IMFM
Institute of Mathematics, Physics and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series
Vol. 51 (2013), 1188
ISSN 2232-2094

INVERSE LIMITS IN THE CATEGORY OF COMPACT HAUSDORFF SPACES AND UPPER SEMICONTINUOUS FUNCTIONS

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Ljubljana, May 17, 2013

Inverse limits in the category of compact Hausdorff spaces and upper semicontinuous functions

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Abstract

We investigate inverse limits in the category $\mathcal{CHU}$ of compact Hausdorff spaces with upper semicontinuous (usc) functions. We introduce the notion of weak inverse limits in this category and show that the inverse limits with upper semicontinuous set-valued bonding functions (as they were defined in [15]) together with the projections are not necessarily inverse limits in $\mathcal{CHU}$ but they are always weak inverse limits in this category. This is a realization of our categorical approach to solving a problem stated by W. T. Ingram in [14].

Keywords: Upper semi-continuous functions, Inverse limits, Weak inverse limits

2000 Mathematics Subject Classification: primary 54C60; secondary 54B30

1 Introduction

W. T. Ingram in his book [14] states the following problem:

> **Problem 6.63.** What can be said about inverse limits with set-valued functions if the underlying directed set is not a sequence of integers?

In this paper we present a categorical approach to solving the above problem.

Consider an inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ of compact Hausdorff spaces and continuous bonding functions. It is a well-known fact that the space

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) =$$

Preprint series, IMFM, ISSN 2232-2094, no. 1188, May 17, 2013

$$\{(x_\gamma)_{\gamma\in A} \in \prod_{\gamma\in A} X_\alpha \mid \text{for all } \alpha,\beta \in A, \alpha < \beta, x_\alpha = f_{\alpha\beta}(x_\beta)\}$$

together with the projection mappings $p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) \to X_\gamma$, $p_\gamma((x_\alpha)_{\alpha\in A}) = x_\gamma$, is in fact an inverse limit in the category $\mathcal{CHC}$ of compact Hausdorff spaces with continuous functions.

In present paper we extend the category $\mathcal{CHC}$ to the category $\mathcal{CHU}$ of compact Hausdorff spaces with usc functions in such a way that $\mathcal{CHC}$ is interpreted as a proper subcategory of $\mathcal{CHU}$. This can be done since every continuous function between compact Hausdorff spaces can be interpreted as a usc function.

As one of our main results we show that the inverse limits with upper semicontinuous set-valued bonding functions

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) =$$

$$\{(x_\gamma)_{\gamma\in A} \in \prod_{\gamma\in A} X_\alpha \mid \text{for all } \alpha,\beta \in A, \alpha < \beta, x_\alpha \in f_{\alpha\beta}(x_\beta)\}$$

together with the projections

$$p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) \to X_\gamma,$$

$$p_\gamma((x_\alpha)_{\alpha\in A}) = \{x_\gamma\},$$

are not necessarily inverse limits in the category but they are always so called weak inverse limits in $\mathcal{CHU}$.

In the second section we give the basic definitions that are used in the paper.

In the third section we give a detailed description of the category $\mathcal{CHU}$ of compact Hausdorff spaces with usc bonding functions.

In the fourth section we give results about inverse limits in the category $\mathcal{CHU}$.

In the last section we define objects in category $\mathcal{CHU}$ that are called weak inverse limits in this category. We also show that for any inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ in $\mathcal{CHU}$, the corresponding inverse limit with upper semicontinuous set-valued bonding functions together with projections is always a weak inverse limit in category $\mathcal{CHU}$.

2 Definitions and notation

For any category $\mathcal{K}$ the class of objects of $\mathcal{K}$ will be denoted by $Ob(\mathcal{K})$, the class of morphisms of $\mathcal{K}$ by $Mor(\mathcal{K})$, and the partial binary associative operation (composition of morphisms) by $\circ$. For any $X \in Ob(\mathcal{K})$ the identity morphism on X will be denoted by $1_X : X \to X$.

For a directed set A (A is nonempty and equipped with a reflexive and transitive binary relation $\leq$ with the property that every pair of elements has an upper bound), a family of objects $\{X_\alpha \mid \alpha \in A\}$ of $\mathcal{K}$, and a family of morphisms $\{f_{\alpha\beta} : X_\beta \to X_\alpha \mid \alpha, \beta \in A, \alpha \leq \beta\}$ of $\mathcal{K}$, such that

1. for each $\alpha \in A$, $f_{\alpha\alpha} = 1_{X_\alpha}$,
2. for each $\alpha, \beta, \gamma \in A$, from $\alpha \leq \beta \leq \gamma$ it follows that $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$,

we call an inverse system (in $\mathcal{K}$) and denote it by

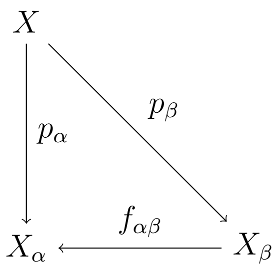
$$(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A}).$$

We assume throughout the paper that A is cofinite, i.e. every $\alpha \in A$ has at most finitely many predecessors.

Next we define inverse limits in $\mathcal{K}$.

Definition 2.1. *An object $X \in Ob(\mathcal{K})$, together with morphisms $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$ is an inverse limit of an inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$ in the category $\mathcal{K}$, if*

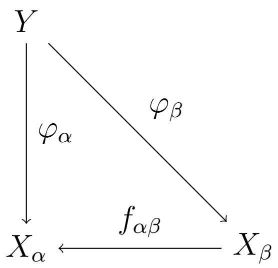
1. *for all $\alpha, \beta \in A$, from $\alpha \leq \beta$ it follows that the diagram*



(1)

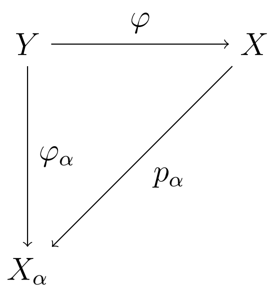
   commutes;

2. *for any object $Y \in \mathcal{K}$ and any family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ it follows that if the diagram*



(2)

commutes, then there is a unique morphism $\varphi : Y \to X$ *such that for each* $\alpha \in A$ *the diagram*



(3)

commutes.

A *map* or *mapping* is a continuous function.

If X is a compact Hausdorff space, then 2^X denotes the set of all nonempty closed subsets of X.

The *graph* $\Gamma(f)$ of a function $f : X \to 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

A function $f : X \to 2^Y$ is *upper semi-continuous* function if for each $x \in X$ and for each open set $U \subseteq Y$ such that $f(x) \subseteq U$ there is an open set V in X such that

1. $x \in V$;

2. for all $v \in V$ it holds that $f(v) \subseteq U$.

The following is a well-known characterization of usc functions between Hausdorff compacta (see [15, p. 120, Theorem 2.1]).

Theorem 2.2. *Let* X *and* Y *be compact Hausdorff spaces and* $f : X \to 2^Y$ *a function. Then* f *is usc if and only if its graph* $\Gamma(f)$ *is closed in* $X \times Y$.

At the end of this section we introduce the notion of inverse limits with usc set-valued bonding functions as it was introduced by Mahavier in [19] and Ingram and Mahavier in [15]. In the last section we use this notion as a motivation for defining inverse limits with usc set-valued bonding functions for arbitrary inverse systems.

An *inverse sequence* of compact Hausdorff spaces X_k with usc bonding functions f_k is a sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $f_k : X_{k+1} \to 2^{X_k}$ is usc for each k.

The *inverse limit with usc set-valued bonding functions* of an inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ is defined to be the subspace of the product space

$\prod_{k=1}^{\infty} X_k$ of all $x = (x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X_k$, such that $x_k \in f_k(x_{k+1})$ for each k. The inverse limit of $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted by $\varprojlim\{X_k, f_k\}_{k=1}^{\infty}$.

Since the introduction of such inverse limits, there has been much interest in the subject and many papers appeared [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 17, 18, 22, 23, 24, 25, 26].

3 The category $\mathcal{CHU}$

The category $\mathcal{CHU}$ of compact Hausdorff spaces and usc functions consists of the following objects and morphisms:

1. $Ob(\mathcal{CHU})$: compact Hausdorff spaces;
2. $Mor(\mathcal{CHU})$: the usc functions from X to Y is the set of morphisms from X to Y, denoted by $Mor(\mathcal{CHU})(X, Y)$.

We also define the partial binary operation $\circ$ (the composition) as follows. For each $f \in Mor(\mathcal{CHU})(X, Y)$ and each $g \in Mor(\mathcal{CHU})(Y, Z)$ we define $g \circ f \in Mor(\mathcal{CHU})(X, Z)$ by

$$(g \circ f)(x) = g(f(x)) = \bigcup_{y \in f(x)} g(y)$$

for each $x \in X$.

Theorem 3.1. *$\mathcal{CHU}$ is a category.*

Proof. First we show that $\circ$ is well-defined. Let $f : X \to Y$ and $g : Y \to Z$ be any morphisms. Let also $x \in X$ be arbitrary and let U be an open set in Z such that $(g \circ f)(x) \subseteq U$. Since g is usc and $f(x) \subseteq Y$, it holds that for each $y \in f(x)$ there is an open set W_y in Y such that

1. $y \in W_y$;
2. for all $w \in W_y$ it holds that $g(w) \subseteq U$.

Let $W = \bigcup_{y \in f(x)} W_y$. Since W is open in Y, $f(x) \subseteq W$, and since f is usc, there is an open set V in X such that

1. $x \in V$;
2. for all $v \in V$ it holds that $f(v) \subseteq W$.

Let $v \in V$ be arbitrary. Then

$$(g \circ f)(v) = g(f(v)) = \bigcup_{z \in f(v)} g(z) \subseteq U$$

since for each $z \in f(v)$, it holds that $g(z) \subseteq U$. Therefore $\circ$ is well-defined.

It is obvious that the composition $\circ$ of usc functions is an associative operation.

All that is left to show is that for each $X \in Ob(\mathcal{CHU})$ there is a morphism $1_X : X \to X$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any morphisms $f : Y \to X$ and $g : X \to Z$. We easily see that the identity map $1_X : X \to X$, defined by $1_X(x) = \{x\}$ for each $x \in X$, is the usc function satisfying the above conditions. □

4 Inverse limits in $\mathcal{CHU}$

In this section we show that if $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$ is an inverse system of compact Hausdorff spaces and usc set-valued bonding functions, then

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$$

(see Definition 4.1) together with the projections is not necessarily an inverse limit in the category $\mathcal{CHU}$.

Motivated by [15, 19], we define in Definition 4.1 objects in $\mathcal{CHU}$, that are called inverse limits with usc set-valued bonding functions. Since such object were first introduced by Mahavier in [19] and Ingram and Mahavier in [15], where they call them the inverse limits with usc set-valued bonding functions, we continue to use the same name for them.

Definition 4.1. *Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$ be any inverse system in $\mathcal{CHU}$. We call the object*

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A}) = \{x \in \prod_{\alpha \in A} X_\alpha \mid \textit{for all } \alpha < \beta,\ x_\alpha \in f_{\alpha\beta}(x_\beta)\}$$

an inverse limit with usc set-valued bonding functions.

In the following theorem we prove that $\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$ is really an object of $\mathcal{CHU}$.

Theorem 4.2. *Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$ be any inverse system in $\mathcal{CHU}$. Then the inverse limit with usc set-valued bonding functions*

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$$

is a compact Hausdorff space.

Proof. For each $\gamma \in A$, X_γ is a compact Hausdorff space, therefore the product $\prod_{\gamma\in A} X_\gamma$ is a compact Hausdorff space. Since $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is a subspace of the Hausdorff space, it is also a Hausdorff space. We show that $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is a closed subset of the compact space $\prod_{\gamma\in A} X_\gamma$ to show that it is compact.

Let for all $\alpha, \beta \in A$, $\alpha < \beta$,

$$G_{\alpha\beta} = \Gamma(f_{\alpha\beta}) \times \prod_{\gamma\in A\setminus\{\alpha,\beta\}} X_\gamma = \{x \in \prod_{\gamma\in A} X_\gamma \mid x_\alpha \in f_{\alpha\beta}(x_\beta)\}.$$

Since the graph $\Gamma(f_{\alpha\beta})$ of $f_{\alpha\beta}$ is by Theorem 2.2 a closed subset of $X_\beta \times X_\alpha$, $G_{\alpha\beta}$ is also a closed subset of $\prod_{\gamma\in A} X_\gamma$. It is obvious that

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) = \bigcap_{\alpha,\beta\in A,\alpha<\beta} G_{\alpha\beta}$$

and hence $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is a a closed subset of $\prod_{\gamma\in A} X_\gamma$. □

In the following example we construct an inverse limit with usc set-valued bonding functions that is not an inverse limit in $\mathcal{CHU}$ regardless of the choice of morphisms $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$.

Example 4.3. *Let $A = \mathbb{N}$, $X_k = [0,1]$, and let $f_{k(k+1)} = f$ for each $k \in \mathbb{N}$, where $f : [0,1] \to 2^{[0,1]}$ is the function on $[0,1]$ defined by its graph*

$$\Gamma(f) = \{(t,t) \in [0,1] \times [0,1] \mid t \in [0,1]\} \cup (\{1\} \times [0,1]).$$

Also let $X = \varprojlim(\mathbb{N}, \{[0,1]\}_{k\in\mathbb{N}}, \{f_{k\ell}\}_{k,\ell\in\mathbb{N}})$ and let $\{p_i : X \to X_i \mid i \in \mathbb{N}\}$ be any set of morphisms in $\mathcal{CHU}$, such that the diagrams (1) always commute. We show that X with $\{p_i : X \to X_i \mid i \in \mathbb{N}\}$ is not an inverse limit of $(\mathbb{N}, \{[0,1]\}_{k\in\mathbb{N}}, \{f_{k\ell}\}_{k,\ell\in\mathbb{N}})$ in $\mathcal{CHU}$. Let $Y = [0,1]$ be an object in $\mathcal{CHU}$ and let $\{\varphi_k : Y \to X_k \mid k \in \mathbb{N}\}$ be the family of morphisms where $\varphi_k(t) = [0,1]$ for each k and each $t \in Y$. The diagram (2) always commutes. We distinguish the following two cases.

1. *If there is a positive integer i_0, such that $1 \notin p_{i_0}(x)$ for each $x \in X$, then suppose that Φ is any morphism $Y \to X$. Then $\varphi_{i_0}(t) = [0,1]$ but $1 \notin p_{i_0}(\Phi(t))$ for any $t \in Y$. Therefore the diagram (3) does not commute for $\alpha = i_0$.*

2. *If for each positive integer i there is $x^i \in X$ such that $1 \in p_i(x^i)$, then let $s \in X$ be an accumulation point of the sequence $\{x^i\}_{i=1}^\infty$. We show*

first that $p_i(s) = [0,1]$ *for each* i*. Let* k *be any positive integer. Then for each* $\ell > k$*, it follows from*

$$[0,1] \supseteq p_k(x^\ell) = f_{k\ell}(p_\ell(x^\ell)) \supseteq f_{k\ell}(1) \supseteq [0,1]$$

that $p_k(x^\ell) = [0,1]$*. Let* $\{n_i\}_{i=1}^\infty$ *be any increasing sequence of positive integers such that*

- $n_i > k$ *for each* i*;*
- $\lim_{i\to\infty} x^{n_i} = s$*.*

It follows from $p_k(x^{n_i}) = [0,1]$ *that* $\{x^{n_i}\} \times [0,1] \subseteq \Gamma(p_k)$ *for each* i*. This means that for each* $t \in [0,1]$*, the point* $(x^{n_i}, t) \in \Gamma(p_k)$*. Therefore* $\lim_{i\to\infty}(x^{n_i}, t) = (s,t) \in \Gamma(p_k)$ *for each* t*, since* $\Gamma(p_k)$ *is a closed subset of* $X \times [0,1]$*. It follows that* $\{s\} \times [0,1] \subseteq \Gamma(p_k)$ *and hence* $p_k(s) = [0,1]$*.*

Next, let $\Phi, \Psi : Y \to X$ *be the morphisms in* $\mathcal{CHU}$*, defined by*

$$\Phi(t) = X,$$

$$\Psi(t) = \{s\}$$

for each $t \in Y$*. It follows from*

$$p_k(\Phi(t)) = p_k(X) = [0,1] = \varphi_k(t)$$

and

$$p_k(\Psi(t)) = p_k(\{s\}) = [0,1] = \varphi_k(t)$$

that the diagram (3) commutes for both $\varphi = \Phi$ *and* $\varphi = \Psi$*. Therefore there is no unique morphism* φ *such that all diagrams (3) commute.*

Note that in the second part of Example 4.3, $\Psi(t) \subseteq \Phi(t) = (\prod_{k=1}^\infty \varphi_k(t)) \cap X$ holds true for each $t \in Y$. The following lemma shows that such an inclusion is not accidental. It will be used in the proof of Theorem 5.5.

Lemma 4.4. *Let* $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ *be any inverse system in* $\mathcal{CHU}$ *and let* $X = \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$*. Suppose that for an object* Y *of* $\mathcal{CHU}$ *and a family of morphisms* $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ *the diagram (2) commutes for any* α *and* β*,* $\alpha < \beta$*. Then* $\varphi : Y \to X$*, defined by* $\varphi(y) = (\prod_{\gamma\in A} \varphi_\gamma(y)) \cap X$ *for each* $y \in Y$*, is a morphism in* $\mathcal{CHU}$ *such that for each* $\alpha \in A$ *the diagram (3) commutes. Even more, for any morphism* $\Psi : Y \to X$ *such that* $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$ *for each* $\alpha \in A$ *and for each* $y \in Y$*,* $\Psi(y) \subseteq \varphi(y)$ *holds true for all* $y \in Y$*.*

Proof. We show that φ satisfies all the conditions in the following steps.

1. The set $\prod_{\gamma\in A}\varphi_\gamma(y)$ is a closed subset of $\prod_{\alpha\in A}X_\alpha$, therefore $\varphi(y)$ is a closed subset of X for any $y\in Y$.

2. Next we show that for any $y\in Y$, the set $\varphi(y)$ is nonempty. Let $y\in Y$ be arbitrarily chosen. Next, let for each positive integer n, $A_n\subseteq A$ be the set of all elements $\alpha\in A$ that have exactly $n-1$ predecessors. For any $\alpha\in A_1$ we arbitrarily choose $t_\alpha\in\varphi_\alpha(y)$. For any $\beta\in A_2$ there is an $\alpha\in A_1$ such that $\alpha<\beta$. For any such α and β it follows from $t_\alpha\in\varphi_\alpha(y)\subseteq f_{\alpha\beta}(\varphi_\beta(y))$ that there is $t_\beta\in\varphi_\beta(y)$ such that $t_\alpha\in f_{\alpha\beta}(t_\beta)$. We choose and fix such t_β for each $\beta\in A_2$. Suppose that we have already constructed $t_\alpha\in\varphi_\alpha(y)$ for all $\alpha\in A_n$. Then for any $\beta\in A_{n+1}$ there is an $\alpha\in A_n$ such that $\alpha<\beta$. For any such α and β it follows from $t_\alpha\in\varphi_\alpha(y)\subseteq f_{\alpha\beta}(\varphi_\beta(y))$ that there is $t_\beta\in\varphi_\beta(y)$ such that $t_\alpha\in f_{\alpha\beta}(t_\beta)$. We choose and fix such t_β for each $\beta\in A_{n+1}$.

   Then obviously $x=(t_\alpha)_{\alpha\in A}\in\varphi(y)$ and therefore $\varphi(y)$ is nonempty.

3. We show that φ is a usc function. Let $y\in Y$ be arbitrary point and let

   $$U=(U_{\gamma_1}\times U_{\gamma_2}\times U_{\gamma_3}\times\cdots\times U_{\gamma_n})\times\prod_{\gamma\in A\setminus\{\gamma_1,\gamma_2,\ldots,\gamma_n\}}X_\gamma$$

   be an open set in X such that $\varphi(y)\subseteq U$, where for each $i=1,2,3,\ldots,n$, U_{γ_i} is an open set in X_{γ_i}. It follows from the definitions of φ and U that $\varphi_{\gamma_i}(y)\subseteq U_{\gamma_i}$ for each $i=1,2,3,\ldots,n$. Since each φ_{γ_i} is usc, there is an open set V_i in Y such that

   (a) $y\in V_i$;

   (b) for each $x\in V_i$, it holds that $\varphi_{\gamma_i}(x)\subseteq U_{\gamma_i}$

   for each i. We define $V=\bigcap_{i=1}^n V_i$. Then V is an open set in Y for which

   (a) $y\in V$;

   (b) for each $x\in V$, it holds that $\varphi(x)=\prod_{\gamma\in A}\varphi_\gamma(x)\subseteq U$

   holds true. Therefore φ is a usc function and so it is a morphism from Y to X.

4. Next we show that the diagram (3) commutes, i.e. for any $\alpha\in A$ and any $y\in Y$, $\varphi_\alpha(y)=(p_\alpha\circ\varphi)(y)$ holds true. Choose any $\alpha\in A$ and any

$y \in Y$. Obviously

$$p_\alpha(\varphi(y)) = p_\alpha((\prod_{\gamma\in A} \varphi_\gamma(y)) \cap X) \subseteq p_\alpha(\prod_{\gamma\in A} \varphi_\gamma(y)) = \varphi_\alpha(y).$$

Next we show that $\varphi_\alpha(y) \subseteq p_\alpha(\varphi(y))$. Let $z \in \varphi_\alpha(y)$ be arbitrarily chosen. We show that $z \in p_\alpha(\varphi(y))$ by showing that there is a point $x \in \varphi(y)$ such that $z \in p_\alpha(x)$. Let k be the positive integer such that $\alpha \in A_k$. For each $\gamma \in A_k \setminus \{\alpha\}$ let $t_\gamma \in \varphi_\gamma(y)$ be arbitrary and let $t_\alpha = z$. For each $\gamma \in A_{k-1}$ we choose $t_\gamma \in \varphi_\gamma(y)$ such that if $\alpha \in A_{k-1}$, $\beta \in A_k$, and $\alpha < \beta$, then $t_\alpha \in f_{\alpha\beta}(t_\beta)$. This can be done since $f_{\alpha\beta}(\varphi_\beta(y)) = \varphi_\alpha(y)$ and therefore $f_{\alpha\beta}(t_\beta) \subseteq \varphi_\alpha(y)$.

Continuing in the same fashion we choose for each $i = 1, 2, 3, \dots, k-1$ and each $\gamma \in A_i$ an element $t_\gamma \in \varphi_\gamma(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$ for each $\alpha \in A_i$, $\beta \in A_{i+1}$, $\alpha < \beta$.

Next, for each $\beta \in A_{k+1}$ and for each $\alpha \in A_k$ such that $\beta > \alpha$, since $t_\alpha \in \varphi_\alpha(y) = f_{\alpha\beta}(\varphi_\beta(y))$, there is $t_\beta \in \varphi_\beta(y)$, such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$.

We continue inductively in the same fashion and choose for each $i = k+1, k+2, k+3, \dots$ and each $\beta \in A_{i+1}$ an element $t_\beta \in \varphi_\alpha(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$ for each $\alpha \in A_i$, such that $\alpha < \beta$.

Let $x \in \prod_{\gamma\in A} X_\gamma$ be such an element that $p_\gamma(x) = \{t_\gamma\}$ for each $\gamma \in A$. It follows from the construction of x that $x \in \varphi(y)$ and $z \in p_\alpha(x)$.

5. Suppose that $\psi : Y \to X$ is a morphism in $\mathcal{CHU}$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$. Let $y \in Y$ be arbitrary and let $z \in \psi(y)$. Obviously $z \in X$ since ψ is a morphism from Y to X. It follows from $p_\alpha(z) \subseteq p_\alpha(\psi(y)) = \varphi_\alpha(y)$ (for each α) that $z \in \prod_{\gamma\in A} \varphi_\gamma(y)$. Therefore $z \in \varphi(y)$ and hence $\psi(y) \subseteq \varphi(y)$.

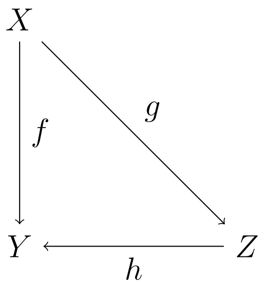
□

5 Weak inverse limits in $\mathcal{CHU}$

In this section we introduce the notion of weak inverse limits in $\mathcal{CHU}$ and show that $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ (together with the projections) is always a weak inverse limit in $\mathcal{CHU}$.

In Definition 5.1 we define a weak commutation of a diagram in the category $\mathcal{CHU}$.

Definition 5.1. *Let $X, Y, Z \in Ob(\mathcal{CHU})$ and let $f : X \to Y$, $g : X \to Z$ and $h : Z \to Y$ be any morphisms in $\mathcal{CHU}$. The diagram*

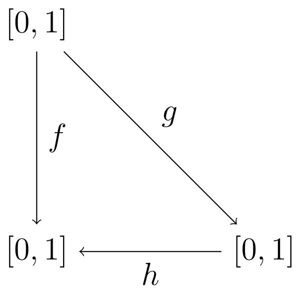


weakly commutes, if for any $x \in X$, $f(x) \subseteq (h \circ g)(x)$.

Example 5.2. *Let* $f : [0,1] \to 2^{[0,1]}$, $g : [0,1] \to 2^{[0,1]}$ *be identity functions on* $[0,1]$ *and let* $h : [0,1] \to 2^{[0,1]}$ *be defined by*

$$h(x) = [0,1]$$

for all $x \in [0,1]$. *Then the diagram*



weakly commutes but does not commute.

In the following definition we generalize the notion of inverse limits in $\mathcal{CHU}$.

Definition 5.3. *An object* $X \in Ob(\mathcal{CHU})$, *together with morphisms* $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$, *is a weak inverse limit of an inverse system*

$$(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})$$

in $\mathcal{CHU}$, *if*

1. *for all* $\alpha, \beta \in A$, *from* $\alpha \leq \beta$ *it follows that the diagram (1) weakly commutes;*

2. *for any object* $Y \in \mathcal{CHU}$ *and any family of morphisms* $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ *it follows that if the diagram (2) commutes, then for any morphism* $\Psi : Y \to X$ *such that for each* $\alpha \in A$ *and for each* $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X$ *holds true for all* $y \in Y$.

Note that each inverse limit in $\mathcal{CHU}$ is always a weak inverse limit in $\mathcal{CHU}$.

Example 5.4. *Let* $X = \varprojlim(\mathbb{N}, \{[0,1]\}_{k\in\mathbb{N}}, \{f_{k\ell}\}_{k,\ell\in\mathbb{N}})$ *be the inverse limit with usc set-valued bonding functions that we defined in Example 4.3. Then* X*, together with the projection mappings, is obviously not an inverse limit but it is a weak inverse limit in* $\mathcal{CHU}$*.*

We show in the following theorem that the inverse limits with upper semicontinuous set-valued bonding functions together with projections are always weak inverse limits in $\mathcal{CHU}$.

Theorem 5.5. *Let* $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ *be any inverse system in* $\mathcal{CHU}$*. Then the inverse limit with usc set-valued bonding functions*

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}),$$

together with projections

$$p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) \to X_\gamma,$$

$$p_\gamma((x_\alpha)_{\alpha\in A}) = \{x_\gamma\},$$

is a weak inverse limit of the inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ *in* $\mathcal{CHU}$*.*

Proof. Let $X = \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$. First we prove that the diagram (1) weakly commutes. Choose any $x \in X$ and let $\alpha < \beta$. Then $p_\alpha(x) = \{x_\alpha\} \subseteq f_{\alpha\beta}(\{x_\beta\}) = (f_{\alpha\beta} \circ p_\beta)(x)$.

Next, suppose that for an object $Y \in \mathcal{CHU}$ and a family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ the diagram (2) commutes. By Lemma 4.4, for any morphism $\Psi : Y \to X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma\in A} \varphi_\gamma(y)) \cap X$ holds true for all $y \in Y$. □

Acknowledgments

The authors thank Uroš Milutinović for constructive discussion.

This work was supported in part by the Slovenian Research Agency, under Grant P1-0285.

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