a^\alpha)$  and in the middle as  $(\tilde{a}_1 - \tilde{a}_2 + a^\alpha)$ , and since the operators  $\sum_{+,-} \tilde{\tau}^{1\Box} \tilde{A}^{1\Box}$  and  $\sum_{+,-} \tilde{N}_L^{\Box} \tilde{A}^{N_L \Box}$  appear in the opposite order as well. Summing all this six terms for each of four matrix elements ( $\langle 1 | \dots | 4 \rangle$ ,  $\langle 2 | \dots | 3 \rangle$ ,  $\langle 3 | \dots | 2 \rangle$ ,  $\langle 4 | \dots | 1 \rangle$ ) we find:

$$\begin{aligned} \langle 1 | \dots | 4 \rangle &= 6a^\alpha \langle \tilde{A}^{1\Box} \rangle \langle \tilde{A}^{N_L \Box} \rangle, \\ \langle 2 | \dots | 3 \rangle &= 6a^\alpha \langle \tilde{A}^{1\Box} \rangle \langle \tilde{A}^{N_L \Box} \rangle, \\ \langle 3 | \dots | 2 \rangle &= 6a^\alpha \langle \tilde{A}^{1\Box} \rangle \langle \tilde{A}^{N_L \Box} \rangle, \\ \langle 4 | \dots | 1 \rangle &= 6a^\alpha \langle \tilde{A}^{1\Box} \rangle \langle \tilde{A}^{N_L \Box} \rangle. \end{aligned} \quad (13.14)$$

These matrix elements are presented in Eq. (13.15).

$$\begin{aligned} {}^\alpha \mathcal{M}_{(o)} = & \begin{pmatrix} -\tilde{a}_1 - \tilde{a}_2 + a^\alpha & \langle \tilde{A}^{N_L \Box} \rangle & \langle \tilde{A}^{1\Box} \rangle & 6a^\alpha \langle \tilde{A}^{1\Box} \tilde{A}^{N_L \Box} \rangle \\ \langle \tilde{A}^{N_L \Box} \rangle & -\tilde{a}_1 + \tilde{a}_2 + a^\alpha & 6a^\alpha \langle \tilde{A}^{1\Box} \tilde{A}^{N_L \Box} \rangle & \langle \tilde{A}^{1\Box} \rangle \\ \langle \tilde{A}^{1\Box} \rangle & 6a^\alpha \langle \tilde{A}^{1\Box} \tilde{A}^{N_L \Box} \rangle & \tilde{a}_1 - \tilde{a}_2 + a^\alpha & \langle \tilde{A}^{N_L \Box} \rangle \\ 6a^\alpha \langle \tilde{A}^{1\Box} \tilde{A}^{N_L \Box} \rangle & \langle \tilde{A}^{1\Box} \rangle & \langle \tilde{A}^{N_L \Box} \rangle & \tilde{a}_1 + \tilde{a}_2 + a^\alpha \end{pmatrix} \end{aligned} \quad (13.15)$$

One notices that the diagonal terms have on the tree level the symmetry  $\langle \psi^1 | \dots | \psi^1 \rangle + \langle \psi^4 | \dots | \psi^4 \rangle = a^\alpha = \langle \psi^2 | \dots | \psi^2 \rangle + \langle \psi^3 | \dots | \psi^3 \rangle$  and that in

the off diagonal elements in next order to zero the contribution of the fields, which depend on particular family member  $\alpha = (u, d, \nu, e)$  enter. We also notice that  $\langle \psi^i | \dots | \psi^j \rangle^\dagger = \langle \psi^j | \dots | \psi^i \rangle$ . In the case that  $\langle \tilde{A}^{\tilde{1}\Box} \rangle = \langle \tilde{A}^{\tilde{1}\Box} \rangle = e$  and  $\langle \tilde{A}^{\tilde{N}_L\Box} \rangle = \langle \tilde{A}^{\tilde{N}_L\Box} \rangle = d$ , which would mean that all the matrix elements are real, the mass matrix simplifies to

$$\mathcal{M}_{(o)}^\alpha = \begin{pmatrix} -\tilde{a}_1 - \tilde{a}_2 + a^\alpha & d & e & 6a^\alpha ed \\ d & -\tilde{a}_1 + \tilde{a}_2 + a^\alpha & 6a^\alpha ed & e \\ e & 6a^\alpha ed & \tilde{a}_1 - \tilde{a}_2 + a^\alpha & d \\ 6a^\alpha ed & e & d & \tilde{a}_1 + \tilde{a}_2 + a^\alpha \end{pmatrix}. \quad (13.16)$$

### 13.2.1 Mass matrices beyond the tree level

To make a proof that the symmetry  $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$  of the mass matrix, presented in Eq. (13.15), is kept in all orders of loop corrections, we need to proof only that at each order the matrix element, let say,  $\langle 1 | \dots | 2 \rangle$  (in Eq. (13.15) this matrix element is equal to  $\langle \tilde{A}^{\tilde{N}_L\Box} \rangle$ ) remains equal to  $\langle 3 | \dots | 4 \rangle$  in all orders, while  $\langle 2 | \dots | 1 \rangle$  remains to be equal to  $\langle 1 | \dots | 2 \rangle^\dagger = \langle 4 | \dots | 3 \rangle (= \langle \tilde{A}^{\tilde{N}_L\Box} \rangle)$ . These should be done for all the matrix elements appearing in Eq. (13.15).

**a.** It is not difficult to see that each of the diagonal terms ( $\tilde{\tau}^{\tilde{1}3} < \tilde{A}^{\tilde{1}3} \rangle$ ,  $\tilde{N}_L^3 < \tilde{A}^{\tilde{N}_L3} \rangle$ ,  $\tau^A < A^A \rangle$ , with  $\tau^A = \tau^4, \tau^{13}, \tau^{23}$ ) have the property that the sum of the contributions  $x + xxx + xxxxx + \dots$  (in all orders) keeps the symmetry of the tree level. Let us check for  $\tilde{\tau}^{\tilde{1}3} < \tilde{A}^{\tilde{1}3} \rangle$ . One obtains for each of the four families  $i = [1, 2, 3, 4]$  the values  $[-\tilde{a}^1(1 + (-\tilde{a}^1)^2 + (-\tilde{a}^1)^4 + \dots), -\tilde{a}^1(1 + (-\tilde{a}^1)^2 + (-\tilde{a}^1)^4 + \dots), \tilde{a}^1(1 + (\tilde{a}^1)^2 + (\tilde{a}^1)^4 + \dots), \tilde{a}^1(1 + (\tilde{a}^1)^2 + (\tilde{a}^1)^4 + \dots)]$ , which we call  $[-\tilde{a}^1, -\tilde{a}^1, \tilde{a}^1, \tilde{a}^1]$  for the four families  $i = [1, 2, 3, 4]$ , respectively. Correspondingly one finds for the same kind of diagrams for  $\tilde{N}_L^3 < \tilde{A}^{\tilde{N}_L3} \rangle$  the four values  $[-\tilde{a}^2, \tilde{a}^2, -\tilde{a}^2, \tilde{a}^2]$  for the four families  $i = [1, 2, 3, 4]$ , respectively. While for  $\tau^A A^A$  we obtain, when summing over the diagrams  $x + xxx + xxxxx + \dots$ , the same value  $a^\alpha$  for a particular family member  $\alpha = (u, d, \nu, e)$  all four families. Family members properties enter in the left/right basis  $\frac{1}{\sqrt{2}}(|\psi_L^i \rangle + |\psi_R^i \rangle)$  into the mass matrix only through  $a^\alpha$ .

One reproduces that the sum of  $\langle 1 | \dots | 1 \rangle + \langle 4 | \dots | 4 \rangle = \langle 2 | \dots | 2 \rangle + \langle 3 | \dots | 3 \rangle$

Correspondingly it is not difficult to see that all the matrix elements, not only diagonal but also off diagonal, keep the symmetry of the mass matrix of Eq. (13.15) in all orders of corrections, provided that the matrix elements of the kind  $\alpha \tilde{a}_1 + \beta \tilde{a}_2 + a^\alpha$  — or of the kind in the  $\alpha \tilde{a}_1 + \beta \tilde{a}_2 + a^\alpha$  — appears in the diagrams in first power only. Here  $(\alpha, \beta)$  are  $\pm 1$ , they are determined by the eigenvalues of the operators  $\tilde{\tau}^{\tilde{1}3}$  (for  $\tilde{a}_1$ ) and  $\tilde{N}_L^3$  (for  $\tilde{a}_2$ ), respectively, on a particular family, Eq. (13.13).

**b.** Let us add to the diagonal terms the loop corrections. Let us evaluate, using the massless basis  $|\psi^i\rangle = \frac{1}{\sqrt{2}}(|\psi_L^i\rangle + |\psi_R^i\rangle)$ , the contribution:

$$\begin{aligned} & \langle \psi^i | \sum_{-,+, \boxplus, \boxminus, j} \gamma^0 (\pm) [\tilde{N}_L^{\boxplus} \tilde{A}^{\tilde{N}_L \boxplus} + \tilde{\tau}^{\boxplus} \tilde{A}^{\tilde{I} \boxplus}] |\psi^j\rangle \\ & \quad \langle \psi^j | \sum_{-,+} \gamma^0 (\pm) [\tilde{N}_L^3 \tilde{A}^{\tilde{N}_L 3} + \tilde{\tau}^{13} \tilde{A}^{\tilde{I} 3} + \sum_A \tau^A A^A] |\psi^j\rangle \\ & \quad \langle \psi^j | \sum_{-,+, \boxplus, \boxminus} \gamma^0 (\pm) [\tilde{N}_L^{\boxplus} \tilde{A}^{\tilde{N}_L \boxplus} + \tilde{\tau}^{\boxplus} \tilde{A}^{\tilde{I} \boxplus}] |\psi^i\rangle. \end{aligned} \quad (13.17)$$

One finds for  $i = [1, 2, 3, 4]$  the values  $[\tilde{A}^{\tilde{I} \boxplus} \tilde{A}^{\tilde{I} \boxminus} (\tilde{a}^1 - \tilde{a}^2 + a^\alpha) + \tilde{A}^{\tilde{N}_L \boxplus} \tilde{A}^{\tilde{N}_L \boxminus} (-\tilde{a}^1 + \tilde{a}^2 + a^\alpha), \tilde{A}^{\tilde{I} \boxplus} \tilde{A}^{\tilde{I} \boxminus} (\tilde{a}^1 + \tilde{a}^2 + a^\alpha) + \tilde{A}^{\tilde{N}_L \boxplus} \tilde{A}^{\tilde{N}_L \boxminus} (-\tilde{a}^1 - \tilde{a}^2 + a^\alpha), \tilde{A}^{\tilde{I} \boxplus} \tilde{A}^{\tilde{I} \boxminus} (-\tilde{a}^1 - \tilde{a}^2 + a^\alpha) + \tilde{A}^{\tilde{N}_L \boxplus} \tilde{A}^{\tilde{N}_L \boxminus} (\tilde{a}^1 + \tilde{a}^2 + a^\alpha), \tilde{A}^{\tilde{I} \boxplus} \tilde{A}^{\tilde{I} \boxminus} (-\tilde{a}^1 + \tilde{a}^2 + a^\alpha) + \tilde{A}^{\tilde{N}_L \boxplus} \tilde{A}^{\tilde{N}_L \boxminus} (\tilde{a}^1 - \tilde{a}^2 + a^\alpha)]$ , respectively, which again has the symmetry of the tree level state  $\langle 1|\dots|1\rangle + \langle 4|\dots|4\rangle = \langle 2|\dots|2\rangle + \langle 3|\dots|3\rangle$ .

One can make three such loops, or any kind of loops in any order of loop corrections with one  $(\alpha \tilde{a}^1 + \beta \tilde{a}^2 + a^\alpha)$  and the symmetry of tree level state  $\langle 1|\dots|1\rangle + \langle 4|\dots|4\rangle = \langle 2|\dots|2\rangle + \langle 3|\dots|3\rangle$  is manifested.

**c.** Let us look at the loop corrections to the off diagonal terms  $\langle 1|\dots|2\rangle, \langle 1|\dots|3\rangle, \langle 2|\dots|4\rangle, \langle 3|\dots|4\rangle$ , as well as their complex conjugate values.

Let us evaluate, using the massless basis  $|\psi^i\rangle = \frac{1}{\sqrt{2}}(|\psi_L^i\rangle + |\psi_R^i\rangle)$ , the contribution:

$$\begin{aligned} & \langle \psi^4 | \sum_{-,+, \boxplus, \boxminus, j, k} \gamma^0 (\pm) [\tilde{N}_L^{\boxplus} \tilde{A}^{\tilde{N}_L \boxplus} + \tilde{\tau}^{\boxplus} \tilde{A}^{\tilde{I} \boxplus}] |\psi^j\rangle \\ & \quad \langle \psi^j | \sum_{-,+} \gamma^0 (\pm) [\tilde{N}_L^{\boxplus} \tilde{A}^{\tilde{N}_L \boxplus} + \tilde{\tau}^{\boxplus} \tilde{A}^{\tilde{I} \boxplus}] |\psi^k\rangle \\ & \quad \langle \psi^k | \sum_{-,+, \boxplus, \boxminus} \gamma^0 (\pm) [\tilde{N}_L^{\boxplus} \tilde{A}^{\tilde{N}_L \boxplus} + \tilde{\tau}^{\boxplus} \tilde{A}^{\tilde{I} \boxplus}] |\psi^2\rangle \\ & \quad + \langle \psi^4 | \sum_{-,+, \boxplus, \boxminus, j} \gamma^0 (\pm) [\tilde{N}_L^3 \tilde{A}^{\tilde{N}_L 3} + \tilde{\tau}^{13} \tilde{A}^{\tilde{I} 3} + \sum_A \tau^A A^A] |\psi^4\rangle \\ & \quad \langle \psi^4 | \sum_{-,+, \boxplus, \boxminus} \gamma^0 (\pm) [\tilde{N}_L^{\boxplus} \tilde{A}^{\tilde{N}_L \boxplus} + \tilde{\tau}^{\boxplus} \tilde{A}^{\tilde{I} \boxplus}] |\psi^j\rangle \\ & \quad \langle \psi^j | \sum_{-,+, \boxplus, \boxminus} \gamma^0 (\pm) [\tilde{N}_L^3 \tilde{A}^{\tilde{N}_L 3} + \tilde{\tau}^{13} \tilde{A}^{\tilde{I} 3} + \sum_A \tau^A A^A] |\psi^2\rangle. \end{aligned} \quad (13.18)$$

One obtains for this term  $\langle 4|\dots|2\rangle = \langle \tilde{A}^{\tilde{I} \boxplus} \rangle \{ \tilde{A}^{\tilde{N}_L \boxplus} \tilde{A}^{\tilde{N}_L \boxminus} + |\tilde{A}^{\tilde{N}_L 3}|^2 + |\tilde{A}^{\tilde{I} 3}|^2 + |\tau^A A^A|^2 \}$ , which is equal to the equivalent loop correction term for the matrix element  $\langle 3|\dots|1\rangle$ .

Checking the loop corrections for the off diagonal elements  $\langle 1|\dots|2\rangle, \langle 1|\dots|3\rangle, \langle 2|\dots|4\rangle, \langle 3|\dots|4\rangle$  in all loop corrections one finds that the symmetry of these off diagonal terms is kept in all orders.

d. There are still the terms  $\langle 1|\dots|4 \rangle, \langle 2|\dots|3 \rangle, \langle 3|\dots|2 \rangle$  and  $\langle 4|\dots|1 \rangle$  to be checked in loop corrections. Adding loop corrections in the way we did in c. we find that also these matrix elements keep the symmetry of Eq. (13.15).

### 13.3 Conclusions

We demonstrate in this contribution on several cases that the matrix elements of mass matrices  $4 \times 4$ , predicted by the *spin-charge-family* theory for each family member  $\alpha = (u, d, \nu, e)$  to have the symmetry  $\widetilde{\text{SU}}(2)_{\widetilde{\text{SO}}(4)_{1+3}} \times \widetilde{\text{SU}}(2)_{\widetilde{\text{SO}}(4)_{\text{weak}}} \times \text{U}(1)$  on the tree level, keeps this symmetry in all loop corrections. The first to groups concern the family groups, the last one concern the family members group.

The only dependence of the mass matrix on the family member ( $\alpha = (u, d, \nu, e)$ ) quantum numbers is on the tree level through the vacuum expectation values of the operators  $\gamma^0 (\pm) Q A_{\pm}^Q, \gamma^0 (\pm) Q' A_{\pm}^{Q'}$  and  $\gamma^0 (\pm) \tau^4 A_{\pm}^4$ , appearing on a tree level in the diagonal terms of the mass matrix only and are the same for each of four families —  $I_{4 \times 4} \alpha^\alpha$ ,  $I$  is the unite matrix. In the loop corrections these operators enter into all the off diagonal matrix elements, causing the difference in the masses of the family members. The right handed neutrino, which is the regular member of the four families, Table 13.3, has the nonzero value of the operator  $\tau^4 A_{\pm}^4$  only (while the family part of the mass matrix is on the tree level the same for all the members).

We demonstrate on several cases, why does the symmetry of the mass matrix, which shows up on the tree level, remain in the loop corrections in all orders.

Although we are not (yet) able to calculate these matrix elements, the predicted symmetry will enable to predict masses of the fourth family (to the observed three), since the  $3 \times 3$  submatrix of the  $4 \times 4$  matrix determines  $4 \times 4$  matrix uniquely [19,4]. We only must wait for accurate enough data for mixing matrices of quarks and leptons to predict, using the symmetry of mass matrices predicted by the *spin-charge-family*, the masses of the fourth family quarks and leptons.

### 13.4 APPENDIX: Short presentation of the *spin-charge-family* theory

This subsection follows similar sections in Refs. [1,4–7].

The *spin-charge-family* theory [1–11,14–22] assumes:

A. A simple action (Eq. (13.20)) in an even dimensional space ( $d = 2n$ ,  $d > 5$ ),  $d$  is chosen to be  $(13 + 1)$ . This choice makes that the action manifests in  $d = (3 + 1)$  in the low energy regime all the observed degrees of freedom, explaining all the assumptions of the *standard model*, as well as other observed phenomena.

There are two kinds of the Clifford algebra objects,  $\gamma^a$ 's and  $\tilde{\gamma}^a$ 's in this theory with the properties.

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}, \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}, \quad , \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0. \quad (13.19)$$

Fermions interact with the vielbeins  $f^\alpha_a$  and the two kinds of the spin-connection fields -  $\omega_{ab\alpha}$  and  $\tilde{\omega}_{ab\alpha}$  - the gauge fields of  $S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$  and  $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$ , respectively.

The action

$$\mathcal{A} = \int d^d x \, E \, \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + \text{h.c.} + \int d^d x \, E \, (\alpha R + \tilde{\alpha} \tilde{R}), \quad (13.20)$$

in which  $p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\}_-$ ,  $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ , and

$$R = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\omega_{ab\alpha, \beta} - \omega_{c a \alpha} \omega^c_{b \beta})\} + \text{h.c.},$$

$$\tilde{R} = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\tilde{\omega}_{ab\alpha, \beta} - \tilde{\omega}_{c a \alpha} \tilde{\omega}^c_{b \beta})\} + \text{h.c.}$$

<sup>4</sup>, introduces two kinds of the Clifford algebra objects,  $\gamma^a$  and  $\tilde{\gamma}^a$ ,  $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$ .  $f^\alpha_a$  are vielbeins inverted to  $e^a_\alpha$ , Latin letters (a, b, ...) denote flat indices, Greek letters ( $\alpha, \beta, \dots$ ) are Einstein indices, (m, n, ...) and ( $\mu, \nu, \dots$ ) denote the corresponding indices in (0, 1, 2, 3), while (s, t, ...) and ( $\sigma, \tau, \dots$ ) denote the corresponding indices in  $d \geq 5$ :

$$e^a_\alpha f^\beta_a = \delta^\beta_\alpha, \quad e^a_\alpha f^\alpha_b = \delta^a_b, \quad (13.21)$$

$E = \det(e^a_\alpha)$ .

**B.** The *spin-charge-family* theory assumes in addition that the manifold  $M^{(13+1)}$  breaks first into  $M^{(7+1)} \times M^{(6)}$  (which manifests as  $SO(7, 1) \times SU(3) \times U(1)$ ), affecting both internal degrees of freedom - the one represented by  $\gamma^a$  and the one represented by  $\tilde{\gamma}^a$ . Since the left handed (with respect to  $M^{(7+1)}$ ) spinors couple differently to scalar (with respect to  $M^{(7+1)}$ ) fields than the right handed ones, the break can leave massless and mass protected  $2^{((7+1)/2-1)}$  families [34]. The rest of families get heavy masses <sup>5</sup>.

**C.** There is additional breaking of symmetry: The manifold  $M^{(7+1)}$  breaks further into  $M^{(3+1)} \times M^{(4)}$ .

**D.** There is a scalar condensate (Table 13.5) of two right handed neutrinos with the family quantum numbers of the upper four families, bringing masses of the scale  $\propto 10^{16}$  GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [5].

**E.** There are the scalar fields with the space index (7, 8) carrying the weak ( $\tau^{1i}$ ) and the hyper charges ( $Y = \tau^{23} + \tau^4$ ,  $\tau^{1i}$  and  $\tau^{2i}$  are generators of the subgroups of

<sup>4</sup> Whenever two indexes are equal the summation over these two is meant.

<sup>5</sup> A toy model [34,35] was studied in  $d = (5 + 1)$  with the same action as in Eq. (13.20). The break from  $d = (5 + 1)$  to  $d = (3 + 1) \times$  an almost  $S^2$  was studied. For a particular choice of vielbeins and for a class of spin connection fields the manifold  $M^{(5+1)}$  breaks into  $M^{(3+1)}$  times an almost  $S^2$ , while  $2^{((3+1)/2-1)}$  families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made in the  $d = (13 + 1)$  case. This study is in progress.

$SO(4)$ ,  $\tau^4$  and  $\tau^{3i}$  are the generators of  $U(1)_{II}$  and  $SU(3)$ , respectively, which are subgroups of  $SO(6)$ , which with their nonzero vacuum expectation values change the properties of the vacuum and break the weak charge and the hyper charge. Interacting with fermions and with the weak and hyper bosons, they bring masses to heavy bosons and to twice four groups of families. Carrying no electromagnetic ( $Q = \tau^{13} + Y$ ) and colour ( $\tau^{3i}$ ) charges and no  $SO(3, 1)$  spin, the scalar fields leave the electromagnetic, colour and gravity fields in  $d = (3 + 1)$  massless.

The assumed action  $\mathcal{A}$  and the assumptions offer the explanation for the origin and all the properties **o.** of the observed fermions:

**o.i.** of the family members, on Table 13.3 the family members, belonging to one Weyl (fundamental) representation of massless spinors of the group  $SO(13, 1)$  are presented in the "technique" [9–11, 14–16, 12, 13] and analyzed with respect to the subgroups  $SO(3, 1)$ ,  $SU(2)_I$ ,  $SU(2)_{II}$ ,  $SU(3)$ ,  $U(1)_{II}$ , Eqs. (13.22, 13.23, 13.24), with the generators  $\tau^{Ai} = \sum_{s,t} c^{Ai}_{st} S^{st}$ ,

**o.ii.** of the families analyzed with respect to the subgroups  $\widetilde{SO}(3, 1)$ ,  $\widetilde{SU}(2)_I$ ,  $\widetilde{SU}(2)_{II}$ ,  $\widetilde{U}(1)_{II}$ , with the generators  $\tilde{\tau}^{Ai} = \sum_{a,b} c^{Ai}_{ab} \tilde{S}^{st}$ , Eqs. (13.25, 13.26, 13.27), are presented on Table 13.4, all the families are singlets with respect to  $\widetilde{SU}(3)$ ,

**oo.i.** of the observed vector gauge fields of the charges

$$SU(2)_I, SU(2)_{II}, SU(3), U(1)_{II}$$

discussed in Refs. ([1, 4, 2], and the references therein), all the vector gauge fields are the superposition of the  $\omega_{stm}$ ,  $A_m^{Ai} = \sum_{s,t} c^{Ai}_{st} \omega_{stm}$ , Eq. vect

**oo.ii.** of the Higgs's scalar and of the Yukawa couplings, explainable with the scalar fields with the space index (7, 8), there are two groups of two triplets, which are scalar gauge fields of the charges  $\tilde{\tau}^{Ai}$ , expressible with the superposition of the  $\tilde{\omega}_{abs}$ ,  $A_m^{Ai} = \sum_{a,b} c^{Ai}_{ab} \omega_{abs}$  and three singlets, the gauge fields of  $Q$ ,  $Q'$ ,  $S'$ , Eqs. (13.28), all with the weak and the hyper charges as assumed by the *standard model* for the Higgs's scalars,

**oo.iii.** of the scalar fields explaining the origin of the matter-antimatter asymmetry, Ref. [5],

**oo.iv.** of the appearance of the dark matter, there are two decoupled groups of four families, carrying family charges  $(\vec{N}_L, \vec{\tau}^1)$  and  $(\vec{N}_R, \vec{\tau}^2)$ , Eqs. (13.25, 13.26), both groups carry also the family members charges  $(Q, Q', Y')$ , Eq. (13.28).

The *standard model* groups of spins and charges are the subgroups of the  $SO(13, 1)$  group with the generator of the infinitesimal transformations expressible with  $S^{ab} (= \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a))$ ,  $\{S^{ab}, S^{cd}\}_- = -i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac})$  for the spin

$$\vec{N}_{\pm} (= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \quad (13.22)$$

for the weak charge,  $SU(2)_I$ , and the second  $SU(2)_{II}$ , these two groups are the invariant subgroups of  $SO(4)$ ,

$$\begin{aligned}\tilde{\tau}^1 &:= \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}), \\ \tilde{\tau}^2 &:= \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}),\end{aligned}\quad (13.23)$$

for the colour charge  $SU(3)$  and for the "fermion charge"  $U(1)_{II}$ , these two groups are subgroups of  $SO(6)$ ,

$$\begin{aligned}\tilde{\tau}^3 &:= \frac{1}{2}\{S^{9\ 12} - S^{10\ 11}, S^{9\ 11} + S^{10\ 12}, S^{9\ 10} - S^{11\ 12}, \\ &\quad S^{9\ 14} - S^{10\ 13}, S^{9\ 13} + S^{10\ 14}, S^{11\ 14} - S^{12\ 13}, \\ &\quad S^{11\ 13} + S^{12\ 14}, \frac{1}{\sqrt{3}}(S^{9\ 10} + S^{11\ 12} - 2S^{13\ 14})\}, \\ \tau^4 &:= -\frac{1}{3}(S^{9\ 10} + S^{11\ 12} + S^{13\ 14}),\end{aligned}\quad (13.24)$$

$\tau^4$  is the "fermion charge", while the hyper charge  $Y = \tau^{23} + \tau^4$ .

The generators of the family quantum numbers are the superposition of the generators  $\tilde{S}^{ab}$  ( $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{Y}^a, \tilde{Y}^b\}_-$ ,  $\{\tilde{S}^{ab}, \tilde{S}^{cd}\}_- = -i(\eta^{ad}\tilde{S}^{bc} + \eta^{bc}\tilde{S}^{ad} - \eta^{ac}\tilde{S}^{bd} - \eta^{bd}\tilde{S}^{ac})$ ,  $\{\tilde{S}^{ab}, S^{cd}\}_- = 0$ ). One correspondingly finds the generators of the subgroups of  $\widetilde{SO}(7, 1)$ ,

$$\tilde{N}_{L,R} := \frac{1}{2}(\tilde{S}^{23} \pm i\tilde{S}^{01}, \tilde{S}^{31} \pm i\tilde{S}^{02}, \tilde{S}^{12} \pm i\tilde{S}^{03}), \quad (13.25)$$

which determine representations of the two  $\widetilde{SU}(2)$  invariant subgroups of  $\widetilde{SO}(3, 1)$ , while

$$\begin{aligned}\tilde{\tau}^1 &:= \frac{1}{2}(\tilde{S}^{58} - \tilde{S}^{67}, \tilde{S}^{57} + \tilde{S}^{68}, \tilde{S}^{56} - \tilde{S}^{78}), \\ \tilde{\tau}^2 &:= \frac{1}{2}(\tilde{S}^{58} + \tilde{S}^{67}, \tilde{S}^{57} - \tilde{S}^{68}, \tilde{S}^{56} + \tilde{S}^{78}),\end{aligned}\quad (13.26)$$

determine representations of  $\widetilde{SU}(2)_I \times \widetilde{SU}(2)_{II}$  of  $\widetilde{SO}(4)$ . Both,  $\widetilde{SO}(3, 1)$  and  $\widetilde{SO}(4)$ , are the subgroups of  $\widetilde{SO}(7, 1)$ . One finds for the infinitesimal generator  $\tilde{\tau}^4$  of  $\widetilde{U}(1)$  originating in  $\widetilde{SO}(6)$  the expression

$$\tilde{\tau}^4 := -\frac{1}{3}(\tilde{S}^{9\ 10} + \tilde{S}^{11\ 12} + \tilde{S}^{13\ 14}). \quad (13.27)$$

The operators for the charges  $Y$  and  $Q$  of the *standard model*, together with  $Q'$  and  $Y'$ , and the corresponding operators of the family charges  $\tilde{Y}$ ,  $\tilde{Y}'$ ,  $\tilde{Q}$ ,  $\tilde{Q}'$  are defined as follows:

$$\begin{aligned}Y &:= \tau^4 + \tau^{23}, \quad Y' := -\tau^4 \tan^2 \vartheta_2 + \tau^{23}, \quad Q := \tau^{13} + Y, \quad Q' := -Y \tan^2 \vartheta_1 + \tau^{13}, \\ \tilde{Y} &:= \tilde{\tau}^4 + \tilde{\tau}^{23}, \quad \tilde{Y}' := -\tilde{\tau}^4 \tan^2 \vartheta_2 + \tilde{\tau}^{23}, \quad \tilde{Q} := \tilde{Y} + \tilde{\tau}^{13}, \quad \tilde{Q}' := -\tilde{Y} \tan^2 \vartheta_1 + \tilde{\tau}^{13}.\end{aligned}\quad (13.28)$$

The families split into two groups of four families, each manifesting the

$$\widetilde{\text{SU}}(2) \times \widetilde{\text{SU}}(2) \times \text{U}(1),$$

with the generators of the infinitesimal transformations  $(\vec{N}_L, \vec{\tau}^1, Q, Q', Y')$  and  $(\vec{N}_R, \vec{\tau}^2, Q, Q', Y')$ , respectively. The generators of  $\text{U}(1)$  group  $(Q, Q', Y')$ , Eq. 13.28, distinguish among family members and are the same for both groups of four families, presented on Table 13.4, taken from Ref. [4].

The vector gauge fields of the charges  $\vec{\tau}^1, \vec{\tau}^2, \vec{\tau}^3$  and  $\tau^4$  follow from the requirement  $\sum_{Ai} \tau^{Ai} A_m^{Ai} = \sum_{s,t} \frac{1}{2} S^{st} \omega_{stm}$  and the requirement that  $\tau^{Ai} = \sum_{a,b} c^{Ai}_{ab} S^{ab}$ , Eq. (13.2), fulfilling the commutation relations  $\{\tau^{Ai}, \tau^{Bj}\}_- = i\delta^{AB} f^{Aijk} \tau^{Ak}$ , Eq. (13.3). Correspondingly we find  $A_m^{Ai} = \sum_{s,t} c^{Ai}_{st} \omega^{st}_m$ , Eq. (13.4), with  $(s, t)$  either in  $(5, 6, 7, 8)$  or in  $(9, \dots, 14)$ .

The explicit expressions for these vector gauge fields in terms of  $\omega_{stm}$  are as follows

$$\begin{aligned} \vec{A}_m^1 &= (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m}), \\ \vec{A}_m^2 &= (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m}), \\ A_m^Q &= \omega_{56m} - (\omega_{910m} + \omega_{1112m} + \omega_{1314m}), \\ A_m^Y &= (\omega_{56m} + \omega_{78m}) - (\omega_{910m} + \omega_{1112m} + \omega_{1314m}), \\ \vec{A}_m^3 &= (\omega_{912m} - \omega_{1011m}, \omega_{911m} + \omega_{1012m}, \omega_{910m} - \omega_{1112m}, \\ &\quad \omega_{914m} - \omega_{1013m}, \omega_{913m} + \omega_{1014m}, \omega_{1114m} - \omega_{1213m}, \\ &\quad \omega_{1113m} + \omega_{1214m}, \frac{1}{\sqrt{3}}(\omega_{910m} + \omega_{1112m} - 2\omega_{1314m})), \\ A_m^4 &= (\omega_{910m} + \omega_{1112m} + \omega_{1314m}). \end{aligned} \quad (13.29)$$

All  $\omega_{stm}$  vector gauge fields are real fields. Here the fields contain the coupling constants which are not necessarily the same for all of them. In the case that the coupling constants would be the same, than the angles  $\theta_2^2$  and  $\theta_1^2$  would be equal to one, which is not the case (at least  $\sin_1^2 \approx 0.22$ .)

One obtains in a similar way the scalar gauge fields, which determine mass matrices of family members. They carry the space index  $s = (7, 8)$ .

$$\begin{aligned} \vec{\tilde{A}}_s^1 &= (\tilde{\omega}_{58s} - \tilde{\omega}_{67s}, \tilde{\omega}_{57s} + \tilde{\omega}_{68s}, \tilde{\omega}_{56s} - \tilde{\omega}_{78s}), \\ \vec{\tilde{A}}_s^2 &= (\tilde{\omega}_{58s} + \tilde{\omega}_{67s}, \tilde{\omega}_{57s} - \tilde{\omega}_{68s}, \tilde{\omega}_{56s} + \tilde{\omega}_{78s}), \\ \vec{\tilde{A}}_{Ls}^N &= (\tilde{\omega}_{23s} + i\tilde{\omega}_{01s}, \tilde{\omega}_{31s} + i\tilde{\omega}_{02s}, \tilde{\omega}_{12s} + \tilde{\omega}_{03s}), \\ \vec{\tilde{A}}_{Rs}^N &= (\tilde{\omega}_{23s} - i\tilde{\omega}_{01s}, \tilde{\omega}_{31s} - i\tilde{\omega}_{02s}, \tilde{\omega}_{12s} - i\tilde{\omega}_{03s}), \\ A_s^Q &= \omega_{56s} - (\omega_{910s} + \omega_{1112s} + \omega_{1314s}), \\ A_s^Y &= (\omega_{56s} + \omega_{78s}) - (\omega_{910s} + \omega_{1112s} + \omega_{1314s}), \\ A_s^4 &= -(\omega_{910s} + \omega_{1112s} + \omega_{1314s}). \end{aligned} \quad (13.30)$$

All  $\omega_{sts'}$ ,  $\tilde{\omega}_{sts'}$ ,  $(s, t, s') = (5, \cdot, 14)$ ,  $\tilde{\omega}_{i,j,s'}$  and  $i\tilde{\omega}_{0,s'}$ ,  $(i, j) = (1, 2, 3)$  scalar gauge fields are real fields.



		$03$	$12$	$56$	$78$	$910$	$1112$	$1314$			$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\tau^{13}$	$\tau^{23}$	$N^1_L$	$N^3_R$	
I	$u_{R1}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R1}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$
I	$u_{R2}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R2}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$
I	$u_{R3}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R3}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$
I	$u_{R4}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R4}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$
II	$u_{R5}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R5}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$0$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$	
II	$u_{R6}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R6}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$0$	$-\frac{1}{2}$	$0$	$-\frac{1}{2}$	
II	$u_{R7}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R7}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$0$	$\frac{1}{2}$	$0$	$-\frac{1}{2}$	
II	$u_{R8}^{c1}$	$03$	$12$	$56$	$78$	$910$	$1112$	$1314$	$\nu_{R8}$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$(+)$	$0$	$\frac{1}{2}$	$0$	$-\frac{1}{2}$	

**Table 13.4.** Eight families of the right handed  $u_R^1$  (13.3) quark with spin  $\frac{1}{2}$ , the colour charge ( $\tau^{33} = 1/2$ ,  $\tau^{38} = 1/(2\sqrt{3})$ ) (the definition of the operators is presented in Eqs. (13.23,13.24), a few examples of these operators on the states can be found in Subsect. 13.5. The definition of the operators, expressible with  $\hat{S}^{ab}$  is:  $\vec{N}_{L,R} = \frac{1}{2}(\hat{S}^{23} \pm i\hat{S}^{01}, \hat{S}^{31} \pm i\hat{S}^{02}, \hat{S}^{12} \pm i\hat{S}^{03})$ ,  $\vec{\tau}^1 = \frac{1}{2}(\hat{S}^{58} - \hat{S}^{67}, \hat{S}^{57} + \hat{S}^{68} - \hat{S}^{78})$ ,  $\vec{\tau}^2 = \frac{1}{2}(\hat{S}^{58} + \hat{S}^{67} - \hat{S}^{68}, \hat{S}^{56} + \hat{S}^{78})$  and  $\vec{\tau}^4 = -\frac{1}{3}(\hat{S}^{9\ 10} + \hat{S}^{11\ 12} + \hat{S}^{13\ 14})$ , and of the colourless right handed neutrino  $\nu_R$  of spin  $\frac{1}{2}$  are presented in the left and in the right column, respectively. They belong to two groups of four families, one (II) is a doublet with respect to  $(\vec{N}_L$  and  $\vec{\tau}^1)$  and a singlet with respect to  $(\vec{N}_R$  and  $\vec{\tau}^2)$ , the other (I) is a singlet with respect to  $(\vec{N}_L$  and  $\vec{\tau}^1)$  and a doublet with respect to  $(\vec{N}_R$  and  $\vec{\tau}^2)$ . All the families follow from the starting one by the application of the operators  $(\vec{N}_{\pm R,L}^{\tau^{(2,1)\pm}}, \text{Eq. (13.45)})$ . The generators  $(N_{\pm R,L}^{\tau^{(2,1)\pm}}, \text{Eq. (13.45)})$  transform  $u_{R,i}, i = (1, \dots, 8)$ , to all the members of the same colour of the family. The same generators transform equivalently the right handed neutrino  $\nu_{R,i}, i = (1, \dots, 8)$ , to all the colourless members of the  $i^{\text{th}}$  family.

The theory predicts, due to commutation relations of generators of the infinitesimal transformations of the family groups,  $\widetilde{\text{SU}}(2)_I \times \widetilde{\text{SU}}(2)_I$  and  $\widetilde{\text{SU}}(2)_{II} \times \widetilde{\text{SU}}(2)_{II}$ , the first one with the generators  $\vec{N}_L$  and  $\vec{\tau}^1$ , and the second one with the generators  $\vec{N}_R$  and  $\vec{\tau}^2$ , Eqs. (13.25,13.26), two groups of four families.

The theory offers (so far) several predictions:

- i. several new scalars, those coupled to the lower group of four families — two triplets and three singlets, the superposition of  $(\vec{A}_s^1, \vec{A}_{Ls}^N$  and  $A_s^Q, A_s^Y, A_s^4$ , Eq. (13.30) — some of them to be observed at the LHC ([1,5,4],
- ii. the fourth family to the observed three to be observed at the LHC ([1,5,4] and the references therein),
- iii. new nuclear force among nucleons built from the quarks of the upper four families.

The theory offers also the explanation for several phenomena, like it is the “miraculous” cancellation of the *standard model* triangle anomalies [3].

The breaks of the symmetries, manifesting in Eqs. (13.22, 13.25, 13.23, 13.26, 13.24, 13.27), are in the *spin-charge-family* theory caused by the scalar condensate of the two right handed neutrinos belonging to one group of four families, Table 13.5, and by the nonzero vacuum expectation values of the scalar fields carrying the space index (7, 8) (Refs. [4,1] and the references therein). The space breaks first to  $\text{SO}(7, 1) \times \text{SU}(3) \times \text{U}(1)_{II}$  and then further to  $\text{SO}(3, 1) \times \text{SU}(2)_I \times \text{U}(1)_I \times \text{SU}(3) \times \text{U}(1)_{II}$ , what explains the connections between the weak and the hyper charges and the handedness of spinors [3].

state	$S^{03}$	$S^{12}$	$\tau^{13}$	$\tau^{23}$	$\tau^4$	$Y$	$Q$	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{\tau}^4$	$\tilde{Y}$	$\tilde{Q}$	$\tilde{N}_L^3$	$\tilde{N}_R^3$
$( \nu_{1R}^{\text{VIII}}\rangle >_1   \nu_{2R}^{\text{VIII}}\rangle >_2)$	0	0	0	1	-1	0	0	0	1	-1	0	0	0	1
$( \nu_{1R}^{\text{VIII}}\rangle >_1   e_{2R}^{\text{VIII}}\rangle >_2)$	0	0	0	0	-1	-1	-1	0	1	-1	0	0	0	1
$( e_{1R}^{\text{VIII}}\rangle >_1   e_{2R}^{\text{VIII}}\rangle >_2)$	0	0	0	-1	-1	-2	-2	0	1	-1	0	0	0	1

**Table 13.5.** This table is taken from [5]. The condensate of the two right handed neutrinos  $\nu_R$ , with the VIII<sup>th</sup> family quantum numbers, coupled to spin zero and belonging to a triplet with respect to the generators  $\tau^{2i}$ , is presented together with its two partners. The right handed neutrino has  $Q = 0 = Y$ . The triplet carries  $\tau^4 = -1$ ,  $\tilde{\tau}^{23} = 1$ ,  $\tilde{\tau}^4 = -1$ ,  $\tilde{N}_R^3 = 1$ ,  $\tilde{N}_L^3 = 0$ ,  $\tilde{Y} = 0$ ,  $\tilde{Q} = 0$ . The  $\tilde{\tau}^{31} = 0$ . The family quantum numbers are presented in Table 13.4.

The stable of the upper four families is the candidate for the dark matter, the fourth of the lower four families is predicted to be measured at the LHC.

### 13.5 APPENDIX: Short presentation of spinor technique [1,4,10,12,13]

This appendix is a short review (taken from [4]) of the technique [10,40,12,13], initiated and developed in Ref. [10] by one of the authors (N.S.M.B.), while proposing the *spin-charge-family* theory [2,4,5,7,8,1,14,15,9–11,16–22]. All the internal degrees

of freedom of spinors, with family quantum numbers included, are describable with two kinds of the Clifford algebra objects, besides with  $\gamma^a$ 's, used in this theory to describe spins and all the charges of fermions, also with  $\tilde{\gamma}^a$ 's, used in this theory to describe families of spinors:

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}, \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}, \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0. \quad (13.31)$$

We assume the "Hermiticity" property for  $\gamma^a$ 's (and  $\tilde{\gamma}^a$ 's)  $\gamma^{a\dagger} = \eta^{aa}\gamma^a$  (and  $\tilde{\gamma}^{a\dagger} = \eta^{aa}\tilde{\gamma}^a$ ), in order that  $\gamma^a$  (and  $\tilde{\gamma}^a$ ) are compatible with (13.31) and formally unitary, i.e.  $\gamma^{a\dagger}\gamma^a = I$  (and  $\tilde{\gamma}^{a\dagger}\tilde{\gamma}^a = I$ ). One correspondingly finds that  $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$  (and  $(\tilde{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\tilde{S}^{ab}$ ).

Spinor states are represented as products of nilpotents and projectors, formed as odd and even objects of  $\gamma^a$ 's, respectively, chosen to be the eigenstates of a Cartan subalgebra of the Lorentz groups defined by  $\gamma^a$ 's

$$(\bar{k}) := \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \quad [\bar{k}] := \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \quad (13.32)$$

where  $k^2 = \eta^{aa}\eta^{bb}$ . We further have [4]

$$\begin{aligned} \gamma^a (\bar{k}) &:= \frac{1}{2}(\gamma^a\gamma^a + \frac{\eta^{aa}}{ik}\gamma^a\gamma^b) = \eta^{aa} [\bar{k}], \\ \gamma^a [\bar{k}] &:= \frac{1}{2}(\gamma^a + \frac{i}{k}\gamma^a\gamma^a\gamma^b) = (-\bar{k}), \\ \tilde{\gamma}^a (\bar{k}) &:= -i\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\gamma^a = -i\eta^{aa} [\bar{k}], \\ \tilde{\gamma}^a [\bar{k}] &:= i\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\gamma^a = -i (\bar{k}), \end{aligned} \quad (13.33)$$

where we assume that all the operators apply on the vacuum state  $|\psi_0\rangle$ . We define a vacuum state  $|\psi_0\rangle$  so that one finds  $\langle (\bar{k}) (\bar{k}) \rangle = 1$ ,  $\langle [\bar{k}] [\bar{k}] \rangle = 1$ .

We recognize that  $\gamma^a$  transform  $(\bar{k})$  into  $[\bar{k}]$ , never to  $[\bar{k}]$ , while  $\tilde{\gamma}^a$  transform  $(\bar{k})$  into  $[\bar{k}]$ , never to  $[\bar{k}]$

$$\begin{aligned} \gamma^a (\bar{k}) &= \eta^{aa} [\bar{k}], \quad \gamma^b (\bar{k}) = -ik [\bar{k}], \quad \gamma^a [\bar{k}] = (-\bar{k}), \quad \gamma^b [\bar{k}] = -ik\eta^{aa} (-\bar{k}), \\ \tilde{\gamma}^a (\bar{k}) &= -i\eta^{aa} [\bar{k}], \quad \tilde{\gamma}^b (\bar{k}) = -k [\bar{k}], \quad \tilde{\gamma}^a [\bar{k}] = i (\bar{k}), \quad \tilde{\gamma}^b [\bar{k}] = -k\eta^{aa} (\bar{k}). \end{aligned} \quad (13.34)$$

The Clifford algebra objects  $S^{ab}$  and  $\tilde{S}^{ab}$  close the algebra of the Lorentz group

$$\begin{aligned} S^{ab} &:= (i/4)(\gamma^a\gamma^b - \gamma^b\gamma^a), \\ \tilde{S}^{ab} &:= (i/4)(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a), \end{aligned} \quad (13.35)$$

$$\{S^{ab}, \tilde{S}^{cd}\}_- = 0, \{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac}), \{\tilde{S}^{ab}, \tilde{S}^{cd}\}_- = i(\eta^{ad}\tilde{S}^{bc} + \eta^{bc}\tilde{S}^{ad} - \eta^{ac}\tilde{S}^{bd} - \eta^{bd}\tilde{S}^{ac}).$$

One can easily check that the nilpotent  $\overset{ab}{(k)}$  and the projector  $\overset{ab}{[k]}$  are "eigenstates" of  $S^{ab}$  and  $\tilde{S}^{ab}$

$$\begin{aligned} S^{ab} \overset{ab}{(k)} &= \frac{1}{2} k \overset{ab}{(k)}, & S^{ab} \overset{ab}{[k]} &= \frac{1}{2} k \overset{ab}{[k]}, \\ \tilde{S}^{ab} \overset{ab}{(k)} &= \frac{1}{2} k \overset{ab}{(k)}, & \tilde{S}^{ab} \overset{ab}{[k]} &= -\frac{1}{2} k \overset{ab}{[k]}, \end{aligned} \quad (13.36)$$

where the vacuum state  $|\psi_0\rangle$  is meant to stay on the right hand sides of projectors and nilpotents. This means that multiplication of nilpotents  $\overset{ab}{(k)}$  and projectors  $\overset{ab}{[k]}$  by  $S^{ab}$  get the same objects back multiplied by the constant  $\frac{1}{2}k$ , while  $\tilde{S}^{ab}$  multiply  $\overset{ab}{(k)}$  by  $\frac{k}{2}$  and  $\overset{ab}{[k]}$  by  $(-\frac{k}{2})$  (rather than by  $\frac{k}{2}$ ). This also means that when  $\overset{ab}{(k)}$  and  $\overset{ab}{[k]}$  act from the left hand side on a vacuum state  $|\psi_0\rangle$  the obtained states are the eigenvectors of  $S^{ab}$ .

The technique can be used to construct a spinor basis for any dimension  $d$  and any signature in an easy and transparent way. Equipped with nilpotents and projectors of Eq. (13.32), the technique offers an elegant way to see all the quantum numbers of states with respect to the two Lorentz groups, as well as transformation properties of the states under the application of any Clifford algebra object.

Recognizing from Eq.(13.35) that the two Clifford algebra objects ( $S^{ab}, S^{cd}$ ) with all indexes different commute (and equivalently for  $(\tilde{S}^{ab}, \tilde{S}^{cd})$ ), we select the Cartan subalgebra of the algebra of the two groups, which form equivalent representations with respect to one another

$$\begin{aligned} S^{03}, S^{12}, S^{56}, \dots, S^{d-1 \ d}, & \quad \text{if } d = 2n \geq 4, \\ \tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1 \ d}, & \quad \text{if } d = 2n \geq 4. \end{aligned} \quad (13.37)$$

The choice of the Cartan subalgebra in  $d < 4$  is straightforward. It is useful to define one of the Casimirs of the Lorentz group — the handedness  $\Gamma$  ( $\{\Gamma, S^{ab}\}_- = 0$ ) (as well as  $\tilde{\Gamma}$ ) in any  $d = 2n$

$$\begin{aligned} \Gamma^{(d)} &:= (i)^{d/2} \prod_a (\sqrt{\eta^{aa}} \gamma^a), & \text{if } d = 2n, \\ \tilde{\Gamma}^{(d)} &:= (i)^{(d-1)/2} \prod_a (\sqrt{\eta^{aa}} \tilde{\gamma}^a), & \text{if } d = 2n. \end{aligned} \quad (13.38)$$

We understand the product of  $\gamma^a$ 's in the ascending order with respect to the index  $a$ :  $\gamma^0 \gamma^1 \dots \gamma^d$ . It follows from the Hermiticity properties of  $\gamma^a$  for any choice of the signature  $\eta^{aa}$  that  $\Gamma^\dagger = \Gamma$ ,  $\Gamma^2 = I$ . (Equivalent relations are valid for  $\tilde{\Gamma}$ .) We also find that for  $d$  even the handedness anticommutes with the Clifford algebra objects  $\gamma^a$  ( $\{\gamma^a, \Gamma\}_+ = 0$ ) (while for  $d$  odd it commutes with  $\gamma^a$  ( $\{\gamma^a, \Gamma\}_- = 0$ )).

Taking into account the above equations it is easy to find a Weyl spinor irreducible representation for  $d$ -dimensional space, with  $d$  even or odd <sup>6</sup>. For  $d$  even we simply make a starting state as a product of  $d/2$ , let us say, only

<sup>6</sup> For  $d$  odd the basic states are products of  $(d-1)/2$  nilpotents and a factor  $(1 \pm \Gamma)$ .

nilpotents  $(k)^{ab}$ , one for each  $S^{ab}$  of the Cartan subalgebra elements (Eqs.(13.37, 13.35)), applying it on an (unimportant) vacuum state. Then the generators  $S^{ab}$ , which do not belong to the Cartan subalgebra, being applied on the starting state from the left hand side, generate all the members of one Weyl spinor.

$$\begin{aligned}
 & (k_{0d})^{0d} (k_{12})^{12} (k_{35})^{35} \cdots (k_{d-1 \ d-2})^{d-1 \ d-2} |\psi_0\rangle > \\
 & [-k_{0d}]^{0d} [-k_{12}]^{12} (k_{35})^{35} \cdots (k_{d-1 \ d-2})^{d-1 \ d-2} |\psi_0\rangle > \\
 & [-k_{0d}]^{0d} (k_{12})^{12} [-k_{35}]^{35} \cdots (k_{d-1 \ d-2})^{d-1 \ d-2} |\psi_0\rangle > \\
 & \vdots \\
 & [-k_{0d}]^{0d} (k_{12})^{12} (k_{35})^{35} \cdots [-k_{d-1 \ d-2}]^{d-1 \ d-2} |\psi_0\rangle > \\
 & (k_{0d})^{0d} [-k_{12}]^{12} [-k_{35}]^{35} \cdots (k_{d-1 \ d-2})^{d-1 \ d-2} |\psi_0\rangle > \\
 & \vdots
 \end{aligned} \tag{13.39}$$

All the states have the same handedness  $\Gamma$ , since  $\{\Gamma, S^{ab}\}_- = 0$ . States, belonging to one multiplet with respect to the group  $SO(q, d - q)$ , that is to one irreducible representation of spinors (one Weyl spinor), can have any phase. We could make a choice of the simplest one, taking all phases equal to one. (In order to have the usual transformation properties for spinors under the rotation of spin and under  $\mathcal{C}_N \mathcal{P}_N$ , some of the states must be multiplied by  $(-1)$ .)

The above representation demonstrates that for  $d$  even all the states of one irreducible Weyl representation of a definite handedness follow from a starting state, which is, for example, a product of nilpotents  $(k_{ab})^{ab}$ , by transforming all possible pairs of  $(k_{ab})^{ab} (k_{mn})^{mn}$  into  $[-k_{ab}]^{ab} [-k_{mn}]^{mn}$ . There are  $S^{am}, S^{an}, S^{bm}, S^{bn}$ , which do this. The procedure gives  $2^{(d/2-1)}$  states. A Clifford algebra object  $\gamma^a$  being applied from the left hand side, transforms a Weyl spinor of one handedness into a Weyl spinor of the opposite handedness.

We shall speak about left handedness when  $\Gamma = -1$  and about right handedness when  $\Gamma = 1$ .

While  $S^{ab}$ , which do not belong to the Cartan subalgebra (Eq. (13.37)), generate all the states of one representation,  $\tilde{S}^{ab}$ , which do not belong to the Cartan subalgebra (Eq. (13.37)), generate the states of  $2^{d/2-1}$  equivalent representations.

Making a choice of the Cartan subalgebra set (Eq. (13.37)) of the algebra  $S^{ab}$  and  $\tilde{S}^{ab}$ :  $(S^{03}, S^{12}, S^{56}, S^{78}, S^{9 \ 10}, S^{11 \ 12}, S^{13 \ 14})$ ,  $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \tilde{S}^{78}, \tilde{S}^{9 \ 10}, \tilde{S}^{11 \ 12}, \tilde{S}^{13 \ 14})$ , a left handed ( $\Gamma^{(13,1)} = -1$ ) eigenstate of all the members of the Cartan subalgebra, representing a weak chargeless  $u_R$ -quark with spin up, hyper charge  $(2/3)$  and colour  $(1/2, 1/(2\sqrt{3}))$ , for example, can be written as

$$\begin{aligned}
 & (+i)^{03} (+)^{12} (+)^{56} (+)^{78} (+)^{9 \ 10} (+)^{11 \ 12} (+)^{13 \ 14} |\psi_0\rangle = \\
 & \frac{1}{2^7} (\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6)(\gamma^7 + i\gamma^8) \\
 & (\gamma^9 + i\gamma^{10})(\gamma^{11} - i\gamma^{12})(\gamma^{13} - i\gamma^{14}) |\psi_0\rangle.
 \end{aligned} \tag{13.40}$$

This state is an eigenstate of all  $S^{ab}$  and  $\tilde{S}^{ab}$  which are members of the Cartan subalgebra (Eq. (13.37)).

The operators  $\tilde{S}^{ab}$ , which do not belong to the Cartan subalgebra (Eq. (13.37)), generate families from the starting  $u_R$  quark, transforming the  $u_R$  quark from Eq. (13.40) to the  $u_R$  of another family, keeping all of the properties with respect to  $S^{ab}$  unchanged. In particular,  $\tilde{S}^{01}$  applied on a right handed  $u_R$ -quark from Eq. (13.40) generates a state which is again a right handed  $u_R$ -quark, weak chargeless, with spin up, hyper charge (2/3) and the colour charge (1/2, 1/(2√3))

$$\tilde{S}^{01} \begin{smallmatrix} 03 & 12 & 56 & 78 & 91011121314 \\ (+i)(+) & | & (+)(+) & || & (+)(-)(-) \end{smallmatrix} = -\frac{i}{2} \begin{smallmatrix} 03 & 12 & 56 & 78 & 91011121314 \\ [+i][+] & | & (+)(+) & || & (+)(-)(-) \end{smallmatrix} . \quad (13.41)$$

One can find both states in Table 13.4, the first  $u_R$  as  $u_{R8}$  in the eighth line of this table, the second one as  $u_{R7}$  in the seventh line of this table.

Below some useful relations follow. From Eq.(13.34) one has

$$\begin{aligned} S^{ac} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \eta^{cc} \begin{smallmatrix} ab & cd \\ [-k][-k] \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix} &= \frac{i}{2} \eta^{aa} \eta^{cc} \begin{smallmatrix} ab & cd \\ [k][k] \end{smallmatrix}, \\ S^{ac} \begin{smallmatrix} ab & cd \\ [k][k] \end{smallmatrix} &= \frac{i}{2} \begin{smallmatrix} ab & cd \\ (-k)(-k) \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ [k][k] \end{smallmatrix} &= -\frac{i}{2} \begin{smallmatrix} ab & cd \\ (k)(k) \end{smallmatrix}, \\ S^{ac} \begin{smallmatrix} ab & cd \\ (k)[k] \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \begin{smallmatrix} ab & cd \\ [-k](-k) \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ (k)[k] \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \begin{smallmatrix} ab & cd \\ [k](k) \end{smallmatrix}, \\ S^{ac} \begin{smallmatrix} ab & cd \\ [k](k) \end{smallmatrix} &= \frac{i}{2} \eta^{cc} \begin{smallmatrix} ab & cd \\ (-k)[-k] \end{smallmatrix}, & \tilde{S}^{ac} \begin{smallmatrix} ab & cd \\ [k](k) \end{smallmatrix} &= \frac{i}{2} \eta^{cc} \begin{smallmatrix} ab & cd \\ (k)[k] \end{smallmatrix} . \end{aligned} \quad (13.42)$$

We conclude from the above equation that  $\tilde{S}^{ab}$  generate the equivalent representations with respect to  $S^{ab}$  and opposite.

We recognize in Eq. (13.43) the demonstration of the nilpotent and the projector character of the Clifford algebra objects  $\begin{smallmatrix} ab \\ (k) \end{smallmatrix}$  and  $\begin{smallmatrix} ab \\ [k] \end{smallmatrix}$ , respectively.

$$\begin{aligned} \begin{smallmatrix} ab & ab \\ (k)(k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ (k)(-k) \end{smallmatrix} &= \eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)(k) \end{smallmatrix} &= \eta^{aa} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)(-k) \end{smallmatrix} &= 0, \\ \begin{smallmatrix} ab & ab \\ [k][k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ [k][-k] \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k][k] \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k][-k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, \\ \begin{smallmatrix} ab & ab \\ (k)[k] \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [k](k) \end{smallmatrix} &= \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)[k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ (-k) \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ (-k)[-k] \end{smallmatrix} &= 0, \\ \begin{smallmatrix} ab & ab \\ (k)[-k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, & \begin{smallmatrix} ab & ab \\ [k](-k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k](k) \end{smallmatrix} &= 0, & \begin{smallmatrix} ab & ab \\ [-k](-k) \end{smallmatrix} &= \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} . \end{aligned} \quad (13.43)$$

Defining

$$\begin{aligned} \begin{smallmatrix} ab \\ (\pm i) \end{smallmatrix} &= \frac{1}{2} (\tilde{\gamma}^a \mp \tilde{\gamma}^b), & \begin{smallmatrix} ab \\ (\pm 1) \end{smallmatrix} &= \frac{1}{2} (\tilde{\gamma}^a \pm i \tilde{\gamma}^b), \\ \begin{smallmatrix} ab \\ [\pm i] \end{smallmatrix} &= \frac{1}{2} (1 \pm \tilde{\gamma}^a \tilde{\gamma}^b), & \begin{smallmatrix} ab \\ [\pm 1] \end{smallmatrix} &= \frac{1}{2} (1 \pm i \tilde{\gamma}^a \tilde{\gamma}^b). \end{aligned}$$

one recognizes that

$$\begin{aligned} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = 0, \quad \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = -i\eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = i \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} = 0. \end{aligned} \quad (13.44)$$

Below some more useful relations [14] are presented:

$$\begin{aligned} N_{+}^{\pm} &= N_{+}^1 \pm i N_{+}^2 = - \begin{smallmatrix} 03 & 12 \\ (\mp i) & (\pm) \end{smallmatrix}, \quad N_{-}^{\pm} = N_{-}^1 \pm i N_{-}^2 = \begin{smallmatrix} 03 & 12 \\ (\pm i) & (\pm) \end{smallmatrix}, \\ \tilde{N}_{+}^{\pm} &= - \begin{smallmatrix} 03 & 12 \\ (\mp i) & (\pm) \end{smallmatrix}, \quad \tilde{N}_{-}^{\pm} = \begin{smallmatrix} 03 & 12 \\ (\pm i) & (\pm) \end{smallmatrix}, \\ \tau^{1\pm} &= \begin{smallmatrix} 56 & 78 \\ (\mp) & (\pm) \end{smallmatrix} \begin{smallmatrix} 56 & 78 \\ (\mp) & (\mp) \end{smallmatrix}, \quad \tau^{2\mp} = \begin{smallmatrix} 56 & 78 \\ (\mp) & (\mp) \end{smallmatrix} \begin{smallmatrix} 56 & 78 \\ (\mp) & (\mp) \end{smallmatrix}, \\ \tilde{\tau}^{1\pm} &= \begin{smallmatrix} 56 & 78 \\ (\mp) & (\pm) \end{smallmatrix} \begin{smallmatrix} 56 & 78 \\ (\mp) & (\mp) \end{smallmatrix}, \quad \tilde{\tau}^{2\mp} = \begin{smallmatrix} 56 & 78 \\ (\mp) & (\mp) \end{smallmatrix} \begin{smallmatrix} 56 & 78 \\ (\mp) & (\mp) \end{smallmatrix}. \end{aligned} \quad (13.45)$$

In Table 13.4 [4] the eight families of the first member in Table 13.3 (member number 1) of the eight-plet of quarks and the 25<sup>th</sup> member in Table 13.3 of the eight-plet of leptons are presented as an example. The eight families of the right handed  $u_{1R}$  quark are presented in the left column of Table 13.4 [4]. In the right column of the same table the equivalent eight-plet of the right handed neutrinos  $\nu_{1R}$  are presented. All the other members of any of the eight families of quarks or leptons follow from any member of a particular family by the application of the operators  $N_{R,L}^{\pm}$  and  $\tau^{(2,1)\pm}$ , Eq. (13.45) on this particular member.

The eight-plets separate into two group of four families: One group contains doublets with respect to  $\vec{N}_R$  and  $\vec{\tau}^2$ , these families are singlets with respect to  $\vec{N}_L$  and  $\vec{\tau}^1$ . Another group of families contains doublets with respect to  $\vec{N}_L$  and  $\vec{\tau}^1$ , these families are singlets with respect to  $\vec{N}_R$  and  $\vec{\tau}^2$ .

The scalar fields which are the gauge scalars of  $\vec{N}_R$  and  $\vec{\tau}^2$  couple only to the four families which are doublets with respect to these two groups. The scalar fields which are the gauge scalars of  $\vec{N}_L$  and  $\vec{\tau}^1$  couple only to the four families which are doublets with respect to these last two groups.

After the electroweak phase transition, caused by the scalar fields with the space index (7, 8), the two groups of four families become massive. The lowest of the two groups of four families contains the observed three, while the fourth remains to be measured. The lowest of the upper four families is the candidate for the dark matter [1].

## References

1. N.S. Mankoč Borštnik, "Spin-charge-family theory is offering next step in understanding elementary particles and fields and correspondingly universe", Proceedings to the Conference on Cosmology, Gravitational Waves and Particles, IARD conferences, Ljubljana, 6-9 June 2016, The 10<sup>th</sup> Biennial Conference on Classical and Quantum Relativistic Dynamics of articles and Fields, J. Phys.: Conf. Ser. 845 (2017) 012017
2. D. Lukman, N.S. Mankoč Borštnik, Eur. Phys. J. C **77** (2017) 231.

3. N.S. Mankoč Borštnik, H.B.F. Nielsen, Fortschritte der Physik, Progress of Physics (2017) 1700046.
4. N.S. Mankoč Borštnik, J. of Mod. Phys. **6** (2015) 2244-2274.
5. N.S. Mankoč Borštnik, Phys. Rev. D **91** (2015) 6, 065004 ID: 0703013. doi:10.1103/[arxiv:1409.7791, arXiv:1502.06786v1].
6. N.S. Mankoč Borštnik, J. of Modern Phys. **4**, 823-847 (2013) [arxiv:1312.1542].
7. N.S. Mankoč Borštnik, "Spin-charge-family theory is explaining appearance of families of quarks and leptons, of Higgs and Yukawa couplings", in *Proceedings to the 16th Workshop "What comes beyond the standard models"*, Bled, 14-21 of July, 2013, eds. N.S. Mankoč Borštnik, H.B. Nielsen and D. Lukman (DMFA Založništvo, Ljubljana, December 2013) p.113 -142, [arxiv:1312.1542].
8. N.S. Mankoč Borštnik, "Do we have the explanation for the Higgs and Yukawa couplings of the *standard model*", <http://arxiv.org/abs/1212.3184v2>, (<http://arxiv.org/abs/1207.6233>), in *Proceedings to the 15th Workshop "What comes beyond the standard models"*, Bled, 9-19 of July, 2012, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2012, p.56-71, [arxiv:1302.4305].
9. N.S. Mankoč Borštnik, Phys. Lett. **B 292**, 25-29 (1992).
10. N.S. Mankoč Borštnik, J. Math. Phys. **34**, 3731-3745 (1993).
11. N.S. Mankoč Borštnik, Int. J. Theor. Phys. **40**, 315-338 (2001).
12. N.S. Mankoč Borštnik, H.B.F. Nielsen, J. of Math. Phys. **43**, 5782 (2002) [hep-th/0111257].
13. N.S. Mankoč Borštnik, H.B.F. Nielsen, J. of Math. Phys. **44** 4817 (2003) [hep-th/0303224].
14. A. Borštnik Bračič and N.S. Mankoč Borštnik, Phys. Rev. **D 74**, 073013 (2006) [hep-ph/0301029; hep-ph/9905357, p. 52-57; hep-ph/0512062, p.17-31; hep-ph/0401043, p. 31-57].
15. A. Borštnik Bračič, N.S. Mankoč Borštnik, "The approach Unifying Spins and Charges and Its Predictions", Proceedings to the Euroconference on Symmetries Beyond the Standard Model, Portorož, July 12 - 17, 2003, Ed. by N.S. Mankoč Borštnik, H.B. Nielsen, C. Froggatt, D. Lukman, DMFA Založništvo, Ljubljana December 2003, p. 31-57, [hep-ph/0401043, hep-ph/0401055].
16. N.S. Mankoč Borštnik, Modern Phys. Lett. **A 10**, 587 -595 (1995).
17. G. Bregar, M. Breskvar, D. Lukman and N.S. Mankoč Borštnik, New J. of Phys. **10**, 093002 (2008), [arXiv:0606159, arXiv:07082846, arXiv:0612250, p.25-50].
18. G. Bregar and N.S. Mankoč Borštnik, Phys. Rev. **D 80**, 083534 (2009) 1-16.
19. G. Bregar, N.S. Mankoč Borštnik, "Can we predict the fourth family masses for quarks and leptons?", Proceedings to the 16th Workshop "What comes beyond the standard models", Bled, 14-21 of July, 2013, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2013, p. 31-51, [arxiv:1403.4441].
20. G. Bregar, N.S. Mankoč Borštnik, "The new experimental data for the quarks mixing matrix are in better agreement with the *spin-charge-family* theory predictions", Proceedings to the 17th Workshop "What comes beyond the standard models", Bled, 20-28 of July, 2014, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2014, p.20-45 [arXiv:1502.06786v1] [arxiv:1412.5866]
21. N.S. Mankoč Borštnik, "Do we have the explanation for the Higgs and Yukawa couplings of the *standard model*", [arxiv:1212.3184, arxiv:1011.5765].
22. N.S. Mankoč Borštnik, "The *spin-charge-family* theory explains why the scalar Higgs carries the weak charge  $\pm \frac{1}{2}$  and the hyper charge  $\mp \frac{1}{2}$ ", Proceedings to the 17th Workshop "What Comes Beyond the Standard Models", Bled, July 20 - 28, 2014, p.163-182, [arxiv:1409.7791, arxiv:1212.4055].
23. A. Ali in discussions and in private communication at the Singapore Conference on New Physics at the Large Hadron Collider, 29 February - 4 March 2016.
24. M. Neubert, in discussions at the Singapore.



25. A. Lenz, *Advances in High Energy Physics* **2013**, ID 910275.
26. N.S. Mankoč Borštnik, H.B.F. Nielsen, "Do the present experiments exclude the existence of the fourth family members?", Proceedings to the 19<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 11-19 of July, 2016, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2016, p.128-146 [arXiv:1703.09699].
27. T. P. Cheng and Marc Sher, *Phys. Rev. D* **35** (1987), 3484.
28. F. Mahmoudi and O. Stål, *Phys. Rev. D* **81** (2010) 035016.
29. C. Anastasioua, R. Boughezalb and F. Petriello, *J. of High Energy Phys.* **04** (2009) 003.
30. C. Patrignani et al. (Particle Data Group), *Chin. Phys. C* **40** (2016) 100001.
31. A. Hoecker, "Physics at the LHC Run-2 and beyond", [arXiv:1611v1[hep-ex]].
32. N.S. Mankoč Borštnik, "The Spin-Charge-Family theory offers the explanation for all the assumptions of the Standard model, for the Dark matter, for the Matter-antimatter asymmetry, making several predictions", Proceedings to the Conference on New Physics at the Large Hadron Collider, 29 Februar - 4 March, **2016**, Nanyang Executive Centre, NTU, Singapore, to be published, [arXiv: 1607.01618v1].
33. N.S. Mankoč Borštnik, D. Lukman, "Vector and scalar gauge fields with respect to  $d = (3 + 1)$  in Kaluza-Klein theories and in the *spin-charge-family theory*", Proceedings to the 18<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 11-19 of July, 2015, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2015, p. 158-164 [arXiv:1604.00675].
34. D. Lukman, N.S. Mankoč Borštnik, H.B. Nielsen, *New J. Phys.* **13** (2011) 103027, hep-th/1001.4679v5.
35. D. Lukman and N.S. Mankoč Borštnik, *J. Phys. A: Math. Theor.* **45**, 465401 (2012), [arxiv:1205.1714, arxiv:1312.541, hep-ph/0412208 p.64-84].
36. R. Franceschini, G.F. Giudice, J.F. Kamenik, M. McCullough, A.Pomarol, R. Rattazzi, M. Redi, F. Riva, A. Strumia, R. Torre, ArXiv:1512.04933.
37. CMS Collaboration, CMS-PAS-EXO-12-045.
38. CMS Collaboration, *Phys. Rev. D* **92** (2015) 032004.
39. N.S. Mankoč Borštnik, D. Lukman, "Vector and scalar gauge fields with respect to  $d = (3 + 1)$  in Kaluza-Klein theories and in the *spin-charge-family theory*", Proceedings to the 18<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 11-19 of July, 2015, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2015, p. 158-164 [arXiv:1604.00675].
40. N.S. Mankoč Borštnik, H. B. Nielsen, *Phys. Rev. D* **62**, 044010, (2000)1-14, hep-th/9911032.
41. N.S. Mankoč Borštnik, H.B. Nielsen, *Jour. of High Energy Phys.* **04** (2014)165-174 [arXiv:1212.2362v3].
42. T. Troha, D. Lukman and N.S. Mankoč Borštnik, *Int. J. of Mod. Phys. A* **29** 1450124 (2014), [arXiv:1312.1541].
43. N.S. Mankoč Borštnik, H.B.F. Nielsen, "Fermionization in an arbitrary number of dimensions", Proceedings to the 18<sup>th</sup> Workshop "What comes beyond the standard models", Bled, 11-19 of July, 2015, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2015, p. 111-128 [arXiv:1602.03175].
44. T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys.* **96** (1921) 69, O. Klein, *Z.Phys.* **37** (1926) 895.
45. The authors of the works presented in *An introduction to Kaluza-Klein theories*, Ed. by H. C. Lee, World Scientific, Singapore 1983, T. Appelquist, A. Chodos, P.G.O. Freund (Eds.), *Modern Kaluza-Klein Theories*, Reading, USA: Addison Wesley, 1987.
46. D. Lukman, N. S. Mankoč Borštnik, H. B. Nielsen, *New J. Phys.* **13** (2011) 10302 [arXiv:1001.4679v4].



## 14 Fermionization, Number of Families

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**Abstract.** We investigate bosonization/fermionization for free massless fermions being equivalent to free massless bosons with the purpose of checking and correcting the old rule by Aratyn and one of us (H.B.F.N.) for the number of boson species relative to the number of fermion species which is required to have bosonization possible. An important application of such a counting of degrees of freedom relation would be to invoke restrictions on the number of families that could be possible under the assumption, that all the fermions in nature are the result of fermionizing a system of boson species. Since a theory of fundamental fermions can be accused for not being properly local because of having anticommutativity at space like distances rather than commutation as is more physically reasonable to require, it is in fact called for to have all fermions arising from fermionization of bosons. To make a realistic scenario with the fermions all coming from fermionizing some bosons we should still have at least some not fermionized bosons and we are driven towards that being a gravitational field, that is not fermionized. Essentially we reach the spin-charge-families theory by one of us (N.S.M.B.) with the detail that the number of fermion components and therefore of families get determined from what possibilities for fermionization will finally turn out to exist. The spin-charge-family theory has long been plagued by predicting 4 families rather than the phenomenologically more favoured 3. Unfortunately we do not yet understand well enough the unphysical negative norm square components in the system of bosons that can fermionize in higher dimensions because we have no working high dimensional case of fermionization. But suspecting they involve gauge fields with complicated unphysical state systems the corrections from such states could putatively improve the family number prediction.

**Povzetek.** Avtorja diskutirata bozonizacijo/fermionizacijo za proste brezmasne fermione, ki jih obravnavata kot ekvivalentne prostim brezmasnim bozonom. Namen je preveriti in popraviti staro pravilo Aratyna in H.B.F.N. za število vrst bozonov glede na število vrst fermionov kot pogoj za obstoj fermionizacije bozonov. Pomembna uporaba takega pravila bi bil pogoj na število možnih družin, če privzamemo, da so vsi fermioni v naravi rezultat fermionizacije vrst bozonov. Teoriji fermionov kot fundamentalnih delcev lahko očitamo, da nima pravilne lokalnosti, ker zahtevamo za fermione antikomutativnost, ne pa komutativnosti. Temu očitku bi se lahko izognili, če vsi fermioni izhajajo iz fermionizacije bozonov. Za realističen opis v modelu, v katerem fermione dobimo s fermionizacijo bozonov, mora vsaj nekaj bozonov ostati nefermioniziranih. Avtorja predlagata, da so ti nefermionizirani bozoni gravitacijska polja. Želita na ta način reproducirati teorijo Spinabojev-družin enega od avtorjev (S.N.M.B.), kjer bi število fermionskih komponent in

\* H.B.F. Nielsen presented the talk.

število družin določale možnosti za fermionizacijo. Teorija spina-nabojev-družin napove 4 družine, namesto doslej opaženih 3. Avtorja še ne razumeta dovolj dobro nefizikalnih komponent z negativnim kvadratom norm v sistemu bozonov, ki se fermionizira v višjih dimenzijah, ker jima fermionizacije v višjih dimenzijah še ni uspelo zares izpeljati. Domnevata, da bodo spoznanja o vlogi nefizikalnih stanj pomagala pri napovedi števila družin.

Keywords: Fermionization, Bosonization, Number of families

## 14.1 Introduction

One of the general requirements for quantum field theories is microcausality [1,2], the requirement of causality, which in its form as suggested from tensor product deduction says, that for two relative to each other spacelike placed events  $x_1$  and  $x_2$  in Minkowski space-time a couple of quantum field operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$  taken at these events will **commute**

$$\{\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)\}_- = 0 \text{ for spacelike } x_1 - x_2. \quad (14.1)$$

This is so for the  $\mathcal{O}_1$  and  $\mathcal{O}_2$  being boson fields, but if they are both fermion fields, one would have instead to let them **anticommute**

$$\{\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)\}_+ = 0 \text{ for spacelike } x_1 - x_2. \quad (14.2)$$

Such **anticommutation** is, however, from the tensor product way of arguing for the relation completely wrong. We could therefore claim that it is not truly allowed to have fermions in the usual way, because it leads to a “crazy” locality axiom. It is one of the purposes of the present proceedings article to suggest to investigate the consequences of such an attitude, that fermions as fundamental particles are not good, but that one should rather seek to obtain fermions, not as fundamental, but rather only by fermionization of some boson fields instead. But then it becomes very important what combinations - what systems - of fermions can be obtained from appropriate bosonic models. For the existence of quite nontrivial restrictions on the number of fermions, we can expect to be obtainable by fermionization from a system bosons, the theorem [3] by Aratyn and one of us (H.B.F.N.) is quite suggestive. In fact this theorem tells, that the ratio between the number of fermion spin components for all the species (families) counted together and the corresponding number of boson spin components counted together must be  $\frac{2^{d_{\text{spatial}}}}{2^{d_{\text{spatial}}-1}}$ . A priori this theorem seems to enforce that in say the experimental number of dimensions,  $d_{\text{spatial}} = 3$  and 1 time, the collective number of fermions components must be divisible by  $2^3 = 8$ . If we count the components as real fields a Weyl fermion has  $2 \cdot 2 = 4$  such real components, and thus the number of Weyl fermions must be divisible by  $8/4 = 2$ .

Let us immediately include the remark, that although we shall below mainly go for the presumably simplest case of non-interacting massless bosons - presumably Kalb-Ramond fields - being fermionized into also free massless fermions, that does NOT mean that we seriously suggest Nature to have no interactions. Rather

the hope is that gravitational degrees of freedom couple in a way specified alone by the flow of energy and momentum, so that we can hope that having a free theory it should be very easy and almost unique how to add a gravitational interaction. Let us say, that by the *spin-charge-family* theory by one of us, the interacting fields are the gravitational ones (vielbeins and spin connections) [5–7] only, but in  $d > (1 + 3)$ , fermions manifesting in  $(1 + 3)$  as spins and all the observed charges, as well as families, gravity manifesting all the observed gauge fields as well as the scalar fields, explaining higgs and the Yukawa couplings.

In analogy to, how one sometimes says that the electromagnetic interaction is added to a system of particles or fields with a global charge is “minimally coupled”, if one essentially just replace the derivatives by the corresponding covariant ones, we shall imagine that our free theory, which has energy and momentum as global charges could be made to contain gravity by some sort of “minimal coupling”. To introduce other extra interactions than just gravity is, however, expected to be much more complicated: Especially higher order Kalb-Ramond fields couple naturally to strings and branes, which in any case would tend to have disappeared in the present status of the universe. So effectively to day the Kalb-Ramond fields [13] should be free except for their “minimal coupling to gravity”. This would mean that allowing such a “later” rather trivial inclusion of gravity, which should be relatively easy, would make our at first free model be precisely the since long beloved model of one of us, the-spin-charge-family theory [5–7]. Fundamentally we have thus in our picture some series of Kalb-Ramond fields together with gravity coupling to them in the minimal way. Then we fermionize only this series of Kalb-Ramond fields, but keep bosonic the gravitational field, which probably cannot be fermionized even, if we wanted to. The resulting theory thus becomes precisely of similar type as the one by one of us, the spin-charge-family theory.

Now, however, the Kalb-Ramond fields are plagued by a lot of gauge symmetry and “unphysical” degrees of freedom, some of which even show up with even negative norm squared inner products. In principle these unphysical degrees of freedom must also somehow be treated in the fermionization procedure. Especially, if we want to use our theorem of counting degrees of freedom under bosonization [3], we should have such a theorem allowed to be used also when the “unphysical” d.o.f. are present.

In fact it is the main new point in the present article, that we put forward a slightly more complicated Aratyn-Nielsen-theorem - an extended Aratyn-Nielsen theorem -, allowing for theories with negative norm squared normalizations.

It is the true motivation of the present work, that once when we shall find some genuinely working case(s) of theories that bosonize/fermionize into each other in high dimensions, they will almost certainly turn out to involve gauge theories on the bosonic side. That is to say it will be combinations of various Kalb-Ramond fields [13] (among which we can formally count also electromagnetic fields and even a scalar field), and such Kalb-Ramond fields often have lots of negative norm square components. Thus once we know what is the boson theory that can be fermionized we need an extended Aratyn-Nielsen theorem to calculate the correct number of fermion components matching the fermionization correspondance.

Well really, if we know it well, we can just read off how many fermion components there are. It is namely this number of fermion components, that translates into the number of families, on which they are to be distributed. It means that knowing the detailed form of the boson system and the rule - the extended Aratyn-Nielsen theorem - for translating the number of boson components into the number of fermion components is crucial for obtaining the correct number of families. Will so to speak the number of fermion-families remain 4 as claimed by one of us in her model, which has reminiscent of being a fermionization, or will it be corrected somehow from the true bosonization requirement including the negative norm square components for the bosons? The reliability of the model would of course - according to the judgement of one of us (H.B.F.N.) - be much bigger, if it turned out that the true prediction were 3 families rather than the 4 as usually claimed, except, of course, if the fourth family, predicted by the spin-charge-family theory, will be measured.

With the old Aratyn-Nielsen theorem (the unextended version) it does crudely not look promising to get the number 3 rather than 4 as H.B.F.N would hope phenomenologically in as far this version implies that the number of fermion components is divisible by a rather high power of the number 2. Such a number-theoretic property of the number of families seems a priori to favor 4 much over 3.

The works of major importance for the present talk are:

- [Aratyn & Nielsen](#) We made a theorem [3] about the ratio of the number of bosons needed to represent a number of fermions based on statistical mechanics in the free case, under the provision that a bosonization exists.
- [Kovner & Kurzepa](#) They[8,9] present an explicit bosonization of two complex fermion fields in 2+1 dimensions being equivalent to QED<sub>3</sub> meaning 2+1 dimensional quantum electrodynamics.
- [Mankoč-Borštnik](#) [5–7] The spin-charge-family unification theory explains the number of families from the number of fermion components appeared in this theory.

In the next section 14.2 we put forward the main hope or point of view of our application of bosonization to make prediction of the number of families. In section 14.3 we give a loose argument for what we think should our picture for nature to cope with the investigations in the present article. Then we shall in section 14.4 and 14.5 review both Kalb Ramond fields and and our old Aratyn-Nielsen theorem about the number fermion components needed to make an equivalent theory with a number of boson components. In section 14.6 we look at the problem, that the components of a Kalb-Ramond field with an index being 0 are on the one hand to be a conjugate momentum to the other components and on the the other hand, if we use Lorentz invariance, have to lead to states with negative norm square. The latter is of course simply a reflection of the signature of the Minkowski metric tensor. It is for the application on such negative norm square components - the components with an index 0 - that our extension of our Aratyn-Nielsen theorem to negative norm square components become relevant.

In section 14.8 we review the work by Kovner and Kurzepa[8], who proposed a concrete bosonization including explicit expressions for the fermion fields in

terms of the boson fields - actually simply electrodynamics - in the case of 2 space dimensions and one time,  $1+2$ . Next in 14.9 we seek to check our Aratyn-Nielsen theorem on this special case of  $1+2$  both by counting the particle species including spin states 14.10 and by counting the fields 14.11.

Towards the end, section 14.12, we seek to reduce away some of the degrees of freedom from the Kovner and Kurzepa model to obtain a reduced case with fewer particles on which we - if it is also a case of bosonization - would be again able to check our counting theorem (Aratyn-Nielsen).

## 14.2 Hope

### Use of Bosonization/Fermionization Justifying Number of Families

The governing philosophy and motivation for the present study is:

- Fermions do NOT exist fundamentally (because they do not have proper causal/local property) .
- Some boson degrees of freedom are rewritten by bosonization (better fermionization) to fermionic ones, which then make up the fermions in the world, we see. (but some other boson degrees of freedom, hopefully gravity, are not bosonized).
- We work here only with an at first free theory - for our presentation, it might be best if only bosonization worked for FREE theories in higher dimensions - i.e. free bosons can be rewritten as free fermions.
- We though suggest - hope- that exterior to both bosons and/or fermions, we can add a GRAVITATIONAL theory. So fundamentally: gravity with matter bosons. It gets rewritten to fermions in a gravitational field, just similar to the theory [5–7] of one of us called spin-charge-family unification theory.

Let us be more specific about the dream or hope behind the present project:

By using say ideas from the below discussed paper by Kovner and Kurzepa [8] or by our own earlier article in last years Bled Proceedings about bosonization, we hope to find at least a case of fermionizing some series of Kalb-Ramond fields (i.e. Boson fields) - and electrodynamics is of course considered here a special Kalb-Ramond one - into some system of fermions. Presumably it is easiest - and perhaps only possible - for free theories or only theories interacting in a very special way. We therefore are most eagerly going for such a free and even massless case.

But now if indeed we can find such a case, or if exists, then it is very likely that we can extend it to interact with gravity in a minimal way. In fact we all the time require our hoped for fermionization cases to have the same energy and momentum for the bosonic and the fermionic theories that shall be equivalent. Thus the fermionization procedure, if it exist at all, is compatible with energy and momentum.

If we therefore let our boson-theory interact with gravity, that couples to the energy and momentum - specifically to the energy momentum tensor  $T_{\mu\nu}(x)$  - we have some hope that this coupling of the boson fields to gravity will simply transfer to a coupling of the fermionized theory, too.

As procedure we might have in mind writing the free massless fermionization procedure in arbitrary coordinates. That should of course be possible, but although the theory would now look as a gravitational theory, it would only have been derived for the case of the gravitational fields having zero curvature, i.e. for the Riemann tensor being zero all over. However, if the fermionization procedure could be described by a local expression for the Kalb-Ramond fields - or other boson fields - expressed in terms of fermion currents or the like, then the correspondence would in that formulation be local and lead to the energy momentum tensor being also related in such simple local way. I.e. we would have in this speculation

$$T_{\mu\nu}(x)|_{\text{boson}} = T_{\mu\nu}(x)|_{\text{fermion}}. \quad (14.3)$$

Here of course the two energy momentum tensors are the ones in respectively the fermion and the boson theory being equivalent by the dream for fermionization.

It is further our hope for further calculation that we may argue that in general it is very difficult to have interaction with Kalb-Ramond fields except for

- The appropriate branes,
- Some general gauge-theory coupling to the charges (think of global ones) conserved by the Kalb-Ramond- theory in question. But since the always conserved global charges are the energy and momentum this suggests the coupling to gravitational field.

We thus want to say that this starting form fundamental Kalb-Ramond fields supposedly difficult to make interact points towards a theory at the end with gravity as the only interaction. Gravity namely is suggested to be hard to exclude as possibility even for otherwise difficult to make interact Kalb-Ramond fields.

If we manage to fermionize the Kalb-Ramond fields as just suggested, we therefore tend to end up with the spin-charge-family unification model of one of us in the sense that we get ONLY gravity interaction, and otherwise a free theory.

But it shall of course be understood here that we only fermionize some of the boson fields in as far as we leave the assumed fundamental gravity field non fermionized.

### 14.3 Guiding and Motivation

The reader might ask why we choose - and suppose Nature to choose - these Kalb-Ramond-type fields which are to be explained a bit more below in section (14.5). Let us therefore put forward a few wish-thinking arguments for our bosonic fundamental model:

- We have no way to make fermionization/bosonization conserving angular-momentum truly (at the same time keeping the spin statistics theorem): The bosons namely necessarily can only produce Fock space states with integer angular momentum, but the fermion sates should for an odd number of fermions in the the Fock state have half integer angular momentum. So clearly fermionization/bosonization conserving angular momentum is *impossible*!

- The trick to overcome this angular-momentum-problem is to *reinterpret a spin 1/2 index on the fermions as a family index instead*. That is to say we accept at first that the fermions come out of the fermionization with bosonic integer spin index combination, and then seek to reinterpret part of the spin polarisation information as instead being a family information.
- In fact we shall be inspired by the spin-charge-family unification model to go for that the fermions come out from the fermionization at first with *two* spinor indices, so that they have indeed formally at this stage *integer* spin. Then we make the interpretation that one of these spinor indices is indeed a *family index*. That of course means, that we let one of the two indices be taken as a scalar index i.e. being not transformed under Lorentz transformations.
- So we decide to go for a system of fermions at the “first interpretation” being a two-spinor-indexed field. But now such a field  $B_{\alpha\beta}$ , where  $\alpha$  and  $\beta$  are the spinor indices, is indeed a Clifford algebra element, or we could say a Dirac matrix (or a Weyl matrix only if we use only the Weyl components). In any case we can expand it on antisymmetrized products of gamma-matrices:

$$B_{\alpha\beta} = (a\mathbf{1} + a_\mu\gamma^\mu + \dots + a_{\mu\nu\dots\rho}\gamma^\mu\gamma^\nu\dots\gamma^\rho + \dots + a_{0,1,\dots,(d-1)}\gamma^0\gamma^1\dots\gamma^{(d-1)})_{\alpha\beta}, \quad (14.4)$$

and thus the boson fields suggested to by fermionization leading to such fermion fields should be a series of antisymmetric tensor fields of all the different orders from the scalar  $a$  and the  $d$ -vector  $a_\mu$  all the way up to the maximal antisymmetric order tensor  $a_{0,1,\dots,(d-1)}$ .

- With random coefficients on a Lagrange-density expansion for a theory with boson fields, which have  $d$ -vectorial indices one unavoidably loose the bottom in the Hamiltonian as one can see from e.g. just a term like

$$c * (\partial_\mu \dots \partial_\nu a_{\rho\dots\tau})^2. \quad (14.5)$$

Think for instance on the terms for which the series of the derivative indices are spatial so that we have to do with a potential energy term. If the coefficient  $c$  is adjusted to let the contribution with the indices on  $a_{\rho\dots\tau}$  being spatial to the Hamiltonian be positive, then the contributions with a 0 among these indices will from Lorentz invariance have to be of the wrong sign. So it is at best exceedingly hard to organize a positive definite Hamiltonian density.

Consider only the free part - meaning bilinear part in the field  $a_{\rho\dots\tau}$  - in the Lagrangian. For simplicity let us consider the situation of a field  $a_{\rho\dots\tau}$  being constant as function of the time coordinate  $x^0$ , and that the number of derivatives acting on the field is so low that some of the indices- say  $\rho$  - on the  $a_{\rho\dots\tau}$  has to be contracted with another one or the same index on this field in order to cope with Lorentz invariance. Then if this (sum of) squares of the field in some combination shall get a for the hamiltonian positive contribution from a spatial value of the index  $\rho$ , it will get the opposite sign for  $\rho = 0$ . So it looks that we cannot avoid the Hamiltonian having both signs for a “free term” in the Lagrangian, unless all the indices on  $a_{\rho\dots\tau}$  are in the term contracted with derivatives. But with the antisymmetry this would be zero for more than



one index on  $a_{\rho\dots\tau}$ . So indeed it seems that unless one gets the fields restricted in some way, so that these fields or their conjugate variables are somehow not allowed to take independent values, then the Hamiltonian will loose its bottom and (presumably infinite) negative energy values will be allowed.

- We are thus driven towards theories with constraints!
- Such constraints are typically obtained by means of some gauge symmetry, and thus we are driven towards theories with gauge symmetry, if we want to uphold a positive definite Hamiltonian for the by the constraints allowed states of the field and its conjugate momenta.
- The obvious candidate for such a gauge theory with antisymmetrised tensor fields is of course the Kalb-Ramond fields. (Personally we suspect, that we can show that ONLY Kalb-Ramond-fields will solve this problem of positivity by providing enough constraints.)
- Thus it seems that it is very hard to hope for our to be used fermionization unless we make use of presumably a whole series of Kalb-Ramond fields!

## 14.4 Review

In theoretical condensed matter physics and particle physics, Bosonization/fermionization is a mathematical procedure by which a system of possibly interacting fermions in (1+1) dimensions can be transformed to a system of massless, non-interacting bosons. In the present article we shall dream about extending such bosonization to higher dimensions, and we shall be most interested in the case when even the fermions do not interact. The method of bosonization was conceived independently by particle physicists Sidney Coleman and Stanley Mandelstam; and condensed matter physicists Daniel Mattis and Alan Luther in 1975. [4] The progress to higher dimensions has been less developed [11] than the 1+1 dimensional case, but there has been some works also on higher dimensions. especially we shall below review a bit a work[8] by Kovner and Kurzepa for the next to simplest case, namely 2+1 dimensions. There has also been developments based on Chern-Simon type action[11], but we suspect that the type of bosonization we are hoping for in the present article should rather be of the Kovner Kurzepa type than of the Chern-Simon one, although we have difficulty in explaining rationally why we believe so.

When we have such transformation and thus two equivalent theories, one with fermions and one with bosons, one will of course expect that the number of degrees of freedom should in some way be the same for the boson and for the fermion theory. Otherwise of course they could not be equivalent. In the most studied case of 1+1 dimensions it has turned out in the cases known that there are two fermion components per boson component. This ratio is in accord with the theorem by Aratyn and one of us [3] - which we call the Aratyn-Nielsen theorem - in as far as this theorem predicts the ratio to be  $\frac{2^{d_{\text{spatial}}}}{2^{d_{\text{spatial}}-1}}$  where  $d_{\text{spatial}}$  is the dimension of space ( not including time) so that we talk about the dimension  $d_{\text{spatial}} + 1$ . In fact of course for the case 1+1 we have thus  $d_{\text{spatial}} = 1$  and the fraction predicted becomes  $\frac{2^1}{2^{1-1}} = 2$  times as many fermion components as boson components. This is really assuming that a “component” corresponds to a

polarization state of a particle. What we - one of us and Aratyn - really derived was that for a theory with massless interacting there had to be the mentioned ratio between the number of polarization states for the fermion(s) relative to that for the bosons. It were namely the contributions of such polarization states to the average energy in a Boltzmann distribution calculation that was used to derive the theorem. Although derived for this non-interacting massless case there could be reasons to believe that by taking a couple of limits in an imagined case of interacting and perhaps massive bosonization it could be argued, that the theorem of ours would have to hold anyway. For instance going to a very small distance scale approximation an approximately massless theory would arrive and the theorem should be applicable even if there is a mass. Since we are concerned in this theorem really with a counting of degrees of freedom a very general validity is in fact, what would be expected. As already said, we are, however, in the present article more concentrating on the generalization to include some unphysical degrees of freedom with possibly wrong signature,

## 14.5 AratynN

### Aratyn-Nielsen Theorem for massless free Bosonization

If there exist two free massless quantum field theories respectively with Boson, and Fermion particles and they are equivalent w.r.t. to the number of states of given momenta and energies, **then** the two theories must have the same average energy densities for a given temperature  $T$ , or simply same average energies, if we take them with the same infrared cut off (a quantisation volume  $V$ ):

$$\langle U_{\text{boson}} \rangle = \langle U_{\text{fermion}} \rangle \text{ where} \quad (14.6)$$

$$\langle U_{\text{boson}} \rangle = \sum_{\vec{p}} \frac{E(\vec{p})}{1 - \exp(E(\vec{p})/T)} \quad (14.7)$$

$$\langle U_{\text{fermion}} \rangle = \sum_{\vec{p}} \frac{E(\vec{p})}{1 + \exp(E(\vec{p})/T)}. \quad (14.8)$$

$$(14.9)$$

Here  $\vec{p}$  runs through the by the infrared cut off allowed momentum eigenstates, and  $E(\vec{p})$  are the corresponding single particle energies. Of course the single particle energy for a mass-less theory is

$$E(\vec{p}) = |\vec{p}|, \quad (14.10)$$

when  $c=1$ , and in  $d_{\text{spatial}}$  dimensions and with an infrared cut off spatial volume  $V$  the sum gets replaced in the continuum limit by the integral

$$\sum_{\vec{p}} \dots \rightarrow \int \sum_{\text{components}} \dots \frac{V}{(2\pi)^{d_{\text{spatial}}}}, \quad (14.11)$$

where  $\sum_{\text{components}} \dots$  stands for the sum over the different polarization components of the particles in question. So effectively in the simplest case of all the

particles having the same “spin”/the same set of components we have the replacement

$$\sum_{\text{components}} \dots \rightarrow N_{\text{families}} * N_c \dots \quad (14.12)$$

where  $N_c$  is the number of components for each particle and  $N_{\text{families}}$  is the number of families. **Some formulas for deriving Aratyn-Nielsen**

$$\langle U_{\text{boson}} \rangle = \sum_{\vec{p}} \frac{E(\vec{p})}{1 - \exp(E(\vec{p})/T)} \quad (14.13)$$

$$= "N_{\text{families}} * N_c'' * V / (2\pi)^{d_{\text{spatial}}} * \quad (14.14)$$

$$\int O(d_{\text{spatial}}) |\vec{p}|^{d_{\text{spatial}}} E(\vec{p}) \sum_{n=0,1,\dots} \exp(nE(\vec{p})/T d|\vec{p}|) \quad (14.15)$$

**Simple Aratyn-Nielsen Relation** For a given temperature must the average energies of respectively the boson and the with it equivalent fermion theories

### Our Realization Suggestion

- **Fermions**

For the fermions we shall use the needed number of say Weyl fermions, i.e. we must adjust the number of families hoping that we get an integer number.

- **Bosons**

For the bosons we let the number  $2^{d_{\text{spatial}}} - 1$  suggest that we take a series of all Kalb-Ramond fields, one combination of fields for each value of the number  $p$  of indices on the “potential field”  $A_{ab\dots k}$  (where then there are just  $p$  symbols in the chain  $ab\dots k$ ). At first we take these symbols  $a, b, \dots, k$  to be only spatial coordinate numbers.

**Free Kalb-Ramond** A Kalb-Ramond field[13] with  $p$  indices on the “potential” and  $p+1$  indices on the strength

$$F_{\mu\nu\rho\dots\tau}(x) = \partial_{[\mu} A_{\nu\rho\dots\tau]}(x), \quad (14.16)$$

where [...] means antisymmetrizing, and the “potential”  $A_{\nu\rho\dots\tau}$  is antisymmetric in its  $p$  indices  $\nu\rho\dots\tau$ , is defined to have an action invariant under the gauge transformation:

$$A_{\nu\rho\dots\tau}(x) \rightarrow A_{\nu\rho\dots\tau}(x) + \partial_{[\nu} \lambda_{\rho\dots\tau]}(x) \quad (14.17)$$

for any arbitrary antisymmetric gauge function  $\lambda_{\rho\dots\tau}(x)$  with  $p-1$  indices.

### Free Kalb-Ramond Action:

Note that the strength  $F_{\mu\nu\rho\dots\tau} = \partial_{[\mu} A_{\nu\rho\dots\tau]}$  is gauge invariant, and that thus we could have a gauge invariant Lagrangian density as a square of this field strength

$$\mathcal{L}(x) = F_{\mu\nu\dots\tau} F_{\mu'\nu'\dots\tau'} g^{\mu\mu'} * g^{\nu\nu'} * \dots * g^{\tau\tau'}. \quad (14.18)$$

Then the conjugate momentum of the potential becomes(formally):

$$\begin{aligned}\Pi_{\nu\rho\dots\tau} &= \Pi_{A_{\nu\mu\dots\tau}} = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_{\nu\rho\dots\tau})} \\ &= F_{0\nu\rho\dots\tau}.\end{aligned}\quad (14.19)$$

**A Lorentz gauge choice:**

$$\partial_\mu A_{\nu\rho\dots\tau} g^{\mu\nu} = 0, \quad (14.20)$$

allows to write the Lagrange density instead as

$$\mathcal{L}_{\text{modified}}(x) = 1/2 * \partial_\mu A_{\mu\nu\dots\tau} \partial_{\mu'} A_{\mu'\nu'\dots\tau'} * g^{\mu\mu'} g^{\nu\nu'} \dots g^{\tau\tau'}, \quad (14.21)$$

which leads to the very simple equations of motion letting each component of the “potential”  $A_{\nu\rho\dots\tau}$  independently obey the Dalambertian equation of motion

$$g^{\mu\mu'} \partial_\mu \partial_{\mu'} A_{\nu\rho\dots\tau} = 0. \quad (14.22)$$

### **Lorentz Invariance Requires Indefinite Inner Product!:**

Lorentz invariant norm square for the states generated by the creation operators  $a_{\nu\rho\dots\tau}^\dagger(p)$ , i.e.  $a_{\nu\rho\dots\tau}^\dagger(p)|0\rangle$ , must have different sign of the norm square depending on whether there is an even (i.e. no) 0's among the indices or whether there is an odd number (i.e. 1). A priori we are tempted to take

$$\begin{aligned}<0|a_{\nu\rho\dots\tau}(p)a_{\nu\rho\dots\tau}^\dagger(p)|0\rangle > > 0 \text{ for no } 0 \text{ among the indices,} \\ <0|a_{\nu\rho\dots\tau}(p)a_{\nu\rho\dots\tau}^\dagger(p)|0\rangle > < 0 \text{ for one } 0 \text{ among the indices,}\end{aligned}\quad (14.23)$$

## **14.6 Time-index**

### **Problem with Components with the time index 0:**

But full Kalb-Ramond fields require also components a 0 among the indices.(This is the main new thing in the present article to treat this problem of the components with one 0 among the indices.)

Remember about these components with a 0 index:

- Using a usual Minkowskian metric tensor  $g^{\mu\nu}$  in constructing an inner product between Kalb-Ramond fields, say

$$g^{\mu\nu} g^{\rho\sigma} \dots g^{\tau\kappa} A_{\mu\rho\dots\tau} (\text{potentially an } \partial^0) A_{\nu\sigma\dots\kappa}, \quad (14.24)$$

we get the opposite signature (=sign of the square norm) depending on whether there is a 0 or not!

**This means that if particles produced by the components without the 0 index have normal positive norm square, then those produced by the ones with the 0 have negative norm-square!**

**Good Luck We Removed the Kalb-Ramond A with  $p = 0$  Indices!** We could namely not have replaced on A one among its indices by a 0 because it has no indices. So we would not have known what to do for the fields A with 0 indices.

We correspondingly also have to leave out the Kalb-Ramond-field with  $p = d_{\text{spatial}} + 1$  indices, because for that there would be no components without an index 0.

For the unexceptional index numbers  $p = 1, 2, \dots, d_{\text{spatial}}$  there are some components both with and without the 0.

For the two exceptions  $p = 0$  and  $d = d_{\text{spatial}} + 1$  we have chosen not to have a Kalb-Ramond-field in our scheme, using it to get the  $-1$  in the from Aratyn-Nielsen required  $2^{d_{\text{spatial}}} - 1$ .

### Simplest (Naive) Norm Square Assignment

Note that for each Kalb-Ramond-field we can choose an overall extra sign on the inner product, because we simply can define the overall inner product with an extra minus sign, if we so choose. But the simplest choice is to just let the particles corresponding to fields with **only spatial indices** (i.e. all  $p$  indices different from 0) to have positive **norm square**, while then **those with one 0** have **negative norm square**.

**This simple rule would lead to equally many components/particles with positive as with negative norm square, so that dreaming about imposing a constraint that removes equally many negative and positive norm square at a time would leave us with nothing.**

**Numbers of Components with and without 0.** An of course totally antisymmetric field  $A_{\mu\nu\dots\tau}$  with  $p$  indices has

$$\begin{aligned}\# \text{ components}_{\text{KR } p\text{ indices}} &= \binom{d}{p} = \binom{d_{\text{spatial}} + 1}{p} \\ \# \text{ no } 0 \text{ components}_{\text{KR } p\text{ indices}} &= \binom{d_{\text{spatial}}}{p} = \binom{d-1}{p} \\ \# \text{ cmps. with } 0 \text{ \& } p-1 \text{ non-0}_{\text{KR } p\text{ indices}} &= \binom{d_{\text{spatial}}}{p-1} = \binom{d-1}{p-1}.\end{aligned}$$

and so one must have as is easily checked

$$\binom{d}{p} = \binom{d-1}{p} + \binom{d-1}{p-1}$$

corresponding to

“All components” = “Without 0” + “With 0”

**Using ONLY the Components WITHOUT 0 would fit  $2^{d_{\text{spatial}}}$  Nicely !** Having decided to leave out the number of indices  $p$  values  $p = 0$  and  $p = d$  the number of components **without** any component indices being 0 just makes up

$$\# \text{ without } 0 \text{ for all } p = 1, 2, \dots, d-1 = \sum_{p=1,2,\dots,d-1} \binom{d-1}{p} = 2^{d-1} - 1$$

so these “only with spatial indices components” could elegantly correspond to  $2^{d-1} = 2^{d_{\text{spatial}}}$  fermion components.

**But problem: Kalb- Ramond fields need also the component with an index being 0.**

**Using ONLY the Components WITH 0 could also fit  $2^{d_{\text{spatial}}}$  Nicely !** Having decided to leave out the number of indices  $p$  values  $p = 0$  and  $p = d$  the number of components **with** the 0 just makes up

$$\begin{aligned} \# \text{ with } 0 \text{ for all } p = 1, 2, \dots, d-1 &= \sum_{p=1,2,\dots,d-1} \binom{d-1}{p-1} \\ &= 2^{d-1} - 1 \end{aligned}$$

also, so these “only with 0 index components” could elegantly correspond to  $2^{d-1} = 2^{d_{\text{spatial}}}$  fermion components, also!

**But problem: Kalb- Ramond fields need also the components without an index being 0, and these with 0 usually come with wrong norm square.**

**The Trick Suggested is to use for Some KR-fields Opposite Hilbert Norm Square**

In other words we shall look along the chain of all the allowed  $p$ -values  $p = 1, 2, \dots, d-1$ ; and for each of these  $p$ -values we can choose whether

- **Normal:** The states associated with the polarization components **without** the 0 among the indices shall be of **positive norm square**, as usual, and then from Lorentz invariance essentially the ones **with the 0 shall have negative norm square**, or
- **Opposite** The states **with 0 shall have positive** norm square, while the components **without 0 negative** norm square.

**Our proposal: Choose so that we get the largest number of positive norm square components. How to get Maximal Number of Positive over Negative Norm Square Single Boson States**

For each value of  $p$  (=the number of indices on the Kalb Ramond “potential”)  $p = 1, 2, \dots, d_{\text{space}}$  decided to be used in the bosonization ansatz a priori, we investigate whether the number of (independent) components **with** or **without** a 0 is the bigger:

$$\begin{aligned} \# \text{ no } 0 \text{ components}_{\text{KR } p \text{ indices}} &= \binom{d_{\text{spatial}}}{p} = \binom{d-1}{p} \\ \# \text{ cmps. with } 0 \text{ \& } p-1 \text{ non-0}_{\text{KR } p \text{ indices}} &= \binom{d_{\text{spatial}}}{p-1} = \binom{d-1}{p-1}. \end{aligned}$$

So if there are **most components without 0**, i.e. if  $\binom{d_{\text{spatial}}}{p-1} < \binom{d_{\text{spatial}}}{p}$ , then we give the particle states corresponding to the **without 0** “potentials” have **positive norm square**. And opposite if  $\binom{d_{\text{spatial}}}{p-1} > \binom{d_{\text{spatial}}}{p}$ .

But if there are **most components with 0**, i.e. if  $\binom{d_{\text{spatial}}}{p-1} > \binom{d_{\text{spatial}}}{p}$ , then we give the particle states corresponding to the **with 0** “potentials” have **positive norm square**.

**To Maximize Positive Norm Square we Choose:**

- When  $p < \frac{d}{2}$ , choose **without 0 positive norm** squared, while “with 0” negative;
- but when  $p > \frac{d}{2}$ , choose **with 0 positive norm** squared, while “without 0” negative;

For e.g.  $p < d/2$  the excess of positive norm square “components ” over the negative norm ones becomes:

$$\begin{aligned} \binom{d_{\text{space}}}{p} - \binom{d_{\text{space}}}{p-1} &= \binom{d_{\text{space}}}{p} \left(1 - \frac{p}{d_{\text{space}} - p + 1}\right) \\ &= \frac{d_{\text{space}}!(d_{\text{space}} + 1 - 2p)}{(d_{\text{space}} - p + 1)!p!} = \frac{(d-1)!(d-2p)}{(d-p)!p!}, \end{aligned} \quad (14.25)$$

However, for  $p > d/2$  the excess is

$$\begin{aligned} \binom{d_{\text{space}}}{p-1} - \binom{d_{\text{space}}}{p} &= \binom{d_{\text{space}}}{p-1} \left(1 - \frac{d_{\text{space}} - p + 1}{p}\right) \\ &= \frac{d_{\text{space}}!(2p - d_{\text{space}} - 1)}{(d_{\text{space}} - p + 1)!p!} = \frac{(d-1)!(2p-d)}{(d-p)!p!}, \end{aligned} \quad (14.26)$$

#### Adding up Positive Norm Square over Negative Excess:

The sums over  $p$  “telescopes” from each of the two cases of  $p$  bigger or smaller than  $d/2$ , and gives by symmetry the same excess of positive over negative norm square states, namely for each for say  $d$  even (i.e.  $d_{\text{space}}$  odd)

$$\binom{d-1}{d/2-1} - 1 = \frac{(d-1)!}{(d/2-1)!(d/2+1)!} - 1, \quad (14.27)$$

where we used that the middle value  $p = d/2$  contribution vanishes. Including as we shall both “sides” smaller than  $d/2$  and also bigger than  $d/2$  we get the double of this.

#### Example Excesses States for even $d$ for Bosons

$$\begin{aligned} \text{Excess}(d=2) &= 2\left(\binom{2-1}{2/2-1} - 1\right) = 0 \\ \text{Excess}(d=4) &= 2\left(\binom{4-1}{4/2-1} - 1\right) = 2 \\ \text{Excess}(d=6) &= 2\left(\binom{6-1}{6/2-1} - 1\right) = 18 \\ \text{Excess}(d=14) &= 2\left(\binom{14-1}{14/2-1} - 1\right) \end{aligned} \quad (14.28)$$

#### Contribution from a Negative Norm square Component

One shall count the Hilbert space states with the negative norm square into the Boltzmann weighted averaging with a minus extra.

This extra minus for a negative norm square boson functions accidentally just like the fermi-statistics versus bose statistics. And thus e.g. a small  $p$  timelike polarization contributes to the average energy just like a fermion, though with an over all minus sign.

## 14.7 Extension of Our Theorem on Counting

It is a major purpose of the present talk to present an extension of the Aratyn-Nielsen theorem[3] on the numbers of bosons versus fermions in a bosonization to include the just above discussed negative norm square states associated with the Kalb-Ramond components having an index 0. Since such states obtaining at first negative norm squares are seemingly enforced by Lorentz invariance, it seems to be important to extend our Aratyn-Nielsen theorem to the case, where some of the components of the fields are quantized with a negative norm square.

We take such a negative norm square mode to mean, that whenever there in a Fock space state is an *odd* number of particles with the component in question, then such a Fock-space basis vector is in the “Hilbert norm” given a negative norm square. Of course that means that strictly speaking our Fock space is no longer a genuine Hilbert space, but rather just an (infinite dimensional) space with an *indefinite* inner product, |, giving the inner product between two Focks,  $|a\rangle$  and  $|b\rangle$  say, as  $\langle b|a\rangle$ . But now the point is just that *we have no sign restriction on*  $\langle a|a\rangle$ ; *it can easily be negative*.

The in usual Hilbert spaces used expansion on an orthonormal basis

$$1 = \sum_a |a\rangle\langle a| \text{ (usual),} \quad (14.29)$$

cannot now be applied. Now we rather have to use

$$1 = \sum_a (-1)^{N_{neg}(a)} |a\rangle\langle a| \text{ (with negative norm square also),} \quad (14.30)$$

where  $N_{neg}(a)$  denotes the number of particles in the various negative norm square single particle states together. If for instance a basis state  $|a\rangle$  for the Fock space has 3 particle in the states with 0 index all together (and we have used the choice of letting the components with a 0-index be the ones with negative norm, rather than the more complicated possibilities discussed above),  $N_{neg}(a) = 3$  and thus such a state would come with a minus sign in the expansion of the unit operator **1**.

Let us now calculate the average energy for a system described by a Fock space with only one single particle state present, so it really is the system with only one single particle state, that may be filled or empty according the rule for it being bosonic or fermionic and having negative or positive norm square. For this purpose we have to think about how one shall define the concept of a trace - which goes into the average procedure to provide us with such a an average of the energy, and we claim that we must indeed in the case with negative norm square states take the trace definition:

$$\text{Tr}(\mathbf{O}) = \sum_a (-1)^{N_{neg}(a)} \langle a|\mathbf{O}|a\rangle. \quad (14.31)$$



With this definition we easily check some usual rule for traces:

$$\text{Tr}(\mathbf{OP}) = \sum_a (-1)^{N_{\text{neg}}(a)} \langle a | \mathbf{OP} | a \rangle \quad (14.32)$$

$$= \sum_a \sum_b (-1)^{N_{\text{neg}}(a)} \langle a | \mathbf{O} | b \rangle \langle b | (-1)^{N_{\text{neg}}(b)} \mathbf{P} | a \rangle \quad (14.33)$$

$$= \text{Tr}(\mathbf{PO}). \quad (14.34)$$

Using this definition of the trace  $\text{Tr}$  we can then put in the quite analogous way to the usual case for Boltzmann distribution in quantum mechanics

$$\langle E \rangle = \frac{\text{Tr}(\exp(-H/T)H)}{\text{Tr}(\exp(-H/T))}, \quad (14.35)$$

where the Boltzmann-Constant  $k$  has been absorbed into the temperature  $T$ , and where now we use in the case of negative norm square the expression (14.31). Let us enumerate the single particle states with the letter  $n$  and denote the single particle energy of the state  $n$  as  $E_n$ . Then the free Hamiltonian  $H$  is given by means of the number operators

$$N_n = a_n^\dagger a_n \quad (14.36)$$

as

$$H = \sum_n E_n N_n = \sum_n E_n a_n^\dagger a_n, \quad (14.37)$$

and we immediately see that

$$\begin{aligned} \langle E_n N_n \rangle |_{\text{boson pos.}} &= \frac{\sum_{N_n=0,1,\dots} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n=0,1,2,\dots} \exp(-E_n N_n/T)} \\ &= \frac{-\frac{d}{d(1/T)} \left( \frac{1}{1 - \exp(-E_n/T)} \right)}{\frac{1}{1 - \exp(-E_n/T)}} = \frac{E_n}{\exp(E_n/T) - 1} (\text{boson; pos. norm sq.}) \end{aligned} \quad (14.38)$$

$$\begin{aligned} \langle E_n N_n \rangle |_{\text{boson neg.}} &= \frac{\sum_{N_n=0,1,\dots} (-1)^{N_n} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n} (-1)^{N_n} \exp(-E_n N_n/T)} \\ &= \frac{-\frac{d}{d(1/T)} \left( \frac{1}{1 + \exp(-E_n/T)} \right)}{\frac{1}{1 + \exp(-E_n/T)}} = -\frac{E_n}{\exp(E_n/T) + 1} (\text{boson; neg. norm sq.}) \end{aligned} \quad (14.39)$$

$$\begin{aligned} \langle E_n N_n \rangle |_{\text{fermion pos.}} &= \frac{\sum_{N_n=0,1} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n=0,1} \exp(-E_n N_n/T)} \\ &= \frac{-\frac{d}{d(1/T)} (1 + \exp(-E_n/T))}{1 + \exp(-E_n/T)} = \frac{E_n}{\exp(E_n/T) + 1} (\text{fermion; pos. norm sq.}) \end{aligned} \quad (14.40)$$

$$\begin{aligned} \langle E_n N_n \rangle |_{\text{fermion neg.}} &= \frac{\sum_{N_n=0,1} (-1)^{N_n} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n=0,1} (-1)^{N_n} \exp(-E_n N_n/T)} \\ &= \frac{-\frac{d}{d(1/T)} (1 - \exp(-E_n/T))}{1 - \exp(-E_n/T)} = -\frac{E_n}{\exp(E_n/T) - 1} (\text{fermion; neg. norm sq.}) \end{aligned} \quad (14.41)$$

We notice that - by accident - the contribution from a negative norm square fermion mode happens to be just the opposite of that of a positive norm square boson mode with the same energy  $E_n$ . And also the positive fermion mode contribution is just minus one time the negative boson contribution. Thus we can get the requirement for the theory of fermions and that of bosons to provide the same average energy:

$$\sum_{\substack{E'_n \text{ for (pos.) fermions} \\ \text{plus neg. bosons}}} \frac{E_n}{\exp(E_n/T) + 1} = \sum_{\substack{E'_n \text{ for (pos.) bosons} \\ \text{plus neg. fermions}}} . \quad (14.42)$$

#### 14.7.1 Free Massless

The simplest case to consider is the one in which both the fermions and the bosons - on their respective sides of the identification of the theories - are supposed to be both free and massless relativistic particles. In this case - which is the one we shall keep to in the present article - we introduce for definiteness an infra red cut off so that we get discretized momentum eigenstates, and the above  $n$  now really becomes a pair of a discretized momentum  $\vec{p}$  and an index denoting the component, which means typically the vector or spinor index including also the family index, all put together say to  $t$ , standing for the word "total component", meaning that both family and genuine component is included. The number of possible values for this total component enumeration is of course for what we are indeed obtaining restrictions for. Let us therefore immediately define the four numbers

$$\begin{aligned} N_{t \text{ ferm pos.}} &= N_{\text{families ferm pos.}} * N_{c \text{ ferm pos.}}, \\ N_{t \text{ ferm neg.}} &= N_{\text{families ferm neg.}} * N_{c \text{ ferm neg.}}, \\ N_{t \text{ boson pos.}} &= N_{\text{families boson pos.}} * N_{c \text{ boson pos.}}, \\ N_{t \text{ boson neg.}} &= N_{\text{families boson neg.}} * N_{c \text{ boson neg.}}, \end{aligned}$$

to denote the total numbers of components of the respective types of particles w.r.t. statistics and normsquare sign.

One technique for calculating the integrals over the momentum space consists in first Taylor expanding the expressions to be integrated

$$\begin{aligned} \frac{E_n}{\exp(E_n/T) - 1} &= \frac{E_n}{\exp(E_n/T)} * (1 + \exp(-E_n/T) + \exp(-2E_n/T) + \dots) \\ &= E_n \left( \sum_{j=1,2,\dots} \exp(-jE_n/T) \right) \end{aligned} \quad (14.43)$$

$$\begin{aligned} \frac{E_n}{\exp(E_n/T) + 1} &= \frac{E_n}{\exp(E_n/T)} * (1 - \exp(-E_n/T) + \exp(-2E_n/T) - \dots) \\ &= E_n \left( \sum_{j=1,2,\dots} (-1)^{j-1} \exp(-jE_n/T) \right), \end{aligned} \quad (14.44)$$

and then using

$$\sum_{\vec{l} \in \text{integer lattice}} \exp(-j|\vec{l}| * 2\pi/L) = \int \exp(-j|\vec{x}|2\pi/L) d^{d_{\text{spatial}}} \vec{x} \quad (14.45)$$

$$= \left( \frac{L}{2\pi * j} \right)^{d_{\text{spatial}}} \int \exp(-|\vec{x}|) d^{d_{\text{spatial}}} \vec{x} \quad (14.46)$$

$$= \left( \frac{L}{2\pi * j} \right)^{d_{\text{spatial}}} \mathcal{O}(d_{\text{spatial}} - 1) \int_0^\infty \exp(-x) x^{d_{\text{spatial}}} dx \quad (14.47)$$

$$= \left( \frac{L}{2\pi * j} \right)^{d_{\text{spatial}}} \mathcal{O}(d_{\text{spatial}} - 1) / d_{\text{spatial}}!. \quad (14.48)$$

Here we denoted the surface area of the unit sphere in  $d_{\text{spatial}}$  dimensions by  $\mathcal{O}(d_{\text{spatial}} - 1)$  because this surface then has the dimension  $d_{\text{spatial}} - 1$ . In fact

$$\mathcal{O}(d_{\text{surface}}) = \frac{2\pi^{d_{\text{surface}}/2}}{\Gamma(d_{\text{surface}}/2)}. \quad (14.49)$$

We then finally shall use

$$\zeta(d_{\text{spatial}}) = \sum_{j=0,1,2,\dots} \frac{1}{j^{d_{\text{spatial}}}}. \quad (14.50)$$

$$\zeta(d_{\text{spatial}}) \left( 1 - \frac{1}{2^{d_{\text{spatial}}}} \right) = \sum_{j=0,1,2,\dots} \frac{(-1)^j}{j^{d_{\text{spatial}}}}. \quad (14.51)$$

When we compare the different expressions for bosons versus for fermions, most factors drop out and the only important factor is the factor  $\left( 1 - \frac{1}{2^{d_{\text{spatial}}}} \right)$ . It is then easy to see that we obtain the extended Aratyn-Nielsen theorem:

$$N_{\text{t ferm pos.}} + N_{\text{t boson neg.}} = \frac{2^{d_{\text{spatial}}}}{2^{d_{\text{spatial}}} - 1} * (N_{\text{t boson pos.}} + N_{\text{t ferm neg.}}). \quad (14.52)$$

### 14.7.2 Properties and Examples

Let us first of all call attention to that this extended Aratyn-Nielsen theorem like the original one has the property of “additivity” meaning that if we have two cases of functioning bosonization - i.e. two cases of a system of fermions being equivalent to a system of bosons - and thus by combining them formally a system with both sets of bosons making up its set of bosons and similarly construct a set of fermions by combing the fermions then the combined system will automatically - just algebraically - come to obey the requirement from our theorem.

Let us also remark that the old Aratyn-Nielsen theorem[3] just is the special case, in which there are no negative norm square components.

In the Bled workshop in 2015 [12] we presented speculations, that one could make a free massless case of bosonization/fermionization in an arbitrary number of dimensions. This attempt were indeed already strongly inspired from our

theorem and counted just  $2^{d_{\text{spatial}}} - 1$  boson particle components and  $2^{d_{\text{spatial}}}$  fermionic components. There were no negative norm square components and the there suggested case of bosonization should thus be an example on the use of the “old” Aratyn-Nielsen theorem. The ratio of the number  $2^{d_{\text{spatial}}}$  of fermion components equivalent to  $2^{d_{\text{spatial}}} - 1$  bosonic components is namely of course just equal to  $\frac{2^{d_{\text{spatial}}}}{2^{d_{\text{spatial}}}-1}$  as it should according to our theorem(s). The special feature of that proposal [12] was that we imagined having chosen such infrared cut off periodicity or antiperiodicity conditions, that these (anti)periodicity conditions specified the components of the fields. Indeed there were just one fermion component for each combination of a choice of periodicity versus antiperiodicity for each of the  $d_{\text{spatial}}$  spatial dimensions. That makes up of course  $2^{d_{\text{spatial}}}$  combinations of periodicity antiperiodicity choices and thus so many fermion components. Similarly almost all such combinations gave rise to a boson component, except that we deleted so to speak the boson components, that should have corresponded to being periodic in all  $d_{\text{spatial}}$  coordinates (taken with infrared cut off). Thus there were just  $2^{d_{\text{spatial}}} - 1$  boson components in the in this Bled proceeding speculated case of bosonization.

### 14.7.3 A speculative semi-trivial example

Starting from the example[12] we would now highly suggestively - but really a bit speculatively - construct a not completely trivial although not so very physically interesting at first example with negative norm square components. Since we have anyway broken in this model full rotational invariance, it is no longer a catastrophe to treat one of coordinate axis - say  $x^1$  in a different way from the other ones.

We modify the model in the 2015 Bled proceedings by:

- On the fermionic side we take all the components specified by having odd momentum along say the  $x^1$ -axis or equivalently have antiperiodic boundary condition in  $x^1$  to have **negative norm square**. They make up just half - and thus  $2^{d_{\text{spatial}}-1}$  - of all the fermionic components.
- On the bosonic side we also change the norm-square to be negative for the components antiperiodic in the  $x^1$ -coordinate. This is for even more than half of the components in as far as it is again for  $2^{d_{\text{spatial}}-1}$ , but now only out of the  $2^{d_{\text{spatial}}} - 1$  bosonic components.

Both of these two modifications have in the Fock-space the same effect in as far as they both just lead to shifting the norm square from positive to negative for all the states with the **total  $p^1$ -momentum odd**. So the two modifications suggested for respectively the bosons and the fermions seem to be the same one in the Fock space. At least speculatively then we expect, that the modified model will have functioning bosonization - provided we trust that the original model from the Bled 2015 proceeding were indeed consistently a case of bosonization.

Now we want to test, if this suggestive speculative case of bosonization will obey our extended Aratyn-Nielsen requirement(14.52):

We have in this modified model/case of bosonization:

- We are left with  $2^{d_{\text{spatial}}-1} - 1$  bosonic positive norm square components, i.e.  $N_{\text{t boson pos.}} = 2^{d_{\text{spatial}}-1} - 1$ .
- While  $2^{d_{\text{spatial}}-1}$  of the bosonic components were made to have negative norm squared. So  $N_{\text{t boson neg.}} = 2^{d_{\text{spatial}}-1}$ .
- Of the fermionic components  $2^{d_{\text{spatial}}-1}$  remained of positive norm-square; so  $N_{\text{t ferm pos.}} = 2^{d_{\text{spatial}}-1}$ .
- Also  $2^{d_{\text{spatial}}-1}$  components had the odd momentum in the  $x^1$ -direction and were made to have negative norm square. So  $N_{\text{t ferm neg.}} = 2^{d_{\text{spatial}}-1}$ .

Inserting these numbers of components into (14.52) is easily seen to make it satisfied. The point really is, that we made the same number of boson components and of fermion components negative norm square. This sign of norm square in our formula makes them move from one side to the other, but since the two groups were of the same number at the end nothing were changed and the formula still satisfied.

## 14.8 Kovner...

**Kovner and Kurzepa made 2+1** The article by these authors [8] contains the expression

$$\psi_{\alpha}(x) = k \Lambda V_{\alpha}(x) \Phi(x) U_{\alpha}(x) \quad (14.53)$$

for the fermion fields expressed in terms of the boson fields in their fermionization in 2+1 dimensions. Here the expressions  $V_{\alpha}(x)$ ,  $\Phi(x)$ , and  $U_{\alpha}(x)$  are exponentials of integrals over the boson field, which are indeed electromagnetic fields in 2+1 dimensions. The variants of expressions are denoted by the index  $\alpha$ , which takes two values. There are thus (a priori) two complex fermion fields defined here.

## 14.9 Match?

**Does the Kovner Kurzepa Bosonization Match with the Aratyn-Nielsen Counting Rule?**

**First look at number of hermitean counted fields:** Kovner and Kurzepa gets two complex meaning 4 real fermion fields  $\text{Re}\psi_1(x)$ ,  $\text{Im}\psi_1(x)$ ,  $\text{Re}\psi_2(x)$ , and  $\text{Im}\psi_2(x)$  out of the for the construction relevant boson-fields  $A_1(x)$ ,  $A_2(x)$ ,  $\partial_1 E_i = \partial_1 E_1 + \partial_2 E_2$ . This looks agreeing with the Aratyn Nielsen prediction that the ratio shall be

$$\frac{\#\text{bosons}}{\#\text{fermions}} = \frac{2^{d_s} - 1}{2^{d_s}} = \frac{2^2 - 1}{2^2} \text{ for the spatial dimension being } d_s = d - 1 = 2 \quad (14.54)$$

**Four real fermion fields bosonize to three real boson-fields! o.k.**

**What about the conjugate momenta to the fields?** While the fermion fields are normally each others conjugate variables(fields) in as far as they anticommute with each other having only no-zero anticommutators with themselves, the boson-fields typically are taken each to have associated an extra field - its conjugate -

with which it does not commute, while of course any variable must commute with itself. But a field, that depends on an  $x$ -point or on a momentum, need NOT to commute with itself, though.

But then the question: Shall we for bosons somehow also count the conjugate momentum fields, when we shall compare the number of fermion and boson fields equivalent through bosonization? For the fermions the conjugate fields are unavoidable already included into the set of fields describing the fermions, because the it is the field in question itself, but for bosons we could easily get the number of fields *doubled*, if we include for each field also its conjugate.

### **Conjugate Momentum Fields NOT to be Included in Counting.**

Let us argue that it is enough in the counting to count the number of fields, from which you by Fourier resolution can extract the annihilation and creation operators needed to annihilate or create the particles, the species of which are to be counted:

- Normally we could extract the conjugate field by differentiating w.r.t. to time the field because usually you can replace the fields and their conjugate by the fields and their time derivatives.
- Using equations of motion these time derivatives can in turn be obtained by some way - also some sort of differentiation - from the field itself.
- Thus at the end the information on the conjugate is extractable from the field itself!

### **Further Support for NOT including also Conjugate Momentum Fields**

We could very easily construct linear (or more complicated) combinations of boson fields and their conjugate fields. Such combinations would like the fermion fields typically not commute/anticommute with themselves.

So provided we can extra the particle creation and annihilation operators from the combined field we would have no rule to tell that we should include more. Thus we would need only the combined field, and with that rule have quite analogy to the fermion case.

### **Meaning of NOT Counting also the Conjugate Field**

In QED<sub>3</sub> say  $A_1(x)$  and  $A_2(x)$  would be enough to represent both longitudinal and transversely polarized photons. It would NOT be needed also to have the essentially conjugate electric fields  $E_1(x)$  and  $E_2(x)$ .

The field  $\partial_i E_i$  is in fact the conjugate  $A_0$  so that we - having the symmetry between a field and its conjugate, it being conjugate of its conjugate - can consider that timelike photons are described by this  $\partial_i E_i$  field combination.

## **14.10 Particles**

### **But in terms of Particles, How??**

Usually one thinks of electrodynamics in 2+1 dimensions as having only one particle polarisation, since there is only *one* transversely polarisation for a photon. So seemingly only one component of boson. This transversely polarized photon is even its own antiparticle, so even the anti-particle is not new.

On the contrary the fermions after the fermionization counts two complex fields meaning two different fermion components ( $\psi_1$  and  $\psi_2$ ) each with an a priori different antiparticle in as far as the fields  $\psi_1$  and  $\psi_2$  both are complex(non-Hermitean). That seems NOT to match!

Where have the two missing photon-polarizations gone?

### **Suggestion for How 3 photons.**

To count independently both  $A_i$  ( $i=1,2$ .) as real fields, we need to consider it that we have not only the transverse photon, but also a **longitudinal photon** !

The third of the real fields  $\partial_i E_i = \text{div} \vec{E}$  is actually the conjugate variable to the time component  $A_0(x)$  of the fourcomponent photon field. So if we take it that conjugate or not does not matter it could correspond to the **timelike polarized photon**.

This would mean that we could hope for interpreting the three photon polarizations as being

- 1) The transverse photon.
- 2) The longitudinal photon.
- 3) The time-like photon.

**But the time like photon has wrong signature ?!**

### **Better Suggestion for the 3 particles ?**

To avoid the problem with the ltime-like photon form Lorentz invariance having the signature with negative norm square states we can instead take a further scalar. If so we could have 3 bosons corresponding to the four (real) fermions.

In any case if we want a fermion system with positive definite Hilbert space we better have the bosons also give positive definite Hilbert space if they shall match in their Hilbert spaces.

## **14.11 Fields**

### **How to count Hermitean Boson fields ?**

To exercise we shall for the moment even begin with a 1+1 dimensional only right moving Hermitean field constructed as a superposition of momentum state creation  $a^\dagger(p)$  and annihilation operators  $a(p)$  for say a series discretized momentum values, which we for "elegance"( and later interest) shall take to be odd integers in some unit:

$$\begin{aligned}\phi(x) &= \sum_{p \text{ odd}, p>0} \sqrt{p} a(p) \exp(ipx) + \sum_{p \text{ odd}, p<0} \sqrt{|p|} a^\dagger(|p|) \exp(ipx) \\ &= \sum_{p \text{ odd}} \sqrt{|p|} a(p),\end{aligned}\tag{14.55}$$

where we have put

$$a(p) = a^\dagger(-p) \text{ for all the odd } p\tag{14.56}$$

**Properties of the Hermitean field** A Hermitean field of the form (in 1+1 dimension say)

$$\begin{aligned}\phi(x) &= \sum_{p \text{ odd}, p>0} \sqrt{p} a(p) \exp(ipx) + \sum_{p \text{ odd}, p<0} \sqrt{|p|} a^\dagger(|p|) \exp(ipx) \\ &= \sum_{p \text{ odd}} \sqrt{|p|} a(p)\end{aligned}\quad (14.57)$$

obeys

$$\phi(x)^\dagger = \phi(x) \text{ (Hermiticity) and} \quad (14.58)$$

$$[\phi(x), \phi(y)] = \sum_{p \text{ odd}} \sum_{p' \text{ odd}} \sqrt{|p|} \sqrt{|p'|} [a(p), a(p')] \exp(ipx + ip'y) \quad (14.59)$$

$$= \sum_{p \text{ odd}} p \exp(ip(x-y)) = 2\pi \frac{d}{id(x-y)} \delta(x-y) \quad (14.60)$$

$$= -i2\pi \delta(x-y) \text{ (local commutation rule).} \quad (14.61)$$

## 14.12 New

### New, Reduce the Kovner Kurzepa model.

We claim, that in a way the Kovner and Kurzepa bosonization in 2 + 1 dimensions has included a kind of “funny extra bosonic degree of freedom” the charge density compared to our own plan of doing a completely free model.

Really we want to say: In a truly free electrodynamics “free QED<sub>3</sub>” (in 2 + 1 dimensions) the divergence of the electric field is zero:

$$\partial_i E_i \approx 0 \text{ (on physical states).} \quad (14.62)$$

When we use  $\approx$  instead of  $=$  it is because we may need the divergence  $\partial_i E_i$  as an operator even though we may take it to be zero on the “physical states”.

#### Reduction of Kovner Kurzepa model w.r.t. degrees of freedom

Inserting formally our claim of a constraint equation

$$\partial_i E_i \approx 0 \text{ (on physical states).} \quad (14.63)$$

into the expressions of Kovner and Kurzepa

$$V_1(x) = -i \exp\left(\frac{i}{2e} \int (\theta(x-y) - \pi) \partial_i E_i\right) \quad (14.64)$$

$$U_1(x) = \exp\left(-\frac{i}{2e} \theta(y-x) \partial_i E_i\right) \quad (14.65)$$

we get

$$V_1(x) \approx -i \quad (14.66)$$

$$U_1(x) \approx 1. \quad (14.67)$$



**Using the constraint equation formally on Kovner and Kurzepa** In Kovner and Kurzepa one finds

$$\psi_\alpha(x) = k \Lambda V_\alpha(x) \Phi(x) U_\alpha(x) \quad (14.68)$$

$$\Phi(x) = \exp \left( i e \int e_i(y-x) A_i(y) d^2 y \right); e_i(y-x) = \frac{y_i - x_i}{(y-x)^2} \quad (14.69)$$

$$V_1(x) = -i \exp \left( \frac{i}{2e} \int (\theta(x-y) - \pi) \partial_i E_i \right); V_2(x) = -i V_1^\dagger(x) \quad (14.70)$$

$$U_1(x) = \exp \left( -\frac{i}{2e} \theta(y-x) \partial_i E_i \right); U_2(x) = V_1^\dagger(x) \quad (14.71)$$

and thus with the constraint formally included

$$\psi_2(x) \approx i \psi_1(x) \quad (14.72)$$

**Our Constraint would Spoil Rotation Symmetry** A constraint equation

$$\psi_2(x) \approx i \psi_1(x) \quad (14.73)$$

would *not* be consistent with the rotation symmetry and the transformation property for the fermion field suggested in Kovner and Kurzepa

$$\psi_1 \rightarrow \exp(i\phi/2) \psi_1; \psi_2 \rightarrow \exp(-i\phi/2) \psi_2. \quad (14.74)$$

So including the constraint would make the bosonization/fermionization become *non-rotational invariant*. But it is our philosophy not to take that as a so serious problem, because it is in any case *impossible* to get in a rotational invariant way spin 1/2 fermions from a purely bosonic theory with only integer spin!

**Rotation symmetry broken in reduced model!**

## 14.13 Conclusion

We have extended the previous “Aratyn-Nielsen-theorem” relating the number of degrees of freedom / number of components / number of particle (orthogonal) polarizations for a set of bosons that by bosonization/fermionization is in correspondance with each other. The extension consists in also allowing negative norm square single particle states. We only considered yet the case of massless noninteracting both bosons and fermions, but expect that by thinking of the limit of small distances the relation of the theorem would also have to hold for massive particles. If there existed a common for both bosons and fermions weak interaction limit you would also expect that the noninteraction assumption could be avoided.

The main result is the relation (14.52):

$$N_{\text{t ferm pos.}} + N_{\text{t boson neg.}} = \frac{2^{d_{\text{spatial}}}}{2^{d_{\text{spatial}}} - 1} * (N_{\text{t boson pos.}} + N_{\text{t ferm neg.}}),$$

where the “normal” boson and fermion component numbers are denoted with  $N_{\text{t boson pos.}}$  and  $N_{\text{t ferm pos.}}$  respectively for bosons and for fermions, and

where the corresponding numbers of components with negative norm square are  $N_{t \text{ boson neg.}}$  and  $N_{t \text{ ferm neg.}}$ .

We have also looked at some examples where one might apply and test our theorem, but the problem is that we do not know the higher dimensional examples so well. Basically the dimension limit where the examples basically stop is not high. Googling you find mainly at most 2+1. The case 3+1 is very rare.

### 14.13.1 Outlook Dream

Our motivation, which has not quite ran out to be realized yet is that we shall find in literature or develop bosonization case(s) for the dimensions of interest as dimension of the space time, such as the experimental dimension 3+1 or the in the spin-charge-family theory practical starting dimension 13 +1. That is to say we hope to find a set of boson fields that is equivalent to a set of fermion fields in the bosonization way. If we have a valid theorem as the one we just extended we strictly speaking only need to know one side, i.e. either the bosons or the fermions, because then we can calculate the number of components for the other side. Without the “extension” of our theorem it looks that the number of fermion components must always be a number divisible by  $2^{d_{\text{spatial}}}$ , which e.g. for the case of the experimental dimension is  $2^3 = 8$ . It makes it especially difficult to avoid the number of families being even, because if we think of Weyl fermions at least and even count real components so that we get twice as many as if we used complex components, we still need a multiplum of 2 families of Weyl particle. With Dirac fermions we could use up a factor 2 more and we would get no prediction than just the number of families being integer. But in the Standard model we know that we have the weak interactions and the components put together to Dirac fermions have separate gauge quantum numbers are are hardly suitable for coming from the same fermionization.

With an extended theorem relating the two sides fermions and bosons, however, the situation gets less clear and the hope for even getting somehow a phenomenologically good number is not excluded yet.

## Acknowledgement

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## References

1. Daniel Ranard, *An introduction to rigorous formulations of quantum field theory*, Perimeter Institute for Theoretical Physics, Advisors: Ryszard Kostecki and Lucien Hardy May 2015.
2. D. Iagolnitzer, “Causality in Local Quantum Field Theory: Some General Results”, *Commun. Math. Phys.* 144,235-255 (1992) Springer-Verlag 1992.

3. Aratyn, H.B. Nielsen, "Constraints On Bosonization In Higher Dimensions", NBI-HE-83-36, Conference: C83-10-10.2 (Ahrenschoop Sympos.1983:0260), p.0260 Proceedings.
4. Coleman, S. (1975). "Quantum sine-Gordon equation as the massive Thirring model" Physical Review D11 2088; Witten, E. (1984). "Non-abelian bosonization in two dimensions", Communications in Mathematical Physics 92 455-472. E. Lieb, D. Mattis, J. Math. Phys. 6 (1965), 304; S. Coleman, Phys. Rev. D11 (1975) 2088; A. Luther, I. Peschel, Phys. Rev. B12 (1975), 3908; S. Mandelstam, Phys. Rev. D11 (1975), 3026.
5. N.S. Mankoč Borštnik, "Spin-charge-family theory is offering next step in understanding elementary particles and fields and correspondingly universe", Proceedings to the Conference on Cosmology, Gravitational Waves and Particles, IARD conferences, Ljubljana, 6-9 June 2016, The 10<sup>th</sup> Biennial Conference on Classical and Quantum Relativistic Dynamics of articles and Fields, J. Phys.: Conf. Ser. 845 012017 [arXiv:1607.01618v2], and the references therein.
6. N. Mankoč Borštnik, "Spin connection as a superpartner of a vielbein", Phys. Lett. B 292 (1992) 25-29.  
N. Mankoč Borštnik, "Spinor and vector representations in four dimensional Grassmann space", J. of Math. Phys. 34 (1993), 3731-3745. N.S. Mankoč Borštnik, "Matter-antimatter asymmetry in the *spin-charge-family* theory", Phys. Rev. D 91 065004 (2015) [arxiv:1409.7791].
7. N.S. Mankoč Borštnik, "The explanation for the origin of the Higgs scalar and for the Yukawa couplings by the *spin-charge-family* theory", J.of Mod. Physics 6 (2015) 2244-2274, <http://dx.org./10.4236/jmp.2015.615230> [<http://arxiv.org/abs/1409.4981>], and the references therein.  
N.S. Mankoč Borštnik, D. Lukman, "Vector and scalar gauge fields with respect to  $d = (3 + 1)$  in Kaluza-Klein theories and in the *spin-charge-family* theory", Eur. Phys. J. C 77 (2017) 231.  
N.S. Mankoč Borštnik, H.B.F. Nielsen, "The spin-charge-family theory offers understanding of the triangle anomalies cancellation in the standard model", Fortschritte der Physik, Progress of Physics (2017) 1700046.
8. A. Kovner and P. S. Kurzepa, "Fermions from photons: Bosonization of QED in (2+1)-dimensions," Int. J. Mod. Phys. A 9 (1994) 4669 doi:10.1142/S0217751X94001862 [hep-th/9306065].
9. A. Kovner and P. Kurzepa, "Bosonization in (2+1)-dimensions without Chern-Simons attached," Phys. Lett. B 321 (1994) 129 doi:10.1016/0370-2693(94)90338-7 [hep-th/9303145].
10. Andrea Cappelli, INFN Florence, "Topological Insulators in 3D and Bosonization"
11. Phys.Lett.B336:18-24,1994, M. Lüscher, Nuclear Physics B Volume 326, Issue 3, 13 November 1989, Pages 557-582 Nuclear Physics B "Bosonization in 2 + 1 dimensions"
12. "Fermionization in an Arbitrary Number of Dimensions" N.S. Mankoč Borštnik and H.B.F. Nielsen in Proceedings, 18th Workshop on "What Comes Beyond the Standard Models?" : Bled, Slovenia, July 11-19, 2015 Norma Susana Mankoč Borštnik (ed.) , Holger Bech Nielsen (ed.) , Dragan Lukman (ed.)(2015)
13. M. Kalb and Pierre Ramond, "Classical direct interstring action." Phys. Rev. D 9 (1974), 2273-2284.



# **Virtual Institute of Astroparticle Physics Presentation**





## 15 Scientific-Educational Platform of Virtual Institute of Astroparticle Physics and Studies of Physics Beyond the Standard Model

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**Abstract.** Being a unique multi-functional complex of science and education online, Virtual Institute of Astroparticle Physics (VIA) operates on website <http://viavca.in2p3.fr/site.html>. It supports presentation online for the most interesting theoretical and experimental results, participation online in conferences and meetings, various forms of collaborative scientific work as well as programs of education at distance, combining online videoconferences with extensive library of records of previous meetings and Discussions on Forum. Since 2014 VIA online lectures combined with individual work on Forum acquired the form of Open Online Courses. Aimed to individual work with students it is not Massive, but the account for the number of visits to VIA site converts VIA in a specific tool for MOOC activity. VIA sessions are now a traditional part of Bled Workshops' programme. At XX Bled Workshop it provided a world-wide discussion of the open questions of physics beyond the standard model, supporting world-wide propagation of the main ideas, presented at this meeting.

**Povzetek.** Virtual Institute of Astroparticle Physics (VIA) je večnamensko spletišče za znanost in izobraževanje na naslovu <http://viavca.in2p3.fr/site.html>. Podpira neposredne predstavitve najbolj zanimivih teoretičnih in eksperimentalnih rezultatov, sodelovanje v neposrednih konferencah in srečanjih, podporo za različne oblike znanstvenega sodelovanja, programe za izobraževanje na daljavo, pri čemer ponuja kombinacije videokonferenc z obširno knjižnico zapisov prejšnjih srečanj in diskusije na Forumu. Po letu 2014 so predavanja VIA na daljavo, kombinirana z individualnim delom na forumu, dobila obliko odprtih tečajev na daljavo. Ker cilja na individualno delo s posameznimi študenti, ni množicna, vendar je, glede na število obiskov spletišča VIA, le to postalo orodje za množične aktivnosti učenja na daljavo (MOOC). Seje VIA so postale tradicionalen del programa te blejske delavnice. Na letošni (jubilejni) dvajseti delavnici so omogočila diskusije o odprtih vprašanjih fizike onkraj standardnih modelov za udeležence iz vseh koncev sveta in razširjanje idej, predstavljenih na delavnici, po vsem svetu.

**Keywords:** Astroparticle physics, Physics beyond the Standard model, E-learning, E-science, MOOC

## 15.1 Introduction

Studies in astroparticle physics link astrophysics, cosmology, particle and nuclear physics and involve hundreds of scientific groups linked by regional networks (like ASPERA/ ApPEC [1,2]) and national centers. The exciting progress in these studies will have impact on the knowledge on the structure of microworld and Universe in their fundamental relationship and on the basic, still unknown, physical laws of Nature (see e.g. [3,4] for review). The progress of precision cosmology and experimental probes of the new physics at the LHC and in nonaccelerator experiments, as well as the extension of various indirect studies of physics beyond the Standard model involve with necessity their nontrivial links. Virtual Institute of Astroparticle Physics (VIA) [5] was organized with the aim to play the role of an unifying and coordinating platform for such studies.

Starting from the January of 2008 the activity of the Institute takes place on its website [6] in a form of regular weekly videoconferences with VIA lectures, covering all the theoretical and experimental activities in astroparticle physics and related topics. The library of records of these lectures, talks and their presentations was accomplished by multi-lingual Forum. Since 2008 there were **180 VIA online lectures**, VIA has supported distant presentations of **92 speakers at 23 Conferences** and provided transmission of talks at **61 APC Colloquiums**.

In 2008 VIA complex was effectively used for the first time for participation at distance in XI Bled Workshop [7]. Since then VIA videoconferences became a natural part of Bled Workshops' programs, opening the virtual room of discussions to the world-wide audience. Its progress was presented in [8–15]. Here the current state-of-art of VIA complex, integrated since 2009 in the structure of APC Laboratory, is presented in order to clarify the way in which discussion of open questions beyond the standard model at the XX Bled Workshop were presented with the of VIA facility to the world-wide audience.

## 15.2 VIA structure and its activity

### 15.2.1 VIA activity

The structure of VIA complex is illustrated by the Fig. 15.1. The home page, presented on this figure, contains the information on the coming and records of the latest VIA events. The menu links to directories (along the upper line from left to right): with general information on VIA (About VIA), entrance to VIA virtual rooms (Rooms), the library of records and presentations (Previous) of VIA Lectures (Previous → Lectures), records of online transmissions of Conferences (Previous → Conferences), APC Colloquiums (Previous → APC Colloquiums), APC Seminars (Previous → APC Seminars) and Events (Previous → Events), Calender of the past and future VIA events (All events) and VIA Forum (Forum). In the upper right angle there are links to Google search engine (Search in site) and to contact information (Contacts). The announcement of the next VIA lecture and VIA online transmission of APC Colloquium occupy the main part of the homepage with the record of the most recent VIA events below. In the announced time of the



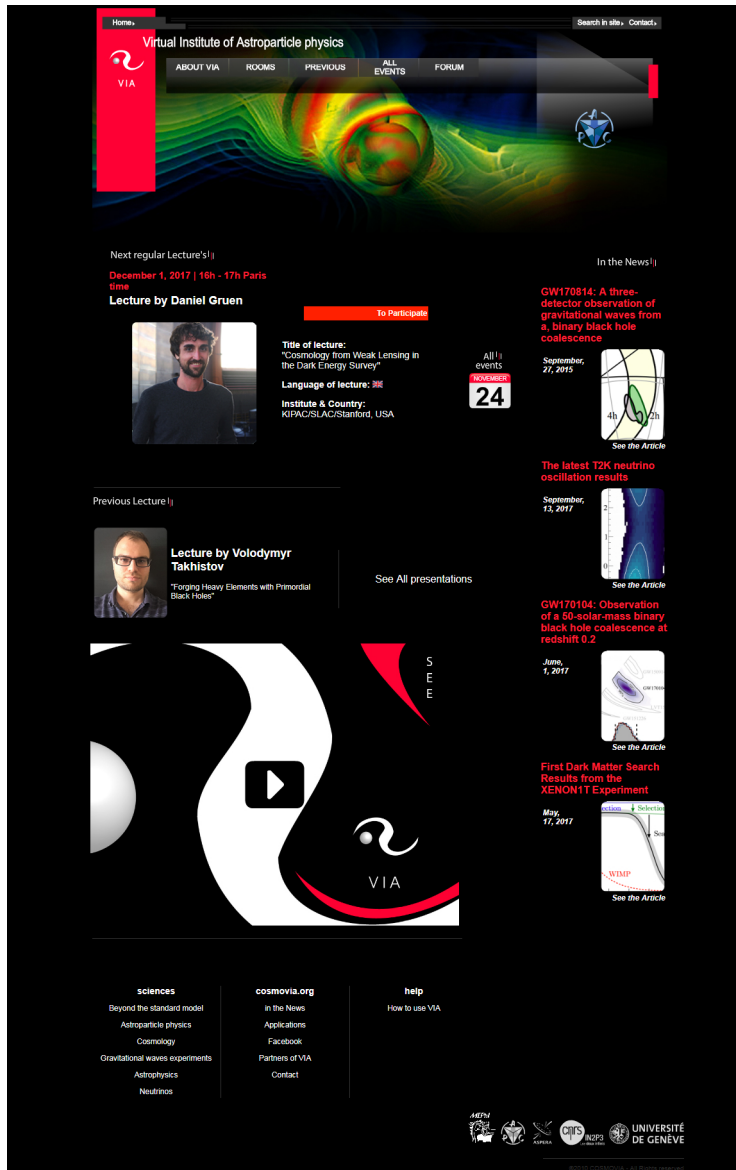


Fig. 15.1. The home page of VIA site

event (VIA lecture or transmitted APC Colloquium) it is sufficient to click on "to participate" on the announcement and to Enter as Guest (printing your name) in the corresponding Virtual room. The Calendar shows the program of future VIA lectures and events. The right column on the VIA homepage lists the announcements of the regularly up-dated hot news of Astroparticle physics and related areas.

In 2010 special COSMOVIA tours were undertaken in Switzerland (Geneva), Belgium (Brussels, Liege) and Italy (Turin, Pisa, Bari, Lecce) in order to test stability of VIA online transmissions from different parts of Europe. Positive results of these tests have proved the stability of VIA system and stimulated this practice at XIII Bled Workshop. The records of the videoconferences at the XIII Bled Workshop are available on VIA site [16].

Since 2011 VIA facility was used for the tasks of the Paris Center of Cosmological Physics (PCCP), chaired by G. Smoot, for the public programme "The two infinities" conveyed by J.L.Robert and for effective support a participation at distance at meetings of the Double Chooz collaboration. In the latter case, the experimentalists, being at shift, took part in the collaboration meeting in such a virtual way.

The simplicity of VIA facility for ordinary users was demonstrated at XIV Bled Workshop in 2011. Videoconferences at this Workshop had no special technical support except for WiFi Internet connection and ordinary laptops with their internal video and audio equipments. This test has proved the ability to use VIA facility at any place with at least decent Internet connection. Of course the quality of records is not as good in this case as with the use of special equipment, but still it is sufficient to support fruitful scientific discussion as can be illustrated by the record of VIA presentation "New physics and its experimental probes" given by John Ellis from his office in CERN (see the records in [17]).

In 2012 VIA facility, regularly used for programs of VIA lectures and transmission of APC Colloquiums, has extended its applications to support M.Khlopov's talk at distance at Astrophysics seminar in Moscow, videoconference in PCCP, participation at distance in APC-Hamburg-Oxford network meeting as well as to provide online transmissions from the lectures at Science Festival 2012 in University Paris7. VIA communication has effectively resolved the problem of referee's attendance at the defence of PhD thesis by Mariana Vargas in APC. The referees made their reports and participated in discussion in the regime of VIA videoconference. In 2012 VIA facility was first used for online transmissions from the Science Festival in the University Paris 7. This tradition was continued in 2013, when the transmissions of meetings at Journées nationales du Développement Logiciel (JDEV2013) at Ecole Polytechnique (Paris) were organized [19].

In 2013 VIA lecture by Prof. Martin Pohl was one of the first places at which the first hand information on the first results of AMS02 experiment was presented [18].

In 2014 the 100th anniversary of one of the founders of Cosmoparticle physics, Ya. B. Zeldovich, was celebrated. With the use of VIA M.Khlopov could contribute the programme of the "Subatomic particles, Nucleons, Atoms, Universe: Processes and Structure International conference in honor of Ya. B. Zeldovich 100th Anniversary" (Minsk, Belarus) by his talk "Cosmoparticle physics: the Universe as a laboratory of elementary particles" [20] and the programme of "Conference YaB-100, dedicated to 100 Anniversary of Yakov Borisovich Zeldovich" (Moscow, Russia) by his talk "Cosmology and particle physics" [21].

In 2015 VIA facility supported the talk at distance at All Moscow Astrophysical seminar "Cosmoparticle physics of dark matter and structures in the Universe"

by Maxim Yu. Khlopov and the work of the Section "Dark matter" of the International Conference on Particle Physics and Astrophysics (Moscow, 5-10 October 2015). Though the conference room was situated in Milan Hotel in Moscow all the presentations at this Section were given at distance (by Rita Bernabei from Rome, Italy; by Juan Jose Gomez-Cadenas, Paterna, University of Valencia, Spain and by Dmitri Semikoz, Martin Bucher and Maxim Khlopov from Paris) and its work was chaired by M. Khlopov from Paris [26]. In the end of 2015 M. Khlopov gave his distant talk "Dark atoms of dark matter" at the Conference "Progress of Russian Astronomy in 2015", held in Sternberg Astronomical Institute of Moscow State University.

In 2016 distant online talks at St. Petersburg Workshop "Dark Ages and White Nights (Spectroscopy of the CMB)" by Khatri Rishi (TIFR, India) "The information hidden in the CMB spectral distortions in Planck data and beyond", E. Kholupenko (Ioffe Institute, Russia) "On recombination dynamics of hydrogen and helium", Jens Chluba (Jodrell Bank Centre for Astrophysics, UK) "Primordial recombination lines of hydrogen and helium", M. Yu. Khlopov (APC and MEPHI, France and Russia) "Nonstandard cosmological scenarios" and P. de Bernardis (La Sapienza University, Italy) "Balloon techniques for CMB spectrum research" were given with the use of VIA system [27]. At the defense of PhD thesis by F. Gregis VIA facility made possible for his referee in California not only to attend at distance at the presentation of the thesis but also to take part in its successive jury evaluation.

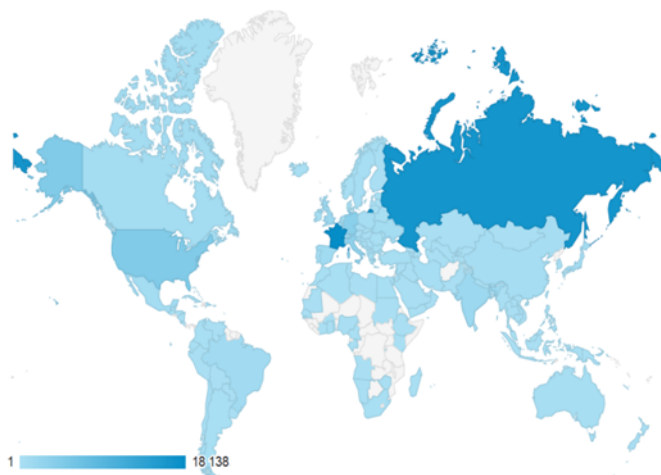
The discussion of questions that were put forward in the interactive VIA events is continued and extended on VIA Forum. Presently activated in English, French and Russian with trivial extension to other languages, the Forum represents a first step on the way to multi-lingual character of VIA complex and its activity. Discussions in English on Forum are arranged along the following directions: beyond the standard model, astroparticle physics, cosmology, gravitational wave experiments, astrophysics, neutrinos. After each VIA lecture its pdf presentation together with link to its record and information on the discussion during it are put in the corresponding post, which offers a platform to continue discussion in replies to this post.

### 15.2.2 VIA e-learning, OOC and MOOC

One of the interesting forms of VIA activity is the educational work at distance. For the last eight years M. Khlopov's course "Introduction to cosmoparticle physics" is given in the form of VIA videoconferences and the records of these lectures and their ppt presentations are put in the corresponding directory of the Forum [22]. Having attended the VIA course of lectures in order to be admitted to exam students should put on Forum a post with their small thesis. In this thesis students are proposed to chose some BSM model and to analyze its cosmological consequences. The list of possible topics for such thesis is proposed to students, but they are also invited to chose themselves any topic of their own on possible links between cosmology and particle physics. Professor's comments and proposed corrections are put in a Post reply so that students should continuously present on Forum improved versions of work until it is accepted as satisfactory. Then they are

admitted to pass their exam. The record of videoconference with their oral exam is also put in the corresponding directory of Forum. Such procedure provides completely transparent way of evaluation of students' knowledge.

Since 2014 the second semester of this course is given in English and converted in an Open Online Course. It was aimed to develop VIA system as a possible accomplishment for Massive Online Open Courses (MOOC) activity [23]. In 2016 not only students from Moscow, but also from France and Sri Lanka attended this course. In 2017 students from Moscow were accompanied by participants from France, Italy, Sri Lanka and India [24]. The students pretending to evaluation of their knowledge must write their small thesis, present it and being admitted to exam pass it in English. The restricted number of online connections to videoconferences with VIA lectures is compensated by the wide-world access to their records on VIA Forum and in the context of MOOC VIA Forum and videoconferencing system can be used for individual online work with advanced participants. Indeed Google analytics shows that since 2008 VIA site was visited by more than 226 thousand visitors from 152 countries, covering all the continents by its geography (Fig. 15.2). According to this statistics more than half of these visitors continued to enter VIA site after the first visit. Still the form of individual educa-



**Fig. 15.2.** Geography of VIA site visits according to Google statistics

tional work makes VIA facility most appropriate for PhD courses and it is planned to be involved in the International PhD program on Fundamental Physics to be in operation on the basis of Russian-French collaborative agreement. In 2017 the test for the ability of VIA to support fully distant education and evaluation of students (as well as for work on PhD thesis and its distant defense) was undertaken. Steve Branchu from France, who attended the Open Online Course and presented on Forum his small thesis has passed exam at distance. The whole procedure, starting from a stochastic choice of number of examination ticket, answers to ticket

questions, discussion by professors in the absence of student and announcement of result of exam to him was recorded and put on VIA Forum [25].

### 15.2.3 Organisation of VIA events and meetings

First tests of VIA system, described in [5,7–9], involved various systems of videoconferencing. They included skype, VRVS, EVO, WEBEX, marratech and adobe Connect. In the result of these tests the adobe Connect system was chosen and properly acquired. Its advantages are: relatively easy use for participants, a possibility to make presentation in a video contact between presenter and audience, a possibility to make high quality records, to use a whiteboard tools for discussions, the option to open desktop and to work online with texts in any format.

Initially the amount of connections to the virtual room at VIA lectures and discussions usually didn't exceed 20. However, the sensational character of the exciting news on superluminal propagation of neutrinos acquired the number of participants, exceeding this allowed upper limit at the talk "OPERA versus Maxwell and Einstein" given by John Ellis from CERN. The complete record of this talk and is available on VIA website [28]. For the first time the problem of necessity in extension of this limit was put forward and it was resolved by creation of a virtual "infinity room", which can host any reasonable amount of participants. Starting from 2013 this room became the only main virtual VIA room, but for specific events, like Collaboration meetings or transmissions from science festivals, special virtual rooms can be created. This solution strongly reduces the price of the licence for the use of the adobeConnect videoconferencing, retaining a possibility for creation of new rooms with the only limit to one administrating Host for all of them.

The ppt or pdf file of presentation is uploaded in the system in advance and then demonstrated in the central window. Video images of presenter and participants appear in the right window, while in the lower left window the list of all the attendees is given. To protect the quality of sound and record, the participants are required to switch out their microphones during presentation and to use the upper left Chat window for immediate comments and urgent questions. The Chat window can be also used by participants, having no microphone, for questions and comments during Discussion. The interactive form of VIA lectures provides oral discussion, comments and questions during the lecture. Participant should use in this case a "raise hand" option, so that presenter gets signal to switch out his microphone and let the participant to speak. In the end of presentation the central window can be used for a whiteboard utility as well as the whole structure of windows can be changed, e.g. by making full screen the window with the images of participants of discussion.

Regular activity of VIA as a part of APC includes online transmissions of all the APC Colloquiums and of some topical APC Seminars, which may be of interest for a wide audience. Online transmissions are arranged in the manner, most convenient for presenters, prepared to give their talk in the conference room in a normal way, projecting slides from their laptop on the screen. Having uploaded in advance these slides in the VIA system, VIA operator, sitting in the

conference room, changes them following presenter, directing simultaneously webcam on the presenter and the audience.

### 15.3 VIA Sessions at XX Bled Workshop

VIA sessions of XX Bled Workshop have developed from the first experience at XI Bled Workshop [7] and their more regular practice at XII, XIII, XIV, XV, XVI, XVII, XVIII and XIX Bled Workshops [8–15]. They became a regular part of the Bled Workshop's program.

In the course of XX Bled Workshop meeting the list of open questions was stipulated, which was proposed for wide discussion with the use of VIA facility. The list of these questions was put on VIA Forum (see [29]) and all the participants of VIA sessions were invited to address them during VIA discussions. During the XX Bled Workshop the announcement of VIA sessions was put on VIA home page, giving an open access to the videoconferences at VIA sessions. Though the experience of previous Workshops principally confirmed a possibility to provide effective interactive online VIA videoconferences even in the absence of any special equipment and qualified personnel at place, VIA Sessions were directed at XX Workshop by M.Khlopov at place. Only laptop with microphone and webcam together with WiFi Internet connection was proved to support not only attendance, but also VIA presentations and discussions.

In the framework of the program of XX Bled Workshop, M. Khlopov, gave his part of talk "Search for double charged particles as direct test for Dark Atom Constituents" (Fig. 15.3), being at Bled, while his co-author Yu.Smirnov continued the talk from Moscow. VIA session also included talk of distant participants of the Workshop A.Djouadi "A deeper probe of new physics scenarii at the LHC" (Fig. 15.4). It provided an additional demonstration of the ability of VIA to support the creative non-formal atmosphere of Bled Workshops (see records in [30]).

The talk "Do no observations so far of the fourth family quarks speak against the spin-charge-family theory?" by Norma Mankoc-Borstnik (Fig. 15.5) was given at Bled, inviting distant participants to join the discussion.

The records of all these lectures and discussions can be found in VIA library [30].

### 15.4 Conclusions

The Scientific-Educational complex of Virtual Institute of Astroparticle physics provides regular communication between different groups and scientists, working in different scientific fields and parts of the world, the first-hand information on the newest scientific results, as well as support for various educational programs at distance. This activity would easily allow finding mutual interest and organizing task forces for different scientific topics of astroparticle physics and related topics. It can help in the elaboration of strategy of experimental particle, nuclear, astrophysical and cosmological studies as well as in proper analysis of experimental data. It can provide young talented people from all over the world to get the

**Nuclear-interacting composite dark matter:  
O-helium « atoms »**

If we have a stable double charged particle  $X^{--}$  in excess over its partner  $X^{++}$  it may create Helium like neutral atom (O-helium) at temperature  $T > T_o$

Where:  $I_o = Z_{He}^2 Z_{\alpha}^2 \alpha^2 m_{He} = 1.6 \text{ MeV}$

$^4\text{He}$  is formed at  $T \sim 100 \text{ keV}$  ( $t \sim 100 \text{ s}$ )  
This means that it would rapidly create a neutral atom, in which all  $X^{--}$  are bound

$$X^{--} + ^4\text{He} \Rightarrow (X \text{ He}) + \gamma$$

The Bohr orbit of O-helium « atom » is of the order of radius of helium nucleus.

$$R_o = 1 / (ZZ_{He} \alpha m_{He}) = 2 \cdot 10^{-13} \text{ cm}$$

**References**  
1. M.Yu. Khlopov, JETP Lett. 83 (2006) 1;  
2. D. Farrier, M. Khlopov, C. Stephens, Class. Quantum Grav. 23 (2006) 7305;  
3. M. Y. Khlopov and C. Kocavut, Phys. Rev. D 77 (2008) 085002

29

**Fig. 15.3.** VIA talk "Search for double charged particles as direct test for Dark Atom Constituents" was started by M.Khlopov in Bled and continued by Yu.Smirnov from Moscow at XX Bled Workshop

**4. Indirect searches for new physics**

Will  $D_{17}$  be the  $g-2$  of the LHC? Yes, if measured at 1% level!

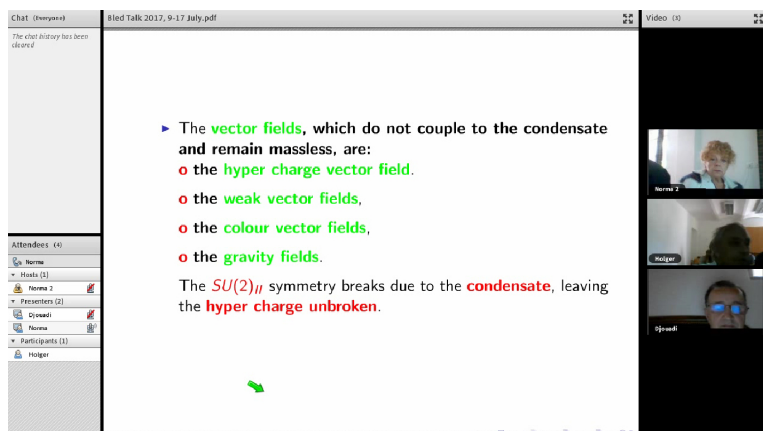
Examples of BSM searches: AD, Quevillon, Vega-Morales, 1509.03913

Bled 12/07/2017 A deeper probe of new physics – A. Djouadi – p.21/22

**Fig. 15.4.** VIA talk from Paris by A.Djouadi "A deeper probe of new physics scenarii at the LHC" at XX Bled Workshop

highest level education, come in direct interactive contact with the world known scientists and to find their place in the fundamental research. These educational aspects of VIA activity is now being evolved in a specific tool for International PhD programme for Fundamental physics. VIA applications can go far beyond the particular tasks of astroparticle physics and give rise to an interactive system of mass media communications.

VIA sessions became a natural part of a program of Bled Workshops, maintaining the platform of discussions of physics beyond the Standard Model for distant participants from all the world. This discussion can continue in posts and post replies on VIA Forum. The experience of VIA applications at Bled Workshops



**Fig. 15.5.** VIA talk "Spin-charge-family theory explains all the assumptions of the standard model, offers explanation for the dark matter, for the matter/antimatter asymmetry, explains miraculous triangle anomaly cancellation,...making several predictions" by N. Mankoč-Borštnik at XX Bled Workshop

plays important role in the development of VIA facility as an effective tool of e-science and e-learning.

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## References

1. <http://www.aspera-eu.org/>
2. <http://www.appec.org/>
3. M.Yu. Khlopov: *Cosmoparticle physics*, World Scientific, New York -London-Hong Kong - Singapore, 1999.
4. M.Yu. Khlopov: *Fundamentals of Cosmic Particle Physics*, CISP-Springer, Cambridge, 2012.
5. M. Y. Khlopov, Project of Virtual Institute of Astroparticle Physics, arXiv:0801.0376 [astro-ph].



6. <http://viavca.in2p3.fr/site.html>
7. M. Y. Khlopov, Scientific-educational complex - virtual institute of astroparticle physics, Bled Workshops in Physics **9** (2008) 81–86.
8. M. Y. Khlopov, Virtual Institute of Astroparticle Physics at Bled Workshop, Bled Workshops in Physics **10** (2009) 177–181.
9. M. Y. Khlopov, VIA Presentation, Bled Workshops in Physics **11** (2010) 225–232.
10. M. Y. Khlopov, VIA Discussions at XIV Bled Workshop, Bled Workshops in Physics **12** (2011) 233–239.
11. M. Y. .Khlopov, Virtual Institute of astroparticle physics: Science and education online, Bled Workshops in Physics **13** (2012) 183–189.
12. M. Y. .Khlopov, Virtual Institute of Astroparticle physics in online discussion of physics beyond the Standard model, Bled Workshops in Physics **14** (2013) 223–231.
13. M. Y. .Khlopov, Virtual Institute of Astroparticle physics and “What comes beyond the Standard model?” in Bled, Bled Workshops in Physics **15** (2014) 285–293.
14. M. Y. .Khlopov, Virtual Institute of Astroparticle physics and discussions at XVIII Bled Workshop, Bled Workshops in Physics **16** (2015) 177–188.
15. M. Y. .Khlopov, Virtual Institute of Astroparticle Physics Scientific-Educational Platform for Physics Beyond the Standard Model Bled Workshops in Physics **17** (2016) 221–231.
16. In <http://viavca.in2p3.fr/> Previous - Conferences - XIII Bled Workshop
17. In <http://viavca.in2p3.fr/> Previous - Conferences - XIV Bled Workshop
18. In <http://viavca.in2p3.fr/> Previous - Lectures - Martin Pohl
19. In <http://viavca.in2p3.fr/> Previous - Events - JDEV 2013
20. In <http://viavca.in2p3.fr/> Previous - Conferences - Subatomic particles, Nucleons, Atoms, Universe: Processes and Structure International conference in honor of Ya. B. Zeldovich 100th Anniversary
21. In <http://viavca.in2p3.fr/> Previous - Conferences - Conference YaB-100, dedicated to 100 Anniversary of Yakov Borisovich Zeldovich
22. In <http://viavca.in2p3.fr/> Forum - Discussion in Russian - Courses on Cosmoparticle physics
23. In <http://viavca.in2p3.fr/> Forum - Education - From VIA to MOOC
24. In <http://viavca.in2p3.fr/> Forum - Education - Lectures of Open Online VIA Course 2017
25. In <http://viavca.in2p3.fr/> Forum - Education - Small thesis and exam of Steve Branchu
26. <http://viavca.in2p3.fr/> Previous - Conferences - The International Conference on Particle Physics and Astrophysics
27. <http://viavca.in2p3.fr/> Previous - Conferences - Dark Ages and White Nights (Spectroscopy of the CMB)
28. In <http://viavca.in2p3.fr/> Previous - Lectures - John Ellis
29. In <http://viavca.in2p3.fr/> Forum - CONFERENCES BEYOND THE STANDARD MODEL - XX Bled Workshop “What comes beyond the Standard model?”
30. In <http://viavca.in2p3.fr/> Previous - Conferences - XX Bled Workshop “What comes beyond the Standard model?”

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